Eigenvalue bounds for the Gramian operator of the heat equation

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Abstract

This paper is concerned with the eigenvalue decay of solution operators to operator Lyapunov equations, a relevant topic in the context of model reduction for parabolic control problems. We mainly focus on the Gramian operator, which arises in the context of control and observation of heat processes in infinite time, which is normally the first step towards observations in a finite time horizon.

By improving existing energy and observability estimates for parabolic equations, we obtain both upper and lower bounds on the convergence rate of the eigenvalues of the Gramian operator towards zero. Both bounds follow the same polynomial decay rate, up to a multiplicative constant, which ensures their optimality. This confirms the slow decay of the eigenvalues and limits the efficiency of model reduction. The theoretical findings are supported by numerical results.

Key words: Heat equation; Gramian operator; eigenvalue decay; infinite-dimensional systems; time-invariant

1 Introduction and the main result

Motivated by recent work on the eigenvalue decay of the solution operator to operator Lyapunov equations, a relevant topic in the context of model reduction for parabolic control problems (see [13] and the references therein), we analyse this issue in the context of the control of the heat equation with controls supported in an open subset of the domain where the heat process evolves.

Let us formulate the problem under consideration more precisely.

Let Ω be an open, bounded domain of $\mathbf{R}^d$ with a Lipschitz boundary. Given $T > 0$ we consider linear parabolic equations of the form

$$\begin{cases} z_t - \Delta z = v 1_\omega & \text{in} \quad \Omega \times (0,T) \\ z = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ z(0) = z_0 & \text{in} \quad \Omega. \end{cases} \tag{1}$$

In (1), $z = z(x,t)$ is the state and $v = v(x,t)$ the control, which acts on the system through the control operator $v \to v1_\omega$, where ω is an open non-empty subset of Ω. Here and in the sequel 1_ω denotes the characteristic function of the set ω.

We shall denote by Q^T the cylinder $\Omega \times (0,T)$ and by Σ^T the lateral boundary $\partial\Omega \times (0,T)$.

We assume that $z_0 \in L^2(\Omega)$ and $v \in L^2(Q^T)$, so that (1) admits a unique solution z in the class

$$z \in C\left([0,T]; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right).$$

It is by now well known that system (1) is null (and con-

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sequently approximate) controllable (see, for instance, [12] and [25]). In other words, given an initial datum $z_0 \in L^2(\Omega)$ there exists a control $v \in L^2(Q^T)$ such that the solution of (1) satisfies

$$z(T) = 0. \tag{2}$$

Accordingly, the *set of admissible controls* $\mathcal{U}^{ad}(z_0)$ below is non-empty (for all $z_0 \in L^2(\Omega)$, $T > 0$ and ω) :

$$\mathcal{U}^{ad}(z_0) = \left\{ v \in L^2(Q^T) : \text{ the solution } z \text{ of (1) satisfies (2)} \right\}. \tag{3}$$

The corresponding, by now well known, null controllability result is based in the dual observability inequality for the adjoint system, which, up to time reversal, reads as :

$$\begin{cases} p_t - \Delta p = 0 & \text{in} \quad Q^T \\ p = 0 & \text{on} \quad \Sigma^T \\ p(0) = p_0 & \text{in} \quad \Omega, \end{cases} \tag{4}$$

with $p_0 \in L^2(\Omega)$.

To be more precise, the null controllability result above is equivalent to the existence of a constant $C(T,\omega)$ such that the following inequality holds:

$$\| p(T) \|^2_{L^2(\Omega)} \leq C(T,\omega) \int_0^T \int_\omega p^2 dx dt \tag{5}$$

for all solutions p of the adjoint system (4).

This inequality was proved to hold in [12] (see also [11]) using Carleman inequalities. The dependence of the observability constant $C(T,\omega)$ with respect to the various ingredients of the control problem (support of the control ω, length of the time-horizon T, etc) was further discussed in [11].

As observed in [11], this estimate, can be improved to obtain a global estimate on p weighted by an exponential vanishing weight at $t = 0$. More precisely, there exist constants $c, C > 0$ depending on T, ω and Ω only, such that

$$\int_0^T \int_\Omega \exp(-c/t)|p|^2 dx dt \leq C \int_0^T \int_\omega p^2 dx dt$$

for any p solving the adjoint system (4). By means of the last estimate and by using the Fourier representation of solutions, (5) can be refined to obtain the following weighted observability inequality:

$$\sum_{j\geq 1} \exp(-a\sqrt{\lambda_j})|\hat{p}_{0,j}|^2 \leq C \int_0^T \int_\omega p^2 dx dt, \tag{6}$$

with a that obviously decreases as T increases. The value of the optimal constant a corresponding to the infinite time horizon is unknown as far as we know.

Here and in the sequel, $\hat{p}_{0,k}$ stand for the Fourier coefficients of the datum p_0 of the adjoint heat equation on the basis of the eigenfunctions of the minus Laplacian that we denote by $\{\phi_k\}_{k\geq 1}$, $\{\lambda_k\}_{k\geq 1}$ being the corresponding eigenvalues, repeated with their multiplicities and arranged in nondecreasing order.

On the other hand, the classical regularizing effects for the solution to the heat equation (4) (e.g. [10, Chapter XVIII, §3]) ensure that whenever $p_0 \in H^{-1}(\Omega)$, then $p \in L^2(\Omega \times (0,\infty))$ and the total energy of the system is given by

$$\int_0^\infty \int_\Omega p^2 dx dt = \frac{1}{2} ||p_0||^2_{H^{-1}(\Omega)}. \tag{7}$$

Here, as usual, $H^{-1}(\Omega)$ stands for the dual of $H^1_0(\Omega)$ and consists of all distributions f that can be written as $\mathsf{div}\, F$ for some $F \in L^2(\Omega; \mathbf{R}^n)$. Its norm is defined as

$$||f||^2_{H^{-1}(\Omega)} = \sup_{\|g\|_{H^1_0(\Omega)}=1} \langle f, g\rangle = \sup_{\|g\|_{H^1_0(\Omega)}=1} \int_\Omega F \cdot \nabla g \, dx,$$

where $\mathsf{div}\, F = f$. In general, by $H^{-k}(\Omega), k \in \mathbf{N}$ we shall denote the dual of $H^k_0(\Omega)$ (for details cf. [1]).

By combining the last estimate with (6), we get the following, two-sided bounds on the observed energy in the infinite time horizon

$$\begin{aligned} \frac{1}{C} \sum_{j\geq 1} \exp(-a\sqrt{\lambda_j})|\hat{p}_{0,j}|^2 &\leq \int_0^\infty \int_\omega p^2 dx dt \\ &\leq \sum_{j\geq 1} \frac{1}{2\lambda_j} |\hat{p}_{0,j}|^2 \end{aligned} \tag{8}$$

where the last inequality is obtained by expressing H^{-1} norm of p_0 in terms of Fourier coefficients.

The both lower and upper bound in (8) are sharp [11]. As we shall see, these estimates suffice to set some first bounds on the decay of the eigenvalues of the infinite time Gramian operator, which is the solution of the Lyapunov equation that, in this particular case, reads

$$\Delta X_\omega + X_\omega \Delta = -1_\omega. \tag{9}$$

Here Δ stands for the Dirichlet Laplacian, which is a negative operator in $L^2(\Omega)$, X_ω is the operator unknown, and the right hand side term takes account of the control operator.

Recall that in the classical setting of a control pair (A, B) the Lyapunov equation reads:

$$AX + XA^* = -BB^*,$$

which, in the context of the heat equation, leads to (9).

The solution of this equation is given by the Gramian in infinite time

$$X = \int_0^\infty \exp(At)BB^* \exp(A^*t)dt,$$

where, in a general setting of an unbounded operator A, the above expression is understood in the sense of operators acting on the domain of A [22, Theorem 5.1.1]. In the context of the specific examples of the heat equation the expression reads

$$X_\omega = \int_0^\infty \exp(\Delta t)1_\omega \exp(\Delta t)dt. \qquad (10)$$

The operator X_ω is a bounded operator on $L^2(\Omega)$ and, as we shall see in the next section, also compact.

The main object of investigation in [13] was the obtention of the upper decay estimates for the eigenvalues of Gramian operators like X. Assuming the control operator B is of finite rank (e.g. a boundary control operator), the authors obtain the exponential decay rate.

This paper is devoted to the case $B = 1_\omega$ which, obviously, is not of finite rank. The existence of the sequence of eigenvalues of the corresponding Gramian X_ω in that case, as well as its properties, are ensured by Lemma 2 in the next section.

As we shall see, one can easily achieve both lower and upper bounds (in terms of λ_k) for the eigenvalues of the Gramian X_ω using the existing observability inequalities as in (8). Consequently, lower bounds obtained in this way will be of the exponential nature, while the upper ones will follow a polynomial law, leaving a large gap in between. Since these bounds follow from the sharp observability inequalities given in (8), one might expect them to be sharp as well. This would imply that there are two subsequences of eigenvalues that decay at different rates, exponentially and polynomially, respectively. However, the main result of the paper, surprisingly, rules out any kind of exponential decay and yields a much slower, polynomial decay along the entire sequence of eigenvalues of the Gramian.

Theorem 1 (The main result)
The sequence of eigenvalues μ_k of the Gramian operator X_ω given by (10) *allow for a two-sided, polynomially decaying bounds of the form*

$$C_1 k^{-2/d} \leq \mu_k \leq C_2 k^{-2/d}, \qquad (11)$$

where $C_{1,2}$ are positive constants that depend on ω and Ω only.

Remark 1 *The paper is devoted to the analysis of the infinite time Gramian, i.e. of observations over the infinite time horizon. However, as we shall see, the main result on polynomial decay of eigenvalues applies both for finite and infinite time Gamians (cf. Theorem 7).*

The paper is organised as follows. The next section contains preliminary bounds on the eigenvalues of the Gramian operator arising from the existing observability inequalities (8). Section 3.1 is devoted to the proof of the main result, Theorem 1. More precise estimates obtained for the case of a rectangular domain, where we exploit the explicit knowledge of the eigenvectors of the Laplacian, are presented in Section 3.2. The relation of the obtained results to control problems for the heat equation is discussed in Section 4. Section 5 contains numerical examples which, surprisingly again, at first sight, seem to contradict the obtained theoretical results. The paper is finalised with some concluding remarks and directions for future research.

2 Preliminary bounds on the eigenvalue decay for the Gramian

We analyse the Gramian operator X_ω for the heat equation given by (10). Note that the quadratic form associated to X_ω reads

$$\langle X_\omega p_0 \mid p_0 \rangle_{L^2(\Omega)} = \int_0^\infty \int_\omega p^2 dx dt \qquad (12)$$

where p is the solution of the heat equation (4) in the infinite horizon $0 < t < \infty$. Thus the quadratic form $\langle X_\omega p_0 \mid p_0 \rangle_{L^2(\Omega)}$ equals the observation of the adjoint state p in ω over the infinite time interval $(0, \infty)$.

For this reason, the study of the eigenvalues of X_ω is directly related to the existing observability estimates for the solution to the heat equation. However, before analysing decay properties of eigenvalues of the Gramian operator, let us first ascertain its spectral decomposition.

Lemma 2 *The Gramian operator X_ω is a compact, self-adjoint operator on $L^2(\Omega)$. Its eigenvalues μ_k constitute a sequence of positive real numbers accumulating at zero and the corresponding eigenfunctions ψ_k form an orthonormal basis of $L^2(\Omega)$.*

Proof: The Dirichlet Laplacian, which is the generator of the semigroup appearing in (10), is a negative, self-adjoint operator on $L^2(\Omega)$ with compact resolvent and the domain $H_0^1(\Omega) \cap H^2(\Omega)$. Consequently, the Gramian X_ω given by (10) is a self-adjoint operator as well. As by

(12) it is also a positive operator, it allows for the unique positive root $\sqrt{X_\omega}$ [14, Theorem V-3.35].

By using the classical energy estimate (7) for the solution to the heat equation, we get

$$\|\sqrt{X_\omega} p_0\|^2_{L^2(\Omega)} = \langle X_\omega p_0 \mid p_0 \rangle_{L^2(\Omega)} \leq \frac{1}{2}\|p_0\|^2_{H^{-1}(\Omega)}.$$

As $L^2(\Omega)$ is compactly embedded into $H^{-1}(\Omega)$, it follows that $\sqrt{X_\omega}$ is a compact, self-adjoint operator on $L^2(\Omega)$. By the spectral theorem it allows for spectral decomposition, i.e. there exists an orthonormal basis $\{\psi_k\}$ of $L^2(\Omega)$ consisting of eigenvectors of $\sqrt{X_\omega}$. The corresponding eigenvalues, denoted by $\sqrt{\mu_k}$, constitute a sequence of positive numbers accumulating at zero and each one has a finite multiplicity.

From here it follows directly that the Gramian X_ω is also a compact operator on $L^2(\Omega)$. In particular, it allows for the spectral decomposition with the eigenpairs (ψ_k, μ_k). □

Remark 2 *The last lemma is proved using classical energy estimates for the solution to the heat equation and is specifically tailored to the setting of this article . In particular, it is based on the special structure of the dynamics operator under consideration, i.e. on $-\Delta$. The compactness (even nuclearity) of Gramian operators in a general setting, for abstract operators A, B can be found in [21]. However, verification of the assumptions posed there (in particular, that $A^{1/2}B$ is a Hilbert-Schmidt operator) is not straightforward (if feasible at all) in the case of an infinite dimensional control space (which is the setting of our paper).*

Based on the last lemma, we can associate two families of eigenpairs to the control system (1): one related to the minus Laplacian, the generator of the dynamics, that we denote by $\{(\phi_k, \lambda_k)\}_{k\geq 1}$, and the other $\{(\psi_k, \mu_k)\}_{k\geq 1}$, corresponding to the infinite time Gramian.

Two sequences of eigenvalues are of a completely opposite nature. While the sequence (λ_k) is known to consists of positive numbers with no finite accumulation point, the eigenvalues of the Gramian accumulate at zero. A precise relation between them will be in our focus for the rest of the paper. By exploiting existing observability results and energy estimates for the heat equation (8), one obtains the following asymptotic bounds on the eigenvalues of the Gramian.

Proposition 3 *The eigenvalues μ_k of the Gramian X_ω satisfy*

$$\frac{1}{C}\exp(-a\sqrt{\lambda_k}) \leq \mu_k \leq \frac{1}{2\lambda_k}, \tag{13}$$

where a and C are positive constants appearing in (6).

Proof: The upper bound in (13) can be easily derived from the existing energy estimate for the heat equation (8), using the min-max characterization of the eigenvalues of symmetric operators (cf. [8, pp 491-492]):

$$\mu_k = \min_{\substack{\dim E = k-1 \\ E\subseteq L^2(\Omega)}} \left\{\max\left\{R_\omega(p_0) : p_0 \in E^\perp \setminus \{0\}\right\}\right\},$$

where R_ω stands for the Rayleigh quotient defined as the ratio of the observed energy and the initial one, i.e.

$$R_\omega(p_0) = \frac{\langle X_\omega p_0 \mid p_0 \rangle_{L^2(\Omega)}}{||p_0||^2_{L^2(\Omega)}} = \frac{\int_0^\infty \int_\omega p^2 dx dt}{\sum_{j\geq 1}|\hat{p}_{0,j}|^2}.$$

In view of this characterisation, and taking $E = \text{span}\{\phi_1, ..., \phi_{k-1}\}$ as the $k-1$-dimensional subspace, namely the one generated by the first $k-1$ eigenfunctions of the Laplacian, and using the upper bound in (8) we easily deduce that

$$\mu_k \leq \max\left\{R_\omega(p_0) : p_0 \perp \phi_1, ..., \phi_{k-1}\right\} \leq \frac{1}{2\lambda_k}.$$

Indeed, for p_0 orthogonal to the span$\{\phi_1, ..., \phi_{k-1}\}$ we have $\hat{p}_{0,j} = \int_\Omega p_0 \phi_j dx = 0$ for $j < k$. Consequently, relation (8) and the monotonicity of the sequence (λ_k) imply

$$R_\omega(p_0) \leq \frac{\sum_{j\geq k}\frac{1}{2\lambda_j}|\hat{p}_{0,j}|^2}{\sum_{j\geq k}|\hat{p}_{0,j}|^2} \leq \frac{1}{2\lambda_k}.$$

Similarly, using the dual characterisation (cf. [8, pp 491-492])

$$\mu_k = \max_{\substack{\dim E = k \\ E\subseteq L^2(\Omega)}} \left\{\min\left\{R_\omega(p_0) : p_0 \in E \setminus \{0\}\right\}\right\}, \tag{14}$$

and taking $E = \text{span}\{\phi_1, ..., \phi_k\}$, the lower bound in (8) implies

$$\begin{aligned}\mu_k &\geq \min_{p_0\in E\setminus\{0\}} R_\omega(p_0) \\ &\geq \frac{\sum_{j\leq k}\exp(-a\sqrt{\lambda_j})|\hat{p}_{0,j}|^2}{C\sum_{j\leq k}|\hat{p}_{0,j}|^2}\mu_k \geq \frac{1}{C}\exp(-a\sqrt{\lambda_k}).\end{aligned}$$

□

Note that, according to Weyl's asymptotic law on the eigenvalues of the Laplacian [23] we know that

$$\lim_k \frac{k}{\lambda_k^{d/2}} = C(\Omega) \tag{15}$$

where the constant $C(\Omega)$ depends on the space dimension and the volume of the domain Ω only. Consequently, there exist positive constants $\tilde{C}_1, \tilde{C}_2$ such that

$$\tilde{C}_1 k^{2/d} \leq \lambda_k \leq \tilde{C}_2 k^{2/d}, \quad k \in \mathbf{N}. \tag{16}$$

Combining the last inequalites with Proposition 3 we obtain the following bounds.

Corollary 4 *The eigenvalues μ_k of the Gramian X_ω satisfy the following bounds*

$$\frac{1}{C} \exp(-\tilde{a} k^{1/d}) \leq \mu_k \leq C_2 k^{-2/d},$$

with suitable positive constants $\tilde{a}, C$ (depending on both ω and Ω) and C_2 (depending on Ω only).

The above estimates provide lower and upper bounds on the convergence rates of the eigenvalues μ_k. Note, however, that they are of different nature. While the first one is of exponential type, the second is polynomial, leaving a huge gap in between. Exploring and filling this gap is the topic of the next section.

Remark 3 *The ideas developed in this section generalize to various parabolic-type problems that allow a spectral decomposition of the elliptic operator generating the dynamics (mainly symmetric problems with time-independent coefficients). In fact, one can automatically transfer the existing results on observability of the heat equation and related models into eigenvalue bounds of the Gramian. Accordingly, the arguments used above can be applied in many other situations in which observability inequalities have been proved, such as time invariant heat equations with lower order potentials [11,12], controls with support in measurable sets (including boundary and lump controls) [3,12,13], systems of heat equations [2], etc.*

Remark 4 *We finalize the section by providing a characterisation of the eigenfunctions ψ_k of the Gramian in terms of the solution to the heat equation* (1) *starting from $z_0 = 0$.*

To this effect, let us first describe $X_\omega p_0$ for an arbitrary $p_0 \in L^2(\Omega)$. Taking into account the expression (10) *we have*

$$\begin{aligned} X_\omega p_0 &= \lim_{T\to\infty} \int_0^T \exp(\Delta t) 1_\omega \exp(\Delta t) dt \; p_0 \\ &= \lim_{T\to\infty} \int_0^T \exp(\Delta(T-t)) 1_\omega \exp(\Delta(T-t)) dt \; p_0 \\ &= \lim_{T\to\infty} z_T(T), \end{aligned} \tag{17}$$

where we observe that the last integral provides the final state $z_T(T)$ of the solution to the control problem (1) *with the zero initial datum and the control $v_T(t) = \exp(\Delta(T-t))p_0 = p(T-t)$, with p being the solution to the system* (4) *with the initial value p_0.*

This allow us to characterize the eigenfunction ψ_k of the Gramian operator X_ω as follows. Let p be the solution to the system (4) *starting from $p_0 = \psi_k$. Then the final state $z_T(T)$ of the solution to the control problem* (1) *with the initial value zero and the control $v_T(t) = p(T-t)$, approximately coincides with ψ_k, up to a multiplicative constant (the corresponding eigenvalue), i.e.*

$$z_T(T) \approx X_\omega \psi_k = \mu_k \psi_k,$$

where the approximation error tends to zero as T approaches infinity. This means that the shape of the eigenvector is preserved when two systems, the adjoint and the control one, are run over the given time horizon, and that the solution of the latter at the final time coincides (approximately, for T large enough) with the initial datum of the first one.

3 Polynomial lower bounds for μ_k

3.1 Proof of the main result

In this section, we aim to reduce the gap between the lower and upper decay rates that results from Proposition 3 and Corollary 4. In contrast to the approach of the previous section, where the obtained bounds were an almost direct consequence of the existing observability and energy estimates (8) for the solution to the heat equation, here we have to apply a different methodology. The first step in that direction consists in improving the observability estimates when the initial datum is supported in the observation region.

Lemma 5 (Improved observability estimates) *Let p be the solution to the heat equation* (4) *with initial datum $p_0 \in H^{-1}(\Omega)$ supported in a compact set $K \subset \omega$. Then for every $T \in \langle 0, \infty]$ there exists constants c_T, C_T independent of p_0, such that the following estimates hold:*

(i)

$$\int_0^T \int_{\omega^c} p^2 dx dt \leq C_T ||p_0||^2_{H^{-2}(\Omega)},$$

(ii)

$$\int_0^T \int_{\omega} p^2 dx dt \geq c_T ||p_0||^2_{H^{-1}(\Omega)}, \tag{18}$$

where ω^c stands for the complement of the control region.

Proof: *(i)* In order to obtain the desired bound let us define a smooth cut-off function $\theta \in C(\Omega)$ such that

$$\theta(x) = \begin{cases} 1, \ x \in \omega^c \\ 0, \ x \in K. \end{cases}$$

Then $q(t,x) := \theta(x)p(t,x)$ solves the problem

$$\begin{cases} q_t - \Delta q = -2\nabla\theta \cdot \nabla p - (\Delta\theta)p & \text{in} \quad Q^T \\ q = 0 & \text{on} \quad \Sigma^T \\ q(0) = 0 & \text{in} \quad \Omega. \end{cases}$$

Using the regularity results for solutions to the heat like equations (e.g. [10, Chapter XVIII, §3]) we obtain

$$\begin{aligned} \|q\|_{L^2(Q^T)} &\leq C_T \|2\nabla\theta \cdot \nabla p + (\Delta\theta)p\|_{L^2(0,T;H^{-2}(\Omega))} \\ &\leq C_T \|p\|_{L^2(0,T;H^{-1}(\Omega))} \leq C_T \|p_0\|_{H^{-2}(\Omega)}, \end{aligned}$$

where C_T denotes a generic constant, independent of p_0. As $p = q$ on ω^c the first claim follows.

(ii) Due to the classical energy estimates and the exponential decay of the solutions $p(t)$ in $H^{-1}(\Omega)$, for every $T > 0$ there exists a constant $\tilde{c}_T > 0$ such that

$$\begin{aligned} \int_0^T \int_\Omega p^2 dxdt &= ||p_0||^2_{H^{-1}(\Omega)} - ||p(T)||^2_{H^{-1}(\Omega)} \\ &\geq \tilde{c}_T ||p_0||^2_{H^{-1}(\Omega)}. \end{aligned}$$

By combining the last inequality with the first part of the lemma we obtain

$$c_T ||p_0||^2_{H^{-1}(\Omega)} \leq \int_0^T \int_\omega p^2 dxdt + ||p_0||^2_{H^{-2}(\Omega)}. \quad (19)$$

By using classical compactness-uniqueness arguments (e.g. [6,16]) one gets rid of the compact reminder in the last estimate, to obtain the desired result with some, not relabelled, positive constant c_T. Indeed, suppose that (18) does not hold. Then there exists a sequence of initial data (p_0^n) such that

$$||p_0^n||^2_{H^{-1}(\Omega)} \geq n \int_0^T \int_\omega (p^n)^2 dxdt, \quad (20)$$

where p^n stands for the solution of the heat equation (4) with initial datum p_0^n.

As the problem is linear, without loosing generality we can suppose that $||p_0^n||^2_{H^{-1}} = 1$ for every n. In that case the contradictory assumption (20) directly implies

$$\int_0^T \int_\omega (p^n)^2 dxdt \longrightarrow 0. \quad (21)$$

In addition, there exist a (nonrelabeled) subsequence such that $p_0^n \rightharpoonup p_0$ weakly in $H^{-1}(\Omega)$. Denoting by p the solution of the heat equation starting from the limit datum p_0, we obtain $p_n \rightharpoonup p$ in $L^2(Q^T)$. In particular, we have

$$\int_0^T \int_\omega p^2 dxdt \leq \liminf_n \int_0^T \int_\omega (p^n)^2 dxdt = 0,$$

where the lower weak semicontinuity of norms and the strong $L^2(\omega \times (0,T))$ convergence of (p_n) given by (21) are used. The observability estimate (5) for the solution of the heat equation ensures that $p = 0$ everywhere. Consequently $p_0 = 0$ as well, and the compact embedding of $H^{-1}(\Omega)$ into $H^{-2}(\Omega)$ implies $||p_0^n||^2_{H^{-2}(\Omega)} \longrightarrow 0$. Combining the obtained convergences with the upper estimate (19) we get that $p_0^n \longrightarrow 0$ in $H^{-1}(\Omega)$, which contradicts the normalization assumption of the same sequence. □

With the aid of the last lemma we are able to improve the lower decay bounds provided by Proposition 3. In particular, we obtain the following result.

Lemma 6 *The sequence of eigenvalues μ_k of the Gramian operator allow for a lower, polynomially decaying bound of the form*

$$\frac{c_\infty}{\lambda_{\omega,k}} \leq \mu_k,$$

where $\lambda_{\omega,k}$ are eigenvalues of the Dirichlet Laplacian on ω and c_∞ is the positive constant from (18) *(for $T = \infty$).*

Proof: Let $\tilde{\omega} \subset \omega$ be an open, non-empty, compactly embedded subset of the control region. Denote by $\{\phi_{\tilde{\omega},k}, k = 1, ..., \infty\}$ the orthonormal basis in $L^2(\tilde{\omega})$ that consists of eigenfunctions of the Dirichlet Laplacian on $\tilde{\omega}$, and let $\lambda_{\tilde{\omega},k}$ be the corresponding eigenvalues.

Without changing notation we extend each $\phi_{\tilde{\omega},k}$ by zero to the whole domain Ω. Let $E_k = [\phi_{\tilde{\omega},1}, ..., \phi_{\tilde{\omega},k}]$ be a k-dimensional subspace of $L^2(\Omega)$ spanned by the first k such extended functions.

Then for $p_0 \in E_k$ the interior regularity estimate (18) implies

$$\int_0^\infty \int_\omega p^2 dxdt \geq c_\infty ||p_0||^2_{H^{-1}(\Omega)} \geq c_\infty ||p_0||^2_{H^{-1}(\tilde{\omega})}. \quad (22)$$

In the next step we observe

$$||p_0||^2_{H^{-1}(\tilde{\omega})} = \sum_1^k \frac{1}{\lambda_{\tilde{\omega},i}} \Big| \int_{\tilde{\omega}} p_0 \phi_{\tilde{\omega},i} dx \Big|^2 \geq \frac{1}{\lambda_{\tilde{\omega},k}} ||p_0||^2_{L^2(\Omega)}$$

where $\int_{\tilde{\omega}} p_0 \phi_{\tilde{\omega},i} dx$ is the Fourier coefficients of p_0 with respect to the eigenfunctions of the Dirichlet Laplacian on $\tilde{\omega}$.

By combining the last estimate with (22) and exploiting the max-min characterisation of the eigenvalues (14), we obtain

$$\mu_k \geq \min \{R_\omega(p_0) : p_0 \in E_k \setminus \{0\}\} \geq \frac{c_\infty}{\lambda_{\tilde{\omega},k}}.$$

The obtained inequality holds for any $\tilde{\omega} \subset\subset \omega$. As the eigenvalues are continuous and nonincreasing functions of the domain (e.g. [4]), we can replace $\lambda_{\tilde{\omega},k}$ by $\lambda_{\omega,k}$, i.e. by the eigenvalues of the Dirichlet Laplacian on ω.

Finally, the Weyl's law implies polynomial growth of the sequence $(\lambda_{\omega,k})$, from which the claim follows. □

The main result, Theorem 1 now follows directly as the consequence of the preceding lemma.

Proof of Theorem 1: Combining the result of the last lemma with the upper bound from (13) we obtain the following improved estimates on the eigenvalues of the Gramian operator

$$\frac{c_\infty}{\lambda_{\omega,k}} \leq \mu_k \leq \frac{1}{2\lambda_k}. \tag{23}$$

Here λ_k and $\lambda_{\omega,k}$ denote eigenvalues of the Dirichlet Laplacian on the domain Ω and control region ω, respectively, while c_∞ is the positive constant appearing in (18), depending on ω and the domain Ω.

By Weyl's law (15), both the eigenvalue sequences (λ_k) and $(\lambda_{\omega,k})$ exhibit the same asymptotic behaviour of the type $k^{2/d}$, up to multiplicative constants $C(\Omega)$ and $C(\omega)$ that depend on the volume of Ω and ω, respectively, as well as on the space dimension. In particular it holds the analogue of the relation (16), i.e. there exist positive constants $\tilde{C}_{1,\omega}, \tilde{C}_{2,\omega}$ such that

$$\tilde{C}_{1,\omega} k^{2/d} \leq \lambda_{\omega,k} \leq \tilde{C}_{2,\omega} k^{2/d}, \quad k \in \mathbf{N}. \tag{24}$$

Combining (16) and (24) with the estimates (23) we obtain the desired polynomial bounds (11), with the constants $C_1 = c_\infty / \tilde{C}_{2,\omega}$ and $C_2 = 1/2\tilde{C}_1$ (the same one appearing in Corollary 4). □

The last result eliminates the gap between these bounds that was present in Proposition 3 and we obtain sharp asymptotic rates for the eigenvalues of the Gramian operator, both from below and above, up to multiplicative constants.

We finalize this subsection by discussing the eigenvalue decays for the finite time Gramian

$$X_\omega^T p_0 = \int_0^T \exp(\Delta t) 1_\omega \exp(\Delta t) dt \, p_0. \tag{25}$$

As already discussed in Remark 1, the infinite time Gramian and its finite time counterpart coincide up to a compact operator. For this reason, it is not surprising that the main result of the paper, obtained in the infinite time horizon, also extends to a finite time framework. More precisely, the following result holds.

Theorem 7 *The eigenvalues μ_k^T of the finite time Gramian X_ω^T given by* (25) *satisfy the following two-sided bounds*

$$\frac{c_T}{\lambda_{\omega,k}} \leq \mu_k^T \leq \frac{1}{2\lambda_k}, \tag{26}$$

where c_T is the positive constant from (18).

In addition, the sequence of eigenvalues follow the same polynomial decay law as their infinite time counterpart

$$C_1^T k^{-2/d} \leq \mu_k^T \leq C_2 k^{-2/d}, \tag{27}$$

where only the constant C_1^T appearing in the lower bound depends on T.

Proof: First, let us note that the spectral decomposition of the operator X_ω^T and the properties of the corresponding eigenvalues μ_k^T are ensured following the same arguments used in Lemma 2 for the infinite time Gramian.

The lower bound in (26) is obtained following the steps of Lemma 6. For this to be possible it is essential that the improved interior observability estimate (18) holds for any $T \in \langle 0, \infty]$.

The upper estimate in (26) follows trivially by bounding the finite time observation by an infinite one and using the energy estimate (7).

Finally, the polynomial decay law (27) is obtained by applying Weyl's law in the same way as in the proof of the main theorem. □

Remark 5 *The eigenvectors of the finite time Gramian operator can be characterised in a similar way as their infinite time counterparts. Indeed, following the arguments of Remark 4, it is easy to show that*

$$X_\omega^T \psi_k^T = \mu_k^T \psi_k^T = z_T(T), \tag{28}$$

where $z_T(T)$ is the final state of the solution to the control problem (1) *with the zero initial datum and the control $v_T(t) = \exp(\Delta(T-t))p_0 = p(T-t)$, with p being the* *solution to the system* (4) *with the initial value* ψ_k. *In this way, each eigenvector* ψ_k^T *can be exactly reached by the control determined by the solution of the adjoint system starting from the same eigenvector (up to a multiplicative constant).*

Note, that unlike in the infinite time framework, in (28) *we have the exact equality (and not just an asymptotic relation). This will be of importance when discussing relation of the obtained decay results to the control problems in Section 4.*

3.2 Precise estimates for rectangular domains

The lower bound in (23) is expressed in terms of an unknown constant c_∞ and eigenvalues $\lambda_{\omega,k}$ of the Laplacian on ω. The value of c_∞ is beyond our reach due to the arguing-by-contradiction procedure used in Lemma 5.

A more precise estimates on the lower bound in (23), with an exact constant value and expressed in terms of eigenvalues λ_k of the Laplacian on Ω only, can be obtained for a rectangular domain Ω. These results are presented in the following theorem.

Theorem 8 *Assume that the domain* Ω *is an open rectangle in* $\mathbf{R}^d$. *Let* $\omega \subseteq \Omega$ *be a control region of a positive measure. Then there exists a positive integer* $n \geq \sqrt[d]{|\Omega|/|\omega|}$ *such that the eigenvalues of the Gramian* X_ω *satisfy*

$$\frac{1}{2n^d\lambda_{kn}} \leq \mu_k \leq \frac{1}{2\lambda_k}. \tag{29}$$

The smallest value of the constant n *for which the lower estimate in* (29) *holds depends in a non-increasing manner on the size of the control region* $|\omega|$, *and it equals to* 1 *when the latter coincides with the whole domain* Ω.

Proof: The upper estimate has already been shown in Proposition 3. In the proof we therefore concentrate just on the lower bound.

For making the arguments more comprehensible and easier to follow, we first prove the desired inequality for a special 1D case, and then we show it holds for a general rectangular domain.

(1) Special 1D case: $\Omega = (0,\pi), \omega = (0,\pi/n)$.
Let Φ_n be the subfamily of eigenfunctions of the Dirichlet Laplacian on Ω consisting of $\phi_{kn}(x) = \phi_k(nx) = \sqrt{2/\pi}\sin(knx), k = 1,...,\infty$. Note that these functions are eigenfunctions of the Dirichlet Laplacian on ω, with the associated eigenvalues $\lambda_{\omega,k} = (kn)^2$. In particular, they are mutually orthogonal with respect to $L^2(\omega)$ scalar product.
Let $E_k = [\phi_n, \phi_{2n}, ..., \phi_{kn}]$ be a k-dimensional subspace of $L^2(\Omega)$ spanned by first k of the above introduced functions. As their squares, ϕ_{jn}^2, are periodic functions with the period π/n, for any $p_0 \in E_k$ we have

$$\int_0^\infty \int_\omega p^2 dxdt = \frac{1}{n}\int_0^\infty \int_\Omega p^2 dxdt = \frac{1}{2n}||p_0||^2_{H^{-1}(\Omega)}, \tag{30}$$

where the last equality is the consequence of the regularity estimate (7).
By expressing H^{-1} norm in terms of the Fourier coefficients of p_0 (with respect to the eigenfunctions of the Dirichlet Laplacian on Ω) we get

$$\int_0^\infty \int_\omega p^2 dxdt = \frac{1}{2n}\sum_1^k \frac{1}{\lambda_{in}}|\hat{p}_{0,in}|^2$$
$$\geq \frac{1}{2n\lambda_{kn}}\sum_1^k |\hat{p}_{0,in}|^2 = \frac{1}{2n\lambda_{kn}}||p_0||^2_{L^2(\Omega)}.$$

By using the max-min characterisation of the eigenvalues, (14), we obtain

$$\mu_k \geq \min\{R_\omega(p_0) : p_0 \in E_k \setminus \{0\}\} = \frac{1}{2n\lambda_{kn}},$$

where the last equality is reached for $p_0 = \phi_{kn}$.

(2) Case 2: General rectangular domain.
Without loosing generality we may assume that $\Omega = \Pi_i\langle 0, L_i\rangle$ for some positive interval lengths L_i. Taking n large enough there exist nonnegative integers $m_i, i = 1,...,d$ such that the rectangle $\tilde{\omega} = \Pi_i(m_iL_i/n, (m_i+1)L_i/n)$ is contained in the control region ω.
Let us consider again the subfamily Φ_n of eigenfunctions of the Dirichlet Laplacian on Ω consisting of $\phi_{kn}(x) = \phi_k(nx), k = 1,...,\infty$ and let us denote by $E_k = [\phi_n, \phi_{2n}, ..., \phi_{kn}]$ a k-dimensional subspace of $L^2(\Omega)$ spanned by the first k elements of the introduced subfamily. As these functions are periodic with period L_i/n in each variable, for any $p_0 \in E_k$ we have

$$\int_0^\infty \int_\omega p^2 dxdt \geq \int_0^\infty \int_{\tilde{\omega}} p^2 dxdt$$
$$= \frac{1}{n^d}\int_0^\infty \int_\Omega p^2 dxdt = \frac{1}{2n^d}||p_0||^2_{H^{-1}(\Omega)}.$$

The rest of the proof goes now as in the first case, just by following the steps from relation (30) onwards.

□

Remark 6 *Unlike the result of Lemma 6, the lower bound in* (29) *does not depend on an unknown multiplicative constant, but provides a precise bound that depends directly on the measure of the observation set* ω.

Note that the lower bounds approach the upper ones by increasing the size of the set ω, and the two bounds coincide in the limit case when the observation is performed on the entire domain Ω. This is in accordance with the fact that the Gramian in that case is just the inverse of the Laplacian operator (up to a multiplicative constant) and its eigenvalues equal $1/(2\lambda_k)$ (cf. Example 1 in the next section).

Obtaining the explicit lower bound (29) *for a general domain Ω remains an open problem.*

Remark 7 *The lower bound obtained in the last theorem is not expected to be optimal. Indeed, its proof is obtained by using periodic initial adjoint datum p_0. According to the improved regularity estimates obtained in Lemma 5, the corresponding observed energy should be the same, up to a compact reminder, as for the system driven by restriction of p_0 to the observability region. Therefore, the Rayleigh quotient in the first case (with periodic p_0) is n^d times smaller than in the latter one (when p_0 is cut off by zero outside the observability region) and we expect the lower bound in* (29) *can be improved by the same factor, i.e. that the estimate $\mu_k \geq \frac{1}{2}\lambda_{kn}$ holds. However, this improved bound resisted several attempts of proof, even in this special case of a rectangular domain, thus leaving the conjecture open.*

4 Relation to control problems

In this section we discuss the connection of the results above on the spectral decomposition and the eigenvalue decay of the Gramian operator, with the control properties of the heat equation. We also investigate the nature of the linear operator transforming the coefficients on the spectral basis of the Gramian into the classical eigenfunctions basis of the Dirichlet Laplacian.

Let us go back to the null control problem (1)+(2) . According to the classical control results (cf. [25] and the references therein), the control with minimal $L^2(\omega \times (0,T))$ norm is of the form $\exp(\Delta(T-t))p_0$, where p_0 is the optimal adjoint datum determined as the solution to

$$X_\omega^T p_0 = -z_H(T) = -\exp(T\Delta)z_0. \tag{31}$$

Here z_H denotes the homogeneous part of the solution to (1), run by the initial datum only, while X_ω^T stands for the finite time Gramian (25). Relation (31) ensures that the null control annihilates the action of the free dynamics, i.e. it steers the system to zero in the prescribed time horizon.

Taking the expansion $z_0 = \sum_j \alpha_j \phi_j$ of the initial datum with respect to eigenvectors of the Laplacian, we obtain

$$z_H(T) = \sum_j \exp(-\lambda_j T)\alpha_j \phi_j = \sum_k \beta_k \psi_k^T, \tag{32}$$

where the last sum is the expansion of the final state $z_H(T)$ with respect to the eigenvectors of the Gramian X_ω^T. Together with the equation (31), this implies that the exact representation of the optimal adjoint datum (i.e. of the one providing the minimal norm control) is of the form

$$p_0 = -\sum_k \frac{\beta_k}{\mu_k^T}\psi_k^T. \tag{33}$$

This is a theoretical result, which is hard to implement in applications. Indeed, the spectral decomposition of the Gramian operator is usually beyond our disposal. Its eigenvectors are much harder to construct than those of the underlying operator (the Dirichlet Laplacian in this paper). However, the last relation opens interesting issues to be discussed.

First of all, according to the classical null controllability results for the heat equation (e.g. [24]), p_0 belongs to the Hilbert space H defined as

$$H = \{p_0 : \int_0^\infty \int_\omega p^2 dxdt < \infty\}$$

and endowed with the canonical norm.

This space, is by construction sharp.

For $p_0 \in H$, the observed energy can be written as

$$\int_0^\infty \int_\omega p^2 dxdt = \langle X_\omega p_0 \mid p_0 \rangle_{L^2(\Omega)} = \sum \mu_k^T |\tilde{p}_{0k}|^2 < +\infty,$$

where $\tilde{p}_{0k}$ are Fourier coefficients of p_0 on the basis of eigenvectors of the Gramian.

Given that the eigenvalues of the Gramian μ_k^T decay polynomially, we observe that when representing the elements p_0 of H on the basis of the Gramian we get a summability condition on its coefficients with polynomial weights. This is in contrast with the inequality in (6), which is also sharp, and that assures that when representing the elements of H on the basis of the eigenfunctions of the Gramian one obtains rather exponentially degenerating weights.

This is a manifestation of the singularity of the transition matrix $M = (m_{kj})$ from one basis, consisting of eigenvectors of the Laplacian, to the other one, consisting of eigenvectors of the Gramian.

5 Numerical examples

Example 1, $\omega = \Omega$

As the Gramian is the solution to the operator Lyapunov equation (9), whose right hand side in this special case is just the identity, it implies $X_\omega = -(2\Delta)^{-1}$. In particular, for the eigenvalues we obtain the exact relation $\mu_k = 1/(2\lambda_k)$, which is in agreement with our bounds obtained in the previos section (cf. Remark 6 above).

Example 2, $\omega = (0, \pi/2)$**,** $\Omega = (0, \pi)$**.**

In order to obtain the eigendecomposition of the Gramian, we reduce the problem to a finite dimensional one. More precisely, instead of the Gramian operator X_ω we consider its finite dimensional approximation X_N defined as

$$X_N := P_{V_N} \circ (X_\omega)_{|V_N},$$

where $V_N = [\phi_1, \ldots, \phi_N] \subset L^2(\Omega)$ is the space spanned by first N eigenvectors of the Dirichlet Laplacian, P_{V_N} is the orthogonal projection from $L^2(\Omega)$ onto V_N, while $(X_\omega)_{|V_N}$ is the restriction of the Gramian to V_N.

According to [8, pp 491-492] the eigenvalues of X_N converge to the eigenvalues of the original operator as N goes to infinity. More precisely, let us denote by μ_k^N the eigenvalues of X_N repeated with their multiplicities and arranged in nonincreasing order. Then for any fixed k the sequence $(\mu_k^N)_N$ is nondecreasing and converges to μ_k as N goes to ∞. This can be shown by using the max-min characterization of the eigenvalues and the fact that the number of subspaces of V_N of a fixed dimension k increases with N.

In order to calculate eigenvalues of X_N, we first determine the matrix representation of X_N (in the basis of V_N consisting of the first N eigenvectors of the Dirichlet Laplacian). Due to the special choice of the observation set and the explicit formula for the eigenfunctions $\phi_k = \sqrt{2/\pi}\sin(kx)$, one obtains the following expressions for the matrix entries

$$\begin{aligned}&(X_N)_{ij}\\ &= \frac{2}{\pi}\int_0^\infty \exp\Big((-i^2-j^2)t\Big)dt \int_0^{\pi/2} \sin(ix)\sin(jx)dx\\ &= \begin{cases} \frac{1}{4i^2}, & i = j\\ \frac{2}{\pi(i^4-j^4)}\Big(-i\cos(\frac{i\pi}{2})\sin(\frac{j\pi}{2}) \\ \quad +j\cos(\frac{j\pi}{2})\sin(\frac{i\pi}{2})\Big), & i \neq j.\end{cases}\end{aligned} \tag{34}$$

Figure 1 depicts eigenvalues of X_N for $N = 100$ (calculated by the Matlab *eig* function) together with the lower and upper polynomial decay bound ensured by Theorem 8. Since ω is exactly one half of the whole domain, the factor n appearing in (29) equals 2, thus implying the lower bound of the form $1/(2n^d\lambda_{kn}) = 1/(16k^2)$.

In contrast to the polynomial decay rates obtained in the previous section, the figure reveals a two-fold behaviour of the eigenvalues of the Gramian. For approximately half of the eigenvalues, the decay rates coincide with the theoretical findings, satisfying both the lower and upper polynomial bound provided by Theorem 8. However, the next group of eigenvalues (represented by a steeply decreasing curve) decays exponentially. The calculation of the smallest twenty ones can not be considered significant due to the numerical resolution constraints.

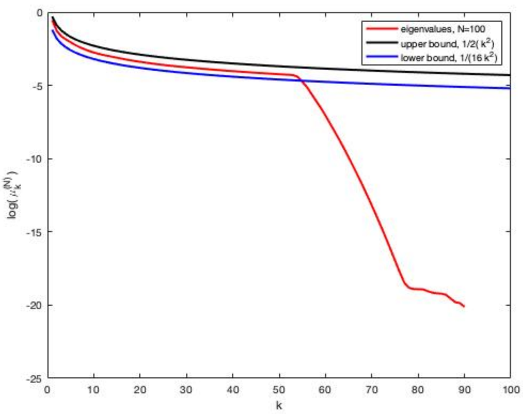
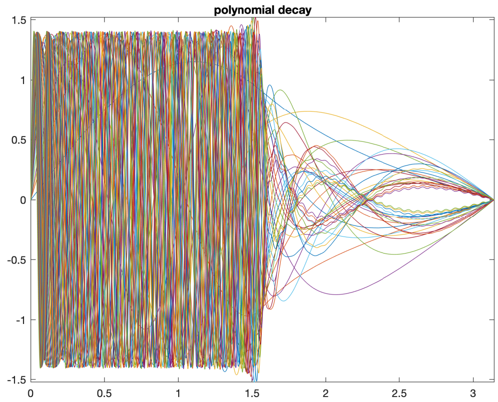


Fig. 1. Eigenvalues decay for the finite dimensional approximation of the Gramian X_N for $N = 100$. Relevant branch obeying the same polynomial law as the spectrum of X for $k \leq 55$, spurious exponential branch for $55 < k < 80$ and irrelevant branch for $k \geq 80$.

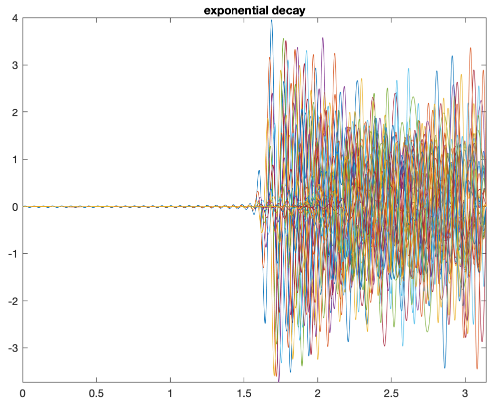
In order to get a better understanding of the phenomena observed in Figure 1 and to explain the departure from the expected polynomially decaying law for high-frequency eigenvalues, we depict the corresponding eigenvectors of the (finite dimensional approximation of the) Gramian X_N (Figure 2). Here, the eigenvectors are separated into two groups: those related to the polynomially decaying eigenvalues are presented in (a) part of the Figure, while the second plot shows the eigenvectors corresponding to the eigenvalues that decay exponentially.

A notable difference between two groups of eigenfunctions is observed immediately: the first group is (mostly) supported within the observation region ω, while the latter on its complement. Such kind of behaviour and the concentration phenomena can be understood by exploiting the min-max characterisation of the eigenvalues again. As it is known, the largest eigenvalue is the one maximizing the Rayleigh quotient:

$$\mu_1^{(N)} = \max_{p_0 \in V_N} \frac{\langle X_N p_0 \mid p_0 \rangle}{\|p_0\|^2_{L^2(\Omega)}} = \max_{p_0 \in V_N} \frac{\int_0^\infty \int_\omega p^2 dx dt}{\|p_0\|^2_{L^2(\Omega)}}, \tag{35}$$



(a)



(b)

Fig. 2. Eigenvectors of the Gramian X_N ($N = 100$) corresponding to (a) polynomially and (b) exponentially decaying eigenvalues.

where p is the solution to the heat equation (4) with the initial datum p_0. Actually, the first eigenvalue equals the maximal ratio of the observed energy compared to the initial one with respect to all possible choices of p_0.

The value of the Rayleigh quotient can be enlarged in two ways. First, by taking an initial datum consisting of low frequencies. Indeed, we know that the observed energy is less than the total one, which, by the classical energy estimates, decreases with frequency by $1/\lambda_k$ (cf. relation (7)). The second mechanism supporting large values of the Rayleigh quotient relies on the dissipation properties of the heat equation. In particular, the solution to the heat equation decays exponentially in space with respect to the distance from the support of its initial datum. For p_0 supported in ω, the amount of unobserved energy should therefore be small, in this way contributing to large values of the Rayleigh quotient.

Thus, in order to maximize the amount of the observed energy, one would like to employ p_0 that is supported on the observation set ω and is a combination of eigenfunctions corresponding to small frequencies. However, these requirements can not be met simultaneously. Indeed, the initial datum consisting of the lowest frequency only is supported on the entire domain. If its support is to be confined within the observation region, it should also incorporate high frequencies. In this way, the maximisation in (35) is obtained as a trade-off between two objectives: the choice of an initial datum p_0 that is supported in the observation region and the choice of p_0 consisting of low frequencies.

If we carefully analyse the eigenfunction corresponding to the largest eigenvalue $\mu_1^{(100)}$, we observe that this is exactly what happens. Indeed, expressed in terms of the eigenfunctions of the Laplacian it reads

$$\psi_1^{(100)} = 0.93\phi_1 + \sum_{k\geq 2} \alpha_k \phi_k \tag{36}$$

where α_k are the corresponding Fourier coefficients. We see that most of the energy corresponds to the lowest frequency, which, as stated above, supports larger values of the Rayleigh quotient. The rest of the expansion in (36) ensures that the largest portion of its mass is contained within the observability region, as it can be observed in Figure 3. This is in accordance to the second above stated mechanism that contributes to large values of the Rayleigh quotient.

The above arguments can be similarly applied to subsequent eigenfunctions: $\psi_2^{(100)}, \psi_3^{(100)}$, etc. Figure 3 depicts the first five of them (left) together with the distribution of their Fourier coefficients (right).

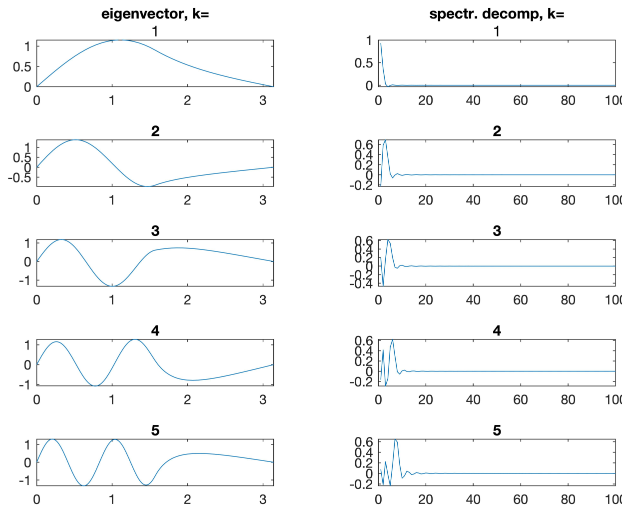


Fig. 3. Eigenvectors $\psi_k^{(100)}$ of the Gramian X_N (left) and their spectral decomposition (right), for $k = 1..5$.

The plotted eigenvectors $\psi_k^{(100)}$ tend to have most of its mass within the control region. Their spectral decomposition is dominated by few (usually two) harmonics whose frequencies shift forward with k. The same behaviour is exhibited by approximatively half of the eigenvectors, after which a new phenomenon occurs. Figure 4 depicts eigenfunctions $\psi_k^{(100)}$ for $k = 51, ..., 55$. For $k = 53$ the dominating frequency reaches the end of the spectrum, and subsequent eigenfunctions have spectral decomposition distributed along the whole spectrum. Moreover, their mass is shifted to the complement of the control region, in accordance to the Figure 2 b).

Such behaviour can be explained as follows. The eigenvectors of X_N have to be mutually orthogonal in $L^2(\Omega)$ and are spanned by the first N eigenfunctions of the Laplacian (since the image of X_N is spanned by them). These N eigenfunctions (of the Laplacian) constitute an orthonormalised set in $L^2(\Omega)$. However, when restricted to the control region (being the first half of the whole domain) just half of them (either even ones or odd ones) will be mutually orthogonal. For that reason, the first half of eigenvectors (of the Gramian X_N), which are mostly supported on the control region, exhaust the set of orthogonal functions supported on the control region and spanned by the first N eigenfunctions of the Laplacian. Consequently, in order to obey with the orthogonality, the support of the second half is shifted outside the control region.

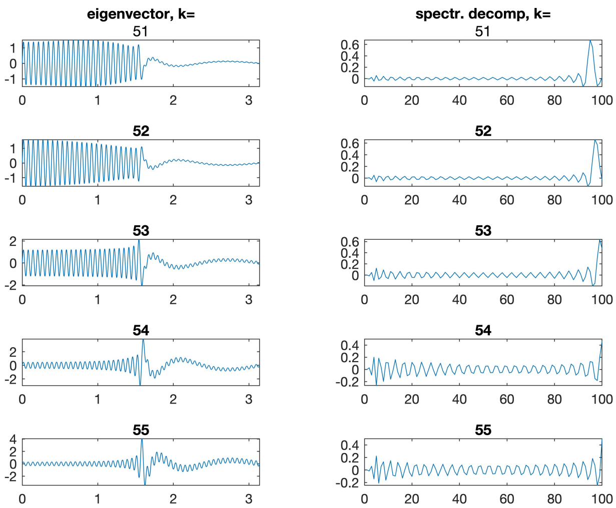


Fig. 4. Eigenvectors $\psi_k^{(100)}$ of the Gramian X_N (left) and their spectral decomposition (right).

The presented numerical results seem to contradict the theoretical bounds obtained in the previous Section. In contrast to the two-fold decay behaviour revealed in Figure 1, Corollary 4 ensures polynomially decay for the whole sequence of eigenvalues of the Gramian operator.

However, this contradiction is just an apparent one and is related to the finite-dimensional approximation of the Gramian defined on an infinite dimensional space. Although finite dimensional approximation allows for a portion of eigenvalues decaying exponentially, this behaviour disappears as the dimension N of the approximation goes to infinity. This can be clearly observed on Figure 5, which depicts eigenvalues for several approximation dimensions, together with the lower and upper polynomial decay bound ensured by Corollary 4. In particular, we note that the eigenvalues μ_k^N approach the predicted decay rate from below as N goes to infinity while keeping k fixed. This is in accordance to the general results for finite dimensional approximations of an operator stating that for a fixed k the sequence $(\mu_k^N)_N$ is nondecreasing and converges to μ_k as N goes to ∞ (cf. [8, pp 491-492]). Consequently, in the limit case all the eigenvalues will follow a polynomial decay law, thus obeying theoretical bounds (29) obtained in the previous section.

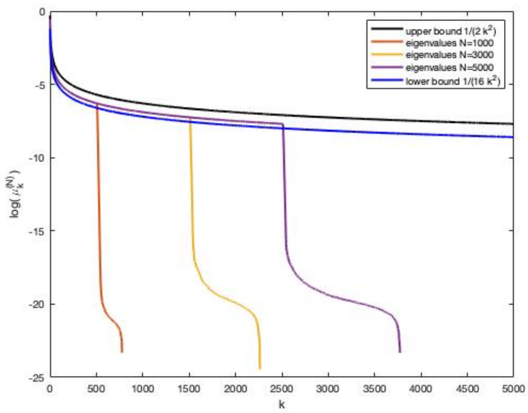


Fig. 5. Eigenvalues decay rates for various approximation dimensions N.

Such behaviour coincides with the one observed for the spectra of a discrete Laplacian operator (e.g. [24]). There it is shown that just a limited portion of the eigenvalues of the original, continuous operator can be approximated by the corresponding eigenvalues of its discrete counterparts. Furthermore, this portion (of the order $N^{-1/3}$) deteriorates as the approximation dimension N goes to infinity. Unlike there, for a fixed control region the percentage of eigenvalues of a discrete Gramian following the polynomial decay seems to be preserved and not influenced by the dimension N. From Figure 5 we note this ratio is close to $1/2$, and this for all approximation dimensions N considered. Of course, the ratio strongly depends on the size of the control region ω and it will increase/decrease by taking ω larger/smaller. In the limit case when the solution is observed on the whole domain Ω, the polynomial decay is obeyed by all the values of the finite dimensional approximation of the Gramian operator. This result is natural having in mind that in this special case the eigenvalues of the finite dimensional approximation $X_N = P_{V_N} \circ (-(2\Delta)^{-1})_{|V_N}$, coincide with the first N eigenvalues of the full Gramian.

Remark 8 *The analysis and results of this example generalize to any domain Ω obtained by dilatation of a rectangular observability region by a finite factor. The only difference is that the portion of eigenvalues of the finite dimensional approximation of the Gramian X_N following polynomial decay changes and should be approximately equal to the ratio $|\omega|/|\Omega|$.*

6 Conclusions and perspectives

The paper focuses on the most classical problem of null control of the heat equation in which the heat process evolves in a bounded domain, with Dirichlet boundary conditions, and the control is localized in an open nonempty subset. The goal was to obtain sharp estimates on the corresponding Grammian operator.

Previous results on the topic provide upper bounds on the eigenvalues, which under appropriate boundedness and finite-rank assumptions on the control operator, imply exponential decay rates [13,19]. However, lower bounds seem to be out of the focus of the research community. In addition, some important cases, like the distributed control one are not covered by the above mentioned assumptions.

The topic was also investigated in a more general context of Hankel singular values, that take into account both observability and controllability Gramian. In the case when two Gramians coincide, we get the framework of this paper. In [18] it was shown that Hankel singular values, and consequently the eigenvalues of the controllability Gramian, converge to zero faster than any polynomial rate, but assuming the control space is finite dimensional. Their exponential decay, under the same assumption, was obtained recently in [20]. The numerical study of the decay of Hankel singular values, in a completely finite dimensional setting, was studied in [7] in the context of the model order reduction.

The analysis performed in this paper is in an essential manner based on the spectral decomposition of the underlying Laplace operator. Employing the min-max characterisation of eigenvalues by Rayleigh quotients, and the existing observability and energy estimates, we have first obtained two-sided bounds on the decay of the eigenvalues of the Gramian operator. As already discussed, such bounds can be generalised to a wide class of parabolic-like equations for which observability inequalities have been proved and that allow a suitable spectral decomposition of the elliptic operator generating the dynamics (cf. Remark 3).

However, the lower and the upper bounds obtained in this way obey different asymptotic laws, an exponential and polynomial one, respectively, leaving a huge gap in between. Although they are derived from observability estimates that are sharp on the basis of eigenfunctions of the Laplacian, they turn not to be optimal when capturing the asymptotic behaviour of the eigenvalues of the Grammian.

Actually, we have then proved that the lower bound can be improved to obey the same polynomial law as the upper one, up to a multiplicative constant. This last result is the most striking one, in clear and unexpected contrast with the exponential decay of the eigenvalues of the Grammian for the boundary control of the $1d$ heat equation (see [13]).

The proof of the optimality of the polynomial bound, however, exploits the fact that the control region is an open subset of the domain where the equation evolves, and therefore does not apply in the context of boundary control or when the control acts in an arbitrary measurable subset of positive measure, two situations in which the null controllability of the heat equation in finite time is guaranteed. It would be interesting to analyze whether in these situations an exponential decay of eigenvalues may occur, as it was shown in [13,19] in the case of $1d$ boundary control.

All in all, for the heat equation with constant coefficients and the control localized in an open subdomain, we have derived sharp polynomial asymptotic rates for the eigenvalues of the corresponding Grammian, which are also confirmed by numerical examples. This provides the slow decay of the eigenvalues, which limits the efficiency of model reduction [5].

The methods introduced and developed in this paper can be applied in other contexts. But they also lead to very interesting and challenging open problems. We mention here some of them.

- More general equations and systems. We could treat for instance a broad class of parabolic-like equations and systems provided we have a suitable spectral decomposition of the elliptic operator generating the dynamics, meaning self-adjoint heat equations with time-independent coefficients in the principal part and lower potential terms [11,12].
  It would be interesting to see if our methods can be adapted to other models, for instance, involving space-time depending potentials or convective terms, for which such a spectral decomposition is not available.
- Boundary control problems [12] extending the $1d$ analysis in [13]. Note however that the nature of the optimal decay rate to be expected is to be clarified since now the control will not be finite-dimensional anymore as in $1d$.
- The heat equation is also known to be null-controllable with controls with support in measurable sets [3].

Whether the optimal decay bounds can be extended to that context is uncertain.

- Pointwise control of the $1d$ heat equation [13] or heat equations on networks [9]. In this case one could expect results aligned with [13]. But this would require further analysis as well.
- Our results could also be extended to symmetric systems of heat equations [2]. But the extension to asymmetric systems for which a spectral decomposition is not available would be an open problem.

Finally let us note, that, just as we have analyzed the relationship between the eigenvalues of the Laplacian operator and the corresponding infinite time Gramian operator, it would be of interest to obtain some kind of connection between the associated sequences of eigenvectors. The latter would lead to the study of the transition matrix from the basis given by the eigenvectors of the Laplacian to the basis consisting of the eigenvectors of the Gramian operator. As mentioned before, this matrix allows us to characterize the space of reachable states by different types of sequences: polynomial and exponential decaying ones. A good understanding of the transition and its properties will help us to understand the deterioration of the observability estimates for fractional systems and their eventual loss for the critical values of the fractional derivative [17].

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