Optimal placement of sensors and actuators for waves

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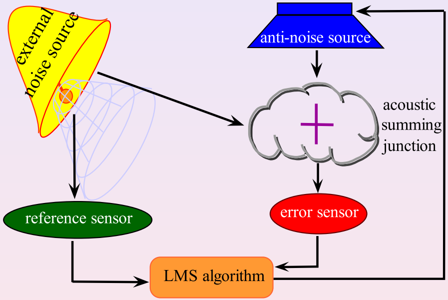
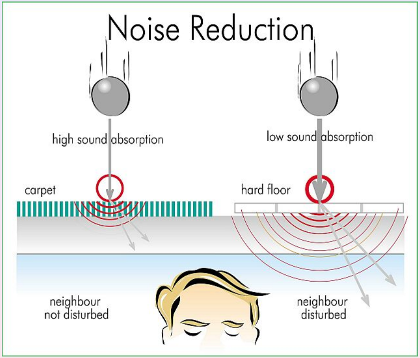
Motivation

Often in engineering and industrial processes an optimal choice and design of sensors and and actuators is required.
Often times, as well, the relevant processes are of a wave-like nature:

- Noise reduction in cavities and vehicles.
- Laser control in Quantum mechanical and molecular systems.
- Seismic waves, earthquakes.
- Flexible structures.
- Environment: Water management.
- Gas networks
- Optimal shape design in aeronautics.
- Human cardiovascular system: the bypass
- ....

An example: noise reduction





Acoustic noise reduction

Active versus passive controllers.

The Geometric Control Condition

The wave equation

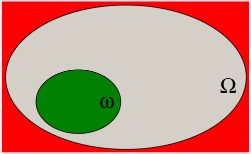
$$\begin{cases} z_{tt} - \Delta z = 0 & \text{in} \quad Q = \Omega \times (0, \infty) \\ z = 0 & \text{on} \quad \Sigma = \Gamma \times (0, \infty) \\ z(x,0) = z^0(x), z_t(x,0) = z^1(x) & \text{in} \quad \Omega. \end{cases}$$

Let ω be an open subset of Ω.

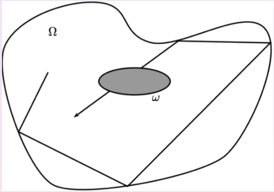
Question: Do they exist $T > 0$ and $C > 0$ such that

$$E(0) = \frac{1}{2}\left(||z^0||^2_{L^2(\Omega)} + ||z^1||_{H^{-1}(\Omega)}\right) \leq C \int_0^T \int_\omega z^2 dxdt,$$

for all solution?

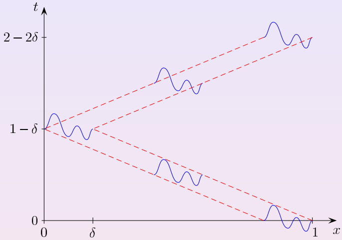


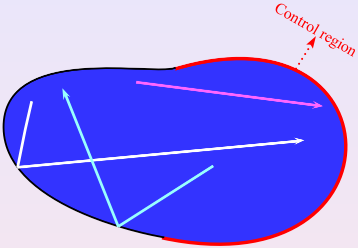
The observability inequality holds if and only if the support ω of the observation set, the location of the sensor, satisfies the so called the Geometric Control Condition (GCC) (Russell, Ralston, Rauch-Taylor, Lions, Bardos-Lebeau-Rauch (1988),...)

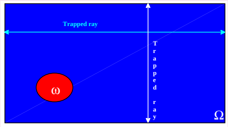


Rays propagating inside the domain Ω following straight lines that are reflected on the boundary according to the laws of Geometric Optics. The control region is the red subset of the boundary. The GCC is satisfied in this case. The proof requires tools from Microlocal Analysis.

The billiards...







Trapped rays escaping the observation region ω. This the observation impossible.

Optimal sensors for the wave equation; Joint work with Y. Privat and E. Trélat (LJLL, Univ. Paris VI) [a]

[a] 2013-present, Annales IHP, J. Fourier Anal. Appl., DCDS, ARMA, JEMS,...

Wave equation:

$$\begin{cases} z_{tt} - \Delta z = 0 & \text{in} \quad Q = \Omega \times (0, T) \\ z = 0 & \text{on} \quad \Sigma = \partial\Omega \times (0, T) \\ z(x,0) = z^0(x), z_t(x,0) = z^1(x) & \text{in} \quad \Omega. \end{cases}$$

Observability inequality:

$$||z^0||^2_{L^2(\Omega)} + ||z^1||^2_{H^{-1}(\Omega)} \leq C(\omega, T) \int_0^T \int_\omega z^2 dxdt.$$

Optimising ω simply means **minimising** the constant $C(\omega, T)$.

The smaller this observability constant is, the bigger the fraction of energy we "see" through ω and, accordingly, the better the observability property.

Spectral formulation

Let us analyse this inequality in terms of the Fourier expansion of solutions, using the eigenvalues λ_k and eigenfunctions ϕ_k of the Laplacian:

$$z(x,t) = \sum_k \hat{z}_k e^{i\sqrt{\lambda_k}t}\phi_k(x).$$

Thus,

$$\int_0^T \int_\omega |z|^2 dxdt = \sum \sum \hat{z}_k \hat{z}_j \int_\omega \phi_k(x)\phi_j(x)dx \int_0^T e^{i(\sqrt{\lambda_k}-\sqrt{\lambda_j})t} dt.$$

What about

$$\int_\omega \phi_k(x)\phi_j(x)dx$$

?

Two methods for mixing

- Infinite Time.

$$\lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T}\int_{\omega} |z|^2 dxdt.$$

$$\lim_{T\to\infty} \frac{1}{T}\int_{0}^{T} e^{i(\sqrt{\lambda_k}-\sqrt{\lambda_j})t} dt = \delta(j,k).$$

- Randomisation. Randomizing initial data and considering the expected observability constant (Zygmund lemma, N. Burq et al.[1])
- In 1-d, $\Omega = (0,\pi)$ in which case solutions are 2π-time periodic.

Cross terms vanish and we are led to the following spectral observability problem:

$$\sum_k |\hat{z}_k|^2 \leq C(\omega) \sum_k |\hat{z}_k|^2 \int_{\omega} \phi_k^2(x) dx.$$

[1] N. Burq, N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations. I. Local theory*, Invent. Math. **173** (2008), no. 3, 449–475.

The problem becomes:

To find ω so that the following minimum is maximized:

$$J(\omega) = \inf_k \int_\omega \phi_k^2(x) dx.$$

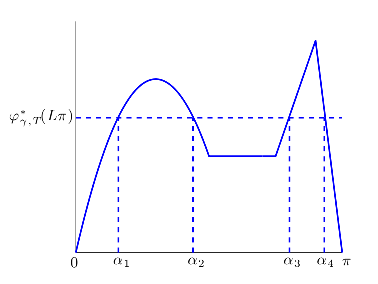
$$I_L = \sup_{|\omega|=L} J(\omega).$$

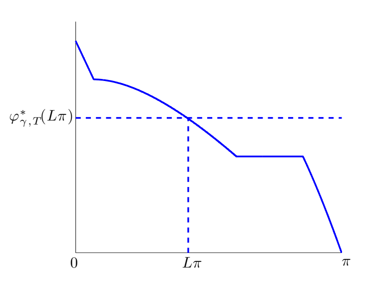
Where:

$$-\Delta \phi_j = \lambda_j \phi_j \text{ in } \Omega; \phi_j = 0 \text{ on } \partial\Omega.$$

Fixed initial data

- For initial data that are analytic (exponential decay of Fourier coefficients), there is a unique minimizer with a finite number of connected components. [2]
- The optimal set always exists but it can be a **Cantor set even for** C^∞ **smooth data**.

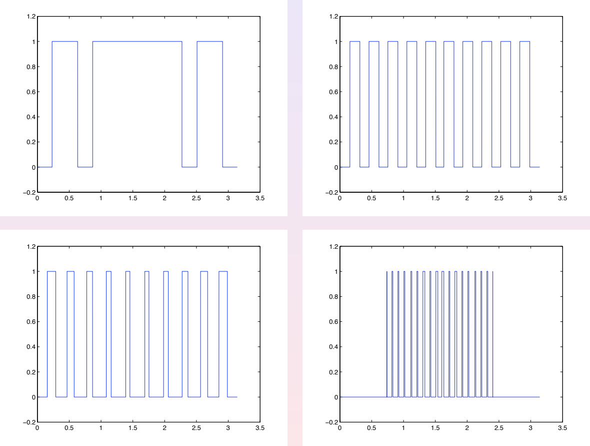




[2]Szolem Mandelbrojt, Sur un problème concernant les séries de Fourier, Boulletin de la SMF, 62 (1934), 143–150.

Warning on numerics

Numerics exhibits often very oscillatory patterns. But it hard to distinguish between Cantorization and relaxation processes!!!!!



Simulations performed using AMPL + IPOPT

Spectral criterium: Relaxation in $1-d$

High frequency relaxation

- Relaxation occurs [3]: the optimum is achieved by a density function $\rho(x)$ so that $\int_0^\pi \rho(x)dx = L$ and not by a measurable set with bang-bang densities (except for $L=\pi/2$). The constant density is optimal but is not the unique one.

- Spillover occurs $(1-d)$: The optimal design for the first N Fourier modes is the worst choice for the $N+1$-th one.

- No gap! The infimum over measurable sets and over densities coincides. The functional is not lower semicontinuous!!!

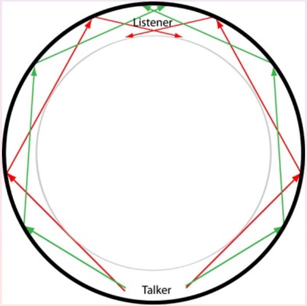
[3] P. Hébrard, A. Henrot, *A spillover phenomenon in the optimal location of actuators*, SIAM J. Control Optim. **44** (2005), 349–366.

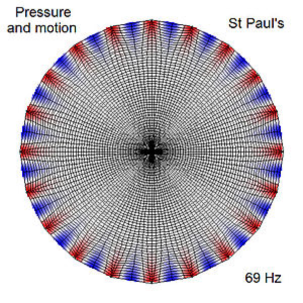
Spectral criterium: multi$-d$

In the multi-dimensional case the problem is much more complex. Spectra do not behave according to our $1-d$ intuition

$$\sin^2(kx) \rightharpoonup 1/2 \quad \text{as} \quad k \to \infty.$$

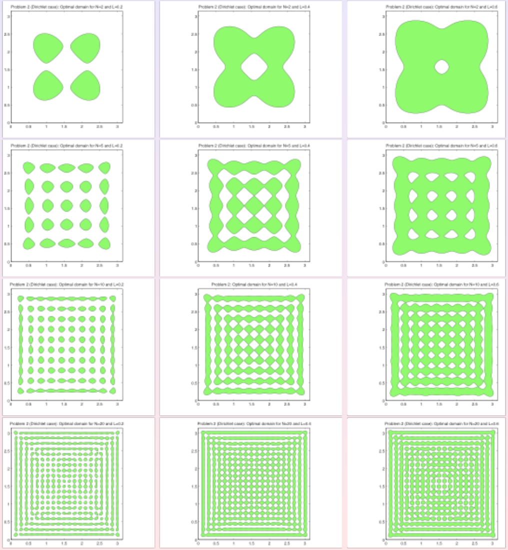
But this is far from being true in multi-d

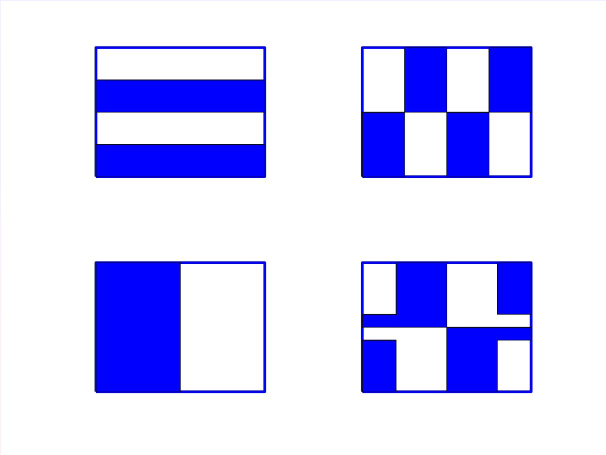




Whispering gallery

In practice, the truncated (in Fourier) criterium, considering the first N modes, is a way of approximating the optimal value.



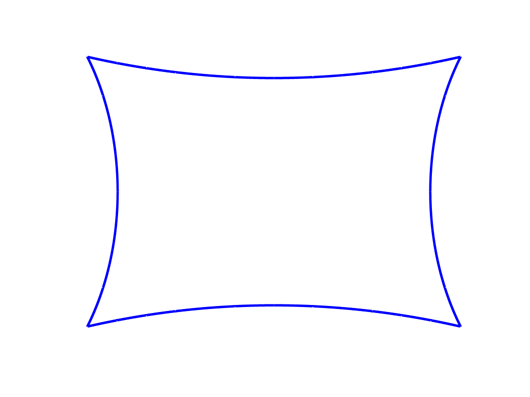


Optimal designs in $2-d$ for the square with volume fraction $1/2$ in the spectral criterium. This is an exceptional case where classical optimal domains exist.

Our work has continued addressing, in the multi-d setting, issues such as:

1. Does a classical solution exist?
2. Is there a gap between the sup for the original problem and the relaxed one (in which the locations of actuators are replaced by densities)?
3. Does the optimal choice for the first N frequencies converge to the global one?
4. What is the role of the geometry of the domain?

Deep issues are still unsolved, related to the link between the dynamics of the billiard, the high frequency equidistribution of the energy, and the shape of the optimal sensors.

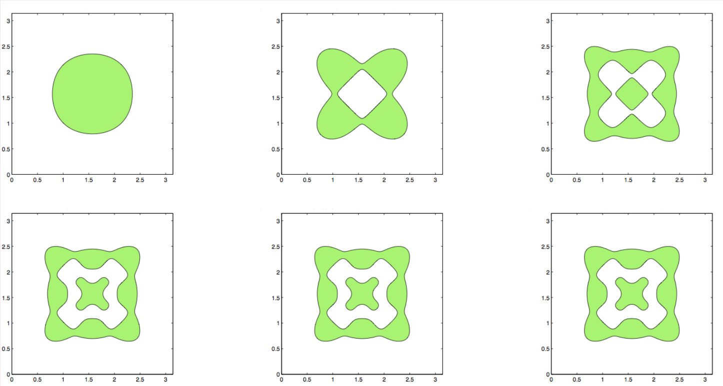


Domain conjectured to satisfy the QUE (Quantum Unique Ergodicity) [4] in A.H. Barnett, Asymptotic rate of quantum ergodicity in chaotic Euclidean billiards, Comm. Pure Appl. Math. 59 (2006), 1457-1488.

[4] Property ensuring that all eigenfunctions get equisdistributed that suffices (together with an uniform L^p-bound ($p > 2$)), roughly, for the non-gap property to hold.

Heat processes

Due to the intrinsic damping in the heat equation, the **high frequency components are penalised/negligible**.
The optimal design is determined by a finite number of eigenfunctions, the relevant number of them diminishing as T increases.



Perspectives

Plenty is to be done:

1. From an analysis perspective:
   1. Deal with the time dependent problem
   2. Develop the billiard perspective (rather than the spectral one)
2. From a computational viewpoint:
   1. Bypass the fact that high frequency eigenfunctions are hard to compute
   2. Develop robust computational tools not requiring spectral decompositions
   3. Address more complex, possibly nonlinear, models