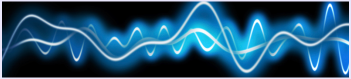

PDE control: Constraints, short and long time horizons

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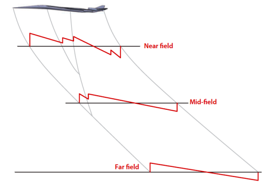
Sonic boom

- Goal: the development of supersonic aircrafts, sufficiently quiet to be allowed to fly supersonically over land.
- The pressure signature created by the aircraft must be such that, when reaching ground, (a) it can barely be perceived by humans, and (b) it results in admissible disturbances to man-made structures.



Juan J. Alonso and Michael R. Colonno, Multidisciplinary Optimization with Applications to Sonic-Boom Minimization, Annu. Rev. Fluid Mech. 2012, 44:505 – 526.

More generally

- It is a well acknowledged fact that, whenever a system admits a property of asymptotic stability in long time, control mechanisms should inherit it.

  Is it really true?
- In particular, it is often "assumed" that, when trajectories of the free dynamics converge to steady states, time evolution control problems should be attracted by the corresponding steady-state version.

  Is it really true?
- How does this fact depend on the model under consideration? Does it depend on the type of control problem?
- Often times optimal shape design problems in aeronautics and elasticity are addressed in a steady-context. Is this model reduction justified?

From a practical viewpoint

- When building and optimal control or design mechanism, it is natural to simplify it to first consider the steady-state version.
- To which extent and in which regimes is the time-evolution control problem approximated by the steady-state optimal control/design?

In this lecture we show that the answer depends very heavily on:

1. Controllability properties,
2. The criterion employed to define optimal controls.

Problem synthetic reformulation

$$T \to \infty + \text{Control} = \text{Control} + T \to \infty \quad ?$$

Note that this commutativity is well-known to fail in some particular instances such as when performing numerical approximations[a].

[a] E. Z. Propagation, observation, and control of waves approximated by finite difference methods. SIAM Review, 47 (2) (2005), 197-243.

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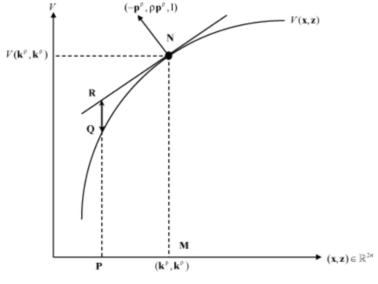
Origins

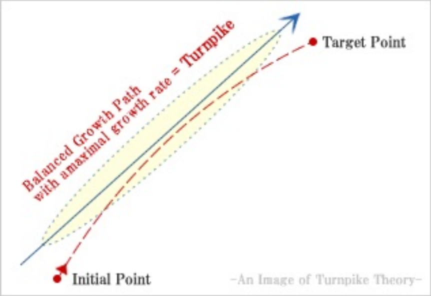
Although the idea goes back to John von Neumann in 1945, Lionel W. McKenzie traces the term to Robert Dorfman, Paul Samuelson, and Robert Solow's "Linear Programming and Economics Analysis" in 1958, referring to an American English word for a Highway:

> *... There is a fastest route between any two points; and if the origin and destination are close together and far from the turnpike, the best route may not touch the turnpike. But if the origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end.*

Mainly motivated by applications to economic models and game theory there is a literature concerned with this kind of stationary behavior in the transient time for long horizon control problems. In that context, such type of result goes under the name of *turnpike theory* which was mostly investigated in the finite dimensional case.



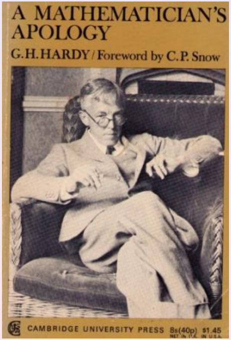
-Figure 1: Value-loss -



A. J. Zaslavski, *Turnpike properties in the calculus of variations and optimal control*. Nonconvex Optimization and its Applications, 80. Springer, New York, 2006.

A mathematician's apology

We are mainly motivated by PDE control and design applications.
A number of model cases have been well understood. But there is still a long way to go...



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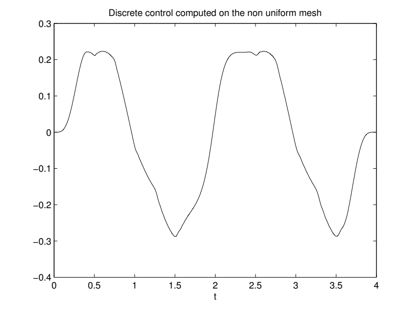
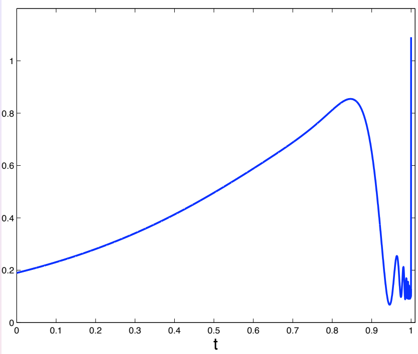
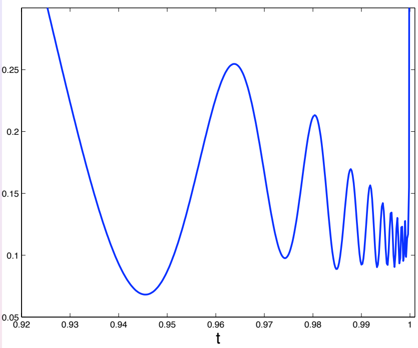
The wave equation



Typical controls for the wave equation exhibit an oscillatory behaviour, and this independently of the length of the control time-horizon.
But nobody would be surprised about this fact. It looks like intrinsically linked to the oscillatory (even periodic in some particular cases) nature of the wave equation solutions.

The heat equation





Typical controls for the heat equation exhibit **unexpected** oscillatory and concentration effects. This was observed by R. Glowinski and J. L. Lions in the 80's in their works in the numerical analysis of controllability problems for heat and wave equations.
Why?

Optimal controls are normally characterised as traces of solutions of the **adjoint problem** through the optimality system or the Pontryagin Maximum Principle, and solutions of the adjoint system of the heat equation

$$-p_t - \Delta p = 0,$$

look precisely this way.

Large and oscillatory near $t = T$ they decay and get smoother when t gets down to $t = 0$. And this is independent of the time control horizon $[0, T]$.

First conclusion:
Typical control problems for wave and heat equations do not seem to exhibit the turnpike property.

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The control problem

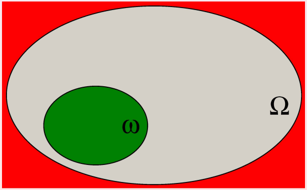
Let $n \geq 1$ and $T > 0$, Ω be a simply connected, bounded domain of $\mathbb{R}^n$ with smooth boundary Γ, $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \Gamma$:

$$\begin{cases} y_t - \Delta y = f 1_\omega & \text{in} \quad Q \\ y = 0 & \text{on} \quad \Sigma \\ y(x,0) = y^0(x) & \text{in} \quad \Omega. \end{cases} \tag{1}$$

1_ω = the characteristic function of ω of Ω where the control is active. We assume that $y^0 \in L^2(\Omega)$ and $f \in L^2(Q)$ so that (15) admits a unique solution

$$y \in C\left([0,T]; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right).$$

$$y = y(x,t) = \textit{solution} = \textit{state}, \ f = f(x,t) = \textit{control}$$



Well known result (Fursikov-Imanuvilov, Lebeau-Robbiano,...) : The system is null-controllable in any time T and from any open non-empty subset ω of Ω.

The control of minimal L^2-norm can be found by minimizing

$$J_0(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \int_\Omega \varphi(0)u^0 dx \tag{2}$$

over the space of solutions of the adjoint system:

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in } Q \\ \varphi = 0 & \text{on } \Sigma \\ \varphi(T,x) = \varphi^0(x) & \text{in } \Omega. \end{cases} \tag{3}$$

Obviously, the functional is continuous and convex from $L^2(\Omega)$ to $\mathbb{R}$ and coercive because of the observability estimate:

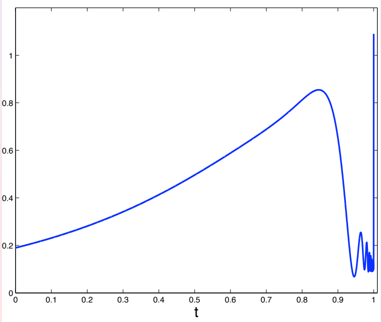
$$\| \varphi(0) \|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega). \tag{4}$$

If $\bar{\varphi}^0$ is the minimiser of the functional J, the needed control is given by

$$f = \bar{\varphi}$$

where $\bar{\varphi}$ is the solution of the adjoint heat equation corresponding to the minimiser $\bar{\varphi}^0$.

And, because of this, we observe the tendency of the control to concentrate all the action in the final time instant $t = T$.



But this is so for the control of minimal L^2-norm for which the Optimality System (OS) reads:

$$y_t - \Delta y = \varphi 1_\omega \text{ in } Q$$

$$y = 0 \text{ on } \Sigma$$

$$y(x,0) = y^0(x) \text{ in } \Omega$$

$$y(x,T) = 0 \text{ in } \Omega$$

$$-\varphi_t - \Delta\varphi = 0 \text{ in } Q$$

$$\varphi = 0 \text{ on } \Sigma.$$

Note that the fact that the adjoint state φ appears isolated as the solution of the adjoint equation induces this unexpected behavior and the tendency to concentrate action at $t = T$.

Better balanced controls

Let us now consider the control f minimising a compromise between the norm of the state and the control among the class of admissible controls:

$$min \frac{1}{2}\Big[\int_0^T\int_\Omega |y|^2 dxdt + \int_0^T\int_\omega |f|^2 dxdt\Big].$$

Then the Optimality System reads

$$y_t - \Delta y = -\varphi 1_\omega \text{ in } Q$$

$$y = 0 \text{ on } \Sigma$$

$$y(x,0) = y^0(x) \text{ in } \Omega$$

$$y(x,T) = 0 \text{ in } \Omega$$

$$-\varphi_t - \Delta\varphi = y \text{ in } Q$$

$$\varphi = 0 \text{ on } \Sigma.$$

We now observe a coupling between φ and y on the adjoint state equation!

New Optimality System Dynamics

What is the dynamic behaviour of solutions of the new fully coupled OS?
For the sake of simplicity, assume $\omega = \Omega$.
The dynamical system now reads

$$y_t - \Delta y = -\varphi$$

$$\varphi_t + \Delta \varphi = -y$$

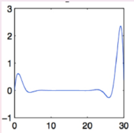
This is a forward-backward parabolic system.
A spectral decomposition exhibits the characteristic values

$$\mu_j^{\pm} = \pm\sqrt{1+\lambda_j^2}$$

where $(\lambda_j)_{j\geq 1}$ are the (positive) eigenvalues of $-\Delta$.
Thus, the system is the superposition of growing + diminishing real exponentials.

The turnpike property for the heat equation

This new dynamic behaviour, combining exponentially stable and unstable branches, is compatible with the turnpike behavior.
Controls and trajectories exhibit the expected dynamics:



The turnpike property for the wave equation

But this relevant fact, so that modifying the optimality criterion for the choice of the control, ensures the turnpike property, is not intrinsic to the heat equation.

The same applies for the **wave equation**: The control and controlled trajectories are close to the steady state ones during most of the time interval of control when $T >> 1$.

M. Gugat, E. Trélat, E. Zuazua, Systems and Control Letters, 90 (2016), 61-70.

What is behind?

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Linear abstract theory. Joint work with A. Porretta, SIAM J. Cont. Optim., 2013.

Consider the finite dimensional dynamics

$$\begin{cases} x_t + Ax = Bu \\ x(0) = x_0 \in \mathbf{R}^N \end{cases} \tag{5}$$

where $A \in M(N, N)$, $B \in M(N, M)$, with control $u \in L^2(0, T; \mathbf{R}^M)$. Given a matrix $C \in M(N, N)$, and some $x^* \in \mathbf{R}^N$, consider the optimal control problem

$$\min_u \; J^T(u) = \frac{1}{2} \int_0^T (|u(t)|^2 + |C(x(t) - x^*)|^2) dt \, .$$

There exists a unique optimal control $u(t)$ in $L^2(0, T; \mathbf{R}^M)$, characterized by the optimality condition

$$u = -B^* p \, , \qquad \begin{cases} -p_t + A^* p = C^* C(x - x^*) \\ p(T) = 0 \end{cases} \tag{6}$$

The steady state control problem

The same problem can be formulated for the steady-state model

$$Ax = Bu.$$

Then there exists a unique minimum $\bar{u}$, and a unique optimal state $\bar{x}$, of the stationary control problem

$$\min_u \; J_s(u) = \frac{1}{2}(|u|^2 + |C(x - x^*)|^2), \qquad Ax = Bu, \tag{7}$$

which is nothing but a constrained minimization in $\mathbf{R}^N$.
The optimal control $\bar{u}$ and state $\bar{x}$ satisfy

$$A\bar{x} = B\bar{u}, \quad \bar{u} = -B^*\bar{p}, \qquad \text{and} \quad A^*\bar{p} = C^*C(\bar{x} - x^*).$$

We assume that

$$\text{The pair } (A, B) \text{ is controllable,} \tag{8}$$

or, equivalently, that the matrices A, B satisfy the Kalman rank condition

$$Rank \left[B \ AB \ A^2B \dots A^{N-1}B \right] = N \, . \tag{9}$$

Concerning the cost functional, we assume that the matrix C is such that (void assumption when $C = Id$)

$$\text{The pair } (A, C) \text{ is observable} \tag{10}$$

which means that the following algebraic condition holds:

$$Rank \left[C \ CA \ CA^2 \dots CA^{N-1} \right] = N \, . \tag{11}$$

Under the above controllability and observability assumptions, we have the following result.

Theorem

For some $\gamma > 0$ *for* $T > 0$ *large enough we have*

$$\int_{aT}^{bT} \left(\left| u - \bar{u} \right|^2 + \left| x - \bar{x} \right|^2 \right) ds \leq K \left(e^{-\gamma a T} + e^{-\gamma (1-b) T} \right)$$

for every $a, b \in [0, 1]$.

Proof

Step 1: A dissipativity identity. We have

$$[(x-\bar{x})(p-\bar{p})]_t = -\Big[B^*(p-\bar{p})|^2 + |C(x-\bar{x})|^2\Big]$$

as a direct consequence of

$$\begin{cases} (x-\bar{x})_t + A(x-\bar{x}) = B(u-\bar{u}) \\ u-\bar{u} = -B^*(p-\bar{p}) \\ -(p-\bar{p})_t + A^*(p-\bar{p}) = C^*C(x-\bar{x}). \end{cases}$$

Step 2. Decay for correlations.

Following [CLLP][1], if B^* and C are coercive[2] we also have

$$|B^*(p-\bar{p})|^2 + |C(x-\bar{x})|^2 \geq \gamma\left(|p-\bar{p}|^2 + |x-\bar{x}|^2\right).$$

Hence

$$[(x-\bar{x})(p-\bar{p})]_t = -|B^*(p-\bar{p})|^2 - |C(x-\bar{x})|^2 \leq -\gamma|(x-\bar{x})(p-\bar{p})|,$$

for some $\gamma > 0$.
Hence,

$$-Ke^{-\gamma(T-t)} \leq [(x-\bar{x})(p-\bar{p})](t) \leq Ke^{-\gamma t}$$

if $(x-\bar{x})(p-\bar{p})$ is bounded at $t=0$ and $t=T$.

[1] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions, A. Porretta, *Long time average of Mean Field Games*, Network Heterogeneous Media, 7 (2), 2012.

[2] These conditions can be relaxed under the controllability-observability conditions above.

Step 3. Convergence of averages.

In fact, the bounds on the extremal values at $t = 0$ and $T = T$ immediately yields the turnpike property in an averaged sense.
Indeed, as a consequence of the identity,

$$\int_0^T \left(|u - \bar{u}|^2 + |C(x - \bar{x})|^2\right) dt = [(x_0 - \bar{x})(p(0) - \bar{p})] - [(x(T) - \bar{x})\bar{p}]$$

and the bounds at the extremal values $t = 0$ and $t = T$ we then have

$$\int_0^T \left(|u - \bar{u}|^2 + |C(x - \bar{x})|^2\right) dt \leq C \qquad (12)$$

with C independent of T and

$$\frac{1}{T} \int_0^T \left(|u - \bar{u}|^2 + |C(x - \bar{x})|^2\right) dt \leq \frac{C}{T} \to 0\,.$$

This, of course, also implies the convergence of the averaged minima to the stationary minimum.

Step 4. Bounds on the extremal values.

Using the observability inequality of the pair (A^*, B^*) we have

$$|(p(0)-\bar{p})| \leq c \left[\left(\int_0^T |C(x-\bar{x})|^2 \, dt \right)^{\frac{1}{2}} + \left(\int_0^T |B^*(p-\bar{p})|^2 \, dt \right)^{\frac{1}{2}} + |\bar{p}| \right]. \tag{13}$$

Similarly, in the equation of $x - \bar{x}$ we use the observability inequality for (A, C) which is ensured by (11):

$$|x(T) - \bar{x}| \leq c \left(\int_0^T |u - \bar{u}|^2 dt + \int_0^T |C(x(t) - \bar{x})|^2 dt + |x_0 - \bar{x}|^2 \right)^{\frac{1}{2}}. \tag{14}$$

This, together with the identity

$$\int_0^T \left(|u - \bar{u}|^2 + |C(x-\bar{x})|^2 \right) dt = [(x_0 - \bar{x})(p(0) - \bar{p})] - [(x(T) - \bar{x})\bar{p}]$$

yields the needed bounds.

Step 5. The exponential turnpike estimate.

The conclusion holds employing the exponential decay of the correlation term and the fact that

$$\int_{aT}^{bT} \left(|u - \bar{u}|^2 + |C(x - \bar{x})|^2\right) dt$$

$$= [(x(aT) - \bar{x})(p(aT) - \bar{p})] - [(x(bT) - \bar{x})(p(bT) - \bar{p})].$$

□

What is the reason?

It is a direct consequence of the hyperbolicity of the underlying dynamics, whose steady state solutions are characterised by the system

$$A\bar{x} + BB^*\bar{p} = 0$$

$$-A^*\bar{p} + C^*C\bar{x} = C^*Cx^*$$

generated by the operator matrix

$$\tilde{A} = \begin{pmatrix} A & BB^* \\ C^*C & -A^* \end{pmatrix}$$

Note however that the hyperbolicity of this matrix operator needs of controllability conditions.

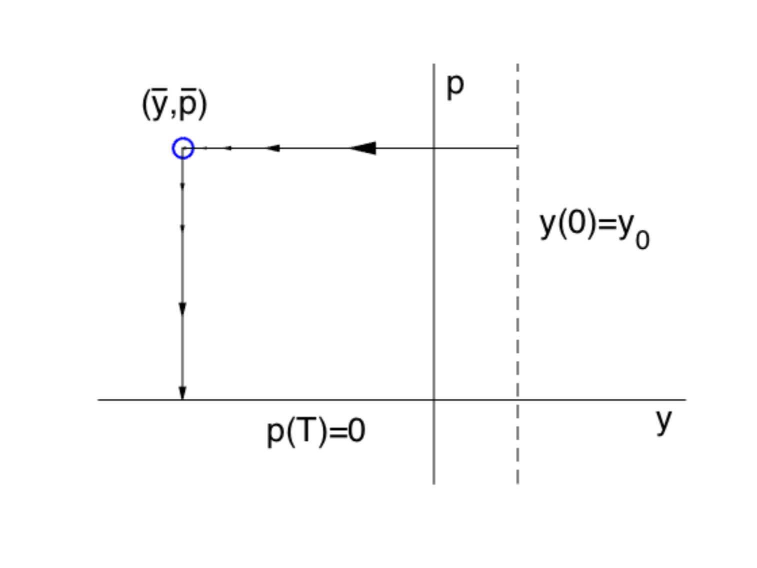
In other words, the fact that the spectrum of the operator matrix $\tilde{A}$ is symmetric to the left and right half complex plane, ensures the stability+unstability pattern.

Two key ingredients are needed for the turnpike property to arise for the optimal control problem:

1. The cost criterion for the optimal control needs to penalise both state and control.
2. The system needs to be controllable.

In particular, it is worth underlying that controllability is needed for the turnpike property to hold !!!

The turnpike path



The turnpike dynamics

Extensions

Some extensions:

1. Extension of this linear finite-dimensional theory to a **linear abstract setting of infinite-dimensional semigroups**, including wave and heat equations.
   Note that, since (null) controllability is required, turnpike holds for the heat equation with any support ω of the control, but that, for the wave equation, ω is required to fulfill the Geometric Control Condition (by Bardos-Lebeau-Rauch).
   When the GCC fails, weaker turnpike properties are achieved, with slower convergence rates (not exponential ones).
   Porretta-Zuazua, SICON, 2013.
2. Nonlinear finite-dimensional systems.
   E. Trélat & E. Zuazua, The turnpike property in finite-dimensional nonlinear optimal control, JDE, 218 (2015) , 81-114.

Strategy of proof for the finite-dimensional nonlinear problem (E. Trélat & E. Z.)

1. Write down the Optimality System (SO) for the nonlinear time evolution problem.
2. Linearise the OS around the steady optimal state-control to get the linearised OS.
3. Check the hyperbolic structure of the linearised OS and its turnpike character.
4. Get back to the nonlinear problem by local perturbation theory.

Some other bibliographical references

1. Turnpike theorems have been derived in the 60's for discrete-time optimal control problems arising in econometry (McKenzie, 1963).
2. Continous versions by Haurie for particular dynamics (economic growth models). See also Carlson-Haurie-Leizarowitz 1991, Zaslavski 2000.
3. More recently, in biology: Rapaport 2005, Coron-Gabriel-Shang 2014; human locomotion: Chitour-Jean-Mason 2012; MPC: Grüne 2012-2014.
4. Rockafellar 1973, Samuelson 1972: saddle point feature of the extremal equations of optimal control.
5. Different point of view by Anderson-Kokotovic (1987), Wilde-Kokotovic (1972):
exponential dichotomy property $\rightarrow$ hyperbolicity phenomenon.
6. Application to a Lotka-Volterra model in population dynamics: A. Ibañez (2016).

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Example in control-affine case (E. Trélat & E. Z.)

$$\dot{x}_1(t) = x_2(t),\ x_1(0) = 1,$$

$$\dot{x}_2(t) = 1 - x_1(t) + x_2(t)^3 + u(t), \qquad x_2(0) = 1$$

$$\min \frac{1}{2}\int_0^T \left((x_1(t)-1)^2 + (x_2(t)-1)^2 + (u(t)-2)^2 \right) dt$$

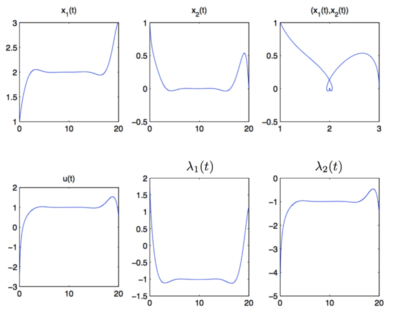
Optimal solution of the static problem:

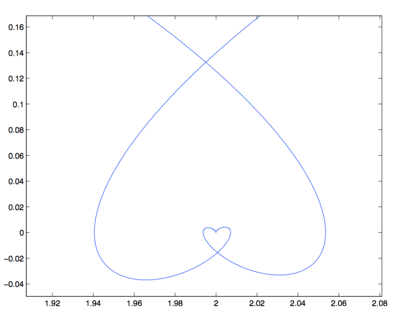
$$\bar{x}_2 = 0, \qquad 1 - \bar{x}_1 + \bar{x}_2^3 + \bar{u} = 0$$

$$\min_{\substack{x_2=0 \\ 1-x_1+x_2^3+u=0}} \left((x_1-1)^2 + (x_2-1)^2 + (u-2)^2 \right)$$

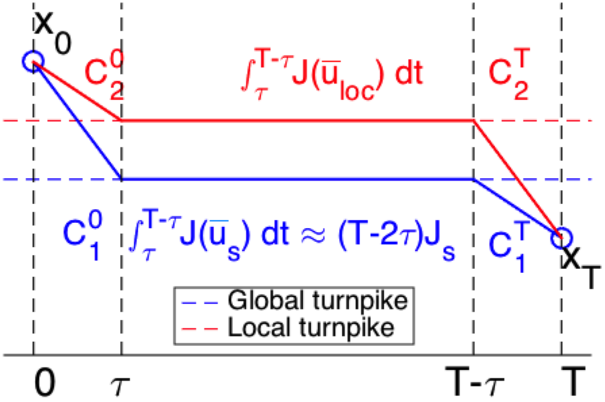
whence

$$\boxed{\bar{x} = (2,0)\,, \quad \bar{u} = 1, \quad \bar{\lambda} = (-1,-1)}$$


x1(t)
x2(t)
(x1(t),x2(t))
u(t)
$\lambda_1(t)$
$\lambda_2(t)$



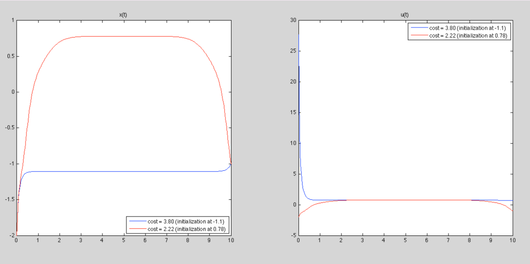
Local versus global turnpike

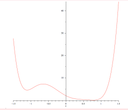


$$\dot{x} = -3x + 3x^3 + u; \; x(0) = x_0.$$

$$\min \int_0^T \left((x(t) - 1)^2 + (u(t) - 1)^2 \right) dt$$

The static optimal problem has a unique global solution $\bar{x} = 0.78$ ($\bar{u} = 0.91$), and a local one $\bar{x}_{loc} = -1.10$ ($\bar{u} = 0.73$).
Local versus global turnpike. Costs 4.19 and 2.61 respectively.





All in all: An advanced simulation

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A. Porretta & E. Z., INdAm, 2019[a]

[a] Extension to 2d NS by S. Zamorano, UChile

Consider now the semilinear heat equation:

$$\begin{cases} y_t - \Delta y + y^3 = f 1_\omega & \text{in} \quad Q \\ y = 0 & \text{on} \quad \Sigma \\ y(x,0) = y^0(x) & \text{in} \quad \Omega. \end{cases} \tag{15}$$

Consider the minimisation problem:

$$\min_f \left[\frac{1}{2} \int_0^T \int_\Omega |y - y_d|^2 dxdt + \int_0^T \int_\omega f^2 dxdt \right].$$

The optimality system reads:

$$y_t - \Delta y + y^3 = -\varphi 1_\omega \text{ in } Q$$

$$y = 0 \text{ on } \Sigma$$

$$y(x, 0) = y^0(x) \text{ in } \Omega$$

$$-\varphi_t - \Delta\varphi + 3y^2\varphi = y - y_d \text{ in } Q$$

$$\varphi = 0 \text{ on } \Sigma$$

$$\varphi(x, T) = 0 \text{ in } \Omega.$$

And the linearised optimality system, around the optimal steady solution $(\bar{y}, \bar{\varphi})$ is as follows:

$$z_t - \Delta z + 3(\bar{y})^2 z = -\psi 1_\omega \text{ in } Q$$

$$z = 0 \text{ on } \Sigma$$

$$z(x,0) = 0 \text{ in } \Omega$$

$$-\psi_t - \Delta \psi + 3(\bar{y})^2 \psi + 6\bar{y}\varphi z = z \text{ in } Q$$

$$\psi = 0 \text{ on } \Sigma$$

$$\psi(x, T) = 0 \text{ in } \Omega.$$

The equations describing the dynamics of the linearised optimality system red as follows:

$$z_t - \Delta z + 3(\bar{y})^2 z = -\psi 1_\omega$$

$$-\psi_t - \Delta\psi + 3(\bar{y})^2\psi = (1 - 6\bar{y}\varphi)z$$

This is the optimality system for a LQ control problem of the model

$$z_t - \Delta z + 3(\bar{y})^2 z = f 1_\omega$$

and the cost

$$\min_f \left[\frac{1}{2}\int_0^T \int_\Omega |z|^2 dxdt + \int_0^T \int_\omega \rho(x) f^2 dxdt\right]$$

with

$$\rho(x) = 1 - 6\bar{y}(x)\varphi(x).$$

And the turnpike property holds as soon as $\rho(x) \geq \delta > 0$.
This holds if $\bar{y}$ and φ are small enough, and this is automatically implied as soon as the target y_d is small enough.

The second order optimality conditions for the minimiser of the steady-state problem guarantee that the functional under consideration is semidefinite positive[3].

Whether this suffices for the turnpike property to hold is under investigation.

[3] E. Casas & M. Mateos, EHF2016 Lecture Notes, 2016.

All in all: An advanced simulation

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Comments

The general picture is rather clear, but there are still some interesting (some of them difficult?) open problems:

• Clarify the turnpike property for nonlinear PDE without smallness conditions on the target exploiting the second order optimality conditions for the steady state optimal pair.

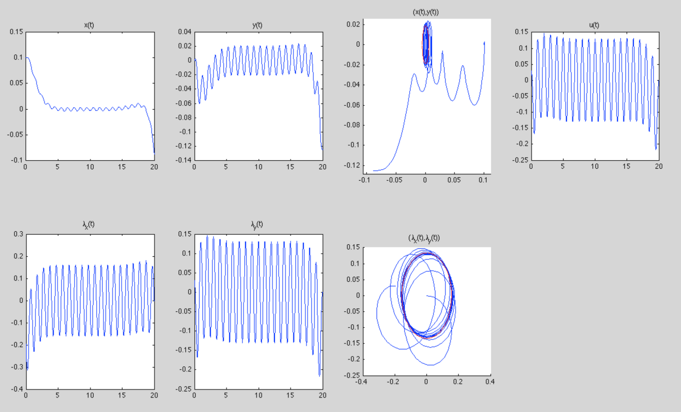
• Extend the theory for optimal control problems on the (diffusivity) coefficients. This has been done by G. Allaire, A. Münch and F. Periago (SICON 2010) for time-independent coefficients, but whether the turnpike property holds for time-evolution control coefficients, is to to be done:

$$y_t - div(\sigma(x,t)\nabla y) = 0$$

the diffusivity $\sigma = (x,t)$ being the control.

• Towards a more global picture: Combine the local analysis presented here, mainly based on the analysis of the SO and its linearised version, and the technical inspired in dissipativity property by L. Grüne et al.

• More general notions such as the periodic turnpike property can also be investigated for, say, periodic non-autonomous evolutions problems (E. Trélat, C. Zhang and E. Z., in progress).



- Development of the turnpike theory for optimal shape design problems.
- Make use of turnpike properties to solve constrained controllability problems (in large time).