Cost of parabolic control

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Table of Contents

The control problem

[1] Let $n \geq 1$ and $T > 0$, Ω be a simply connected, bounded domain of $\mathbb{R}^n$ with smooth boundary Γ, $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \Gamma$:

$$\begin{cases} u_t - \Delta u = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x, 0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{1}$$

$1_\omega =$ characteristic function of the control subset ω of Ω.
We assume that $u^0 \in L^2(\Omega)$ and $f \in L^2(Q)$ so that (1) admits an unique solution

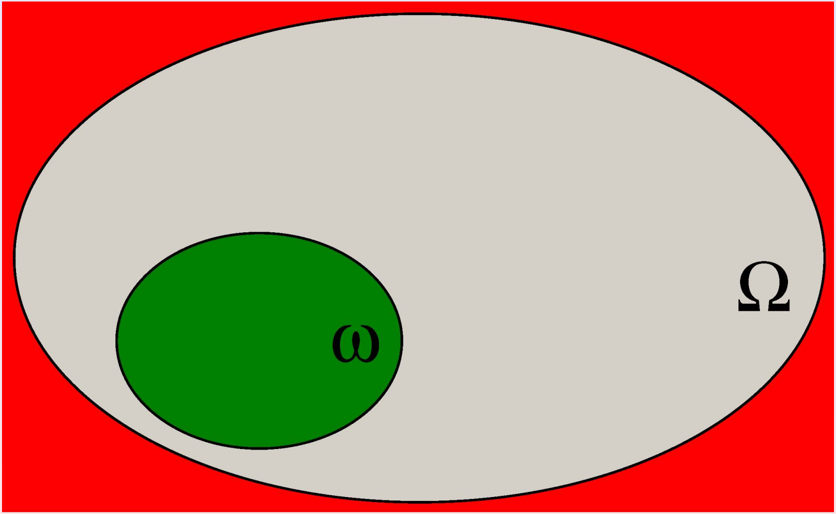
$$u \in C\left([0, T]\,; L^2(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right).$$

$$u = u(x, t) = solution = state, \; f = f(x, t) = control$$

Goal: To produce prescribed deformations on the solution u by means of suitable choices of the control function f s.t.

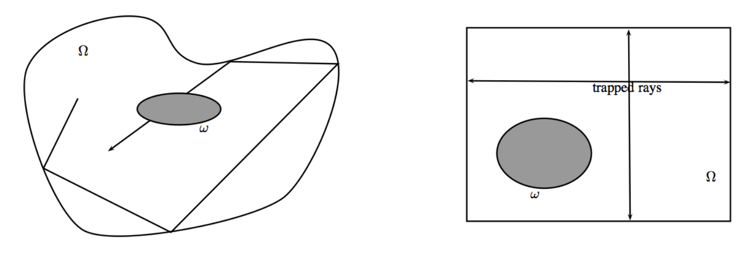
$$u(x, T) \equiv 0.$$

[1] EFC & EZ. ADE, 5 (4-6) (2000), 465–514.


ω
Ω

Due to the intrinsic infinite velocity of propagation of the heat equation we can expect the system to be controllable in any time $T > 0$ and from any open non-empty open subset ω of Ω.

Note that for similar properties to hold for wave equations, typically, one needs to impose geometric conditions on the control subset and the time of control, namely, the so called GCC (Geometric Control Condition) by Bardos-Lebeau-Rauch: It asserts, roughly, that all rays of geometric optics enter the control set ω in time T.



But this kind of Geometric Condition in unnecessary for the heat equation.

Null controllability is equivalent to an observability inequality:[2] More precisely, to an inequality of the form

$$\| \varphi(0) \|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega). \tag{2}$$

for the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega \end{cases} \tag{3}$$

This estimate, the so-called observability property, was proved by Imanuvilov-Fursikov and Lebeau-Robbiano in the 90's using Carleman inequalities. As pointed out by L. L. Lions, the control can be obtained minimising the functional ($f \equiv \varphi$, φ being the minimiser):

$$J(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \int_\Omega \varphi(0) u^0 dx. \tag{4}$$

[2] Norbert Wiener, Cybernetics: "The Science of communication and control in animals and machines" (or Frenchel-Rockafellar's duality).

Consider the heat equation or system with a potential $a = a(t, x)$ in $L^\infty(Q; \mathbb{R}^{N\times N})$[3]:

$$\varphi_t - \Delta\varphi + a\varphi = 0$$

where φ takes values in $\mathbb{R}^N$.

Theorem: (Fursikov+Imanuvilov, 1996, E. Fernández-Cara+E. Zuazua, 2000)

$$\| \varphi(T) \|^2_{(L^2(\Omega))^N} \leq \exp\left(C\left(1 + \frac{1}{T} + T \| a \|_\infty + \| a \|_\infty^{2/3} \right)\right) \int_0^T \int_\omega |\varphi|^2 dxdt,$$

for every solution φ, potential $a \in L^\infty(Q; \mathbb{R}^{N\times N})$ and time $T > 0$.

Note that

$$2/3 \in [1/2, 1]$$

Power 1 would be justified by Gronwall like arguments, and power 2 by the fact that the heat equation is second order in x. But why power 2/3?

[3] One of then main reasons to consider these zero order potential perturbations is to deal with semilinear problems by fixed point arfguments

The following Carleman inequality[4] holds:[5]
There exist positive constants $C_*, s_1 > 0$ such that

$$\frac{1}{s}\int_Q \rho^{-2s} t(T-t)\left[|q_t|^2 + |\Delta q|^2\right] dxdt \tag{5}$$
$$+s\int_Q \rho^{-2s} t^{-1}(T-t)^{-1}|\nabla q|^2 dxdt + s^3\int_Q \rho^{-2s} t^{-3}(T-t)^{-3} q^2 dxdt$$
$$\leq C_*\left[\int_Q \rho^{-2s}|\partial_t q - \Delta q|^2 dxdt + s^3\int_0^T\int_\omega \rho^{-2s} t^{-3}(T-t)^{-3} q^2 dxdt\right]$$

for all $q \in Z$ and $s \geq s_1$.
Moreover, C_* depends only on Ω and ω and s_1 is of the form

$$s_1 = s_0(\Omega,\omega)(T+T^2).$$

[4] T. Carleman, 1939

[5] Fursikov + Imanuvilov, 1996

The optimality of the $2/3$ exponent was proved by Th. Duyckaerts, X. Zhang and EZ (Annales IHP, 2005), based on the following result by V. Z. Meshkov, 1991.

Theorem

(Meshkov, 1991). *Assume that the space dimension is $n = 2$. Then, there exists a nonzero complex-valued bounded potential $q = q(x)$ and a non-trivial complex valued solution $u = u(x)$ of*

$$\Delta u = q(x)u, \qquad \text{in } \mathbb{R}^2, \tag{6}$$

with the property that

$$| u(x) | \leq C \exp(- | x |^{4/3}), \qquad \forall\, x \in \mathbb{R}^2 \tag{7}$$

for some positive constant $C > 0$.

Sketch of the proof of the optimality

Step 1: Construction on $\mathbb{R}^n$.

Given the solution u and potential q given by Meshkov, setting

$$u_R(x) = u(Rx), \qquad a_R(x) = R^2 q(Rx), \tag{8}$$

we get

$$\Delta u_R = a_R(x) u_R, \qquad \text{in } \mathbb{R}^n \tag{9}$$

and

$$| u_R(x) | \leq C \exp\left(-R^{4/3} | x |^{4/3}\right), \qquad \text{in } \mathbb{R}^n. \tag{10}$$

These functions may also be viewed as stationary solutions of the corresponding parabolic systems. Indeed, $\psi_R(t,x) = u_R(x)$, satisfying

$$\psi_{R,t} - \Delta\psi_R + a_R\psi_R = 0, \qquad x \in \mathbb{R}^n,\ t > 0 \tag{11}$$

$$| \psi_R(x,t) | \leq C \exp(-R^{4/3} | x |^{4/3}), \qquad x \in \mathbb{R}^n,\ t > 0. \tag{12}$$

$4/3 : 2 = 2/3$

Step 2: Restriction to Ω.

Without loss of generality (by translation and scaling) we can assume that $B \subset \Omega \backslash \bar{\omega}$ and

$$\begin{cases} \psi_{R,t} - \Delta \psi_R + a_R \psi_R = 0, & \text{in } Q, \\ \psi_R = \varepsilon_R, & \text{on } \Sigma, \end{cases} \tag{13}$$

where $\varepsilon_R = \psi_R \Big|_{\partial\Omega} = u_R \Big|_{\partial\Omega}$.

Taking into account that both ω and $\partial\Omega \subset B^c$ for a suitable C:

$$| \psi_R(t,x) | \leq C \exp\left(-R^{4/3}\right), \quad x \in \omega, \, 0 < t < T,$$

$$| \varepsilon_R(t,x) | \leq C \exp\left(-R^{4/3}\right), \qquad x \in \partial\Omega, \, 0 < t < T$$

$$\| \psi_R(T) \|^2_{L^2(\Omega)} \sim \| \psi_R(T) \|^2_{L^2(\mathbb{R}^n)} = \| u_R \|^2_{L^2(\mathbb{R}^n)} = \frac{1}{R^n} \| u \|^2_{L^2(\mathbb{R}^n)} = \frac{c}{R^n}$$

$$\| a_R \|_{L^\infty(\Omega)} \sim \| a_R \|_{L^\infty(\mathbb{R}^n)} = CR^2.$$

We can then correct these solutions to fulfill the Dirichlet homogeneous boundary condition.

Convective potentials

Equations or systems with convective potentials of the form

$$\varphi_t - \Delta\varphi + W \cdot \nabla\varphi = 0.$$

For these equations the observability inequality reads:[6]

$$\| \varphi(T) \|^2_{(L^2(\Omega))^N} \leq \exp\left(C\left(1 + \frac{1}{T} + T \| W \|_\infty + \| W \|^2_\infty\right)\right) \int_0^T \int_\omega |\varphi|^2 dxdt.$$

Optimality can be shown by a Meshkov like construction:

$$-\Delta u = W(x) \cdot \nabla u$$

with the same u decaying as $\exp\left(-|x|^{4/3}\right)$ and the potential $W(x)$ such that

$$(|x| + 1)^{1/3} |W(x)| \leq C.$$

[6] A. Doubova, EFC, M. González-Burgos & EZ, SICON, 2002

Consider semilinear parabolic equation of the form

$$\begin{cases} y_t - \Delta y + g(y) = f 1_\omega & \text{in} \quad \Omega \times (0, T) \\ y = 0 & \text{on} \quad \partial\Omega \times (0, T) \\ y(x, 0) = y^0(x) & \text{in} \quad \Omega. \end{cases} \tag{14}$$

Theorem

(E. Fernández-Cara + EZ, Annales IHP, 2000) The semilinear system is null controllable if

$$g(s) / \mid s \mid \log^{3/2} \mid s \mid \to 0 \ \text{as} \ \mid s \mid \to \infty. \tag{15}$$

- Note that blow-up phenomena occur if

$$g(s) \sim | s | \log^p(1+ | s |), \text{ as } | s | \to \infty$$

with $p > 1$.

- Thus, in particular, weakly blowing-up equations may be controlled.
- On the other hand, it is also well known that blow-up may not be avoided when $p > 2$ and then control fails.
- Note that in the control process the propagation of energy in the x direction plays a key role. When viewing the underlying elliptic problem $\Delta y + g(y)$ a a second order differential equation in x we see how the critical exponent $p = 2$ arises. For $p > 2$ concentration in space may occur so that the control may not avoid the blow-up to occur outside the control region ω.

Sketch of the proof. Linearization + fixed point.

$$y_t - \Delta y + h(z)y = f1_\omega$$

$$h(z) = g(z)/z.$$

Note that, if $z = y$, $h(z)y = g(y)$. In that case solutions of the linearized system are also solutions of the semilinear one.
The cost of controlling the system is of the form:

$$||f|| \leq ||y^0|| \exp\left(C\left(1 + \frac{1}{T} + T \parallel g(z) \parallel_\infty + \parallel g(z) \parallel_\infty^{2/3}\right)\right).$$

But $g(z) \sim \log^p(z)$. Thus

$$||f|| \leq ||y^0|| \exp\left(C\left(1 + \frac{1}{T} + T \log^p(||z||) + \log^{2p/3}(||z||)\right)\right).$$

When $||z||$ is large the term $\log^p(||z||)$ dominates but can be compensated by taking T small enough.

Control quickly ! ($\sim$ rapidito)

For the heat equation

$$-\varphi_t - \Delta\varphi = 0$$

we have

$$\int_\Omega \exp(-A/(T-t))\varphi^2 dxdt \leq C \int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega)$$

which is much stronger than

$$\| \varphi(0) \|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega).$$

- This inequality is sharp!
- What is the best constant $A > 0$? How does it depend on the geometry of Ω and ω?

This kind of exponential weight is sharp

The " monster"

$$u(x,t) = \cos\left(\frac{Ax_1}{2t}\right) \exp(A^2/4t)\, G(x,t)\,, \tag{16}$$

where G is the fundamental solution of the heat equation in $\mathbf{R}^d$, i.e.

$$G(x,t) = (4\pi t)^{-d/2} \exp(-|x|^2/4t) \quad \forall (x,t) \in \mathbf{R}^d \times \mathbf{R}^+. \tag{17}$$

solves the heat equation:

$$u_t - \Delta u = 0 \;\text{ in }\; \mathbf{R}^d \times \mathbf{R}^+\,. \tag{18}$$

Note that u is the real part of the inverse Fourier transform of $e^{A\xi_1}\widehat{G}$. Accordingly, u can be viewed as a derivative of infinite order of the fundamental solution G.

The monster = $[\text{Kannai}]^{-1}$

The **Kannai transform** allows transferring the results we have obtained for the wave equation to other models and in particular to the heat equation (Y. Kannai, 1977; K. D. Phung, 2001; L. Miller, 2004)

$$e^{t\Delta}\varphi = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-s^2/4t} W(s) ds$$

where $W(x,s)$ solves the corresponding wave equation with data $(\varphi, 0)$.

$$W_{ss} + AW = 0 \quad + \quad K_t - K_{ss} = 0 \quad \to \quad U_t + AU = 0,$$

The reverse Kannai transform:[7] Our proof is based on an inverse Kannai transform that, to the best of our knowledge, was unknown until now:

$$W(s) = \int_{\mathbf{R}_+} \frac{1}{(4\pi t)^{1/2}} \sin\left(\frac{sS}{2t}\right) \exp\left(\frac{s^2 - S^2}{4t}\right) U(t)\, dt.$$

[7] S. Ervedoza & EZ, ARMA, 2011

Null controls for the heat equation are hard to compute because of the very strong irreversibilty of the system, the very weak observability inequalities, etc.
R. Glowinski and J. L. Lions.[8] propose a remedy based on Tychonnoff regularization. It consists on adding a regularizing term to the functional to be minimized (or its discrete version):

$$J_0(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \int_\Omega \varphi(0)u^0 dx. \tag{19}$$

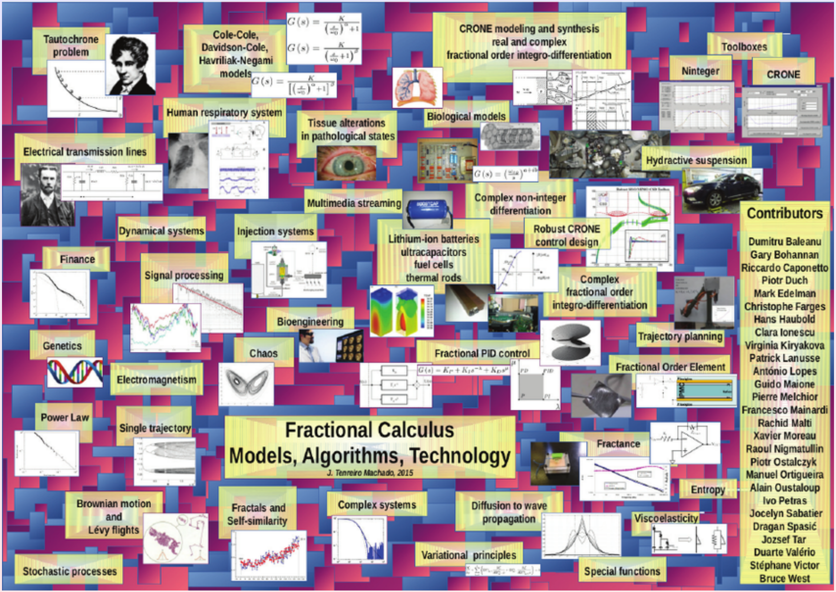
Namely:

$$J_0^\varepsilon(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \frac{\varepsilon}{2}||\varphi^0||_{L^2}^2 + \int_\Omega \varphi(0)u^0 dx. \tag{20}$$

In our paper with EFC we show that the convergence of minimisers as $\varepsilon \to 0$ is very slow: logarithmic convergence rate.

[8] R. Glowinski and J.L. Lions, Acta Numerica, 1996.

Table of Contents


Tautochrone problem
Cole-Cole, Davidson-Cole, Havriliak-Negami models
CRONE modeling and synthesis real and complex fractional order integro-differentiation
Toolboxes
Ninteger
CRONE
Human respiratory system
Tissue alterations in pathological states
Biological models
Electrical transmission lines
Hydractive suspension
Multimedia streaming
Complex non-integer differentiation
Robust CRONE control design
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Ivo Petras
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Dragan Spasić
Jozsef Tar
Duarte Valério
Stéphane Victor
Bruce West
Dynamical systems
Injection systems
Lithium-ion batteries ultracapacitors fuel cells thermal rods
Finance
Signal processing
Complex fractional order integro-differentiation
Bioengineering
Trajectory planning
Genetics
Chaos
Fractional PID control
Fractional Order Element
Electromagnetism
Power Law
Single trajectory
Fractional Calculus
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J. Tenreiro Machado, 2015
Fractance
Entropy
Brownian motion and Lévy flights
Fractals and Self-similarity
Complex systems
Diffusion to wave propagation
Viscoelasticity
Variational principles
Stochastic processes
Special functions

Consider the following controlled heat equation involving nonlocal in space terms:[9]

$$\begin{cases} y_t - \Delta y + \int_\Omega K(x,\xi) y(\xi,t)\, d\xi = v 1_\omega & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x) & \text{in} \quad \Omega \end{cases} \tag{21}$$

And let us analyse its null controllability property, which is equivalent to the observability inequality

$$\left|\phi(\cdot\,,0)\right|^2 \leq C \int_{\omega\times(0,T)} |\phi|^2\, dx\, dt \qquad \forall \phi^T \in L^2(\Omega) \tag{22}$$

for the solutions of the adjoint system

$$\begin{cases} -\phi_t - \Delta\phi + \int_\Omega K(\xi,x) \phi(\xi,t)\, d\xi = 0 & \text{in} \quad Q, \\ \phi = 0 & \text{on} \quad \Sigma, \\ \phi(x,T) = \phi^T(x) & \text{in} \quad \Omega. \end{cases} \tag{23}$$

[9] E. Fernández-Cara, Q. Lü and EZ, SICON, 2016.

Recall that

$$s^3 \int_Q \rho^{-2s} t^{-3}(T-t)^{-3} q^2 dxdt$$

$$\leq C_* \left[\int_Q \rho^{-2s} \left| \partial_t q - \Delta q \right|^2 dxdt + s^3 \int_0^T \int_\omega \rho^{-2s} t^{-3}(T-t)^{-3} q^2 dxdt \right]$$

which, when applied to our nonlocal model yields,

$$s^3 \int_Q \rho^{-2s} t^{-3}(T-t)^{-3} \phi^2 dxdt$$

$$\leq C_* \int_Q \rho^{-2s} \left| \int_\Omega K(\xi, x)\phi(\xi, t)\, d\xi \right|^2 dxdt + \dots.$$

The non-local second term in the right hand side cannot be absorbed by the left hand side.

These difficulties do not arise when dealing with classical potential terms acting locally in space, i. e. for equations of the form

$$-\phi_t - \Delta\phi + K(x, t)\phi(x, t) = 0.$$

Fourier version of the sharp observability inequality

Recall that

$$\int_\Omega \exp(-A/(T-t))\varphi^2 dxdt \leq C \int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega)$$

for the solutions of the adjoint heat equation

$$-\varphi_t - \Delta\varphi = 0.$$

This inequality can be rewritten in terms of the Fourier coefficients $\hat{\varphi}_k^T$ of the final datum φ^T of the adjoint state:[10]

$$||\varphi^T||_F^2 = \sum_{k\geq 1} |\varphi_k^T|^2 \exp(-B\sqrt{\lambda_k}) \leq C \int_0^T \int_\omega \varphi^2 dxdt.$$

This provides the functional setting to work in this nonlocal problem from a Fourier viewpoint.

[10] E. Fernández-Cara & E. Z. in ADE, 2000.

Denote by $\lambda_1, \lambda_2, \dots$ (resp. $w_1, w_2, \dots$) the eigenvalues (resp. the unit L^2 norm eigenfunctions) of the Dirichlet Laplacian in Ω. Recall that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ \cdots$, $\lambda_m \sim m^{2/N}$ as $m \to +\infty$ and $\phi_1 > 0$ in Ω.

We impose the following conditions on the kernel $K \in L^2(\Omega \times \Omega)$:

- $$x \mapsto \int_\Omega K(\xi, x) f(\xi)\, d\xi \text{ is analytic for all } f \in L^2(\Omega) \tag{24}$$
- $$\begin{cases} K(x,\xi) = \sum_{m,j\geq 1} k_{mj} w_m(x) w_j(\xi) \quad \text{in} \quad L^2(\Omega\times\Omega), \quad \text{with} \\ |K|_R^2 \triangleq \sum_{m\geq 1} \left(\sum_{j\geq 1} \lambda_j^{-1} |k_{mj}|^2 \right) \lambda_m^{-1} e^{2R\sqrt{\lambda_m}} < +\infty. \end{cases} \tag{25}$$

The norm $||\cdot||_F$ provides however a functional setting in which the non-local lower order term can be treated as a compact perturbation of the free dynamics.
For any $\phi^T \in L^2(\Omega)$, denote by ϕ the solution to (23) and write

$$\Phi = p + \zeta,$$

where p is the unique solution to

$$\begin{cases} -p_t - \Delta p = 0 & \text{in} \quad Q, \\ p = 0 & \text{on} \quad \Sigma, \\ p(x,T) = \phi^T(x) & \text{in} \quad . \end{cases} \tag{26}$$

and

$$\begin{cases} -\zeta_t - \Delta\zeta + \int_\Omega K(\xi,x)\zeta(\xi,t)d\xi = -\int_\Omega K(\xi,x)p(\xi,t)d\xi & \text{in} \quad Q, \\ \zeta = 0 & \text{on} \quad \Sigma, \\ \zeta(x,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{27}$$

In this functional setting (exponentially weighted Fourier norm) and under the previous analyticity assumptions on the nonlocal potential, the reminder term ζ can be shown to be a compact perturbation.
Compactness-uniqueness arguments can be developed, reducing the observability inequality for the nonlocal problem to an unique continuation problem.
Can one guarantee that the unique eigenfunction

$$-\Delta\Psi + \int_{\Omega} K(\xi, x)\Psi(\xi)d\xi = \lambda\Psi$$

such that

$$\Psi(x) = 0 \quad \text{in} \quad \omega$$

is the null one, $\Psi \equiv 0$?

This can be easily achieved under the assumption that the kernel K depends analytically on x.
What other results can be expected in that respect?

For the wave equation:

$$\begin{cases} y_{tt} - \Delta y + \int_\Omega K(x,\xi) y(\xi,t)\, d\xi = v 1_\omega & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(x,0) = z^0(x), \quad y_t(x,0) = z^1(x) & \text{in} \quad . \end{cases} \tag{28}$$

the same arguments apply but, this time, milder assumptions on the Fourier coefficients of the kernel are needed since the perturbation argument can be developed in the standard energy space.

Note however that the analyticity of the kernel with respect to x is needed for unique continuation to hold.