Analysis and numerical solvability of backward-forward conservation laws

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Abstract

In this paper, we study the problem of initial data identification for weak-entropy solutions of the one-dimensional Burgers equation. This problem consists in identifying the set of initial data evolving to a given target at a final time. Due to the time-irreversibility of the Burgers equation, some target functions are unattainable from solutions of this equation, making the identification problem under consideration ill-posed. To get around this issue, we introduce a non-smooth optimization problem, which consists in minimizing the difference between the predictions of the Burgers equation and the observations of the system at a final time in $L^2(\mathbb{R})$ norm. Here, we characterize the set of minimizers of the aforementioned non-smooth optimization problem. One of the minimizers is the backward entropy solution, constructed using a backward-forward method. Some simulations are given using a wave-front tracking algorithm.

Keywords: Backward-forward method, Identification problems; Conservation Laws; Weak-entropy solutions; Non-smooth optimization problem; Wave-front tracking algorithm.

AMS classification: 35L65, 35F20, 93B30, 35R30.

1 Introduction

1.1 Presentation of the Problem

Initial data identification problems consist in finding the origin of physical phenomena, governed for instance by partial differential equations (PDEs), from a set of observations at a given time. These arise naturally in meteorology, oceanography or climatology [32, 45, 23, 44, 30, 5, 19] to improve the forecasts of a model. Finding optimal positions or shapes of sensors [40, 41, 42] also lead to the study of identification problems.

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Initial identification problems need to be carefully addressed, depending on each type of PDEs.

- In the case of parabolic PDEs, the high and instant regularization effect induces the non-existence of initial data for which the corresponding solution evolves to given not-necessary regular target functions, and causes numerical instabilities when solving the PDE backwards in time. In [34], the authors solve an identification problem for the heat equation with applications in pollution source localization. Note however that, when the target is attainable, the initial datum whose the corresponding trajectory evolves to this target, is unique as seen in [36].
- In the case of nonlinear hyperbolic PDE as (1), the backward uniqueness property fails due to the presence of discontinuities (so-called *shocks*), i.e multiple initial data may evolve to the same attainable target function. Moreover, due to the time-irreversibility of nonlinear hyperbolic PDEs, a target function u^T can be unattainable, that is to say that there is non-existence of initial data leading to the target function u^T.

In this paper, we study the latter case. More precisely, we consider the scalar conservation laws

$$\begin{cases} \partial_t u(t,x) + \frac{1}{2}\partial_x u^2(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u_0(x). \end{cases} \tag{1}$$

Kruzkov's theory [31] provides existence and uniqueness of a weak-entropy solution u of (1) with initial datum $u_0 \in L^\infty(\mathbb{R})$. Let $T > 0$ and $u^T \in L^\infty(\mathbb{R})$ a given function, the goal is to find the set of initial data generating weak-entropy solutions of (1) that are as close as possible to u^T at time T in $L^2(\mathbb{R})$-norm. This leads to solve the following non-smooth optimization problem

$$\inf_{u_0 \in \mathcal{U}^0_{\text{ad}}} J_0(u_0) := \|u^T(\cdot) - u(T,\cdot)\|_{L^2(\mathbb{R})}, \tag{$\mathcal{O}_T$}$$

where u is the weak-entropy solution of (1) and $\mathcal{U}^0_{\text{ad}}$ is the class of admissible initial data in $L^\infty(\mathbb{R})$ with compact essential support (see (10) for more details). The study of initial data identification for (1) is motivated by the minimization of the sonic boom effects generated by supersonic aircrafts, which are modeled by an augmented Burgers equation [15, 3, 2, 35].

1.2 State of the art and main results

Initial data identification problems for (1) in the case of attainable targets have already been studied in [12, 13, 24, 1, 28, 16, 35]. In [16, Theorem 3.1, Corollary 3.2], [28, Corollary 1] or [24], the authors prove that u^T is truly attainable in an exact manner by a solution of (1) if and only if u^T satisfies the one-sided Lipschitz condition [8, 25, 39, 21], i.e

$$\partial_x u^T \leq \frac{1}{T} \text{ in the sense of distributions.} \tag{2}$$

When u^T is an attainable target, the authors in [28] prove that the set of initial data evolving to u^T is a convex set. Later on, the aforementioned set was fully characterized in [16, 22] using the classical Lax-Hopf formula. In [35], an alternative proof is given using backward generalized characteristics.

The optimization problem $(\mathcal{O}_T)$ with attainable targets has also been studied in [12, 13, 1] from numerical points of view. Since the weak-entropy solution u of (1) may contain shocks even if the initial datum is a smooth function, this generates important added difficulties to solve $(\mathcal{O}_T)$ that have been the object of intensive study in the past, see [38, 37, 9, 10, 6, 7, 4] and the references therein. More precisely, in [9, 10, 6, 7], the derivative of the cost function J_0 in $(\mathcal{O}_T)$ is regarded in a weak sense by requiring strong conditions on the set of initial data. This leads to require that weak-entropy solutions of (1) have a finite number of non-interacting jumps. When J_0 is weakly differentiable, gradient descent methods have been implemented in [12, 13, 1] to solve numerically the optimization problem $(\mathcal{O}_T)$. In the cases where it was applied successfully, only one possible initial datum emerges, namely the backward entropy solution, see Remark 1. This is mainly due to the numerical viscosity that numerical schemes introduce to gain stability. To find some multiple minimizers, the authors in [28] use a filtering step in the backward adjoint solution.

In this article, we give a full characterization of the set of minimizers of the optimization problem $(\mathcal{O}_T)$. More precisely, we prove that the backward entropy solution, denoted by $S_T^-(u^T)$, is a minimizer of $(\mathcal{O}_T)$ using a backward-forward method described in Section 2.1. Then, we show that u_0^* is a minimizer of $(\mathcal{O}_T)$ if and only if the weak-entropy solution of (1) with initial datum u_0^* coincides, at time T, with the weak-entropy solution of (1) with initial datum $S_T^-(u^T)$ using variational methods. Contrary to [12, 13, 1], we do not require strong assumptions on the set of initial data, i.e weak-entropy solutions of (1) may have a countable number of interacting jumps. Finally, we construct numerically random minimizers of $(\mathcal{O}_T)$ based on a wave-front tracking algorithm.

1.3 Some related open problems

Let us address some related open questions and possible extensions of this work.

- It would be interesting to study the optimization problem $(\mathcal{O}_T)$ in L^1-norm, which is the natural distance in the framework of conservation laws. This problem leads to additional difficulties since $x \mapsto \|x\|_{L^1(\mathbb{R})}$ is not a differentiable function.

- It would be also interesting to consider a convex-concave function as a flux function in (1) which is, for instance, a more realistic choice to describe the flow of pedestrian [17, 14]. The main difficulty comes from the existence of discontinuities (called non-classical shocks) violating standard admissibility entropy conditions such that the Oleinik inequality.

- We could also study a Burgers equation with source terms. In this case, some suitable conditions on source terms have to be determined to use the backward-froward method described in this paper. For instance, the backward operator $S_t^-(u^T)$ defined in Section 2.1 associated to

$$\begin{cases} \partial_t u(t,x) + \frac{1}{2}\partial_x u^2(t,x) = -u^3(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(T,\cdot) = u^T(x), & x \in \mathbb{R}. \end{cases}$$

  may blow up at time $t < T$.

- We can also investigate systems of conservation laws in one dimension (Euler equations, Saint-Venant equations, Aw-Rascle-Zhang traffic flow model). Note that, as soon as the backward-forward operator $S_T^+(S_T^-)$ is well-defined, $S_T^+(S_T^-)(u^T)$ may give a good candidate to solve initial data identification problems for systems of conservation laws.

- We may consider a multi-dimensional conservation of laws in a numerical point of view. For instance, a fractional steps method [18, 33, 29] (or splitting method) may be implemented to solve an identification problem of a two-dimensional equation of conservation laws.

The article is organized as follows. In section 2, we describe the backward-forward method and we recall some known results on initial data identification problems for (1). In Section 3, we state the main results where the proofs are given in Section 4. In Section 5, we run some simulations using a wave-front tracking method.

2 Notations and comments

We now introduce some notations and recall some known results which will be essential to state the main theorem of this paper. In the sequel, we denote by $BV(\mathbb{R})$ the class of functions of bounded variation, see [21, Definition 1.7.1]. If $g \in BV(\mathbb{R})$, we use the notation $g(x-) := \lim_{\substack{y\to x\\ y<x}} g(y)$ and $g(x+) := \lim_{\substack{y\to x\\ x<y}} g(y)$. Let $f \in BV(\mathbb{R})$, we denote by $X(f)$ the set defined by

$$X(f) := \{x \in \mathbb{R}/f(x-) = f(x+)\}, \tag{3}$$

and $\mathrm{Supp}(f)$ stands for the essential support of the function f. Let Ω be a domain in $\mathbb{R}$, $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$ denotes the set of infinitely differentiable functions with compact support. Let two distributions $T_1, T_2 \in \mathcal{D}'(\Omega)$, we say that $T_1 \leq T_2$ in the sense of distributions if

$$\forall \varphi \in \mathcal{D}(\Omega), \varphi \geq 0, \quad \langle T_1, \varphi\rangle \leq \langle T_2, \varphi\rangle,$$

where $\langle \cdot, \cdot\rangle$ is a duality bracket between $\mathcal{D}'$ and $\mathcal{D}$.

2.1 The backward-forward method

For the sake of completeness, we recall the definition of a weak-entropy solution of (1).

Definition 2.1 *We say that $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}) \cap C^0(\mathbb{R}^+, L^1_{loc}(\mathbb{R}))$ is a weak-entropy solution if for every $k \in \mathbb{R}$, for all $\varphi \in C_c^1(\mathbb{R}^2, \mathbb{R}^+)$,*

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} (|u-k|\partial_t\varphi + \frac{1}{2}\operatorname{sgn}(u-k)(u^2-k^2)\partial_x\varphi)dxdt + \int_{\mathbb{R}} |u_0 - k|\varphi(0,x)dx \geq 0.$$

Kruzkov's theory [31] provides existence and uniqueness of a weak-entropy solution $(t,x) \to S_t^+(u_0)(x)$ of (1) with initial datum $u_0 \in L^\infty(\mathbb{R})$. For a given function u^T, we introduce the function $(t,x) \to S_t^-(u^T)(x)$ as follows: for every $t \in [0,T]$, for a.e $x \in \mathbb{R}$,

$$S_t^-(u^T)(x) = S_t^+(x \to u^T(-x))(-x). \tag{4}$$

Note that $(t,x) \to S_t^-(u^T)(x)$ is the weak-entropy solution of

$$\begin{cases} \partial_t u(t,x) - \frac{1}{2}\partial_x u^2(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R},\\ u(0,x) = u^T(x), & x \in \mathbb{R}. \end{cases}$$

Using the change of variable $t \to T-t$, $(t,x) \to S_t^-(u^T)(x)$ is also the weak-entropy solution of

$$\begin{cases} \partial_t u(t,x) + \frac{1}{2}\partial_x u^2(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R},\\ u(T,x) = u^T(x), & x \in \mathbb{R}. \end{cases}$$

Thus, the backward-forward method consists in solving backward in time the PDE (1) with final target u^T and then solving it forward in time with initial datum $S_T^-(u^T)$.

Remark 1 *The solutions* $S_t^+(u_0)$ *and* $S_t^-(u^T)$ *may be regarded as the zero viscosity limit of the solutions* $S_t^{+,\epsilon}(u_0)$ *and* $S_T^{-,\epsilon}(u^T)$ *respectively where* $S_t^{+,\epsilon}(u_0)$ *and* $S_t^{-,\epsilon}(u^T)$ *are defined as follows:* $S_t^{+,\epsilon}(u_0)$ *is the solution of the following viscous Burgers equation*

$$\begin{cases} \partial_t u(t,x) + \frac{1}{2}\partial_x u^2(t,x) = +\epsilon \partial_{xx}^2 u(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

and $S_t^{-,\epsilon}(u^T)$ *is the solution of the following backward equation*

$$\begin{cases} \partial_t u(t,x) + \frac{1}{2}\partial_x u^2(t,x) = -\epsilon \partial_{xx}^2 u(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(T,x) = u^T(x), & x \in \mathbb{R}. \end{cases}$$

Using the change of variable $(t,x) \to (T-t,-x)$*, we notice that the backward equation above is well-defined. Thus,* $S_T^-(u^T)$ *is called the backward entropy solution. Note that the construction involving* $S_t^{\pm}$ *is also related to scattering theory [27, 26]*

For any attainable target u^T, we have $S_T^+(S_T^-(u^T)) = u^T$ as seen in [16, Theorem 3.1, Corollary 3.2] and [28, Corollary 1]. Note that there exist target functions u^T verifying $S_T^+(S_T^-(u^T)) \neq u^T$. For instance, if $u^T(\cdot) = -\mathbb{1}_{(-\infty,0)}(\cdot) + \mathbb{1}_{(0,\infty)}(\cdot)$ then

$$S_T^+(S_T^-(u^T))(x) = \begin{cases} -1 & \text{if } x < -T, \\ \frac{x}{T} & \text{if } -T \leq x \leq T, \\ 1 & \text{if } T < x. \end{cases}$$

These targets are called unattainable targets.

2.2 Identification problem for attainable targets

Fix $u^T \in L^\infty(\mathbb{R})$, we introduce the set

$$\mathcal{I}^+(u^T) = \{u_0 \in L^\infty(\mathbb{R}) \ / \ S_T^+(u_0) = u^T\}. \tag{5}$$

From [16, Corollary 3.2], $\mathcal{I}^+(u^T) \neq \emptyset$ if and only if a suitable representative of u^T satisfies the Oleinik condition [8, 25, 39, 21], i.e for every $x \in \mathbb{R}$ and $y \in \mathbb{R}^+\backslash\{0\}$,

$$u^T(x+y) - u^T(x) \leq \frac{y}{T}. \tag{6}$$

The following theorem stated in [35, Theorem 1] (see also [16, 22]) gives a full characterization of the set of initial data $u_0 \in L^\infty(\mathbb{R})$ such that $S_T^+(u_0) = u^T$.

Theorem 2.1 ([35]) *Let* $T > 0$ *and let a suitable representation of* $u^T \in L^\infty(\mathbb{R})$ *satisfy the Oleinik condition* (2)*. Then the initial data* $u_0 \in L^\infty(\mathbb{R})$ *satisfies* $S_T^+(u_0) = u^T$ *if and only if the following statements holds. For any* $(x,y) \in X(u^T) \times \mathbb{R}$,

$$\int_{x-Tu^T(x)}^{y} S_T^-(u^T)(s)\, ds \leq \int_{x-Tu^T(x)}^{y} u_0(s)\, ds, \tag{7}$$

For any $(x,y) \in X(u^T)^2$,

$$\int_{x-Tu^T(x)}^{y-Tu^T(y)} S_T^-(u^T)(s)\, ds = \int_{x-Tu^T(x)}^{y-Tu^T(y)} u_0(s)\, ds, \tag{8}$$

where $X(u^T)$ *is defined in* (3) *and* $S_T^-(u^T)$ *is defined in* (4)*.*

Remark 2 *When $u^T \in L^\infty(\mathbb{R})$ satisfies the Oleinik condition* (2)*, then $u^T \in BV_{loc}(\mathbb{R})$. Thus, $X(u^T)$ is well-defined.*

Theorem 2.1 points out the richness and the diversity of initial data evolving to the same target at time T (see Figure 1).

- There exists $u_0 \in L^\infty(\mathbb{R})$ such that $S_T^+(u_0) = u^T$ with $\min_{x\in\mathbb{R}} u_0(x) < \min_{x\in\mathbb{R}} u^T(x)$ and/or $\max_{x\in\mathbb{R}} u^T(x) < \max_{x\in\mathbb{R}} u_0(x)$, see Figure 1.
- The set $\mathcal{I}^+(u^T)$ defined in (5) is a convex cone having as unique extremal point at its vertex the map $S_T^-(u^T)$, see [16, Proposition 5.2]; for any $u_0 \in \mathcal{I}^+(u^T)$, for every $\eta > 0$, $u_0^\eta = S_T^-(u^T) + \eta(u_0 - S_T^-(u^T)) \in \mathcal{I}^+(u^T)$.

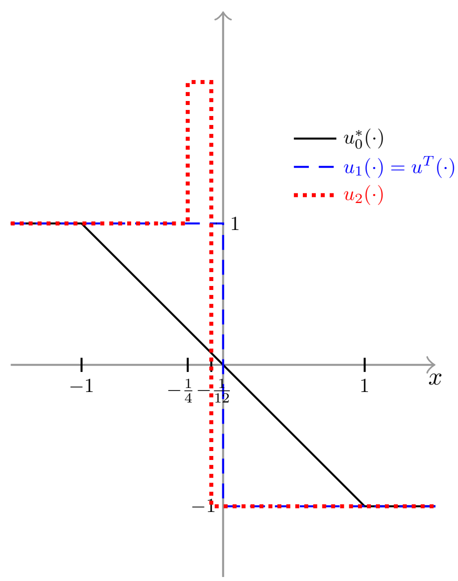


Figure 1: Three initial data $u_0^*(-)$, $u_1(--)$ and $u_2(\cdots)$ leading to an attainable target $u^T(\cdot) := \mathbb{1}_{(-\infty,0)}(\cdot) - \mathbb{1}_{(0,+\infty)}(\cdot)$ at time $T = 1$ along forward entropic evolution.

3 Main results

Let $T > 0$, $C > 0$, $K^T := [a^T, b^T] \subset \mathbb{R}$ be a compact set. We consider a target function $u^T \in L^\infty(\mathbb{R})$ that satisfies

$$\text{Supp}(u^T) \subset K^T \text{ and } \|u^T\|_{L^\infty(\mathbb{R})} \leq C. \tag{9}$$

Let $K_0 := [a_0, b_0]$ verify $[a^T - TC, b^T + TC] \subset K_0$ (see an illustration in Figure 3). We study the non-smooth optimization problem

$$\inf_{u_0 \in \mathcal{U}_{\text{ad}}^0} J_0(u_0) := \|u^T(\cdot) - S_T^+(u_0)(\cdot)\|_{L^2(\mathbb{R})}, \tag{$\mathcal{O}_T$}$$

where S_T^+ is defined in Section 2.1 and $\mathcal{U}_{\text{ad}}^0$ is the class of admissible initial data defined by

$$\mathcal{U}_{\text{ad}}^0 = \{u_0 \in L^\infty(\mathbb{R}) \ / \ \|u_0\|_{L^\infty(\mathbb{R})} \leq C \text{ and } \text{Supp}(u_0) \subset K_0\}. \tag{10}$$

Theorem 3.1 characterizes the set of minimizers of $(\mathcal{O}_T)$ (see an illustration in Figure 2).

Theorem 3.1 *Let $T > 0$, $C > 0$, $K^T := [a^T, b^T] \subset \mathbb{R}$ and let $u^T \in L^\infty(\mathbb{R})$ satisfy (9). The initial datum $u_0^* \in L^\infty(\mathbb{R})$ is a minimizer of $(\mathcal{O}_T)$ if and only if $u_0^* \in \mathcal{U}_{ad}^0$ satisfies $S_T^+(u_0^*) = S_T^+(S_T^-(u^T))$.*

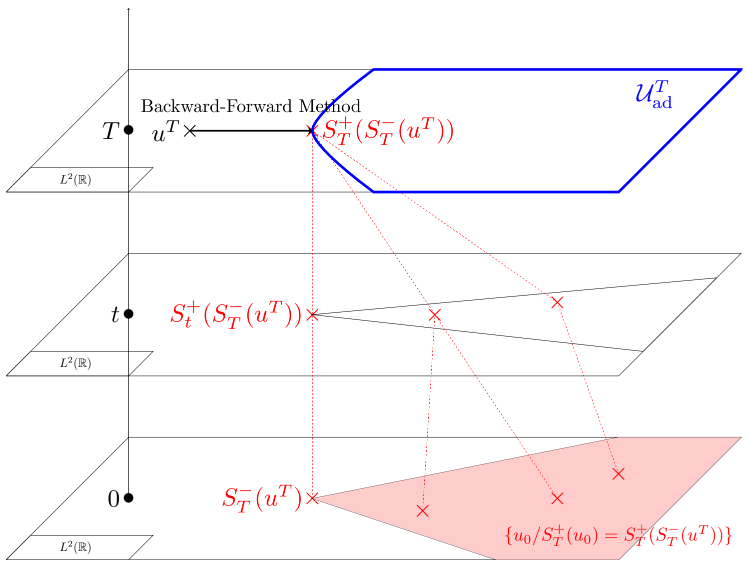


Figure 2: The backward-forward solution $S_T^+(S_T^-(u^T))$ is the projection of u^T onto the set of attainable target functions. The shaded area in red at time $t = 0$ represents the set of minimizers of $(\mathcal{O}_T)$.

Corollary is a direct consequence of Theorem 3.1 and the full characterization of the set $\{u_0 \in L^\infty(\mathbb{R})/S_T^+(u_0) = S_T^+(S_T^-(u^T))\}$ given in Theorem 2.1.

Corollary 3.1 *Let $T > 0$, $C > 0$, $K^T := [a^T, b^T] \subset \mathbb{R}$ and let $u^T \in L^\infty(\mathbb{R})$ satisfy (9). The map $u_0^* \in L^\infty(\mathbb{R})$ is a minimizer of $(\mathcal{O}_T)$ if and only if the following statements hold. For any $(x, y) \in X(S_T^+(S_T^-(u^T))) \times \mathbb{R}$,*

$$\int_{x-TS_T^+(S_T^-(u^T))(x)}^{y} S_T^-(u^T)(s)\, ds \le \int_{x-TS_T^+(S_T^-(u^T))(x)}^{y} u_0^*(s)\, ds, \tag{11}$$

For any $(x, y) \in X(S_T^+(S_T^-(u^T)))^2$,

$$\int_{x-TS_T^+(S_T^-(u^T))(x)}^{y-TS_T^+(S_T^-(u^T))(y)} S_T^-(u^T)(s)\, ds = \int_{x-TS_T^+(S_T^-(u^T))(x)}^{y-TS_T^+(S_T^-(u^T))(y)} u_0^*(s)\, ds, \tag{12}$$

where $X(S_T^+(S_T^-(u^T)))$ is defined in (3) and $S_T^-(u^T)$ is defined in (4).

Remark 3

- *The constraints $\|u_0\|_{L^\infty(\mathbb{R})} \le C$ and $Supp(u_0) \subset K_0$ in (10) are used to guarantee the existence of minimizers of $(\mathcal{O}_T)$. Moreover, the assumption $[a^T - TC, b^T + TC] \subset K_0$ is required to have $S_T^-(u^T) \in \mathcal{U}_{ad}^0$.*
- *From (26), $S_T^+(S_T^-(u^T)) \in BV(\mathbb{R})$. Thus, for any $x \in \mathbb{R}$, $S_T^+(S_T^-(u^T))(x-)$ and $S_T^+(S_T^-(u^T))(x+)$ exist and then $X(S_T^+(S_T^-(u^T)))$ is well-defined.*

- *We assume that the given target u^T is attainable. Since $S_T^+(S_T^-(u^T)) = u^T$, Theorem 3.1 and Corollary 3.1 give a full characterization of initial data generating weak-entropy solutions of* (1) *that coincide with u^T at time T, as in [22, 35]. Note that there exist initial data, generating weak solutions of* (1) *that coincide with u^T at time T, such that the inequalities* (11) *and* (12) *do not hold. For instance, we choose $T = 1$ and $u^T(\cdot) = \mathbb{1}_{(-\infty,0)}(\cdot) - \mathbb{1}_{(0,+\infty)}(\cdot)$ then the weak solution u defined by*

$$u(t,x) = \begin{cases} \mathbb{1}_{(-\infty,4t-2)}(x) + 7\mathbb{1}_{(4t-2,3t-\frac{3}{2})}(x) - \mathbb{1}_{(3t-\frac{3}{2},+\infty)}(x) & \text{if } t < \frac{1}{2}, \\ \mathbb{1}_{(-\infty,0)}(x) - \mathbb{1}_{(0,+\infty)}(x) & \text{if } \frac{1}{2} \leq t, \end{cases}$$

satisfies $u(T,\cdot) = u^T$ and $S_T^+(u(0,\cdot)) \neq u^T$.

The next section is devoted to the proof of Theorem 3.1 that is structured as follows. From [16, Theorem 3.1, Corollary 3.2], [28, Corollary 1] or [24], there exists $u_0 \in L^\infty(\mathbb{R})$ such that $S_T^+(u_0) = q$ if and only if $q \in L^\infty(\mathbb{R})$ satisfies the one-sided Lipschitz condition (2). Let $K_1 \subset \mathbb{R}$ be a set such that $[a_0 - TC, b_0 + TC] \subset K_1$ with $K_0 := [a_0, b_0]$, the optimization problem $(\mathcal{O}_T)$ is equivalent to

$$\min_{q\in\mathcal{U}_{\text{ad}}^T} J_1(q) := \|u^T - q\|_{L^2(\mathbb{R})}, \tag{13}$$

where the admissible set $\mathcal{U}_{\text{ad}}^T$ is defined by

$$\mathcal{U}_{\text{ad}}^T = \{q \in L^\infty(\mathbb{R})/\ \partial_x q \leq \frac{1}{T},\ \|q\|_{L^\infty(\mathbb{R})} \leq C \text{ and } \mathrm{Supp}(q) \subset K_1\}. \tag{14}$$

The optimization problem (13) admits a unique minimizer using Hilbert projection Theorem. Since the cost function J_1 in (13) is a differentiable function (unlike J_0 in $(\mathcal{O}_T)$), we can write down the first-order optimality conditions for (13). Thus, together with the full characterization of the set $\{u_0 \in BV(\mathbb{R})/S_T^-(u_0) = S_T^-(u^T)\}$ (see Theorem 4.1), we prove that $q = S_T^+(S_T^-(u^T))$ is the minimizer of (13). As a consequence, u_0^* is a minimizer of $(\mathcal{O}_T)$ if and only if $S_T^+(u_0^*) = S_T^+(S_T^-(u^T))$.

4 Proof of Theorem 3.1

4.1 On the solution of the optimization problem (13).

The following proposition will be an essential tool to solve (13). Let $u_0 \in L^\infty(\mathbb{R})$, we introduce the set

$$\mathcal{I}^-(u_0) = \{u^T \in L^\infty(\mathbb{R})\ /\ S_T^-(u^T) = u_0\}. \tag{15}$$

From (4) and [16, Corollary 3.2], $\mathcal{I}^-(u_0) \neq \emptyset$ if and only if for every $x \in \mathbb{R}$ and $y \in \mathbb{R}^+\backslash\{0\}$,

$$u_0(x+y) - u_0(x) \geq -\frac{y}{T}. \tag{16}$$

Proposition 4.1 gives a full characterization of the set of functions $u_T \in L^\infty(\mathbb{R})$ such that $S_T^-(u_T) = u_0$.

Proposition 4.1 *Let $T > 0$ and let a suitable representation of $u_0 \in L^\infty(\mathbb{R})$ satisfy* (16)*. Then a map $u_T \in L^\infty(\mathbb{R})$ satisfies $S_T^-(u_T) = u_0$ if and only if the following statements hold. For any $(x,y) \in X(u_0) \times \mathbb{R}$,*

$$\int_{x+Tu_0(x)}^{y} S_T^+(u_0)(s)\,ds \geq \int_{x+Tu_0(x)}^{y} u_T(s)\,ds, \tag{17}$$

For any $(x, y) \in X(u_0)^2$,

$$\int_{x+Tu_0(x)}^{y+Tu_0(y)} S_T^+(u_0)(s)\, ds = \int_{x+Tu_0(x)}^{y+Tu_0(y)} u_T(s)\, ds, \tag{18}$$

where $X(u_0) = \{x \in \mathbb{R} \ / \ u_0(x-) = u_0(x+)\}$.

Proposition 4.1 is a direct consequence of Theorem 2.1 noticing that $S_T^-(u_T) : x \to S_T^+(x \to u_T(-x))(-x)$.

Lemma 4.1 *The optimization problem* (13) *admits a unique minimizer* $S_T^+(S_T^-(u^T))$.

Proof. The proof is divided in two steps.

Step 1: Existence of minimizers of (13)**.**
By definition of J_1 in (13), it is enough to prove that $\mathcal{U}_{\text{ad}}^T$ defined in (14) is a closed convex set of $L^2(\mathbb{R})$ using Hilbert projection Theorem.

- Assuming that $q_1, q_2 \in \mathcal{U}_{\text{ad}}^T$, we immediately have, for every $\alpha \in [0, 1]$, $\alpha q_1 + (1-\alpha) q_2 \in \mathcal{U}_{\text{ad}}^T$. Thus, $\mathcal{U}_{\text{ad}}^T$ is a convex set.
- Assuming that $q_n \in \mathcal{U}_{\text{ad}}^T$ converges to q in $L^2(\mathbb{R})$ then q_n converges to q in the sense of distributions and by passing to the limit in $\partial_x q_n \leq \frac{1}{T}$, we have $\partial_x q \leq \frac{1}{T}$. Since $\|q_n\|_{L^\infty(\mathbb{R})} \leq C$ and using that the closed ball $B_{L^\infty(\mathbb{R})}$ is compact in the weak* topology $\sigma(L^\infty, L^1)$ [11, Theorem 3.16], q_n converges, (up to a subsequence, still denoted by q_n) to $q \in L^\infty(\mathbb{R})$ in the weak* topology of $L^\infty(\mathbb{R})$. Moreover, from [11, Proposition 3.13], $\|q\|_{L^\infty(\mathbb{R})} \leq \liminf_n \|q_n\|_{L^\infty(\mathbb{R})} \leq C$. Using that q_n converges to q in $L^2(\mathbb{R})$, q_n converges a.e to q. Moreover, since $\mathrm{Supp}(q_n) \subset K_1$, we have $q_n(x) = 0$ for a.e $x \in \mathbb{R} \backslash K_1$. Therefore, we have $\mathrm{Supp}(q) \subset K_1$ and we conclude that $q \in \mathcal{U}_{\text{ad}}^T$. Thus, $\mathcal{U}_{\text{ad}}^T$ is a closed set.

Step 2: First-order optimality conditions.
Our aim is to prove that,

$$S_T^+(S_T^-(u^T)) \in \mathcal{U}_{\text{ad}}^T, \tag{19}$$

and for any admissible perturbation $h \in \mathcal{T}_{\mathcal{U}_{\text{ad}}^T}(S_T^+(S_T^-(u^T)))$,

$$-\int_{\mathbb{R}} \left(u^T(x) - S_T^+(S_T^-(u^T))(x)\right) h(x)\, dx \geq 0. \tag{20}$$

Above, $\mathcal{T}_{\mathcal{U}_{\text{ad}}^T}(S_T^+(S_T^-(u^T)))$ is a set of functions $h \in L^\infty(\mathbb{R})$ such that, for any sequence of positive real numbers ϵ_n decreasing to 0, there exists a sequence of functions $h_n \in L^\infty(\mathbb{R})$ converging to h as $n \to \infty$ and $S_T^+(S_T^-(u^T)) + \epsilon_n h_n \in \mathcal{U}_{\text{ad}}^T$ for every $n \in \mathbb{N}$. If (19) and (20) hold then $S_T^+(S_T^-(u^T))$ is a critical point of (13). Since J_1 is a strictly convex function, $S_T^+(S_T^-(u^T))$ is the unique minimizer of (13).

We now prove (19): since u^T satisfies (9), we have $\|u^T\|_{L^\infty(\mathbb{R})} \leq C$. By using the definition of S_T^+ and S_T^- and the maximum principle fulfilled by weak-entropy solutions [43, Theorem 2.3.5], we have

$$\|S_T^-(u^T)\|_{L^\infty(\mathbb{R})} \leq C \text{ and } \|S_T^+(S_T^-(u^T))\|_{L^\infty(\mathbb{R})} \leq C. \tag{21}$$

From (9), by definition of S_T^- and using the finite velocity of propagation, we have $\mathrm{Supp}(S_T^-(u^T)) \subset [a^T - TC, b^T + TC] \subset K_0$ (see Figure 3). Therefore, together with (21),

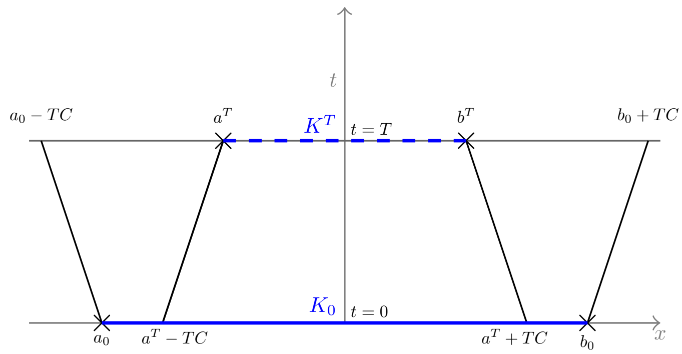


Figure 3: Illustration of K^T(−−) and K_0(—) defined in Section 3.

$$S_T^-(u^T) \in \mathcal{U}_{\text{ad}}^0. \tag{22}$$

Moreover, by definition of S_T^+ and using $[a_0 - TC, b_0 + TC] \subset K_1$ and [21, Theorem 6.2.3], we have

$$S_T^+(u_0) \in \mathcal{U}_{\text{ad}}^T \text{ for any } u_0 \in \mathcal{U}_{\text{ad}}^0, \tag{23}$$

where $\mathcal{U}_{\text{ad}}^T$ is defined in (14). We replace u_0 in (23) by $S_T^-(u^T)$ and we deduce that (19) holds.

We now prove (20) : let $x \in X(S_T^-(u^T))$ with X defined in (3) and we introduce the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F : y \mapsto \int_{x+Tf'(S_T^-(u^T)(x))}^{y} \left(u^T(s) - S_T^+(S_T^-(u^T))(s)\right) ds. \tag{24}$$

Since $u^T \in L^\infty(\mathbb{R})$ satisfies (9), we have

$$u^T \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}). \tag{25}$$

By definition of S_T^- and S_T^+ (see Section 2.1), from [21, Theorem 11.2.2] and $u^T \in L^\infty(\mathbb{R})$, we have $S_T^+(S_T^-(u^T)) \in BV_{\text{loc}}(\mathbb{R})$. Therefore, together with (19), we deduce that for any $T > 0$,

$$S_T^+(S_T^-(u^T)) \in BV(\mathbb{R}) \text{ and } S_T^+(S_T^-(u^T)) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}). \tag{26}$$

From (24), (25) and (26), we have that

$$F \in W^{1,1}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}). \tag{27}$$

and for a.e $y \in X(S_T^+(S_T^-(u^T)))$,

$$F'(y) = u^T(y) - S_T^+(S_T^-(u^T))(y). \tag{28}$$

We now introduce the function $p : X(S_T^+(S_T^-(u^T))) \to \mathbb{R}$ defined by

$$p(y) = y - TS_T^+(S_T^-(u^T))(y). \tag{29}$$

From [21, Theorem 11.1.3], $p(y) = \xi_+(0) = \xi_-(0)$ where ξ_- and ξ_+ denote respectively the minimal and the maximal backward generalized characteristics associated with the solution $S_t^+(S_T^-(u^T))$ emanating from (y, T) (see Figure 4). From (24), (26) and [21, Theorem 1.7.4, Theorem 11.3.4],

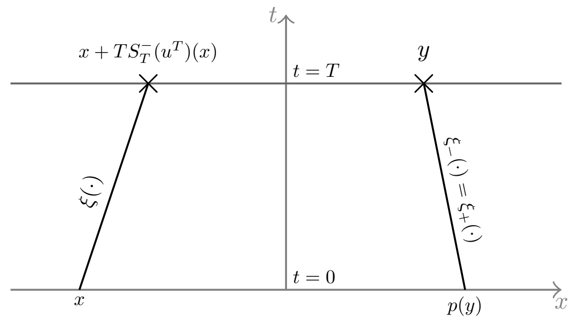


Figure 4: Plotting of the forward generalized characteristic $\xi(\cdot)$ emanating from $(x, 0)$ and the extremal backward generalized characteristics $\xi_-(\cdot)$ and $\xi_+(\cdot)$ emanating from (y, T) associated with the solution $S_t^+(S_T^-(u^T))$. We have $p(y) := y - TS_T^+(S_T^-(u^T))(y) = \xi_+(0) = \xi_-(0)$.

$X(S_T^+(S_T^-(u^T)))$ has full Lebesgue measure and

$$\begin{aligned} -\int_{\mathbb{R}}(u^T(y) - S_T^+(S_T^-(u^T))(y))h(y)\,dy &= -\int_{X(S_T^+(S_T^-(u^T)))}(u^T(y) - S_T^+(S_T^-(u^T))(y))h(y)\,dy, \\ &= -\int_{p^{-1}(X(S_T^-(u^T)))} F'(y)h(y)\,dy \\ &\quad - \int_{X(S_T^+(S_T^-(u^T)))\setminus p^{-1}(X(S_T^-(u^T)))} F'(y)h(y)\,dy. \end{aligned} \tag{30}$$

where $p^{-1}(X(S_T^-(u^T))) := \{y \in X(S_T^+(S_T^-(u^T)))\ /\ p(y) \in X(S_T^-(u^T))\}$ (see an illustration in Figure 5). By definition of S_T^-, using that $u^T \in L^\infty(\mathbb{R})$ satisfies (9) and [21, Theorem 6.2.6], we

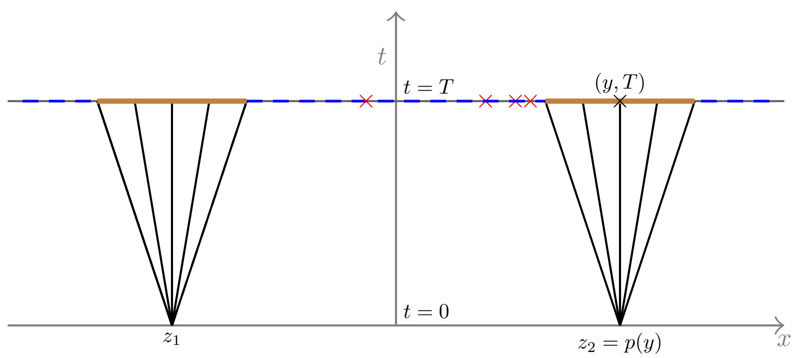


Figure 5: Illustration of $(t, x) \to S_t^+(S_T^-(u^T))(x)$, $p^{-1}(X(S_T^-(u^T)) \subset X(S_T^+(S_T^-(u^T)))(--)$, $X(S_T^+(S_T^-(u^T)))\setminus p^{-1}(X(S_T^-(u^T)))$ (-) and discontinuous points of $S_T^+(S_T^-(u^T))$ ($\times$). Here, $y \in X(S_T^+(S_T^-(u^T)))\setminus p^{-1}(X(S_T^-(u^T)))$, $p(y)$ is defined in (29) and at any discontinuous points $(z_k)_{k\in\mathbb{N}}$ of $S_T^-(u^T)$ verifying $S_T^-(u^T)(z_k-) < S_T^-(u^T)(z_k+)$, a rarefaction wave is created at time $t = 0$.

have $S_T^-(u^T) \in BV(\mathbb{R})$. As a consequence, $S_T^-(u^T)$ has a countable number of discontinuous points $(z_k)_{k\in\mathbb{N}}$ verifying $S_T^-(u^T)(z_k-) < S_T^-(u^T)(z_k+)$. Moreover, if $y \in X(S_T^+(S_T^-(u^T)))\setminus p^{-1}(X(S_T^-(u^T)))$, from [21, Theorem 11.1.3] associated with the solution $S_t^+(S_T^-(u^T))$, we have $S_T^-(u^T)(p(y)-) < S_T^-(u^T)(p(y)+)$. Thus, a rarefaction wave is created at time $t = 0$ and at the position $p(y)$, i.e

$y \in [p(y) + TS_T^-(u^T)(p(y)-), p(y) + TS_T^-(u^T)(p(y)+)]$. We conclude that

$$X(S_T^+(S_T^-(u^T))) \setminus p^{-1}(X(S_T^-(u^T))) = \cup_{k\in\mathbb{N}}[z_k + TS_T^-(u^T)(z_k-), z_k + TS_T^-(u^T)(z_k+)]. \tag{31}$$

Thus, (30) can be written as

$$\begin{aligned} -\int_{\mathbb{R}}(u^T(y) - S_T^+(S_T^-(u^T))(y))h(y)\,dy &= -\int_{p^{-1}(X(S_T^-(u^T)))} F'(y)h(y)\,dy \\ &\quad - \sum_{k\in\mathbb{N}} \int_{\mathcal{I}_k} F'(y)h(y)\,dy, \end{aligned} \tag{32}$$

with

$$\mathcal{I}_k := (z_k + TS_T^-(u^T)(z_k-), z_k + TS_T^-(u^T)(z_k+)). \tag{33}$$

We now study each term on the right side of the equality (32).

- Let $x \in X(S_T^-(u^T))$ and $y \in p^{-1}(X(S_T^-(u^T)))$. Applying Proposition 4.1 with $u_0 = S_T^-(u^T)$ and $u_T = u^T$, the equality (18) holds, i.e for any $(x, p(y)) \in X(S_T^-(u^T))^2$,

$$\int_{x+TS_T^-(u^T)(x)}^{p(y)+TS_T^-(u^T)(p(y))} S_T^+(S_T^-(u^T))(s)\,ds = \int_{x+TS_T^-(u^T)(x)}^{p(y)+TS_T^-(u^T)(p(y))} u^T(s)\,ds, \tag{34}$$

  Using $y \in p^{-1}(X(S_T^-(u^T)))$ and [21, Theorem 11.1.3, Theorem 11.3.2] associated with the solution $S_t^+(S_T^-(u^T))$, there exists a unique forward generalized characteristic $\xi(\cdot)$ emanating from $(p(y), 0)$ and $\xi(T) = p(y) + TS_T^-(u^T)(p(y)) = y$. From (24) and (34), we conclude that for any $y \in p^{-1}(X(S_T^-(u^T)))$, $F(y) = 0$. From (27), F is a continuous function on $\mathbb{R}$ and from (26) the set of discontinuous points of $S_T^+S_T^-(u^T)$ is countable. Then, together with (31), we have for any $y \in \mathbb{R}\setminus(\cup_{k\in\mathbb{N}}\mathcal{I}_k)$,

$$F(y) = 0. \tag{35}$$

  Therefore, for ϵ small enough, for any $y \in \mathbb{R}\setminus\left(\cup_{k\in\mathbb{N}}\overline{\mathcal{I}_k}\right)$, we deduce that

$$0 = \frac{F(y+\epsilon) - F(y)}{\epsilon} = \frac{1}{\epsilon}\int_y^{y+\epsilon} F'(s)ds. \tag{36}$$

  Combining (36) with Lebesgue differentiation Theorem, we have for a.e $y \in p^{-1}(X(S_T^-(u^T)))$

$$F'(y) = 0. \tag{37}$$

  Thus, from (37), for every $h \in \mathcal{T}_{\mathcal{U}_{\text{ad}}^T}(S_T^+(S_T^-(u^T)))$,

$$-\int_{p^{-1}(X(S_T^-(u^T)))} F'(y)h(y)\,dy = 0. \tag{38}$$

- Let $x \in X(S_T^-(u^T))$ and $y \in \cup_{k\in\mathbb{N}}\mathcal{I}_k$ with $\mathcal{I}_k$ defined in (33). Since a rarefaction is created at $(p(y), 0)$ (see Figure 5), we have

$$\partial_y S_T^+(S_T^-(u^T))(y) = \frac{1}{T}. \tag{39}$$

  Since $h \in \mathcal{T}_{\mathcal{U}_{\text{ad}}^T}(S_T^+(S_T^-(u^T)))$ is an admissible perturbation, for every $\epsilon_n > 0$ such that $\epsilon_n \to 0$ when $n \to \infty$ there exists $h_n \in L^\infty(\mathbb{R})$ such that $\lim_{n\to\infty} h_n = h$ in $L^\infty(\mathbb{R})$ and $S_T^+(S_T^-(u^T)) + \epsilon_n h_n \in \mathcal{U}_{\text{ad}}^T$. Thus,

$$\partial_y S_T^+(S_T^-(u^T))(y) + \epsilon_n \partial_y h_n(y) \le \frac{1}{T} \quad \text{in the sense of distributions.} \tag{40}$$

Using (39) and (40), we have $\partial_y h_n(y) \leq 0$ in the sense of distributions. Since $\lim_{n\to\infty} h_n = h$ in $L^\infty(\mathbb{R})$, h_n tends to h in the sense of distributions and we conclude that for any admissible perturbation $h \in \mathcal{T}_{\mathcal{U}_{\rm ad}^T}(S_T^+(S_T^-(u^T)))$,

$$\partial_y h(y) \leq 0 \quad \text{in the sense of distributions.} \tag{41}$$

Applying Proposition 4.1 with $u_0 = S_T^-(u^T)$, $u_T = u^T$, the inequality (11) holds, i.e

$$\int_{x+TS_T^-(u^T)(x)}^{y} u^T(s)\,ds \leq \int_{x+TS_T^-(u^T)(x)}^{y} S_T^+(S_T^-(u^T))(s)\,ds,$$

From (24), we conclude that for any $y \in \cup_k \mathcal{I}_k$

$$F(y) \leq 0. \tag{42}$$

Let $k \in \mathbb{N}$. Using (27) and (35), we have $F \in W_0^{1,1}(\mathcal{I}_k)$. Thus, there exists $F_n \in C_c^\infty(\mathcal{I}_k)$ such that F_n converges to F in $W^{1,1}(\mathcal{I}_k)$. Moreover, $F_n := \rho_n * F$ where ρ_n is a sequence of positive mollifiers (see details in [11, Section 8]). Therefore, together with (42), we have $F_n(y) \leq 0$ for any $y \in \mathcal{I}_k$. Besides, for every $n \in \mathbb{N}$,

$$-\int_{\mathcal{I}_k} F_n'(y)h(y)\,dy = \langle \partial_y h, F_n \rangle, \tag{43}$$

where $\langle \cdot, \cdot \rangle$ is a duality bracket between the distribution $\partial_y h$ and the test function $F_n \in C_c^\infty(\mathcal{I}_k)$. Using (41) and (42), we have $\langle \partial_y h, F_n \rangle \geq 0$. From (43),

$$-\int_{\mathcal{I}_k} F_n'(y)h(y)\,dy \geq 0. \tag{44}$$

Since F_n converges to F in $W^{1,1}(\mathcal{I}_k)$, by passing to the limit in (44), we conclude that, for any admissible perturbation $h \in \mathcal{T}_{\mathcal{U}_{\rm ad}^T}(S_T^+(S_T^-(u^T)))$,

$$-\int_{\mathcal{I}_k} F'(y)h(y)\,dy \geq 0. \tag{45}$$

Using (32), (38) and (45), the inequality (20) holds.

□

4.2 On the solutions of the optimization problem $(\mathcal{O}_T)$.

We are now ready to prove Theorem 3.1. From Lemma 4.1, for every $q \in \mathcal{U}_{\rm ad}^T$, we have

$$\|u^T - S_T^+(S_T^-(u^T))\|_{L^2(\mathbb{R})} \leq \|u^T - q\|_{L^2(\mathbb{R})}. \tag{46}$$

From (23) and (46), we deduce that, for any $u_0 \in \mathcal{U}_{\rm ad}^0$,

$$\|u^T - S_T^+(S_T^-(u^T))\|_{L^2(\mathbb{R})} \leq \|u^T - S_T^+(u_0)\|_{L^2(\mathbb{R})}, \tag{47}$$

with $S_T^-(u^T) \in \mathcal{U}_{\rm ad}^0$ using (22). Thus, $S_T^-(u^T)$ is a minimizer of $(\mathcal{O}_T)$.

- Let u_0^* a minimizer of $(\mathcal{O}_T)$. Then $u_0^* \in \mathcal{U}_{\text{ad}}^0$ and for any $u_0 \in \mathcal{U}_{\text{ad}}^0$ we have

$$\|u^T - S_T^+(u_0^*)\|_{L^2(\mathbb{R})} \leq \|u^T - S_T^+(u_0)\|_{L^2(\mathbb{R})}. \tag{48}$$

  From (47) and (48), we immediately have $S_T^+(u_0^*) = S_T^+(S_T^-(u^T))$.

- Let $u_0^* \in \mathcal{U}_{\text{ad}}^0$ satisfiying $S_T^+(u_0^*) = S_T^+(S_T^-(u^T))$. From (47), for any $u_0 \in \mathcal{U}_{\text{ad}}^0$,

$$\|u^T - S_T^+(u_0^*)\|_{L^2(\mathbb{R})} \leq \|u^T - S_T^+(u_0)\|_{L^2(\mathbb{R})}.$$

  Thus, u_0^* is a minimizer of $(\mathcal{O}_T)$. This concludes the proof of Theorem 3.1.

5 Numerical investigations

5.1 Set of minimizers of $(\mathcal{O}_T)$ when u^T is a classical shock.

In this section, we present how we can construct randomly a minimizer $u_0^* \in BV(\mathbb{R})$ of $(\mathcal{O}_T)$ when u^T is a classical shock, i.e

$$u^T = u_L \mathbb{1}_{(-\infty,\bar{x})} + u_R \mathbb{1}_{(\bar{x},\infty)}, \tag{49}$$

with $u_L > u_R$, $\bar{x} \in \mathbb{R}$, $T > 0$. We introduce the set

$$\Gamma(u_L, u_R, \bar{x}, T) := \left\{\gamma \in W^{1,1}\left([\bar{x} - Tu_L, \bar{x} - Tu_R], \mathbb{R}\right) / \gamma \text{ satisfies } (A1), (A2), (A3) \text{ and } (A4)\right\} \tag{50}$$

with

(A1) $\dot{\gamma} \in BV(\mathbb{R})$

(A2) $\gamma(\bar{x} - Tu_L) = 0$,

(A3) $\gamma(\bar{x} - Tu_R) = \frac{1}{2}T(u_L^2 - u_R^2)$,

(A4) For every $x \in [\bar{x} - Tu_L, \bar{x} - Tu_R]$,

$$\gamma(x) \geq \gamma_*(x) := -T \int_{u_L}^{\frac{\bar{x}-x}{T}} s ds.$$

An illustration of the set $\Gamma(u_L, u_R, \bar{x}, T)$ is given in Figure 6.

Construction. We pick a random path $\gamma \in \Gamma(u_L, u_R, \bar{x}, T)$ and from Theorem 4.1, the initial data $u_0^\gamma \in BV(\mathbb{R})$ defined by

$$u_0^\gamma = \begin{cases} u_L & \text{if } x < \bar{x} - Tu_L \\ \dot{\gamma}(x) & \text{if } \bar{x} - Tu_L < x < \bar{x} - Tu_R \\ u_R & \text{if } \bar{x} - Tu_R < x \end{cases} \tag{51}$$

is a minimizer of $(\mathcal{O}_T)$ (see an illustration in Figure 6).

Example 1 *Let $T = 1$ and $u^T(\cdot) := \mathbb{1}_{(-\infty,0)}(\cdot) - \mathbb{1}_{(0,+\infty)}(\cdot)$. In Figure 1, two different initial data defined by $u_1(x) = \mathbb{1}_{(-\infty,0)}(x) - \mathbb{1}_{(0,+\infty)}(x)$ and $u_2(x) = \mathbb{1}_{(-\infty,-\frac{1}{4})}(x) + 2\mathbb{1}_{(-\frac{1}{4},-\frac{1}{12})}(x) - \mathbb{1}_{(-\frac{1}{12},+\infty)}(x)$ are constructed. The two $\gamma_1 : [-1,1] \to \mathbb{R}$ and $\gamma_2 : [-1,1] \to \mathbb{R}$ defined almost everywhere by $\dot{\gamma}_1(\cdot) = u_1(\cdot)$ and $\dot{\gamma}_2(\cdot) = u_2(\cdot)$ belongs to $\Gamma(1,-1,0,1)$, see Figure 7. From Theorem 4.1, $S_T^+(u_1) = S_T^+(u_2) = u^T$.*

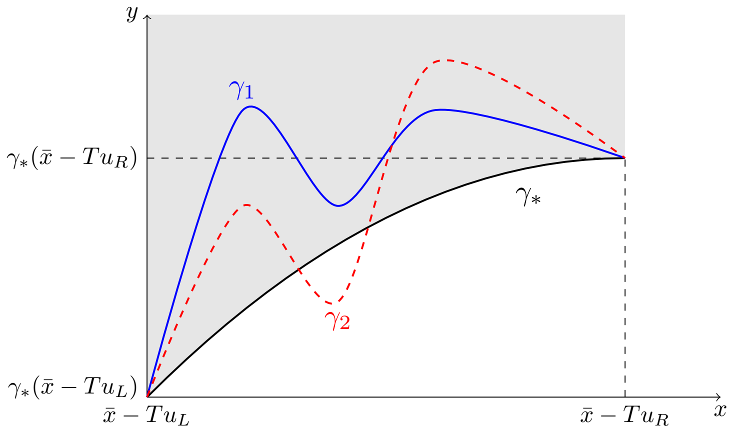


Figure 6: Let u^T defined in (49). The set $\Gamma(u_L, u_R, \bar{x}, T)$ is illustrated by the shaded area. The function γ_* is defined by $\gamma_*(x) = -T\int_{u_L}^{\frac{\bar{x}-x}{T}} s ds = S_T^+(S_T^-(u^T))(x)$ for a.e $x \in [\bar{x}-Tu_L, \bar{x}-Tu_R]$. We have $\gamma_1 \in \Gamma(u_L, u_R, \bar{x}, T)$ and $\gamma_2 \notin \Gamma(u_L, u_R, \bar{x}, T)$. From Theorem 4.1, $u_0^{\gamma_1}$ defined in (51) is a minimizer of $(\mathcal{O}_T)$ while $u_0^{\gamma_2}$ is not.

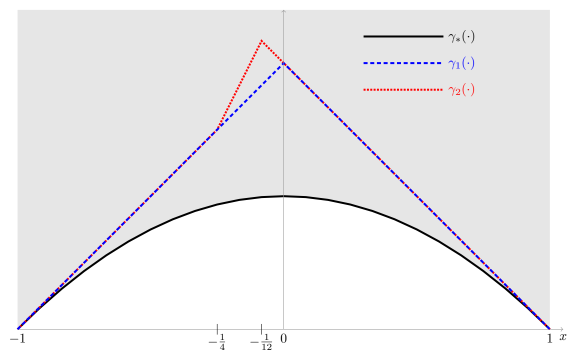


Figure 7: Plotting of γ_1 and γ_2 belonging to $\Gamma(1, -1, 0, 1)$. For a.e $x \in [-1, 1]$, $\dot{\gamma}_*(x) = S_T^+(S_T^-(u^T))(x)$, $\dot{\gamma}_1(x) = u_1(x)$ and $\dot{\gamma}_2(x) = u_2(x)$ where $u^T(\cdot)$, $u_1(\cdot)$ and $u_2(\cdot)$ are defined in Example 1.

5.2 Construction of a set of approximate minimizers for $(\mathcal{O}_T)$

In this section, we solve numerically the optimization problem $(\mathcal{O}_T)$ using Section 5.1.

Algorithm. Construction of an approximate random minimizer u_0^* of $(\mathcal{O}_T)$.
First, we construct numerically the backward-forward solution $S_T^+(S_T^-(u^T))$ using a wave-front tracking algorithm, see [20]. Second, at each discontinuous point $\bar{x}$ of $S_T^+(S_T^-(u^T))$ satisfying $u_L := S_T^+(S_T^-(u^T))(\bar{x}-) > S_T^+(S_T^-(u^T))(\bar{x}+) := u_R$, we pick a random path $\gamma \in \Gamma(u_L, u_R, \bar{x}, T)$ defined in (50) such that $\dot{\gamma}$ belongs to the state mesh generated by the wave-front tracking algorithm. Then for almost every $x \in (\bar{x}-Tu_L, \bar{x}-Tu_R)$, $u_0^*(x) := \dot{\gamma}(x)$. Finally, out of each interval $(\bar{x}-Tu_L, \bar{x}-Tu_R)$, u_0^* coincides with the approximate backward-forward solution $S_T^+(S_T^-(u^T))$.

5.2.1 With attainable target u^T

Let $T = 1$. We consider the target u^T defined by $u^T(\cdot) = 0.6875\mathbb{1}_{(-\infty,4.6)}(\cdot) - \mathbb{1}_{(4.6,\infty)}(\cdot)$ (Figure 8). Since $u^T(4.6+) < u^T(4.6-)$, the inequality (2) holds and so $u^T(\cdot)$ is an attainable function. As a consequence, we have $u^T = S_T^+(S_T^-(u^T))$. In Figure 8 and Figure 9, six approximate minimizers u_0^* of $(\mathcal{O}_T)$ are constructed. Note that in the top left corner of Figure 9, $S_T^-(u^T)$ is plotted with respect to x.

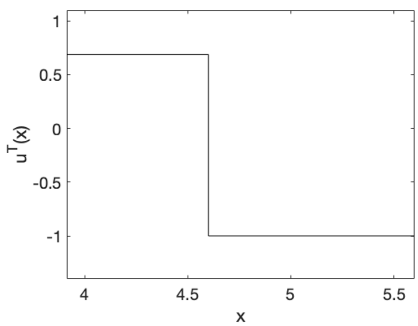


Figure 8: The attainable target $u^T(\cdot) = 0.6875\mathbb{1}_{(-\infty,4.6)}(\cdot) - \mathbb{1}_{(4.6,\infty)}(\cdot)$

5.2.2 With unattainable target u^T

In Example 2 and Example 3, the optimization problem $(\mathcal{O}_T)$ is solved numerically with two unattainable targets.

Example 2 *Let $T = 2$. We consider the target u^T defined by*

$$u^T(x) = \begin{cases} 2 & \text{if } x \in (-0.2, 1.1) \bigcup (2, 3.1) \bigcup (4.1, 5.3) \bigcup (6.1, 7.2), \\ -1 & \text{otherwise.} \end{cases}$$

Since, for every $x \in \{-0.2, 2, 4.1, 6.1\}$, we have $u^T(x-) < u^T(x+)$. Therefore, the inequality (2) does not hold. Thus, u^T is an unattainable target and $S_T^+(S_T^-(u^T)) \neq u^T$.

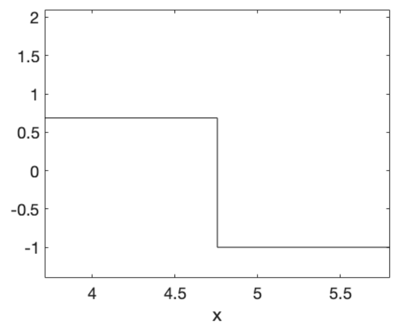
- *In Figure 10a), an approximate function of $(x,t) \to S_t^-(u^T)(-x)$ is plotted.*
- *In Figure 10b), an approximate function of $S_T^-(u^T)$ of $(\mathcal{O}_T)$ is plotted.*
- *In Figure 10c), an approximate function of $(x,t) \to S_t^+(S_T^-(u^T))(x)$ is plotted.*
- *In Figure 10d), the function u^T and an approximate function of $x \to S_T^+(S_T^-(u^T))(x)$ are plotted.*

Four approximate minimizers u_0^* of $(\mathcal{O}_T)$ are constructed in Figure 11.

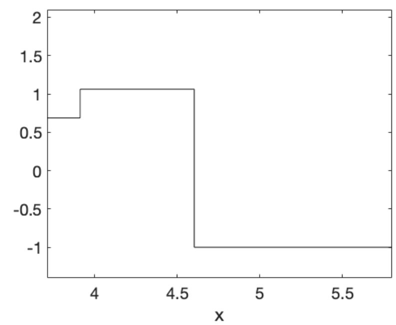
Example 3 *Let $T = 1$. We consider the target u^T defined by*

$$\begin{aligned} u^T = \; & -\mathbb{1}_{(-\infty,-0.2)} + 2\mathbb{1}_{(-0.2,1.1)} + 0.16\mathbb{1}_{(1.1,2)} + 1.33\mathbb{1}_{(2,3.1)} - 0.77\mathbb{1}_{(3.1,4.1)} \\ & -0.42\mathbb{1}_{(4.1,5.3)} - \mathbb{1}_{(5.3,6.1)} + 1.91\mathbb{1}_{(6.1,7.2)} - \mathbb{1}_{(7,2,\infty)}. \end{aligned}$$

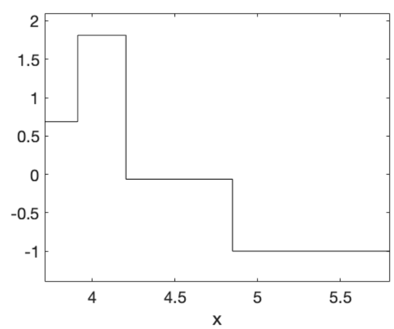
The function u^T is an unattainable target. In Figure 12, the function u^T and $x \mapsto S_T^+(S_T^-(u^T))(x)$ are plotted.



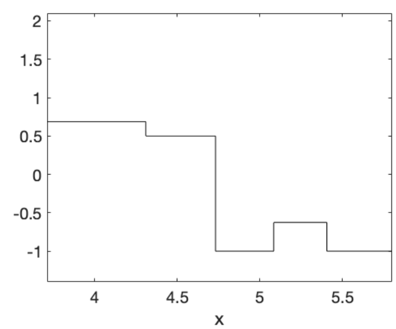
$M = 1$ discontinuous point



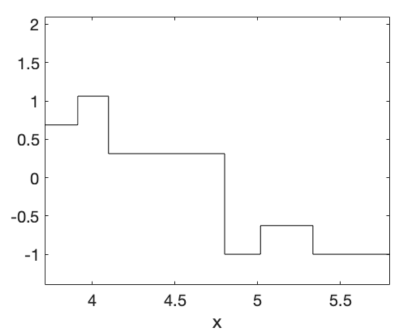
$M = 2$ discontinuous points



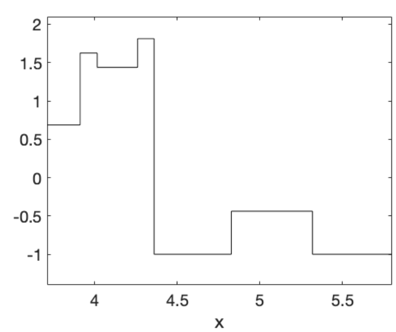
$M = 3$ discontinuous points



$M = 4$ discontinuous points



$M = 5$ discontinuous points



$M = 6$ discontinuous points

Figure 9: Construction of six approximate minimizers u_0^* of $(\mathcal{O}_T)$ having $M \in \{1, \cdots, 6\}$ discontinuous points. $T = 1$, $u^T = u_L \mathbb{1}_{(-\infty,\bar{x})} + u_R \mathbb{1}_{(\bar{x},\infty)}$ with $u_L = 0.6875$, $u_R = -1$, $\bar{x} = 4.6$.

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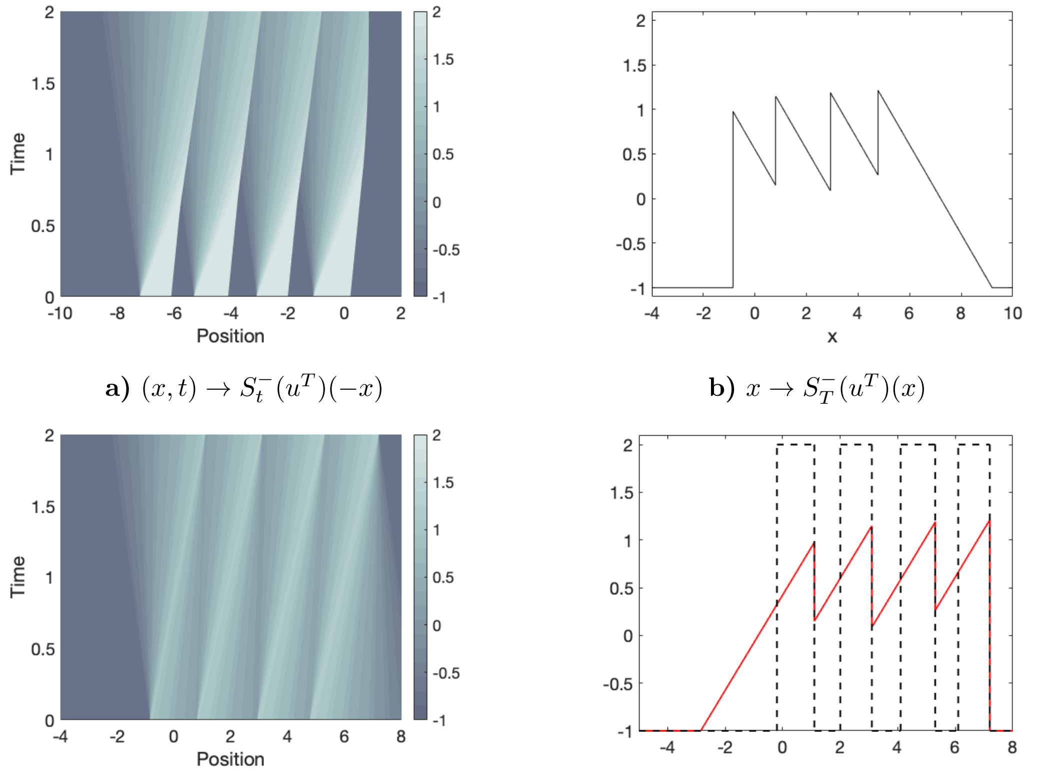

a) $(x,t) \to S_t^-(u^T)(-x)$ **b)** $x \to S_T^-(u^T)(x)$

c) $(x,t) \to S_t^+(S_T^{-,n}(u^T))(x)$ **d)** $u^T(--)$ and $x \to S_T^+(S_T^-(u^T))(x)(-)$

Figure 10: $T = 2$. Construction of an approximate function of $S_T^+(S_T^-(u^T))$ with u^T an unattainable target defined in Example 2 using a wave-front tracking algorithm.

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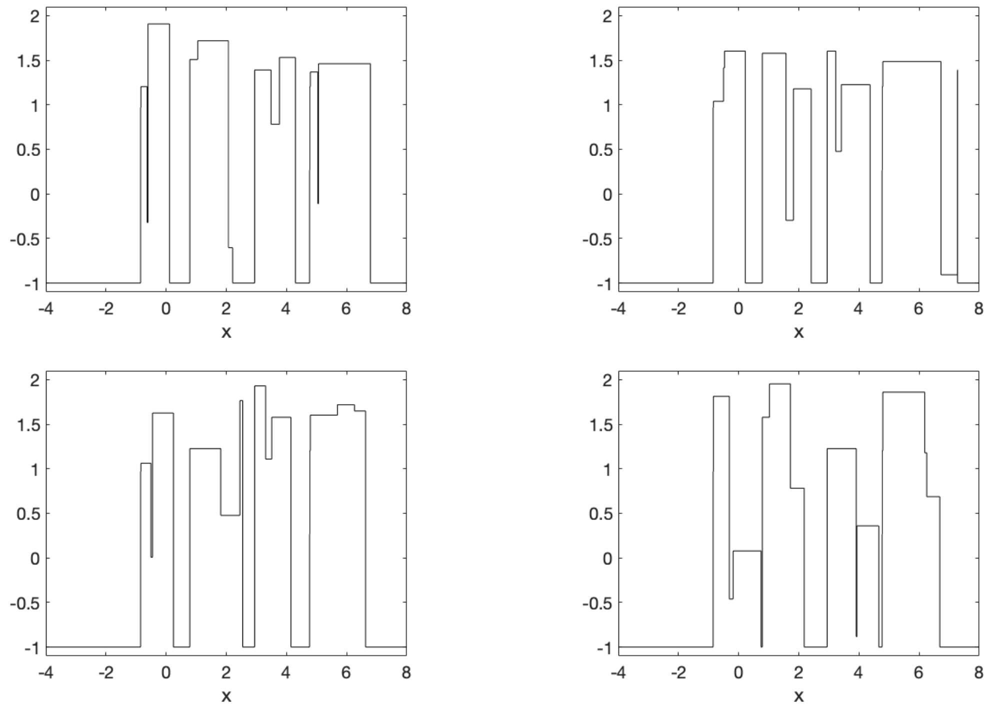

Figure 11: $T = 2$. Four minimizers u_0^* of $(\mathcal{O}_T)$ with u^T defined in Example 2.

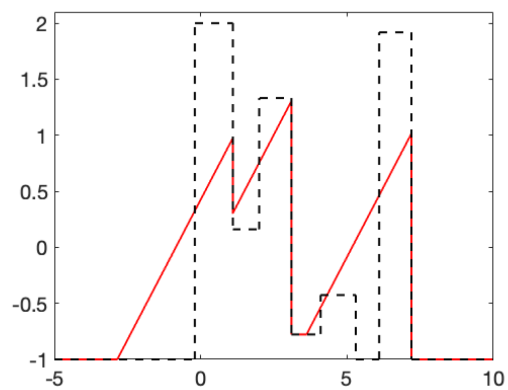

Figure 12: $u^T(--)$ and $x \to S_T^+(S_T^-(u^T))(x)(-)$ with u^T defined in Example 3.

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