SOME COMMENTS ON NONLINEAR CONTROL PROBLEMS

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Our discussion on controllability and optimal controls of linear systems can be extended to nonlinear problems. This is relevant in Mechanics and Machine Learning.

Consider the system

$$\begin{cases} x'(t) = f(x(t), u(t)), & 0 < t < T \\ x(0) = x_0 \end{cases}$$

Here

$$x = x(t) \in \mathbb{R}^n \to \text{state}, \quad u = u(t) \in \mathbb{R}^m \to \text{control}.$$

We assume, to simplify the presentation, that f is globally Lipschitz so that the existence and uniqueness of solutions holds. The goal is to find $u = u(t)$, the control, such that

$$x(T) \sim x_T.$$

This can be done minimizing a functional of the form:

$$\min_{u \in H^1(0,T;\mathbb{R}^m)} \frac{1}{2} \|u\|^2_{H^1(0,T;\mathbb{R}^m)} + \frac{1}{2} \|x(T) - x_T\|^2_{\mathbb{R}^n} = J(u)$$

Thanks to the compactness of the Sobolev embedding

$$H^1(0,T;\mathbb{R}^m) \hookrightarrow L^2(0,T;\mathbb{R}^m)$$

the existence of an optimal control is guaranteed by the DMCV.

Note that, due to the nonlinear dependence "$u \to x$" the functional J is not convex. So the uniqueness of the minimizer cannot be guaranteed. In fact the functional may have also local minima. Therefore Gradient Descent Algorithms can lead to local minima without reaching the global minimum. The later will require the combination of gradient methods and random search procedures.

The uniqueness of the global minimizer is not a major problem in principle, if one is able to compute one of them. But the lack of convexity, obviously, is a major drawback.

Remark. A note on the convexity of compositions.

Assume $\phi : \mathbb{R}^d \to \mathbb{R}$ convex

$$h : \mathbb{R}^d \to \mathbb{R}^d \text{ smooth}$$

Then $\phi \circ h$ can only be guaranteed to be convex when h is linear or when both ϕ and h are convex and ϕ increasing:

$$[(\phi \circ h)(x)]'' = \phi''(h(x))\,|h'(x)|^2 + \phi'(h(x))\,h''(x).$$

Taking into account that the goal is

$$x(T) \sim x_T$$

it is natural to penalize the functional J to be minimized to enhance this goal minimizing

$$J_k(u) = \frac{1}{2}\|u\|^2_{H^1(0,T;\mathbb{R}^m)} + \frac{k}{2}\|x(T) - x_T\|^2$$

with $k >> 1$.

The relation between the behaviour of minimizers u_k as $k \to \infty$ and the controllability of the problem, i.e. the possibility of actually achieving $x(T) = x_T$, is similar as in the linear case.

In practical applications, the uniform boundedness of the controls u_k as $k \to \infty$ can be viewed as a test of the controllability of the system.

Recall that a system that is not controllable takes, in a suitable basis, the form

$$\begin{cases} x_1' + A_1 x_1 = 0 & , 0 < t < T \\ x_1(0) = x_{0,1} \\ x_2' + A_2 x_2 = Bu, & 0 < t < T \\ x_2(0) = x_{0,2} \end{cases}$$

with $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ and $n_1 + n_2 = n$, so that the component x_2 is controllable but x_1 lacks of controllability because the control $u = u(t)$ does not actually act on it.

The same occurs for nonlinear systems that would take the form

$$\begin{cases} x_1'(t) = g(x_1(t)), \quad 0<t<T \\ x_1(0) = x_{0,1} \\ x_2'(t) = h(x_2(t), u(t)), \quad 0<t<T \\ x_2(0) = x_{0,2}. \end{cases}$$

Let us now discuss briefly the controllability of

$$\begin{cases} x'(t) = f(x(t), u(t)), \quad 0<t<T \\ x(0) = x_0. \end{cases}$$

First, using linearization arguments we can analyze local controllability properties. For, we consider any reference trajectory. In other words, given $\bar{u}(t) \in L^2(0,T;\mathbb{R}^m)$ we consider the corresponding solution $\bar{x}(t) \in C([0,T];\mathbb{R}^n)$ with

$$\bar{x}_0 = \bar{x}(0), \quad \bar{x}_T = \bar{x}(T).$$

We now proceed by linearization.

$$u = \bar{u} + \varepsilon v$$

$$x = \bar{x} + \varepsilon y.$$

We then get, as first order approximation of (u, x) when $\varepsilon \to 0$ the linearized system

$$\begin{cases} y' = f_x(\bar{x}(t), \bar{u}(t))y + f_u(\bar{x}(t), \bar{u}(t))v \\ y(0) = y_0 \end{cases}$$

which takes the form

$$\begin{cases} y' = A(t)\,y + B(t)\,v, & 0<t<T \\ y(0) = y_0 \end{cases}$$

with

$$A(t) = f_x(\bar{x}(t), \bar{u}(t)), \quad B(t) = f_u(\bar{x}(t), \bar{u}(t)).$$

This is a linear non-autonomous control system.

When the reference trajectory $(\bar{u}, \bar{x})$ is a steady-state

$$\bar{x}(t) \equiv \bar{x}, \quad \bar{u}(t) \equiv \bar{u} \quad : \quad f(\bar{x}, \bar{u}) = 0$$

then $\quad A(t) \equiv A \quad$ and $\quad B(t) \equiv B.$

If (A, B) fulfill the Kalman rank condition the linearized system is controllable and, accordingly, using perturbation arguments, one deduces that the non-linear one is locally controllable: There exists $\varepsilon_T > 0$ and $\delta_T > 0$ such that the nonlinear system can be driven from any point in an ε_T-neighborhood of $\bar{x}$ to any point on it by control in a δ_T-neighborhood of $\bar{u}$.

But, in general, when the reference trajectory $(\bar{u}(t), \bar{x}(t))$ the linear system under consideration

$$x'(t) = A(t)\,x(t) + B(t)\,u(t)$$

is such that

$A(t)$ = depends on t

$B(t)$ = depends on t

How can be extend the (A, B) controllability theory to these time-dependent systems?

The duality theory applies.

Consider the adjoint system:

$$-\varphi' + A(t)^* \varphi = 0, \quad \varphi(T) = \varphi_T.$$

Assume that the following unique continuation condition

$$B^*(t)\,\varphi(t) = 0, \quad \forall\, 0 < t < T$$

$$\Rightarrow \varphi_T = 0$$

holds.

This can be understood in a purely algebraic manner, for instance, when $A(t)$ and $B(t)$ are analytic with respect to t, using Cauchy theory of power series expansions.

Once this done the minimization of a functional of the form

$$J(\varphi_T) = \frac{1}{2}\int_0^T |B^*\varphi(t)|^2\,dt + \langle x_T, \varphi_T \rangle - \langle x(0), \varphi(0) \rangle$$

leads to the control

$$\hat{u}(t) = B^*\hat{\varphi}(t)$$

of minimal L^2-norm.

This allows to guarantee the controllability of the linearized system.

Nonlinear perturbation arguments allow showing that for the nonlinear system any initial datum in an ε_T-neighborhood of $\bar{x}(0)$ can be driven to any final datum in a ε_T-neighborhood of $\bar{x}(T)$ with controls $u(t)$ in an δ_T-neighborhood of $\bar{u}(t)$.

Iterative arguments can allow to build global results. But we doing that the time-horizon for control need, T, is often large: $T \gg 1$.

Note however that for systems of the form

$$x'(t) = u(t) f(x(t))$$

when the control $u = u(t)$ enters multiplying the nonlinearity $f(x)$, the time of control is irrelevant.

Indeed, by scaling it can be shown that what the control $u = u(t)$ achieves in time $[0, T]$, the control $Tu(t)$ achieves in time $[0, 1]$.