EXPONENTIAL CONVERGENCE TO STEADY-STATES FOR TRAJECTORIES OF A DAMPED DYNAMICAL SYSTEM MODELLING ADHESIVE STRINGS

GIUSEPPE MARIA COCLITE, NICOLA DE NITTI, FRANCESCO MADDALENA, GIANLUCA ORLANDO, AND ENRIQUE ZUAZUA

Abstract. We study the global well-posedness and asymptotic behavior for a semilinear damped wave equation with Neumann boundary conditions, modelling a one-dimensional linearly elastic body interacting with a rigid substrate through an adhesive material. The key feature of of the problem is that the interplay between the nonlinear force and the boundary conditions allows for a continuous set of equilibrium points. We prove an exponential rate of convergence for the solution towards a (uniquely determined) equilibrium point.

Contents

1. Introduction

The qualitative and quantitative study of the long-time behavior of solutions to dynamical systems has always been a challenging mathematical issue [8]. The by now deep-rooted machinery for these problems can now deal with models arising from real phenomena. As usual, taking into account relevant physical features exposes to unavoidable additional difficulties. This is the case for the problem studied in this paper.

Our problem falls into the class of semilinear evolution equations of the form

$$\partial_{tt}^2 u(t,x) + \partial_t u(t,x) - \Delta u(t,x) + f(u(t,x)) = 0\,.$$

In particular, we focus on the global well-posedness and the long-time behavior as $t \to +\infty$ of its solutions. More precisely, we intend to prove compactness of the trajectories in a suitable strong sense and deduce a precise estimate on the rate of the convergence to the limit trajectory.

The peculiar structure of the nonlinearity $f(u)$ treated in the present paper is the novelty and the source of the distinguishing features and the related mathematical difficulties of our problem. These do not arise when polynomial nonlinearities are considered (see [22, 23, 19]).

In modelling the dynamics of an elastic string interacting with a rigid substrate through an adhesive layer, we are lead to analyze a nonlinearity mimicking possible attachment–detachment regimes. To take into account this kind of interactions, one considers a force

2020 *Mathematics Subject Classification.* 35L71, 35B40, 74B20, 74K05.
Key words and phrases. Damped wave equation; adhesion phenomena; long-time asymptotics; decay.

$f(u)$ which vanishes when u overcomes a critical threshold u_* (see [18, 6, 7, 5]). Specifically, we let $f(u) = \Phi'(u)$, where the potential $\Phi(u)$ is depicted in Figure 1; see (2.1) for its precise definition.

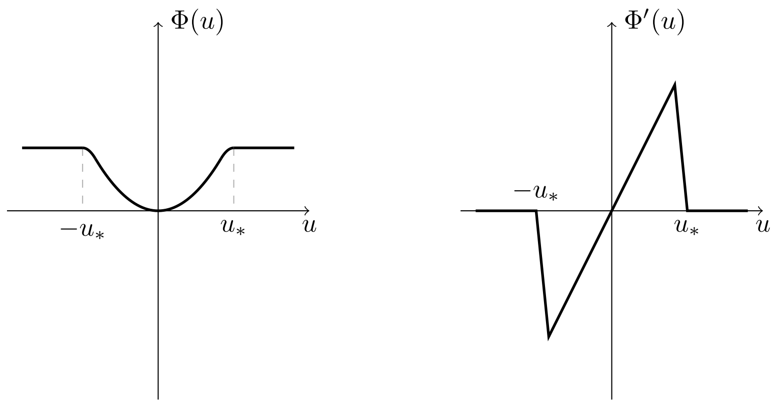

Figure 1. Plot of the potential Φ and the force Φ' as functions of u.

The model under investigation is ruled by the following initial boundary value problem:

$$\begin{cases} \partial^2_{tt} u(t,x) + \partial_t u(t,x) - \partial^2_{xx} u(t,x) + \Phi'(u(t,x)) = 0\,, & (t,x) \in (0,+\infty)\times(0,L)\,, \\ \partial_x u(t,0) = \partial_x u(t,L) = 0\,, & t \in (0,+\infty)\,, \\ u(0,x) = u_0(x)\,,\ \partial_t u(0,x) = v_0(x)\,, & x \in (0,L)\,. \end{cases} \tag{1.1}$$

This describes the damped dynamics of a one-dimensional linearly elastic body, whose reference configuration is $(0,L)$, interacting with a rigid substrate through an adhesive material, acting through the force $\Phi'(u)$. If the displacement u is small compared to u_*, this force is purely elastic; otherwise, if $|u| \geq u_*$, the adhesive material ceases to act on the elastic body. We stress that the adhesion process described here is reversible. In (1.1), initial and Neumann boundary conditions are imposed.

The attachment–detachment process ruled by the nonlinear force $\Phi'(u)$ induces the natural question about the long-time behavior of such dynamics and, more specifically, whether a convergence to a stationary state occurs or switching between the two states persists. The presence of the damping term in (1.1) suggests that the former case should be the reasonable one. We answer this question quantitatively in the present paper as we explain below.

Our first task is to show global well-posedness of (1.1) in the energy space. This is achieved in Theorem 2.1 by standard techniques. Note that the result is, in fact, obtained in arbitrary spatial dimension d.

Our next task is to show convergence of trajectories as $t \to +\infty$. The structure of the dynamical system (1.1) suggests to move our steps in the framework of LaSalle's invariance principle (see [17, 12] and also [4, Theorem 9.2.3]). In this perspective, one expects that $u(t) \to u_\infty$ as $t \to +\infty$, where u_∞ is a solution to the stationary equation

$$\begin{cases} -\partial^2_{xx} u_\infty(x) + \Phi'(u_\infty(x)) = 0\,, & x \in (0,L)\,, \\ \partial_x u_\infty(0) = \partial_x u_\infty(L) = 0\,. \end{cases} \tag{1.2}$$

In fact, the first step in this direction is reached in Proposition 5.3 (proven in any dimension $d \geq 1$), where we show that the accumulation points of the trajectories $\{u(t)\}_{t\geq 0}$ satisfy (1.2) via a compactness argument. We bring the reader's attention to the decisive role played here by the interplay between the set of critical points of Φ and the kernel of the Laplace operator with Neumann boundary conditions, which makes the arguments intriguing. Indeed, if homogeneous Dirichlet conditions were imposed instead, the stationary problem

$$\begin{cases} -\partial^2_{xx} u_\infty(x) + \Phi'(u_\infty(x)) = 0\,, & x \in (0,L)\,, \\ u_\infty(0) = u_\infty(L) = 0\,, \end{cases}$$

would admit a unique trivial solution—the constant value 0. The same would occur in the case of Neumann boundary conditions with a force only vanishing at 0, *e.g.*,

$$\begin{cases} -\partial_{xx}^2 u_\infty(x) + u_\infty |u_\infty|^{p-1} = 0\,, & x \in (0, L)\,, \\ \partial_x u_\infty(0) = \partial_x u_\infty(L) = 0\,, \end{cases} \quad , \quad p > 1\,.$$

In those cases, uniqueness of the accumulation point, combined with the compactness argument, would force the desired convergence; see [22, 23, 19] for results in that direction. The same conclusion holds true also in the case of multiple, yet discrete, rest points, as pointed out in [9, 1, 20, 2]. Here, in contrast, the set of solutions to problem (1.2) is given by constant functions valued in $\{\Phi' = 0\} = (-\infty, -u_*] \cup \{0\} \cup [u_*, +\infty)$, allowing for a continuous set of possible choices for the limit profiles. Therefore, the compactness argument alone does not suffice to infer convergence of the whole trajectory as $t \to +\infty$ to a uniquely determined limit profile. Even when proving this convergence in the case of a continuous set of equilibria is possible, finer techniques are required, see, *e.g.*, [21] for this phenomenon arising in a dissipative system when the Laplacian is replaced by a differential operator with a one-dimensional kernel. Here, in Theorem 2.5, we accomplish to prove that the initial conditions (1.1) enforce the selection of a unique limit profile $u_\infty \in \{\Phi' = 0\}$, thus allowing us to conclude the convergence $u(t) \to u_\infty$ as $t \to +\infty$. The proof of Theorem 2.5 is carried out in any dimension $d \geq 1$ and its main ingredient is the existence of an auxiliary Lyapunov functional, introduced in (2.7), inspired by [13].

Our last, yet fundamental, task is to quantify the rate of convergence for such asymptotic dynamics. The interplay of nonlinearities and differential operators with a nontrivial kernel might affect the rate of decay of dissipative dynamical systems, as shown in some cases with the existence of slow solutions decaying polynomially fast in [15, 11, 10]. In Theorem 2.7, we prove that this phenomenon is ruled out in the model studied in this paper, by showing that the convergence $u(t) \to u_\infty$ as $t \to +\infty$ occurs in an exponential fashion, *i.e.*, $\|u(t, \cdot) - u_\infty\|_{H^1((0,L))} \leq M e^{-\kappa t}$. The arguments used in the proof rely on the compact embedding $H^1((0, L)) \subset\subset C([0, L])$, true only in dimension $d = 1$. This allows us to single out only one of the two possible attachment–detachment regimes for time large enough, thus allowing us to study separately the decay rate of solutions to the equations

$$\begin{aligned} &\partial_{tt}^2 u(t, x) + \partial_t u(t, x) - \partial_{xx}^2 u(t, x) = 0, && (t, x) \in (0, +\infty) \times (0, L)\,, \\ &\partial_{tt}^2 u(t, x) + \partial_t u(t, x) - \partial_{xx}^2 u + 2u(t, x) = 0, && (t, x) \in (0, +\infty) \times (0, L)\,. \end{aligned}$$

Trajectories of both dynamics share the exponential decay rate to the equilibrium points. This result is proved through Grönwall type inequalities on suitable perturbations of the energy functionals, in the spirit of [16], giving a positive answer to the question raised in the paper.

We conclude by commenting on an interesting question concerning the decay rate and the shape of the force $\Phi'(u)$. If the rate of the exponential decay $M e^{-\kappa t}$ is uniform with respect to the slope of the decreasing part of the force $\Phi'(u)$, one can think of treating the degenerate case of discontinuous force, *e.g.*, $\Phi'(u) = 2u$ for $|u| < u_*$ and $\Phi'(u) = 0$ for $|u| \geq u_*$. This models an abrupt attachment–detachment phenomenon. Answering to this question is not in the scope of the paper. However, in this perspective, we study an ODE related to this phenomenon in Appendix A, showing that the uniform exponential decay occurs.

All the precise statements of the results contained in the paper are collected in Section 2.

2. Main results

In this section we present the model and the main results proven in the paper.

2.1. **The model.** Let $d \geq 1$ be the dimension. We will explicitly declare for which results $d = 1$ is needed. We fix $\Omega \subset \mathbb{R}^d$, a bounded, open, and connected set, additionally with C^2 boundary when $d \geq 2$.[1] We let ν denote the exterior normal derivative to $\partial\Omega$.

We give a precise definition of the nonliner term in the equation. Let us fix a threshold $u^* \in (0, +\infty)$ and a potential $\Phi \in C^1(\mathbb{R};\mathbb{R})$ defined as follows:

$$\Phi(u) := \begin{cases} u^2 \,, & \text{if } |u| \leq u_* - \dfrac{1}{\sigma} \,, \\ u_*\Big(u_* - \dfrac{1}{\sigma}\Big) - (u_*\sigma - 1)(u_* - u)^2 \,, & \text{if } u_* - \dfrac{1}{\sigma} \leq u \leq u_* \,, \\ u_*\Big(u_* - \dfrac{1}{\sigma}\Big) - (u_*\sigma - 1)(u_* + u)^2 \,, & \text{if } -u_* \leq u \leq -u_* + \dfrac{1}{\sigma} \,, \\ u_*\Big(u_* - \dfrac{1}{\sigma}\Big) \,, & \text{if } u_* \leq |u| \,; \end{cases} \tag{2.1}$$

see Figure 1. The specific form of the potential Φ is not significant; the following properties are the relevant ones:

- (P1) $|\Phi| \leq u_*^2$;
- (P2) $|\Phi'| \leq 2$ and Φ' is globally Lipschitz with $\mathrm{Lip}(\Phi') \leq 2(u_*\sigma - 1)$;
- (P3) $\Phi(u) - u^2$ is concave;
- (P4) $u\Phi'(u) \geq 0$.

We study the following initial boundary value problem

$$\begin{cases} \partial_{tt}^2 u - \Delta u + \partial_t u + \Phi'(u) = 0 \,, & (t, x) \in (0, +\infty) \times \Omega \,, \\ \nabla u \cdot \nu = 0 \,, & (t, x) \in (0, +\infty) \times \partial\Omega \,, \\ u(0, x) = u_0(x) \,, \ \partial_t u(0, x) = v_0(x) \,, & x \in \Omega \,. \end{cases} \tag{2.2}$$

2.2. **Global well-posedness.** To establish the well-posedness result, it is convenient to interpret (2.2) as the system of ODEs in $H^1(\Omega) \times L^2(\Omega)$

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t}(t) - v(t) = 0 \,, & t \in (0, +\infty) \,, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t}(t) - \Delta u(t) + v(t) + u(t) - u(t) + \Phi'(u(t)) = 0 \,, & t \in (0, +\infty) \,, \\ u(0) = u_0 \,, \ v(0) = v_0 \,. \end{cases} \tag{2.3}$$

In Section 4 we prove the following result.

Theorem 2.1 (Well-posedness with initial data in the energy space)**.** *Let $d \geq 1$. Let $U_0 = (u_0, v_0)$ and assume that*

$$u_0 \in H^1(\Omega) \quad \textit{and} \quad v_0 \in L^2(\Omega) \,. \tag{2.4}$$

Then there exists a unique solution $(u, v) \in C\big([0, +\infty); H^1(\Omega) \times L^2(\Omega)\big)$ to (2.3) with initial datum $u(0) = u_0$ and $v(0) = v_0$ (see Proposition 3.4 below for the precise definition of solution).

Remark 2.2. A solution u in the sense of Theorem 2.1 is also a weak solution to (2.3) in the following sense: $u \in C([0, +\infty); H^1(\Omega)) \cap C^1([0, +\infty); L^2(\Omega))$, and it satisfies the ODE

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t}(t) - v(t) = 0 \,, \quad \text{for } t \in (0, +\infty) \,, \\ u(0) = u_0 \,, \end{cases} \tag{2.5}$$

and

$$0 = \langle v_0, \varphi(0) \rangle_{L^2(\Omega)} + \int_0^{+\infty} \Big\langle v(t), -\frac{\mathrm{d}\varphi}{\mathrm{d}t}(t) + \varphi(t) \Big\rangle_{L^2(\Omega)} \mathrm{d}t + \int_0^{+\infty} \langle \nabla u(t), \nabla \varphi(t) \rangle_{L^2(\Omega)} \,\mathrm{d}t$$
$$+ \int_0^{+\infty} \langle \Phi'(u(t)), \varphi(t) \rangle_{L^2(\Omega)} \,\mathrm{d}t \,,$$

[1]This regularity requirement is motivated by the need to define traces on $\partial\Omega$, exploit the compact embedding $H^1(\Omega) \subset L^2(\Omega)$, and apply elliptic regularity theory up to the boundary.

for every $\varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$. We show this additional fact in the proof of Theorem 2.1.

2.3. **Energy balances.** We stress that a key role in our analysis is played by the *energy functional* $E\colon H^1(\Omega)\times L^2(\Omega) \to [0,+\infty)$ defined by

$$E(u,v) := \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \|\Phi(u)\|_{L^1(\Omega)}\,, \quad \text{for all } (u,v) \in H^1(\Omega)\times L^2(\Omega) \tag{2.6}$$

and the auxiliary functional $J\colon H^1(\Omega)\times L^2(\Omega) \to [0,+\infty)$ defined by

$$J(u,v) := \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \langle u, v\rangle_{L^2(\Omega)}\,, \quad \text{for all } (u,v) \in H^1(\Omega)\times L^2(\Omega)\,. \tag{2.7}$$

In Section 4 we prove the following energy balances.

Theorem 2.3 (Energy balances)**.** *Let $d \geq 1$. Let $U_0 = (u_0, v_0)$ and assume that* (2.4) *is satisfied. Then the unique solution to* (2.2) *obtained in Theorem 2.1 satisfies the following equalities:*

$$E(u(t), \partial_t u(t)) + \int_0^t \|\partial_t u(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s = E(u_0, v_0)\,, \quad \textit{for every } t \in [0,+\infty)\,, \tag{2.8}$$

and

$$\begin{aligned} J(u(t), \partial_t u(t)) &+ \int_0^t \Big(\|\nabla u(s)\|^2_{L^2(\Omega)} + \langle u(s), \Phi'(u(s))\rangle_{L^2(\Omega)}\Big)\,\mathrm{d}s \\ &= J(u_0, v_0) + \int_0^t \|\partial_t u(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s\,, \quad \textit{for every } t \in [0,+\infty)\,. \end{aligned} \tag{2.9}$$

Remark 2.4. Theorem 2.3 shows, in particular, that E is nonnegative and decreasing on trajectories. The same holds true for

$$J(u(t), \partial_t u(t)) + \int_t^{+\infty} \|\partial_t u(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s\,. \tag{2.10}$$

Both of these facts will play a crucial role in the argument of this paper inspired by LaSalle's invariance principle.

2.4. **Long-time asymptotics.** We then turn to studying the asymptotic behavior of the solution. Our first result in this direction is qualitative and is proven in Section 5.

Theorem 2.5 (Long-time asymptotics)**.** *Let $d \geq 1$. Assume that u_0, v_0 satisfy* (2.4)*. Let (u, v) be the unique solution to* (2.2) *obtained in Theorem 2.1. Then there exists u_∞ constant a.e. in Ω with $u_\infty \in \{\Phi' = 0\}$ such that $u(t) \to u_\infty$ in $H^1(\Omega)$ and $\partial_t u(t) \to 0$ in $L^2(\Omega)$ for $t \to +\infty$.*

We remark that constant functions valued in $\{\Phi' = 0\}$ are the weak solutions to the stationary problem

$$\begin{cases} -\Delta u + \Phi'(u) = 0\,, & x \in \Omega\,, \\ \nabla u \cdot \nu = 0\,, & x \in \partial\Omega\,. \end{cases}$$

Remark 2.6. Even though not explicitly quantified, the long-time limit u_∞ in Theorem 2.5 depends only the unique long-time limit of (2.10), which in turn only depends on the initial data.

2.5. **Exponential decay rate.** Our final result is the exponential decay of the solutions proven in Section 6. For this, we need to exploit the compact embedding of $H^1(\Omega) \hookrightarrow C(\Omega)$, which holds in one space-dimension, and exclude the critical case $u_\infty \equiv \pm u_*$.

Theorem 2.7 (Exponential decay rate)**.** *Let us assume that $d = 1$ and $\Omega = (0, L) \subset \mathbb{R}$. Given initial data u_0, v_0 satisfying* (2.4)*, let (u, v) be the unique solution to* (2.2) *obtained in Theorem 2.1. Let u_∞ be as in Theorem 2.5. Let us assume that $u_\infty = 0$ or $|u_\infty| > u_*$. Then*

$$\|u(t) - u_\infty\|_{H^1(\Omega)} + \|\partial_t u(t)\|_{L^2(\Omega)} \leq M_\Phi e^{-\kappa t}\,,$$

for some $M_\Phi > 0$ (possibly depending on the Lipschitz constant of Φ') and $\kappa > 0$.

Remark 2.8 (Dependence of the decay rate on the Lipschitz constant of Φ'). In the proof of Theorem 2.7, the decay rates are given for time $t > T_\sigma$, depending on the constant σ in the definition (2.1). However, it is not clear whether T_σ is bounded independently of σ or $T_\sigma \to +\infty$ as $\sigma \to 0^+$. This issue is addressed in Appendix A for a finite-dimensional system.

3. Preliminary results on abstract evolution problems and semigroups

The well-posedness theory for semilinear evolution problems of the form

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t}(t) + \mathscr{A}U(t) + F(U(t)) = 0\,, \quad t \in (0, +\infty)\,, \\ U(0) = U_0\,, \end{cases} \tag{3.1}$$

where $\mathscr{A}$ is a maximal monotone operator, is based on a well-known machinery (see, *e.g.*, [14, 4, 3]). We collect some details in this section since some tools will be used later.

First, we recall the following definition.

Definition 3.1 (Maximal monotone operators)**.** Let H be a Hilbert space, let $D(\mathscr{A}) \subset H$ be a vector subspace, and let $\mathscr{A}\colon D(\mathscr{A}) \to H$ be a linear operator. The linear operator $\mathscr{A}$ is said to be *maximal monotone* if the following two conditions are satisfied:

1. $\langle \mathscr{A}U, U\rangle_H \geq 0$ for all $U \in D(\mathscr{A})$;
2. there exists $\lambda > 0$ such that $(\mathrm{Id} + \lambda\mathscr{A})D(\mathscr{A}) = H$.

The starting point for the analysis of (3.1) is the well-posedness for the linear homogeneous problem.

Proposition 3.2 (Semigroup of contractions and abstract homogeneous Cauchy problem)**.** *Let $\mathscr{A}\colon D(\mathscr{A}) \to H$ be a maximal monotone operator. Then $\mathscr{A}$ generates a continuous semigroup of contractions $\{S_{\mathscr{A}}(t)\}_{t\geq 0}$, i.e., for every $t \geq 0$, $S_{\mathscr{A}}(t)\colon H \to H$ is a linear operator with $\|S_{\mathscr{A}}(t)\|_H \leq 1$, $S_{\mathscr{A}}(s+t) = S_{\mathscr{A}}(s)S_{\mathscr{A}}(t)$, $S_{\mathscr{A}}(0) = \mathrm{Id}$, and $\lim_{t\to 0^+} \|S_{\mathscr{A}}(t)U_0 - U_0\|_H = 0$ for every $U_0 \in H$. Moreover, if $U_0 \in D(\mathscr{A})$, then $U(t) := S(t)U_0 \in C^1([0,+\infty); H) \cap C([0,+\infty); D(\mathscr{A}))$ is the unique solution to*

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t}(t) + \mathscr{A}U(t) = 0\,, \quad t \in (0, +\infty)\,, \\ U(0) = U_0\,. \end{cases} \tag{3.2}$$

Proof. The result is classical: see, *e.g.*, [14, Theorem I.2.2.1 & Remark I.2.2.3]. □

Owing to the previous result, the curve $U(t) = S_{\mathscr{A}}(t)U_0$ is interpreted as the solution to (3.2) even when the initial datum $U_0 \in H$ (rather than the smaller space $D(\mathscr{A})$).

To deal with the (linear) inhomogeneous problem, we need to resort to *Duhamel's formula.*

Proposition 3.3 (Duhamel's formula and abstract inhomogeneous Cauchy problem)**.** *Let $\mathscr{A}\colon D(\mathscr{A}) \to H$ be a maximal monotone operator and let $\{S_{\mathscr{A}}(t)\}_{t\geq 0}$ be the continuous group of contractions given by Proposition 3.2. Let $T > 0$ and let $f \in W^{1,1}([0,T]; H)$. Let $U_0 \in D(\mathscr{A})$. Then the curve defined by*

$$U(t) := S_{\mathscr{A}}(t)U_0 - \int_0^t S_{\mathscr{A}}(t-s)f(s)\,\mathrm{d}s\,, \quad t \in [0,T]$$

satisfies $U \in C^1([0,T]; H) \cap C([0,T]; D(\mathscr{A}))$ and is the unique solution to

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t}(t) + \mathscr{A}U(t) + f(t) = 0\,, \quad t \in (0, T)\,, \\ U(0) = U_0\,. \end{cases} \tag{3.3}$$

Proof. The result is obtained by approximating the initial datum with initial data in $D(\mathscr{A})$ and applying Proposition 3.2. For details, see, *e.g.*, [4, Proposition 4.1.6]. □

As above, thanks to Duhamel's formula one interprets the curve $U(t) := S_{\mathscr{A}}(t)U_0 - \int_0^t S_{\mathscr{A}}(t-s)f(s)\,\mathrm{d}s$ as the solution to (3.3) even when $U_0 \in H$ (rather than the smaller space $D(\mathscr{A})$).

We conclude the preliminary results by recalling the well-posedness for semilinear problems with a Lipschitz-continuous nonlinearity. Duhamel's formula allows us to give a notion of solution here as well.

Proposition 3.4 (Well-posedness of an abstract semilinear problem)**.** *Let $\mathscr{A}\colon D(\mathscr{A}) \to H$ be a maximal monotone operator and let $\{S_{\mathscr{A}}(t)\}_{t\geq 0}$ be the continuous group of contractions given by Proposition 3.2. Let $F\colon H \to H$ be Lipschitz continuous and let $U_0 \in H$. Then there exists a unique curve $U \in C([0,+\infty);H)$ such that*

$$U(t) = S_{\mathscr{A}}(t)U_0 - \int_0^t S_{\mathscr{A}}(t-s)F(U(s))\,\mathrm{d}s\,, \quad t \in [0,+\infty)\,. \tag{3.4}$$

If, in addition, $U_0 \in D(\mathscr{A})$, then $U \in C^1([0,+\infty);H) \cap C([0,+\infty);D(\mathscr{A}))$ and it satisfies

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t}(t) + \mathscr{A}U(t) + F(U(t)) = 0\,, \quad t \in (0,+\infty)\,,\\ U(0) = U_0\,. \end{cases}$$

Proof. The proof of the existence and uniqueness claim is based on Banach's fixed-point theorem (see, *e.g.*, [4, Proposition 4.3.3]). For completeness, we provide a sketch of the proof below.

Following [3, Theorem 7.3], we note that $C([0,+\infty);H) =: E$ is a Banach space when endowed with the exponentially-weighted norm $\|U\|_E := \sup_{t\geqslant 0} e^{-\gamma t}\|U(t)\|_H$.

We fix $\gamma \geq \mathrm{Lip}(F)$ and apply Banach's fixed point theorem to the map

$$T : V \mapsto S_{\mathscr{A}}(t)U_0 - \int_0^t S_{\mathscr{A}}(t-s)F(V(s))$$

in the space $X := \{V \in C([0,+\infty);H) : \|V\|_E \leq \|U_0\|_H\}$. First, we check that T is a self mapping of X:

$$\begin{aligned} \|V(t)\|_H &\leq \|U_0\|_H + \int_0^t \mathrm{Lip}(F)\|V(s)\|_H\,\mathrm{d}s\\ &\leq \|U_0\|_H \exp(\mathrm{Lip}(F)t)\,, \end{aligned}$$

which yields

$$\|V\|_E \leq \|U_0\|_H \underbrace{\exp((\mathrm{Lip}(F)-\gamma)t)}_{\leq 1}\,.$$

Second, we check that T is a contraction:

$$\begin{aligned} \|T(U)(t) - T(V)(t)\|_H &\leq \int_0^t \underbrace{\|S_{\mathscr{A}}(t-s)\|}_{\leq c<1} \mathrm{Lip}(F)\|U(t)-V(t)\|_H\\ &\leq c\,\mathrm{Lip}(F)\int_0^t e^{\gamma s}\,\mathrm{d}s\,\|U-V\|_E\\ &\leq c\frac{\mathrm{Lip}(F)}{\gamma}(e^{\gamma t}-1)\,\|U-V\|_E, \end{aligned}$$

which yields

$$\|T(U)(t) - T(V)(t)\|_E \leq \underbrace{c\frac{\mathrm{Lip}(F)}{\gamma}(1-e^{\gamma t})}_{\leq c<1}\,\|U-V\|_E.$$

By the arbitrariness of the time-horizon $T > 0$, we may thus deduce the claimed global well-posedness result. The additional regularity for $U_0 \in D(\mathscr{A})$ is proven in [4, Proposition 4.3.9]. ∎

Remark 3.5 (Dynamical system). Let us consider the family of (nonlinear) operators $\{T(t)\}_{t\geq 0}$ where $T(t)\colon H \to H$ defined as follows: For every $U_0 \in H$, let $U(t)$ be the unique solution to (3.4) provided by Proposition 3.4 and let $T(t)(U_0) := U(t)$. It satisfies the following properties:

(D1) $T(0) = \mathrm{Id}$;
(D2) $T(t)$ is continuous for every $t \geq 0$;
(D3) $T(t+s) = T(t) \circ T(s)$ for every $t, s \geq 0$;
(D4) $t \mapsto T(t)(U_0)$ is continous for every $U_0 \in H$.

Following the notation of [4, Definition 9.1.1], we say that $\{T(t)\}_{t\geq 0}$ is a *dynamical system on H*. Properties (D1) and (D4) are consequences of the definition and the continuity in time of solutions to the equation. Property (D3) follows from the uniqueness of solutions. Property (D2) amounts to continuous dependence on the initial data for the solution of the PDE, which follows from the Grönwall-type argument in [4, Proposition 4.3.7].

4. Proof of the well-posedness result

4.1. **ODE in Hilbert space.**

To apply the general well-posedness theory for semilinear evolution problems illustrated in Section 3, it is convenient to recast the Cauchy problem (2.3) in the form (3.1). We let $H = H^1(\Omega)\times L^2(\Omega)$ be endowed with the scalar product

$$\begin{aligned}\langle (u,v),(\widetilde{u},\widetilde{v})\rangle_H &:= \int_\Omega u\widetilde{u}\,\mathrm{d}x + \int_\Omega \nabla u\cdot\nabla\widetilde{u}\,\mathrm{d}x + \int_\Omega v\widetilde{v}\,\mathrm{d}x \\ &= \langle u,\widetilde{u}\rangle_{L^2(\Omega)} + \langle \nabla u,\nabla\widetilde{u}\rangle_{L^2(\Omega)} + \langle v,\widetilde{v}\rangle_{L^2(\Omega)} \quad \text{for } (u,v),(\widetilde{u},\widetilde{v})\in H,\end{aligned}$$

and we consider the linear operator $\mathscr{A}\colon D(\mathscr{A}) \to H$

$$\begin{cases} D(\mathscr{A}) := \{(u,v)\in H \;:\; u\in H^2(\Omega) \text{ with } \mathrm{Tr}(\nabla u\cdot\nu)=0 \text{ on } \partial\Omega,\ v\in H^1(\Omega)\}, \\ \mathscr{A}(u,v) := (-v, -\Delta u + v + u), \text{ for } (u,v)\in D(\mathscr{A}). \end{cases} \tag{4.1}$$

In this framework, problem (2.3) reads

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t}(t) + \mathscr{A}U(t) + F(U(t)) = 0, & t\in(0,+\infty), \\ U(0) = U_0, \end{cases} \tag{4.2}$$

with $U = (u,v)$ and $U_0 = (u_0, v_0)$.

Remark 4.1 ($\mathscr{A}$ is maximal monotone). The operator $\mathscr{A}$ is maximal monotone according to Definition 3.1. To see this, integrating by parts we obtain for all $(u,v)\in D(\mathscr{A})$

$$\begin{aligned}\langle \mathscr{A}(u,v),(u,v)\rangle_H &= \langle(-v,-\Delta u+v+u),(u,v)\rangle_H \\ &= -\int_\Omega vu\,\mathrm{d}x - \int_\Omega \nabla v\cdot\nabla u\,\mathrm{d}x - \int_\Omega \Delta u v\,\mathrm{d}x + \int_\Omega v^2\,\mathrm{d}x + \int_\Omega uv\,\mathrm{d}x \\ &= \int_\Omega v^2\,\mathrm{d}x \geq 0.\end{aligned}$$

Moreover, for every $(f,g)\in H$ there exists $(u,v)\in D(\mathscr{A})$ such that $(u,v)+\mathscr{A}(u,v) = (f,g)$. Indeed, this condition is equivalent to the existence of solutions to the problem

$$\begin{cases} u - v = f, \\ v - \Delta u + v + u = g, \end{cases} \iff \begin{cases} v = u - f, \\ 3u - \Delta u = 2f + g. \end{cases}$$

Since $2f+g\in L^2(\Omega)$, there exists a unique solution to

$$\begin{cases} 3u - \Delta u = 2f + g, & x\in\Omega, \\ \nabla u\cdot\nu = 0, & x\in\partial\Omega, \end{cases}$$

which, by elliptic regularity theory, belongs to $H^2(\Omega)$ and satisfies the previous equation in a classical sense. Then, setting $v = u - f$, we have $v\in H^1(\Omega)$, hence $(u,v)\in D(\mathscr{A})$. This proves that $\mathscr{A}$ is maximal monotone.

Remark 4.2 (Lipschitz continuity of the nonlinearity)*.* We observe that the nonlinear perturbation $F\colon H \to H$ given by

$$F(U) = (0, -u + \Phi'(u))\,, \quad \text{for } U = (u,v) \in H\,, \tag{4.3}$$

is Lipschitz continuous. Indeed, the continuity of Φ' yields

$$\begin{aligned} \|F(U) - F(\widetilde{U})\|_H &= \| -u + \Phi'(u) + \widetilde{u} - \Phi'(\widetilde{u})\|_{L^2(\Omega)} \\ &\le \|u - \widetilde{u}\|_{L^2(\Omega)} + \mathrm{Lip}(\Phi')\|u - \widetilde{u}\|_{L^2(\Omega)} \\ &\le (1 + \mathrm{Lip}(\Phi'))\|U - \widetilde{U}\|_H\,, \end{aligned} \tag{4.4}$$

for every $U = (u,v)$, $\widetilde{U} = (\widetilde{u}, \widetilde{v}) \in H$. Notice that the Lipschitz constant explodes in the limit $k \to +\infty$.

4.2. **Solutions with regular initial data.** In this subsection, we study the well-posedness of (4.2) under higher regularity assumptions on initial data. Specifically, we assume that

$$u_0 \in H^2(\Omega) \text{ with } \mathrm{Tr}(\nabla u_0 \cdot \nu) = 0 \text{ on } \partial\Omega\,, \qquad v_0 \in H^1(\Omega)\,, \tag{4.5}$$

i.e., $(u_0, v_0) \in D(\mathscr{A})$.

Thanks to the observations in Subsection 4.1, we obtain the following result.

Proposition 4.3 (Well-posedness of the problem with regular initial data)**.** *Let* $U_0 = (u_0, v_0)$ *and assume that* u_0, v_0 *satisfy* (4.5)*. Then there exists a unique solution* $U = (u,v) \in C^1([0,+\infty); H) \cap C([0,+\infty); D(\mathscr{A}))$ *to* (4.2) *with initial datum* $u(0) = U_0$*, in the sense of Proposition 3.3. In particular,* $U = (u,v)$ *satisfies* (2.3)*.*

Proof. The result follows by Subsection 4.1 and Proposition 3.4, observing that $U_0 = (u_0, v_0) \in D(\mathscr{A})$. ☐

In the next result, we deduce energy estimates on the unique strong solution to (2.2).

Proposition 4.4 (Energy balance for regular initial data)**.** *Assume that* u_0, v_0 *satisfy* (4.5)*. Let* (u,v) *be the unique strong solution to* (2.2) *with initial data* u_0, v_0 *obtained in Proposition 4.3. Let* E *and* J *be as in* (2.6) *and* (2.7)*, respectively. Then the following energy balances hold true:*

$$E(u(t), \partial_t u(t)) + \int_0^t \|\partial_t u(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s = E(u_0, v_0)\,, \quad \textit{for every } t \in [0, +\infty)\,, \tag{4.6}$$

and

$$\begin{aligned} J(u(t), \partial_t u(t)) &+ \int_0^t \Big(\|\nabla u(s)\|^2_{L^2(\Omega)} + \langle u(s), \Phi'(u(s))\rangle_{L^2(\Omega)}\Big)\,\mathrm{d}s \\ &= J(u_0, v_0) + \int_0^t \|\partial_t u(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s\,, \quad \textit{for every } t \in [0, +\infty)\,. \end{aligned} \tag{4.7}$$

Proof. Let us prove (4.6). By (2.3), by the regularity of u, and integrating by parts, we get that for every $t \in (0, +\infty)$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big(E(u(t), \partial_t u(t))\Big) \\ &= \Big\langle \frac{\mathrm{d}^2 u}{\mathrm{d}t^2}(t), \frac{\mathrm{d}u}{\mathrm{d}t}(t)\Big\rangle_{L^2(\Omega)} + \Big\langle \frac{\mathrm{d}\nabla u}{\mathrm{d}t}(t), \nabla u(t)\Big\rangle_{L^2(\Omega)} + \Big\langle \Phi'(u(t)), \frac{\mathrm{d}u}{\mathrm{d}t}(t)\Big\rangle_{L^2(\Omega)} \\ &= \langle \Delta u(t) - v(t) - \Phi'(u(t)), v(t)\rangle_{L^2(\Omega)} \\ &\quad + \langle \nabla v(t), \nabla u(t)\rangle_{L^2(\Omega)} + \langle \Phi'(u(t)), v(t)\rangle_{L^2(\Omega)} = -\|v(t)\|^2_{L^2(\Omega)} = -\Big\|\frac{\mathrm{d}u}{\mathrm{d}t}(t)\Big\|^2_{L^2(\Omega)}\,. \end{aligned}$$

Integrating in time we get (4.6).

Let us prove (4.7). By (2.3) and integrating by parts, we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(J(u(t),\partial_t u(t))\Big) &= \Big\langle u(t),\frac{\mathrm{d}u}{\mathrm{d}t}(t)\Big\rangle_{L^2(\Omega)} + \Big\|\frac{\mathrm{d}u}{\mathrm{d}t}(t)\Big\|^2_{L^2(\Omega)} + \Big\langle u(t),\frac{\mathrm{d}^2u}{\mathrm{d}t^2}(t)\Big\rangle_{L^2(\Omega)}\\ &= \Big\langle u(t),\frac{\mathrm{d}v}{\mathrm{d}t}(t)+v(t)\Big\rangle_{L^2(\Omega)} + \Big\|\frac{\mathrm{d}u}{\mathrm{d}t}(t)\Big\|^2_{L^2(\Omega)}\\ &= \langle u(t),\Delta u(u(t))-\Phi'(u(t))\rangle_{L^2(\Omega)} + \Big\|\frac{\mathrm{d}u}{\mathrm{d}t}(t)\Big\|^2_{L^2(\Omega)}\\ &= -\|\nabla u(t)\|^2_{L^2(\Omega)} - \langle u(t),\Phi'(u(t))\rangle_{L^2(\Omega)} + \Big\|\frac{\mathrm{d}u}{\mathrm{d}t}(t)\Big\|^2_{L^2(\Omega)}.\end{aligned}$$

Integrating in time we get (4.7). □

4.3. **Solution with initial data in energy space.** In this subsection we drop the higher regularity assumptions (4.5).

Proof of Theorem 2.1. We split the proof into several steps.

Step 1 (Existence and uniqueness of mild solutions) Existence and uniqueness are obtained by Proposition 3.3, which guarantees that $U=(u,v)\in C([0,+\infty);H)$ is the unique curve satisfying

$$U(t) = S_{\mathscr{A}}(t)U_0 - \int_0^t S_{\mathscr{A}}(t-s)F(U(s))\,\mathrm{d}s\,,\quad t\in[0,+\infty)\,. \tag{4.8}$$

Step 2 (Weak solution) We need to show that u is a weak solution in the sense given in the statement. In Subsection 4.1 we have shown that the operator $\mathscr{A}$ defined in (4.1) is maximal monotone. As a consequence, its domain $D(\mathscr{A})$ is dense in $H=H^1(\Omega)\times L^2(\Omega)$, see [14, Proposition I.1.1.2]. Hence, there exists a sequence $U_0^j=(u_0^j,v_0^j)\in D(\mathscr{A})$ such that $U_0^j=(u_0^j,v_0^j)\to U_0=(u_0,v_0)$ in H as $j\to+\infty$. Let $U^j=(u^j,v^j)$ be the unique solution to (2.3) with initial datum $U^j(0)=U_0^j$, in the sense of Proposition 3.4, satisfying

$$U^j(t) = S_{\mathscr{A}}(t)U_0^j - \int_0^t S_{\mathscr{A}}(t-s)F(U^j(s))\,\mathrm{d}s\,,\quad t\in[0,+\infty)\,. \tag{4.9}$$

Notice that, by Proposition 3.4, $U^j=(u^j,v^j)\in C^1([0,+\infty);H)\cap C([0,+\infty);D(\mathscr{A}))$ and it satisfies (4.2) in a strong sense. This is the unique solution provided by Proposition 4.3.

Let us show that $U^j=(u^j,v^j)$ is a Cauchy sequence in $C([0,T];H)$ with respect to j for every $T>0$. To do so, we exploit (4.9), the contraction property of $\{S_{\mathscr{A}}(t)\}_{t\geq 0}$, and (4.4) to estimate for i and j and for every $t\in(0,+\infty)$

$$\begin{aligned}\|U^i(t)-U^j(t)\|_H &= \Big\|S_{\mathscr{A}}(t)(U_0^i-U_0^j) - \int_0^t S_{\mathscr{A}}(t-s)(F(U^i(s))-F(U^j(s)))\,\mathrm{d}s\Big\|_H\\ &\leq \|U_0^i-U_0^j\|_H + \int_0^t (1+\mathrm{Lip}(\Phi'))\|U^i(s)-U^j(s)\|_H\,\mathrm{d}s\,.\end{aligned}$$

By Grönwall's inequality, given $T>0$, we deduce that for every $t\in[0,T]$

$$\|U^i(t)-U^j(t)\|_H \leq \|U_0^i-U_0^j\|_H e^{(1+\mathrm{Lip}(\Phi'))t} \leq \|U_0^i-U_0^j\|_H^2 e^{(1+\mathrm{Lip}(\Phi'))T}\,,$$

Since U_0^j converges, we conclude that U^j is a Cauchy sequence with respect to j in $C([0,T];H)$. Since (4.8) is stable with respect to the uniform convergence, we conclude that $U^j=(u^j,v^j)\to U=(u,v)$ in $C([0,T];H)$ as $j\to+\infty$ for every $T>0$. Furthermore, since $u^j\in C^1([0,+\infty);H^1(\Omega))$ and $\partial_t u^j=v^j\to v$ in $C([0,T];L^2(\Omega))$ as $j\to+\infty$ for every $T>0$, we deduce that $u\in C^1([0,+\infty);L^2(\Omega))$ with derivative $\partial_t u=v$ in $L^2(\Omega)$. Hence $u\in C([0,+\infty);H^1(\Omega))\cap C^1([0,+\infty);L^2(\Omega))$ and (2.5) is satisfied.

Let us prove Remark 2.2. We multiply by $\varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$ the equation (2.3) solved by $U^j = (u^j, v^j)$ in a strong sense and integrate by parts to obtain that for every $t \in (0,+\infty)$

$$\begin{aligned}
0 &= \Big\langle \frac{\mathrm{d}^2 U^j}{\mathrm{d}t^2}(t), \varphi(t) \Big\rangle_{L^2(\Omega)} - \langle \Delta U^j(t), \varphi(t)\rangle_{L^2(\Omega)} + \Big\langle \frac{\mathrm{d}U^j}{\mathrm{d}t}(t), \varphi(t) \Big\rangle_{L^2(\Omega)} + \langle \Phi'(U^j(t)), \varphi(t)\rangle_{L^2(\Omega)} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big\langle \frac{\mathrm{d}U^j}{\mathrm{d}t}(t), \varphi(t) \Big\rangle_{L^2(\Omega)} - \Big\langle \frac{\mathrm{d}U^j}{\mathrm{d}t}(t), \frac{\mathrm{d}\varphi}{\mathrm{d}t}(t) \Big\rangle_{L^2(\Omega)} + \langle \nabla U^j(t), \nabla\varphi(t)\rangle_{L^2(\Omega)} \\
&\quad + \Big\langle \frac{\mathrm{d}U^j}{\mathrm{d}t}(t), \varphi(t) \Big\rangle_{L^2(\Omega)} + \langle \Phi'(U^j(t)), \varphi(t)\rangle_{L^2(\Omega)} \,.
\end{aligned}$$

Integrating in time, we get that

$$\begin{aligned}
0 = \langle v_0, \varphi(0)\rangle_{L^2(\Omega)} &+ \int_0^{+\infty} \Big\langle v^j(t), -\frac{\mathrm{d}\varphi}{\mathrm{d}t}(t) + \varphi(t) \Big\rangle_{L^2(\Omega)} \,\mathrm{d}t + \int_0^{+\infty} \langle \nabla U^j(t), \nabla\varphi(t)\rangle_{L^2(\Omega)} \\
&+ \int_0^{+\infty} \langle \Phi'(U^j(t)), \varphi(t)\rangle_{L^2(\Omega)} \,\mathrm{d}t \,.
\end{aligned}$$

Letting $j \to +\infty$, we conclude that

$$\begin{aligned}
0 = \langle v_0, \varphi(0)\rangle_{L^2(\Omega)} &+ \int_0^{+\infty} \Big\langle v(t), -\frac{\mathrm{d}\varphi}{\mathrm{d}t}(t) + \varphi(t) \Big\rangle_{L^2(\Omega)} \,\mathrm{d}t + \int_0^{+\infty} \langle \nabla u(t), \nabla\varphi(t)\rangle_{L^2(\Omega)} \\
&+ \int_0^{+\infty} \langle \Phi'(u(t)), \varphi(t)\rangle_{L^2(\Omega)} \,\mathrm{d}t \,.
\end{aligned}$$

This concludes the proof.

□

Now we turn to the proof of the energy balance.

Proof of Theorem 2.3. The proof of the energy balance is based on the approximation used in the proof of Theorem 2.1. Let $U_0^j = (u_0^j, v_0^j) \to U_0 = (u_0, v_0)$ in H and let $U^j = (u^j, v^j)$ the corresponding solutions obtained therein satisfying $U^j = (u^j, v^j) \to U = (u, v)$ in $C([0,T]; H)$ as $j \to +\infty$ for every $T > 0$. By Proposition 4.4 we have that

$$E(U^j(t), \partial_t U^j(t)) + \int_0^t \|\partial_t U^j(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s = E(u_0^j, v_0^j) \,, \quad \text{for every } t \in [0,+\infty) \,,$$

and

$$\begin{aligned}
J(U^j(t), \partial_t U^j(t)) &+ \int_0^t \Big(\|\nabla U^j(s)\|^2_{L^2(\Omega)} + \langle U^j(s), \Phi'(U^j(s))\rangle_{L^2(\Omega)} \Big) \,\mathrm{d}s \\
&= J(u_0^j, v_0^j) + \int_0^t \|\partial_t U^j(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s \,, \quad \text{for every } t \in [0,+\infty) \,.
\end{aligned}$$

Exploiting the convergences $U_0^j = (u_0^j, v_0^j) \to U_0 = (u_0, v_0)$ in H, $u^j \to u$ in $C([0,T]; H^1(\Omega))$, and $\partial_t u^j \to \partial_t u$ in $C([0,T]; L^2(\Omega))$, we pass to the limit as $j \to +\infty$ to get (2.8) and (2.9).

□

5. Proof of the long-time asymptotics

In this section, we study the long-time limit of the unique solution obtained in Theorem 2.1.

5.1. **Qualitative result.** In Theorem 2.5, we claimed that the limit $\lim_{t\to+\infty} u(t)$ exists and is a solution to the stationary problem. The proof requires several intermediate results.

Before discussion the case of the semilinear problem, we recall the asymptotic behavior of solutions to the linear homogeneous PDE

$$\begin{cases} \partial^2_{tt}u - \Delta u + \partial_t u + u = 0\,, & (t,x)\in(0,+\infty)\times\Omega\,,\\ \nabla u\cdot\nu = 0\,, & (t,x)\in(0,+\infty)\times\partial\Omega\,,\\ u(0,x) = u_0(x)\,,\ \partial_t u(0,x) = v_0(x)\,, & x\in\Omega\,. \end{cases} \tag{5.1}$$

Proposition 5.1 (Long-time asymptotics for the linear homogeneous problem)**.** *Let $\{S_{\mathscr{A}}(t)\}_{t\geq 0}$ be the continuous semigroup of contractions generated by $\mathscr{A}$, see Proposition 3.2. Then there exists a constant $c>0$ such that*

$$\|S_{\mathscr{A}}(t)\|_{\mathscr{L}(H)} \leq c\,e^{-t/2}\,,$$

where $\|S_{\mathscr{A}}(t)\|_{\mathscr{L}(H)}$ denotes the operator norm of $S_{\mathscr{A}}(t)\colon H\to H$.

The previous result states in an abstract form the following fact: If u_0, v_0 satisfy (2.4) and $(u(t),v(t))$ is the unique solution to (5.1), then $\|u(t)\|_{H^1(\Omega)}\to 0$ and $\|\partial_t u(t)\|_{L^2(\Omega)}\to 0$ exponentially fast. The main result in this paper is about the analogous result for the semilinear problem under investigation. Before going on with the discussion in the semilinear case, we provide some details for the proof of Proposition 5.1, which follows the lines of [4, Lemma 9.5.1].

Proof of Proposition 5.1. We consider the following energy functional

$$G(u,v) = \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\langle u,v\rangle_{L^2(\Omega)}\,.$$

We observe that

$$\frac{1}{2}|\langle u,v\rangle_{L^2(\Omega)}| \leq \frac{1}{4}\|v\|^2_{L^2(\Omega)} + \frac{1}{4}\|u\|^2_{L^2(\Omega)}\,,$$

hence

$$\frac{1}{4}\|u\|^2_{H^1(\Omega)} + \frac{1}{4}\|v\|^2_{L^2(\Omega)} \leq G(u,v) \leq \frac{3}{4}\|u\|^2_{H^1(\Omega)} + \frac{3}{4}\|v\|^2_{L^2(\Omega)}\,. \tag{5.2}$$

Let us fix $U_0 = (u_0,v_0)\in D(\mathscr{A})$ and let $U(t) = (u(t),v(t)) = S_{\mathscr{A}}(t)U_0$. By Proposition 3.2 we have that $U(t)\in C^1([0,+\infty);H)\cap C([0,+\infty);D(\mathscr{A}))$ solves (3.2), *i.e.*,

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t}(t) - v(t) = 0\,, & \text{for } t\in(0,+\infty)\,,\\ \dfrac{\mathrm{d}v}{\mathrm{d}t}(t) - \Delta u(t) + v(t) + u(t) = 0\,, & \text{for } t\in(0,+\infty)\,,\\ u(0) = u_0\,,\ v(0) = v_0\,. \end{cases}$$

Integrating by parts and exploiting the previous equation, we have that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(G(u(t),\partial_t u(t))\Big)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\Big\|\frac{\mathrm{d}u}{\mathrm{d}t}(t)\Big\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\Big\langle u(t),\frac{\mathrm{d}u}{\mathrm{d}t}(t)\Big\rangle_{L^2(\Omega)}\Big)\\
&= \Big\langle \frac{\mathrm{d}^2u}{\mathrm{d}t^2}(t),\frac{\mathrm{d}u}{\mathrm{d}t}(t)\Big\rangle_{L^2(\Omega)} + \Big\langle \frac{\mathrm{d}\nabla u}{\mathrm{d}t}(t),\nabla u(t)\Big\rangle_{L^2(\Omega)} + \Big\langle u(t),\frac{\mathrm{d}u}{\mathrm{d}t}(t)\Big\rangle_{L^2(\Omega)}\\
&\quad + \frac{1}{2}\Big\|\frac{\mathrm{d}u}{\mathrm{d}t}(t)\Big\|^2_{L^2(\Omega)} + \frac{1}{2}\Big\langle u(t),\frac{\mathrm{d}^2u}{\mathrm{d}t^2}(t)\Big\rangle_{L^2(\Omega)}\\
&= \Big\langle \Delta u(t) - \frac{\mathrm{d}u}{\mathrm{d}t}(t) - u(t),\frac{\mathrm{d}u}{\mathrm{d}t}(t)\Big\rangle_{L^2(\Omega)} - \Big\langle \frac{\mathrm{d}u}{\mathrm{d}t}(t),\Delta u(t)\Big\rangle_{L^2(\Omega)} + \Big\langle u(t),\frac{\mathrm{d}u}{\mathrm{d}t}(t)\Big\rangle_{L^2(\Omega)}\\
&\quad + \frac{1}{2}\Big\|\frac{\mathrm{d}u}{\mathrm{d}t}(t)\Big\|^2_{L^2(\Omega)} + \frac{1}{2}\Big\langle u(t),\Delta u(t) - \frac{\mathrm{d}u}{\mathrm{d}t}(t) - u(t)\Big\rangle_{L^2(\Omega)}\\
&= -\frac{1}{2}\Big\|\frac{\mathrm{d}u}{\mathrm{d}t}(t)\Big\|^2_{L^2(\Omega)} - \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\Big\langle u(t),\frac{\mathrm{d}u}{\mathrm{d}t}(t)\Big\rangle_{L^2(\Omega)}\\
&= -G(u(t),\partial_t u(t))\,.
\end{aligned}$$

This implies, together with (5.2), that

$$\begin{aligned}\frac{1}{4}\|U(t)\|_H^2 &= \frac{1}{4}\|u(t)\|_{H^1(\Omega)}^2 + \frac{1}{4}\|\partial_t u(t)\|_{L^2(\Omega)}^2 \le G(u(t), \partial_t u(t)) \\ &= G(u_0, v_0)e^{-t} \le \Big(\frac{3}{4}\|u_0\|_{H^1(\Omega)}^2 + \frac{3}{4}\|v_0\|_{L^2(\Omega)}^2\Big)e^{-t} = \frac{3}{4}\|U_0\|_H^2 e^{-t}\,,\end{aligned}$$

i.e.

$$\|S_{\mathscr{A}}(t)U_0\|_H^2 \le 3\|U_0\|_H^2 e^{-t}\,,$$

By the density of $D(\mathscr{A})$ in H, we conclude the proof. □

We are now in a position to start the asymptotic analysis of solutions to the semilinear problem. The first step is to deduce a bound on $u(t)$ in $H^1(\Omega)$.

Proposition 5.2 (H^1-bound)**.** *Assume that* u_0, v_0 *satisfy* (2.4)*. Let* (u, v) *be the unique solution to* (2.2) *obtained in Theorem 2.1. Then*

$$\sup_{t\ge 0}\|u(t)\|_{H^1(\Omega)} < +\infty\,.$$

Proof. We split the proof of the estimate in two steps.

Step 1 (Estimating $\|\nabla u(t)\|_{L^2(\Omega)}$) By (2.8) and the definition of E in (2.6), we obtain that

$$\frac{1}{2}\|\partial_t u(t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\nabla u(t)\|_{L^2(\Omega)}^2 \le E(u(t), \partial_t u(t)) \le E(u_0, v_0) \le E(u_0, v_0)\,, \tag{5.3}$$

where we used that $\Phi \le \Phi$ in the last inequality. In particular, $\sup_{t\ge 0}\|\nabla u(t)\|_{L^2(\Omega)} < +\infty$.

Step 2 (Estimating $\|u(t)\|_{L^2(\Omega)}$) Estimating $\|u(t)\|_{L^2(\Omega)}$ requires several steps. The first observation is that the functional J defined in (2.7) satisfies that

$$\lim_{t\to+\infty} J(u(t), \partial_t u(t)) = \ell\,, \quad \text{for some } \ell \in \mathbb{R}\,. \tag{5.4}$$

To prove (5.4), we start by observing that $t \mapsto J(u(t), \partial_t u(t))$ is bounded from below, since (5.3) yields

$$\begin{aligned}J(u(t), \partial_t u(t)) &= \frac{1}{2}\|u(t)\|_{L^2(\Omega)}^2 + \langle u(t), \partial_t u(t)\rangle_{L^2(\Omega)} \\ &= \frac{1}{2}\|u(t) + \partial_t u(t)\|_{L^2(\Omega)}^2 - \frac{1}{2}\|\partial_t u(t)\|_{L^2(\Omega)}^2 \\ &\ge -\frac{1}{2}\|\partial_t u(t)\|_{L^2(\Omega)}^2 \ge -E(u_0, v_0)\,.\end{aligned} \tag{5.5}$$

Let us show that $t \mapsto J(u(t), \partial_t u(t)) - \int_0^t \|\partial_t u(r)\|_{L^2(\Omega)}^2\,\mathrm{d}r$ is nonincreasing. By (2.9) and property (P4) of Φ, we have that, for every $s \le t$,

$$\begin{aligned}&J(u(s), \partial_t u(s)) - \int_0^s \|\partial_t u(r)\|_{L^2(\Omega)}^2\,\mathrm{d}r \\ &= J(u(t), \partial_t u(t)) - \int_0^t \|\partial_t u(r)\|_{L^2(\Omega)}^2\,\mathrm{d}r \\ &\quad + \int_s^t \Big(\|\nabla u(r)\|_{L^2(\Omega)}^2 + \langle u(r), \Phi'(u(r))\rangle_{L^2(\Omega)}\Big)\,\mathrm{d}r \\ &\ge J(u(t), \partial_t u(t)) - \int_0^t \|\partial_t u(r)\|_{L^2(\Omega)}^2\,\mathrm{d}r\,.\end{aligned}$$

The term $\int_0^t \|\partial_t u(r)\|_{L^2(\Omega)}^2\,\mathrm{d}r$ admits a limit as $t \to +\infty$. Indeed, equality (2.8) yields

$$\sup_{t\ge 0}\int_0^t \|\partial_t u(s)\|_{L^2(\Omega)}^2\,\mathrm{d}s \le E(u_0, v_0) \le E(u_0, v_0)\,,$$

hence $\partial_t u \in L^2([0, +\infty); L^2(\Omega))$ and

$$\lim_{t\to+\infty}\int_0^t \|\partial_t u(s)\|_{L^2(\Omega)}^2\,\mathrm{d}s = \int_0^{+\infty} \|\partial_t u(s)\|_{L^2(\Omega)}^2\,\mathrm{d}s\,. \tag{5.6}$$

The monotonicity of $t \mapsto J(u(t), \partial_t u(t))$, together with (5.5) and (5.6), gives

We exploit (5.4) to show that

$$\lim_{t\to+\infty} \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} = \ell\,. \tag{5.7}$$

By definition of J we have that

$$\begin{aligned} J(u(t), \partial_t u(t)) &= \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \langle u(t), \partial_t u(t)\rangle_{L^2(\Omega)} \\ &= \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)}\Big)\,, \end{aligned}$$

i.e., the function $t \mapsto \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)}$ solves the Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\Big(\dfrac{1}{2}\|u(t)\|^2_{L^2(\Omega)}\Big) + \dfrac{1}{2}\|u(t)\|^2_{L^2(\Omega)} = J(u(t), \partial_t u(t))\,, & t \in (0, +\infty)\,, \\ \dfrac{1}{2}\|u(0)\|^2_{L^2(\Omega)} = \dfrac{1}{2}\|u_0\|^2_{L^2(\Omega)}\,, \end{cases}$$

hence

$$\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} = \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} e^{-t} + \int_0^t e^{s-t} J(u(s), \partial_t u(s))\,\mathrm{d}s\,.$$

We observe that, by (5.4) and l'Hôpital's rule

$$\begin{aligned} \ell &= \lim_{t\to+\infty} J(u(t), \partial_t u(t)) = \lim_{t\to+\infty} \frac{e^t J(u(t), \partial_t u(t))}{e^t} \\ &= \lim_{t\to+\infty} \frac{\int_0^t e^s J(u(s), \partial_t u(s))\,\mathrm{d}s}{e^t} = \lim_{t\to+\infty} \int_0^t e^{s-t} J(u(s), \partial_t u(s))\,\mathrm{d}s\,, \end{aligned}$$

thus (5.7) follows. In particular, $\sup_{t\geq 0} \|u(t)\|_{L^2(\Omega)} < +\infty$. This concludes the proof. □

Next, we study the accumulation points of $u(t)$ as $t \to +\infty$.

Proposition 5.3 (ω-limit set)**.** *Assume that u_0, v_0 satisfy* (2.4)*. Let (u, v) be the unique solution to* (2.2) *obtained in Theorem 2.1. For every sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \nearrow +\infty$ there exists a subsequence $(t_{n_m})_{m\in\mathbb{N}}$ and $u_\infty \in H^1(\Omega), v_\infty \in L^2(\Omega)$ such that $u(t_{n_m}) \to u_\infty$ strongly in $H^1(\Omega)$ and $\partial_t u(t_{n_m}) \to v_\infty$ strongly in $L^2(\Omega)$ as $m \to +\infty$. Moreover, $v_\infty = 0$ and u_∞ is a weak solution to the stationary problem*

$$\begin{cases} -\Delta u_\infty + \Phi'(u_\infty) = 0\,, & x \in \Omega\,, \\ \nabla u_\infty \cdot \nu = 0\,, & x \in \partial\Omega\,. \end{cases} \tag{5.8}$$

Proof. We divide the proof in two steps.

Step 1 (Compactness) The proof follows the lines of [4, Lemma 9.5.2]. We provide the details for the sake of completeness.

By Theorem 2.1, we have that $u = (u, v) \in C([0, +\infty); H)$ solves

$$u(t) = S_{\mathscr{A}}(t) U_0 - \int_0^t S_{\mathscr{A}}(t-s) F(u(s))\,\mathrm{d}s\,, \quad t \in [0, +\infty)\,, \tag{5.9}$$

where $U_0 = (u_0, v_0) \in H$, F is defined in Remark 4.1, and $\{S_{\mathscr{A}}(t)\}_{t\geq 0}$ is the continuous semigroup of contractions associated to $\mathscr{A}$ (see Proposition 3.2). By Proposition 5.1, we have that $\|S_{\mathscr{A}}(t) U_0\|_H \to 0$, hence

$$\{S_{\mathscr{A}}(t) U_0\}_{t\geq 0} \subset K_1\,, \quad \text{for some compact set } K_1 \subset H\,. \tag{5.10}$$

To conclude, it is enough to show that the family $\{\int_0^t S_{\mathscr{A}}(t-s) F(u(s))\,\mathrm{d}s\}_{t\geq 0}$ is contained in a compact set of $H^1(\Omega)$. Let us set

$$V(t) := \int_0^t S_{\mathscr{A}}(t-s) F(u(s))\,\mathrm{d}s = \int_0^t S_{\mathscr{A}}(s) F(u(t-s))\,\mathrm{d}s\,.$$

Let us fix $\varepsilon > 0$. By Proposition 5.2, we have, in particular, that $\sup_{t\geq 0}\|u(t)\|_{L^2(\Omega)} < +\infty$. Recalling the boundedness of Φ', we deduce that

$$M := \sup_{t\geq 0}\|F(u(t))\|_H \leq \sup_{t\geq 0}(\|u(t)\|_{L^2(\Omega)} + \|\Phi'\|_\infty |\Omega|^{\frac{1}{2}}) < +\infty\,.$$

By the previous bound and by Proposition 5.1, there exists $T > 0$ such that

$$\int_T^{+\infty} \|S_{\mathscr{A}}(s)\|_{\mathscr{L}(H)}\|F(u(t-s))\|_H \,\mathrm{d}s \leq \int_T^{+\infty} c\, e^{-s/2} M \,\mathrm{d}s < \varepsilon\,. \tag{5.11}$$

The curve $t \mapsto V(t) \in H$ is continuous, hence it maps the interval $[0,T]$ into a compact set. This implies that the family $\{V(t)\}_{0\leq t\leq T}$ is compact and, as such,

$$\{V(t)\}_{0\leq t\leq T} \text{ is totally bounded.} \tag{5.12}$$

Let us study the family $\{V(t)\}_{t\geq T}$. Thanks to (5.11), we approximate $V(t)$ with

$$\widetilde{V}(t) := \int_0^T S_{\mathscr{A}}(s)F(u(t-s))\,\mathrm{d}s\,.$$

Indeed, by (5.11), for $t \geq T$ we have that

$$\|V(t) - \widetilde{V}(t)\|_H \leq \int_T^t \|S_{\mathscr{A}}(s)\|_{\mathscr{L}(H)}\|F(u(t-s))\|_H \,\mathrm{d}s < \varepsilon/2\,. \tag{5.13}$$

Let us show that the family $\{\widetilde{V}(t)\}_{t\geq T}$ is totally bounded. We start by observing that the nonlinearity $F\colon H \to H$ maps sets of the form $B\times L^2(\Omega)$, with $B \subset H^1(\Omega)$ bounded, into relatively compact sets, thanks to the compact embedding $H^1(\Omega) \subset\subset L^2(\Omega)$ and the specific expression of F in (4.3). Hence, there exists a compact set $K \subset H$ such that $\{F(u(t))\}_{t\geq 0} \subset K$. It follows that the family $\{S_{\mathscr{A}}(s)F(u(t-s))\}_{0\leq s\leq T}$ is contained in the set $\bigcup_{0\leq s\leq T} S_{\mathscr{A}}(s)K$, which is compact, since it is given by the image of the compact set $[0,T]\times K$ through the jointly continuous map $(t,U) \in [0,+\infty)\times H \mapsto S_{\mathscr{A}}(t)U \in H$. The operator $I\colon C([0,T];H) \to H$ given by $I(U) = \int_0^T U(s)\,\mathrm{d}s$ is continuous, thus it maps the set $\bigcup_{0\leq s\leq T} S_{\mathscr{A}}(s)K$ into a compact set $K_2 \subset H$. We conclude that $\{\widetilde{V}(t)\}_{t\geq T} \subset K_2$, hence $\{\widetilde{V}(t)\}_{t\geq T}$ is totally bounded. In particular, $\{\widetilde{V}(t)\}_{t\geq T}$ can be covered by finitely-many balls with radii $\varepsilon/2$. By (5.13), we infer that $\{V(t)\}_{t\geq T}$ can be covered by finitely-many balls with radii ε. Together with (5.10) and (5.12), this shows that the family $\{u(t)\}_{t\geq 0}$ is totally bounded. In particular, we deduce that for every sequence $t_n \nearrow +\infty$ we can extract a subsequence (that we do not relabel) t_n such that

$$U(t_n) = (u(t_n), \partial_t u(t_n)) \to U_\infty = (u_\infty, v_\infty) \quad \text{in } H \text{ as } n \to +\infty\,.$$

This concludes the proof of compactness.

Step 2 (Stationary problem) We use $U_\infty = (u_\infty, v_\infty)$ as initial datum for the PDE, *i.e.*, we consider the unique solution $w(t)$ to

$$\begin{cases} \partial^2_{tt} w - \Delta w + \partial_t w + \Phi'(w) = 0\,, & \text{in } (0,+\infty)\times\Omega\,,\\ \nabla w\cdot\nu = 0\,, & \text{on } (0,+\infty)\times\partial\Omega\,,\\ w(0,\cdot) = u_\infty\,,\ \partial_t w(0,\cdot) = v_\infty\,, & \text{in } \Omega\,. \end{cases}$$

More rigorously, $W(t) = (w(t), \partial_t w(t)) \in C([0,+\infty);H)$ is the unique curve satisfying

$$W(t) = S_{\mathscr{A}}(t)U_\infty - \int_0^t S_{\mathscr{A}}(s)F(W(t-s))\,\mathrm{d}s\,,$$

provided by Theorem 2.1.

By Remark 3.5, there exists a family of operators $\{T(t)\}_{t\geq 0}$ satisfying Properties (D1)–(D2) and such that $T(t)U_0$ is the unique solution to the evolution problem with initial datum U_0. Adopting this operator, we have that the solution $u(t)$ considered in the previous step and satisfying (5.9) is given by $u(t) = T(t)(U_0)$. Analogously, $W(t) = T(t)(U_\infty)$.

In the previous step we have shown that $T(t_n)(U_0) = u(t_n) \to U_\infty$ in H. By continuity of $T(t)$, we get that

$$u(t+t_n) = T(t+t_n)(U_0) = T(t) \circ T(t_n)(U_0) \to T(t)(U_\infty) = W(t), \quad \text{in } H \text{ as } n \to +\infty. \tag{5.14}$$

Let us now consider the energy functional E defined in (2.6). By Theorem 2.3, we have that for every $s \le t$

$$E(u(t), \partial_t u(t)) + \int_s^t \|\partial_t u(r)\|^2_{L^2(\Omega)} \, \mathrm{d}r = E(u(s), \partial_t u(s)),$$

and, in particular, that $t \mapsto E(u(t), \partial_t u(t))$ is nonincreasing. Being bounded from below, we deduce the existence of $e_\infty \ge 0$ such that $\lim_{t\to+\infty} E(u(t), \partial_t u(t)) = e_\infty$. The functional E is continuous with respect to the convergence $u(t_n) = (u(t_n), \partial_t u(t_n)) \to U_\infty = (u_\infty, v_\infty)$ in H as $n \to +\infty$, hence

$$e_\infty = \lim_{n\to+\infty} E(u(t_n), \partial_t u(t_n)) = E(u_\infty, v_\infty).$$

Analogously, by (5.14) we have that

$$e_\infty = \lim_{n\to+\infty} E(u(t+t_n), \partial_t u(t+t_n)) = E(w(t), \partial_t w(t)).$$

The energy balance obtained in Theorem 2.3 yields that

$$e_\infty + \int_0^t \|\partial_t w(s)\|^2_{L^2(\Omega)} \, \mathrm{d}s = E(w(t), \partial_t w(t)) + \int_0^t \|\partial_t w(s)\|^2_{L^2(\Omega)} \, \mathrm{d}s = E(u_\infty, v_\infty) = e_\infty,$$

for every $t \in [0, +\infty)$, therefore

$$\int_0^t \|\partial_t w(s)\|^2_{L^2(\Omega)} \, \mathrm{d}s = 0,$$

for every $t \in [0, +\infty)$. This implies that $\partial_t w(t) = 0$ for every $t \in (0, +\infty)$, *i.e.*, $w(t)$ is constant in time. In particular, $v_\infty = 0$ and u_∞ is a weak solution to the stationary problem (5.8).

□

In the next proposition, we characterize the solutions to the stationary problem.

Proposition 5.4 (Solutions to the stationary problem)**.** *Let $u_\infty \in H^1(\Omega)$ be a weak solution to the stationary problem* (5.8)*. Then u_∞ is constant a.e. in Ω and $u_\infty \in \{\Phi' = 0\}$.*

Proof. We test the equation with u_∞. We get that

$$\int_\Omega |\nabla u_\infty|^2 \, \mathrm{d}x + \int_\Omega u_\infty \Phi'(u_\infty) \, \mathrm{d}x = 0.$$

By property (P4) of Φ, we have that $u_\infty \Phi'(u_\infty) \ge 0$, hence both terms must equal zero. By the connectedness of Ω we deduce that u_∞ is constant a.e. in Ω. From the condition $u_\infty \Phi'(u_\infty)$ we infer that either $u_\infty = 0$ or $u_\infty \in \{\Phi' = 0\}$. Note that $0 \in \{\Phi' = 0\}$. This concludes the proof. □

The previous result does not guarantee uniqueness of the accumulation points of $u(t)$ for large t, since there are infinitely many values that annihilate Φ'. To obtain this result and therefore existence of the limit for large t (as claimed in Theorem 2.5), we exploit the energy balance (2.9).

Proof of Theorem 2.5. The fact that

$$\partial_t u(t) \to 0 \quad \text{strongly in } L^2(\Omega) \text{ for } t \to +\infty \tag{5.15}$$

follows from the equality $v_\infty = 0$ obtained in Proposition 5.3.

Let us now fix two sequences $t_n \nearrow +\infty$ and $t_m \nearrow +\infty$ and $u_\infty, u'_\infty \in H^1(\Omega)$ such that $u(t_n) \to u_\infty$ strongly in $H^1(\Omega)$ as $n \to +\infty$ and $u(t_m) \to u'_\infty$ strongly in $H^1(\Omega)$ as $m \to +\infty$. By Proposition 5.4, u_∞ and u'_∞ are a.e. constant in Ω and satisfy $u_\infty, u'_\infty \in \{\Phi' = 0\}$.

On the one hand, as a consequence of the energy balance in Theorem 2.1, we have that (see the proof of Proposition 5.2 for the details) the following limit exists:

$$\lim_{t\to+\infty} J(u(t),\partial_t u(t)) = \ell\,.$$

By definition (2.7), $\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \langle u(t),\partial_t u(t)\rangle_{L^2(\Omega)} \to \ell$ as $t\to+\infty$. By the convergence (5.15) and the bound obtained in Proposition 5.2, we get that $\langle u(t),\partial_t u(t)\rangle_{L^2(\Omega)} \to 0$ as $t\to+\infty$. It follows that $\|u(t)\|_{L^2(\Omega)} \to \sqrt{2\ell}$.

On the other hand, the convergences $u(t_n)\to u_\infty$ and $u(t_m)\to u'_\infty$ yield $\|u(t_n)\|_{L^2(\Omega)} \to \|u_\infty\|_{L^2(\Omega)}$ and $\|u(t_m)\|_{L^2(\Omega)} \to \|u'_\infty\|_{L^2(\Omega)}$, therefore

$$|u_\infty| = \sqrt{2\ell}\,|\Omega|^{-\frac{1}{2}} = |u'_\infty|\,, \tag{5.16}$$

i.e., the modulus of the limit does not depend on the sequence of times.

If $\ell=0$, then $|u_\infty| = |u'_\infty| = 0$ independently of the subsequence. Thus $\|u(t)\|_{H^1(\Omega)}\to 0$ as $t\to+\infty$.

If $\ell>0$, let us show that $u_\infty = u'_\infty$ arguing by contradiction. Let us assume that $u_\infty \neq u'_\infty$, *i.e.*, $u'_\infty = -u_\infty$. Let $\delta>0$ be such that the balls $B_\delta(u_\infty), B_\delta(u'_\infty) \subset H^1(\Omega)$ of radius δ centered in u_∞ and u'_∞, respectively, are disjoint, *i.e.*, $B_\delta(u_\infty)\cap B_\delta(u'_\infty) = \emptyset$. By suitably extracting subsequences from $(t_n)_n$ and $(t_m)_m$, we construct a new sequence $(s_j)_j$, $s_j \nearrow +\infty$, such that $u(s_j)\in B_\delta(u_\infty)$ for j even and $u(s_j)\in B_\delta(u'_\infty)$ for j odd. Let us consider two consecutive times s_j and s_{j+1}, assuming, without loss of generality, that j is even. By the connectedness of the image of the continuous curve $u\colon [s_j,s_{j+1}]\to H^1(\Omega)$, there exists a time labelled $s_{j+1/2}\in(s_j,s_{j+1})$ such that $u(s_{j+1/2})\notin B_\delta(u_\infty)\cup B_\delta(u'_\infty)$, *i.e.*,

$$\|u(s_{j+1/2}) - u_\infty\|_{H^1(\Omega)} \geq \delta\,, \quad \|u(s_{j+1/2}) - u'_\infty\|_{H^1(\Omega)} \geq \delta\,. \tag{5.17}$$

By Proposition 5.3 we find u''_∞ and we extract a subsequence (not relabeled) from $s_{j+1/2}$ such that $u(s_{j+1/2})\to u''_\infty$. By Proposition 5.4, u''_∞ is a.e. constant in Ω. Passing to the limit in (5.17), we obtain that

$$|u''_\infty - u_\infty| \geq \delta|\Omega|^{-\frac{1}{2}}\,, \quad |u''_\infty + u_\infty| = |u''_\infty - u'_\infty| \geq \delta|\Omega|^{-\frac{1}{2}}\,,$$

and, in particular, $|u''_\infty| \neq |u_\infty|$. However, in (5.16) we have shown that the modulus of the limit is independent of the sequence of times. This is a contraction, thus necessarily $u_\infty = u'_\infty$. ☐

6. Proof of the exponential decay rate

We now turn to the proof of the convergence rate in Theorem 2.7. We stress that we work in dimension $d=1$ and assume that Ω is the interval $\Omega=(0,L)\subset\mathbb{R}$.

Inspired by [4, 16], we exploit an energy balance for the auxiliary functionals $G_\lambda\colon H^1(\Omega)\times L^2(\Omega)\to[0,+\infty)$ defined by

$$G_\lambda(u,v) = \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \int_\Omega (\Phi(u)-\Phi(u_\infty))\,\mathrm{d}x + \lambda\langle u-u_\infty, v\rangle_{L^2(\Omega)}\,,$$

where $\lambda\in(0,1)$ will be suitably chosen later and $u_\infty\in\{\Phi'=0\}$. This is a perturbation (depending on λ) of $E(u,v)-E(u_\infty,0)$.

Proof of Theorem 2.7. We split the proof in several steps.

Step 1 (Energy estimate for G_λ with regular initial data) In this step, we assume that the initial data u_0, v_0 are more regular and satisfy (4.5). We will relax this assumption in the next step. By Proposition 4.3, this guarantees higher regularity for the solution, *i.e.*, $(u,v)\in C^1([0,+\infty);H)\cap C([0,+\infty);D(\mathscr{A}))$, which justifies the following computations:

$$\begin{aligned} G_\lambda(u(t),\partial_t u(t)) = {} & \frac{1}{2}\|\partial_t u(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega)} + \int_\Omega (\Phi(u(t))-\Phi(u_\infty))\,\mathrm{d}x \\ & + \lambda\langle u(t)-u_\infty, \partial_t u(t)\rangle_{L^2(\Omega)}\,. \end{aligned}$$

Recalling that u_∞ is a.e. constant, exploiting the PDE (2.2), and integrating by parts, we obtain that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(G_\lambda(u(t),\partial_t u(t))\Big) = \langle \partial_t u(t), \partial_{tt} u(t)\rangle_{L^2(\Omega)} + \langle \nabla u(t), \partial_t \nabla u(t)\rangle_{L^2(\Omega)} \\
&\quad + \langle \Phi'(u(t)), \partial_t u(t)\rangle_{L^2(\Omega)} + \lambda\|\partial_t u(t)\|^2_{L^2(\Omega)} + \lambda\langle u(t)-u_\infty, \partial_{tt} u(t)\rangle_{L^2(\Omega)} \\
&= \langle \partial_t u(t), \Delta u(t) - \partial_t u(t) - \Phi'(u(t))\rangle_{L^2(\Omega)} - \langle \Delta u(t), \partial_t u(t)\rangle_{L^2(\Omega)} + \lambda\|\partial_t u(t)\|^2_{L^2(\Omega)} \\
&\quad + \langle \Phi'(u(t)), \partial_t u(t)\rangle_{L^2(\Omega)} + \lambda\langle u(t)-u_\infty, \Delta u(t) - \partial_t u(t) - \Phi'(u(t))\rangle_{L^2(\Omega)} \\
&= -(1-\lambda)\|\partial_t u(t)\|^2_{L^2(\Omega)} - \lambda\|\nabla u(t)\|^2_{L^2(\Omega)} - \lambda\langle u(t)-u_\infty, \Phi'(u(t))\rangle_{L^2(\Omega)} \\
&\quad - \lambda\langle u(t)-u_\infty, \partial_t u(t)\rangle_{L^2(\Omega)}\,, \qquad \text{for } t>0.
\end{aligned} \tag{6.1}$$

To reconstruct G_λ on the right-hand side of the previous expression and thus apply Grönwall's inequality, we need to compare $\int_\Omega(\Phi(u(t)) - \Phi(u_\infty))\,\mathrm{d}x$ and $\langle u(t) - u_\infty, \Phi'(u(t))\rangle_{L^2(\Omega)}$.

We claim that there exists a $T_\sigma > 0$ (possibly depending on σ) such that

$$2\int_\Omega (\Phi(u(t)) - \Phi(u_\infty))\,\mathrm{d}x \le \langle u(t)-u_\infty, \Phi'(u(t))\rangle_{L^2(\Omega)} \quad \text{for } t\ge T_\sigma\,. \tag{6.2}$$

By the convergence $u(t)\to u_\infty$ in $H^1(\Omega)$ as $t\to+\infty$ and the compact embedding in one dimension $H^1(\Omega)\subset\subset C(\overline{\Omega})$, we deduce that $u(t)\to u_\infty$ *uniformly* as $t\to+\infty$. Let us distinguish two cases:

Case 1: $u_\infty = 0$. Then there exists $T_\sigma>0$ such that $|u(t)| < u_* - \frac{1}{\sigma}$ for $t\ge T_\sigma$, *i.e.*, $\Phi(u(t)) = u(t)^2$ (see the definition of Φ in (2.1)). Then, observing that $u(t)\Phi'(u(t)) = 2u(t)^2$, we deduce (6.2).

Case 2: $|u_\infty| > u_*$. Then there exists $T_\sigma>0$ such that $|u(t)|>u_*$ for $t\ge T_\sigma$, and thus, by the definition of Φ in (2.1), both terms in (6.2) vanish.

Combining (6.1)–(6.2), we obtain that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(G_\lambda(u(t),\partial_t u(t))\Big) \le -(1-\lambda)\|\partial_t u(t)\|^2_{L^2(\Omega)} - \lambda\|\nabla u(t)\|^2_{L^2(\Omega)} \\
&\quad - 2\lambda\int_\Omega(\Phi(u(t)) - \Phi(u_\infty))\,\mathrm{d}x - \lambda\langle u(t)-u_\infty, \partial_t u(t)\rangle_{L^2(\Omega)} \\
&\le -\kappa_\lambda G_\lambda(u(t),\partial_t u(t))\,, \qquad \text{for } t>T_\sigma,
\end{aligned}$$

where $\kappa_\lambda>0$ is a constant depending on the choice of $\lambda\in(0,1)$.

Step 2 (Energy estimate for G_λ with general initial data) By a density argument (as in the proof of Theorem 2.1), we can establish the same result for data that are only in the energy space.

Step 3 (Decay of the solution) From the decay of G_λ, we now want to deduce the decay of the energy E and of the solution itself.

We split the analysis into two cases as above.

Case 1: $u_\infty = 0$. Then there exists $T_\sigma>0$ such that $|u(t)| < u_* - \frac{1}{\sigma}$ for $t\ge T:\sigma$, *i.e.*, $\Phi(u(t)) = u(t)^2$. Thus, for $t>T_\sigma$, the problem reduces to a damped Klein–Gordon equation:

$$\begin{cases} \partial^2_{tt}u - \Delta u + \partial_t u + 2u = 0\,, & (t,x)\in(T_\sigma,+\infty)\times\Omega\,,\\ \nabla u\cdot\nu = 0\,, & (t,x)\in(T_\sigma,+\infty)\times\partial\Omega\,,\end{cases}$$

with initial data $u(T_\sigma)$ and $\partial_t u(T_\sigma)$ at time $t=T_\sigma$. Then, we go back to the study of the evolution of G_λ:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}G_\lambda(u(t),\partial_t u(t)) &= -(1-\lambda)\|\partial_t u(t)\|^2_{L^2(\Omega)} - \lambda\|\nabla u(t)\|_{L^2(\Omega)} \\
&\quad - 2\lambda\|u(t)\|^2_{L^2(\Omega)} - \lambda\langle u(t), \partial_t u(t)\rangle.
\end{aligned}$$

Noticing that $2u^2 \geq u^2$ (and thus $-2u^2 \leq -u^2$), choosing $\lambda = 1/2$, and applying Grönwall's inequality, we deduce

$$G_{1/2}(u(t), \partial_t u(t)) \leq e^{-\frac{1}{2}(t-T_\sigma)} G_{1/2}(u(T_\sigma), \partial_t u(T_\sigma)).$$

Using Young's inequality and the monotonicity of the energy, we deduce

$$E(u(t), \partial_t u(t)) \leq C\, e^{-\frac{1}{2}(t-T_\sigma)} E(u(T_\sigma), \partial_t u(T_\sigma)) \leq C\, e^{-\frac{1}{2}(t-T_\sigma)} E(u_0, u_1)\,, \qquad \text{for } t > T_\sigma\,,$$

which, in turn, directly implies

$$\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_{H^1(\Omega)} \leq C e^{-\frac{1}{2}(t-T_\sigma)} =: M_\sigma e^{-t/2}\,, \qquad \text{for } t > T_\sigma\,.$$

Case 2: $|u_\infty| > u_*$. Then there exists $T_\sigma > 0$ such that $|u(t)| > u_*$ for $t \geq T_\sigma$, and thus, by the definition of Φ in (2.1), the problem reduces to a damped wave equation

$$\begin{cases} \partial^2_{tt} u - \Delta u + \partial_t u = 0\,, & (t,x) \in (T_\sigma, +\infty) \times \Omega\,, \\ \nabla u \cdot \nu = 0\,, & (t,x) \in (T_\sigma, +\infty) \times \partial\Omega\,, \end{cases}$$

with initial data $u(T_\sigma)$ and $\partial_t u(T_\sigma)$ at time $t = T_\sigma$. Then, going back to the evolution of G_λ,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} G_\lambda(u(t), \partial_t u(t)) = &-(1-\lambda)\|\partial_t u(t)\|^2_{L^2(\Omega)} - \lambda \|\nabla u(t)\|^2_{L^2(\Omega)} \\ &- \lambda \langle u(t) - u_\infty, \partial_t u(t)\rangle\,. \end{aligned}$$

Choosing $\lambda = 1/2$ and applying Grönwall's inequality and the monotonicity of the energy, we deduce

$$\begin{aligned} &E(u(t), \partial_t u(t)) - E(u_\infty, 0) \\ &\leq C e^{-\frac{1}{2}(t-T_\sigma)} \Big(E(u(T_\sigma), \partial_t u(T_\sigma)) - E(u_\infty, 0) \Big) \\ &\leq e^{-\frac{1}{2}(t-T_\sigma)} \Big(E(u_0, u_1) - E(u_\infty, 0) \Big)\,, \qquad \text{for } t > T_\sigma. \end{aligned}$$

Now, in order to study the decay of the solution itself, we need to apply Poincaré's inequality,

$$c_p \|u(t) - \bar{u}(t)\|^2_{L^2(\Omega)} \leq \|\nabla u(t)\|^2_{L^2(\Omega)},$$

which yields—when combined with Young's inequality—

$$\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|u(t) - \bar{u}(t)\|^2_{H^1(\Omega)} \leq C e^{-\frac{1}{2}(t-T_\sigma)} =: M_\sigma e^{-t/2}\,, \qquad \text{for } t > T_\sigma.$$

On the other hand, the average $\bar{u}$ satisfies the ODE

$$\ddot{\bar{u}}(t) + \dot{\bar{u}}(t) = 0, \qquad t > T_\sigma, \tag{6.3}$$

with initial data $\bar{u}(T_\sigma)$ and $\dot{\bar{u}}(T_\sigma)$ at $t = T_\sigma$, which can be explicitly solved:

$$\bar{u}(t) = \bar{u}(T_\sigma) + \dot{\bar{u}}(T_\sigma) - e^{-(t-T_\sigma)} \dot{\bar{u}}(T_\sigma)$$

In conclusion, we have

$$\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|u(t) - (\bar{u}(T_\sigma) + \dot{\bar{u}}(T_\sigma))\|^2_{H^1(\Omega)} \leq \widetilde{M}_\sigma e^{-\kappa t}\,, \qquad \text{for } t > T_\sigma.$$

Since

$$\|u(t) - u_\infty\|_{H^1(\Omega)} \to 0 \qquad \text{as } t \to +\infty,$$

we have that $u_\infty \equiv (\bar{u}(T_\sigma) + \dot{\bar{u}}(T_\sigma))$. We stress that, integrating (6.3), we have that $\dot{u}(t) + u(t)$ is constant for $t > T_\sigma$; hence, the limit does not depend on the choice of the threshold time T_σ.

□

Appendix A. Analysis of the ODE model

In the following, we assume—without loss of generality—that $u_* = 1$. For every $\sigma \geq 1$ we let Φ be as in (2.1) and we consider the scalar ODE:

$$\begin{cases} \ddot{z}(t) + \dot{z}(t) + \Phi'(z(t)) = 0\,, & t \in (0, +\infty)\,, \\ z(0) = z_0\,, \\ \dot{z}(0) = w_0\,, \end{cases} \tag{A.1}$$

modelling a damped nonlinear spring. We drop the dependence on σ not to overburden the notation, but we invite the reader to pay attention to its role.

By the Cauchy–Lipschitz theorem, there exists a unique C^1 solution to (A.1). We can study the long-time behavior of the trajectories of (A.1) and, in particular, prove an exponential decay rate *independent* of the parameter σ (compare Remark 2.8 for the case of the PDE).

Theorem A.1 (Uniform decay rate for the ODE model)**.** *Let* $(z_0, w_0) \in \mathbb{R}^2$. *Let* $z \in C^1([0, +\infty))$ *the unique solution to* (A.1). *Then there exists constants* σ_0, R, $M > 0$, *depending on the initial data* (z_0, w_0) *such that, for every* $\sigma \geq \sigma_0$ *there exists* $z_\infty \in [-R, R] \subset \mathbb{R}$ *with* $\Phi'(z_\infty) = 0$, *such that*

$$|z(t) - z_\infty| \leq M e^{-\frac{1}{2}t}\,.$$

A.1. **Case-by-case analysis.** We shall study the possible behavior of the trajectory depending on the initial data (z_0, w_0).

Case I: Initial data $z_0 \geq 1$ *and* $w_0 > 0$. (Arguing by symmetry, this is analogous to the case $z_0 \leq -1$ and $w_0 < 0$.) Since $w_0 > 0$, by continuity there exists $\delta > 0$ such that $\dot{z}(t) > 0$ for $t \in (0, \delta)$. This implies that $z(t)$ is strictly increasing for $t \in (0, \delta)$. In particular, $z(t) > z_0 \geq 1$ and $\Phi'(z(t)) = 0$ for $t \in (0, \delta)$. The trajectory $z(t)$ solves the ODE

$$\begin{cases} \ddot{z}(t) + \dot{z}(t) = 0\,, & t \in (0, \delta)\,, \\ z(0) = z_0\,, \\ \dot{z}(0) = w_0\,. \end{cases} \tag{A.2}$$

The unique solution to this problem is given by

$$z(t) = z_0 + w_0(1 - e^{-t})\,, \quad \text{for every } t \in [0, \delta)\,. \tag{A.3}$$

Notice that the function $z_0 + w_0(1 - e^{-t}) \geq 1$ for every $t \geq 0$, since $w_0 > 0$. Thus equality (A.3) is extended to $\delta = +\infty$ and $z(t) \to z_0 + w_0$ as $t \to +\infty$. The rate of convergence is exponential with constants independent of k and depending on the size of the initial data. See Figure 2.

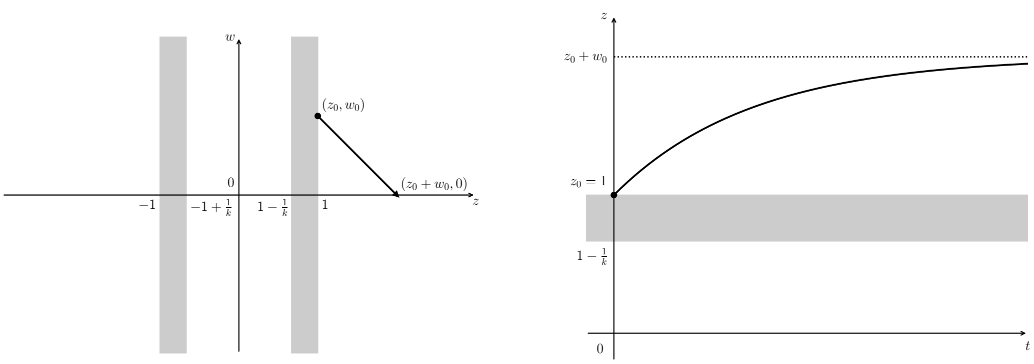

Figure 2. Example of an evolution described in *Case I*. On the left, the trajectory is represented in the phase space. On the right, the trajectory is described as a function of time.

Case II: Initial data $z_0 \geq 1$ *and* $w_0 = 0$. (Arguing by symmetry, this is analogous to the case $z_0 \leq -1$ and $w_0 = 0$.) Since $\Phi'(z_0) = 0$ when $z_0 \geq 1$, the constant trajectory $z(t) \equiv z_0$ is the unique solution to the ODE (A.1). See Figure 3.

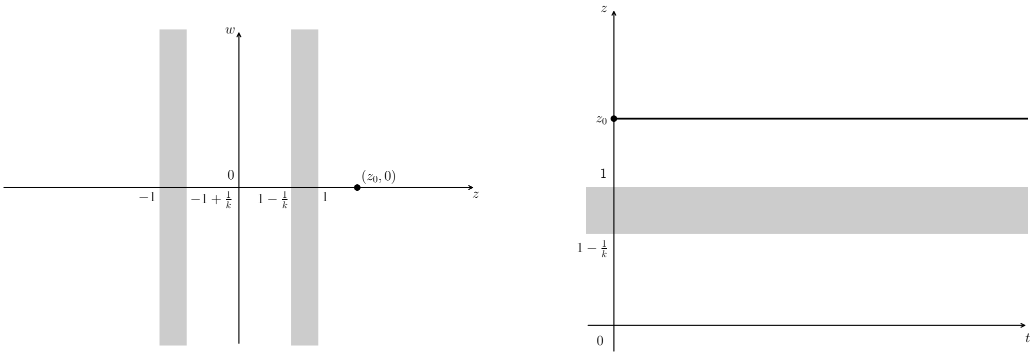

Figure 3. Example of a constant evolution described in *Case II*. On the left, the trajectory (a point) is represented in the phase space. On the right, the trajectory is described as a function of time.

Case III: Initial data $z_0 > 1$ *and* $w_0 < 0$ *with* $1 - z_0 \leq w_0$. (Arguing by symmetry, this is analogous to the case $z_0 < -1$ and $w_0 > 0$ with $w_0 \leq -1 - z_0$.) By continuity, there exists $\delta > 0$ such that $z(t) > 1$ for $t \in (0, \delta)$. This implies that $\Phi'(z(t)) = 0$ for $t \in (0, \delta)$. The trajectory $z(t)$ solves the ODE (A.2). The unique solution to this problem is given by

$$z(t) = z_0 + w_0(1 - e^{-t}) = z_0 - |w_0|(1 - e^{-t}), \quad \text{for every } t \in [0, \delta). \tag{A.4}$$

We observe that for every $t \geq 0$

$$z_0 - |w_0|(1 - e^{-t}) \geq 1 \iff e^{-t} \geq \frac{|w_0| + 1 - z_0}{|w_0|}$$

and the last inequality holds true since $1 - z_0 \leq w_0$. This implies that (A.4) can be extended up to $\delta = +\infty$ and $z(t) \to z_0 - |w_0|$ as $t \to +\infty$, with $z_0 - |w_0| \geq 1$. The rate of convergence is exponential with constants independent of k. The rate of convergence is exponential with constants independent of k and depending on the size of the initial data. See Figure 4.

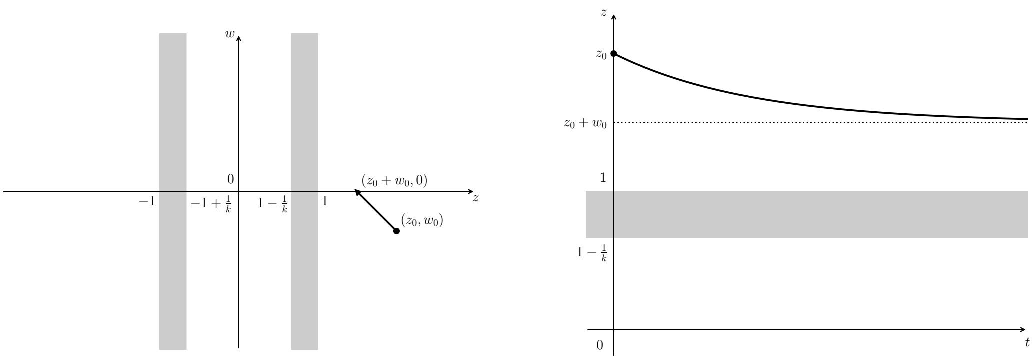

Figure 4. Example of an evolution described in *Case III*. On the left, the trajectory is represented in the phase space. On the right, the trajectory is described as a function of time.

Case IV: Initial data $z_0 > 1$ *and* $w_0 < 0$ *with* $1 - z_0 > w_0$. (Arguing by symmetry, this is analogous to the case $z_0 < -1$ and $w_0 > 0$ with $w_0 > -1 - z_0$.) As in the previous case, there exists $\delta > 0$ such that

$$z(t) = z_0 + w_0(1 - e^{-t}) = z_0 - |w_0|(1 - e^{-t}), \quad \text{for every } t \in [0, \delta). \tag{A.5}$$

Let us find the first time T_{IV} at which the function $z_0 - |w_0|(1 - e^{-t})$ reaches the value 1 by solving

$$z_0 - |w_0|(1 - e^{-T_{IV}}) = 1 \iff e^{-T_{IV}} = \frac{|w_0| + 1 - z_0}{|w_0|} \iff T_{IV} = \log\Big(\frac{|w_0|}{|w_0| + 1 - z_0}\Big).$$

This implies that (A.5) can be extended up to $\delta = T_{IV} = \log\Big(\frac{|w_0|}{|w_0|+1-z_0}\Big) > 0$. We stress that T_{IV} depends only on the size of the initial data and is independent of k. The exiting speed at time $T_{IV} = \log\Big(\frac{|w_0|}{|w_0|+1-z_0}\Big)$ is $\dot z(T_{IV}) = -|w_0|e^{-T_{IV}} < 0$. The future of the evolution after time T_{IV} is obtained by restarting the evolution at time T_{IV} and by considering *Case V*. See Figure 5.

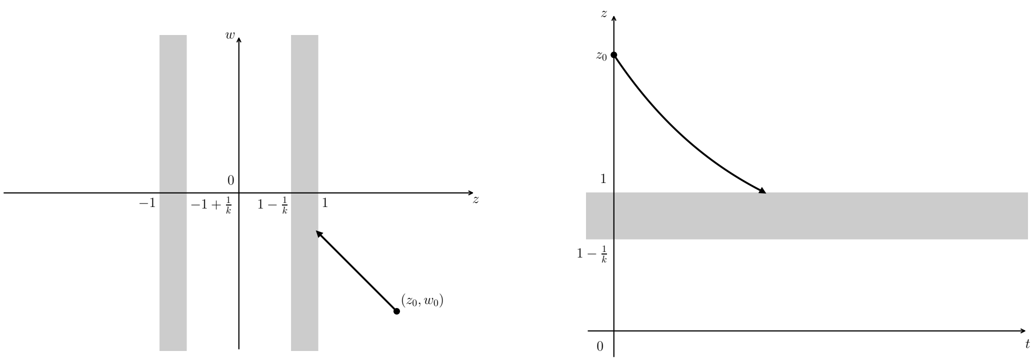

Figure 5. Example of an evolution described in *Case IV*. On the left, the trajectory is represented in the phase space. On the right, the trajectory is described as a function of time.

Case V: Initial data $z_0 = 1$ *and* $w_0 < 0$. (Arguing by symmetry, this is analogous to the case $z_0 = -1$ and $w_0 > 0$.) By continuity there exists $\delta > 0$ such that $\dot z(t) < 0$ for $t \in (0, \delta)$. This implies that $z(t)$ is strictly decreasing for $t \in (0, \delta)$. For $\delta > 0$ small enough, we have, in particular, that $1 - \frac{1}{\sigma} < z(t) < z_0 = 1$. This implies $\Phi'(z(t)) = 2(\sigma - 1)(1 - z(t))$ for $t \in (0, \delta)$ and the trajectory $z(t)$ solves the ODE

$$\begin{cases} \ddot z(t) + \dot z(t) + 2(\sigma - 1)(1 - z(t)) = 0, & t \in (0, \delta), \\ z(0) = z_0 = 1, \\ \dot z(0) = w_0. \end{cases}$$

The unique solution to this problem is given by

$$z(t) = 1 + \frac{|w_0|}{\lambda_\sigma + \mu} e^{-\lambda_\sigma t} - \frac{|w_0|}{\lambda_\sigma + \mu} e^{\mu t}, \quad \text{for } t \in [0, \delta), \tag{A.6}$$

where

$$\lambda_\sigma = \frac{1 + \sqrt{1 + 8(k-1)}}{2} > 0, \quad \mu = \frac{-1 + \sqrt{1 + 8(k-1)}}{2} > 0. \tag{A.7}$$

Let us estimate the first time T_V^σ such that the function defined in (A.6) reaches the value $1 - \frac{1}{\sigma}$. Let us observe that

$$1 + \frac{|w_0|}{\lambda_\sigma + \mu} e^{-\lambda_\sigma t} - \frac{|w_0|}{\lambda_\sigma + \mu} e^{\mu t} < 1 + \frac{|w_0|}{\lambda_\sigma + \mu} - \frac{|w_0|}{\lambda_\sigma + \mu} e^{\mu t}$$

for every $t > 0$. At time $t = 0$ both sides of the inequality are equal to 1. Let us solve

$$1 + \frac{|w_0|}{\lambda_\sigma + \mu} - \frac{|w_0|}{\lambda_\sigma + \mu} e^{\mu t} = 1 - \frac{1}{\sigma} \iff t = \frac{1}{\mu} \log\Big(1 + \frac{\lambda_\sigma + \mu}{k|w_0|}\Big) .$$

This implies that the function $1 + \frac{|w_0|}{\lambda_\sigma+\mu} e^{-\lambda_\sigma t} - \frac{|w_0|}{\lambda_\sigma+\mu} e^{\mu t}$ reaches $1 - \frac{1}{\sigma}$ in a time T_V^σ satisfying

$$0 < T_V^\sigma < \frac{1}{\mu} \log\Big(1 + \frac{\lambda_\sigma + \mu}{k|w_0|}\Big) .$$

We remark that $\frac{1}{\mu} \log(1 + \frac{\lambda_\sigma+\mu}{k|w_0|}) \to 0$ as $k \to +\infty$. In particular, $T_V^\sigma < 1$ for k large enough, depending only on the size of the initial data. The exiting speed at time T_V^σ is $\dot z(T_V^\sigma) = -\frac{\lambda_\sigma |w_0|}{\lambda_\sigma+\mu} e^{-\lambda_\sigma T_V^\sigma} - \frac{\mu|w_0|}{\lambda_\sigma+\mu} e^{\mu T_V^\sigma} < 0$. See Figure 6. The future of the evolution after time T_V^σ is obtained by restarting the evolution at time T_V^σ and by considering *Case VI.*

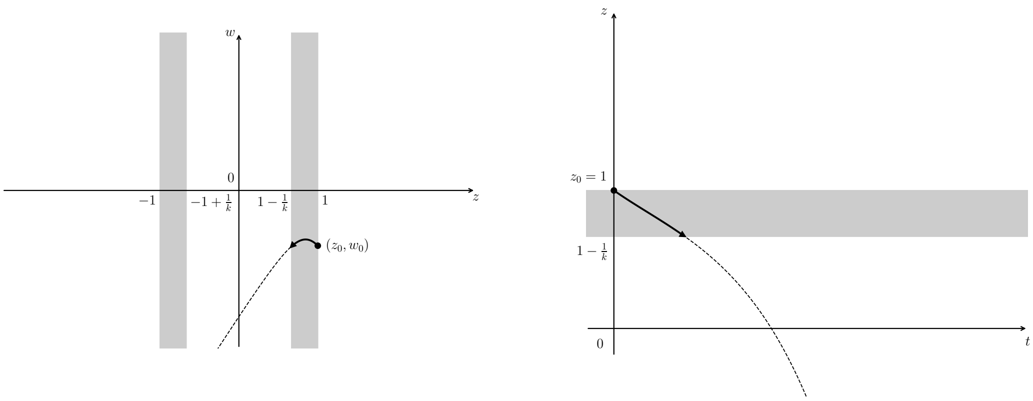

Figure 6. Example of an evolution described in *Case V*. On the left, the trajectory is represented in the phase space. On the right, the trajectory is described as a function of time. The solid line depicts $z(t)$ while the dashed line represents the function $1 + \frac{|w_0|}{\lambda_\sigma+\mu} e^{-\lambda_\sigma t} - \frac{|w_0|}{\lambda_\sigma+\mu} e^{\mu t}$ beyond time T_V^σ.

We stress that the cases including the initial data satisfying $1 - \frac{1}{\sigma} \leq z_0 < 1$ or $-1 < z_0 \leq -1 + \frac{1}{\sigma}$ are unnecessary, since for k large enough (depending on the initial data), these are not satisfied. Thus, we consider directly the case $-1 + \frac{1}{\sigma} < z_0 < 1 - \frac{1}{\sigma}$.

Case VI: Initial data $-1 + \frac{1}{\sigma} \leq z_0 \leq 1 - \frac{1}{\sigma}$ and $w_0 \in \mathbb{R}$. Several possibilities might occur:

- If $-1 + \frac{1}{\sigma} < z_0 < 1 - \frac{1}{\sigma}$, by continuity there exists $\delta > 0$ such that $-1 + \frac{1}{\sigma} < z(t) < 1 - \frac{1}{\sigma}$.
- If $z_0 = 1 - \frac{1}{\sigma}$ and $w_0 > 0$, we fall into *Case VII* and we do not study this case here as the trajectory immediately exits from the interval $[-1 + \frac{1}{\sigma}, 1 - \frac{1}{\sigma}]$.
- If $z_0 = 1 - \frac{1}{\sigma}$ and $w_0 = 0$, then there exists $\delta > 0$ such that $\dot z(t) \neq 0$ for $t \in (0, \delta)$ (for, otherwise, $1 - \frac{1}{\sigma}$ would be an equilibrium point). It cannot occur that $\dot z(t) > 0$. Indeed, this would imply that $1 - \frac{1}{\sigma} < z(t) < 1$ for $t \in (0, \delta)$ with δ small enough. This would give that $\Phi'(z(t)) = 2(k-1)(1 - z(t))$ for $t \in (0, \delta)$ and the trajectory $z(t)$ would solve the ODE

$$\begin{cases} \ddot z(t) + \dot z(t) + 2(k-1)(1 - z(t)) = 0 , & t \in (0, \delta) , \\ z(0) = z_0 = 1 - \frac{1}{\sigma} , \\ \dot z(0) = 0 . \end{cases}$$

The explicit solution to this problem is

$$z(t) = 1 - \frac{\frac{\mu}{\sigma}}{\lambda_\sigma + \mu} e^{-\lambda_\sigma t} - \frac{\frac{\lambda_\sigma}{\sigma}}{\lambda_\sigma + \mu} e^{\mu t} , \quad \text{for } t \in [0, \delta) ,$$

where λ_σ and μ are defined as in (A.7). However, the derivative of this function is

$$\dot{z}(t) = \frac{\frac{\lambda_\sigma \mu}{\sigma}}{\lambda_\sigma + \mu}\left(e^{-\lambda_\sigma t} - e^{\mu t}\right) < 0\,, \quad \text{for } t \in (0,\delta)\,,$$

which would lead to a contradiction. We conclude that $\dot{z}(t) < 0$ for $t \in (0,\delta)$, hence $z(t)$ is strictly decreasing and, in particular, $-1+\frac{1}{\sigma} < z(t) < z_0 = 1-\frac{1}{\sigma}$ for $t \in (0,\delta)$ with δ small enough.

- If $z_0 = 1-\frac{1}{\sigma}$ and $w_0 < 0$, then by continuity there exists $\delta > 0$ such that $\dot{z}(t) < 0$ for $t \in (0,\delta)$. Hence $z(t)$ is strictly decreasing and, in particular, $-1+\frac{1}{\sigma} < z(t) < z_0 = 1-\frac{1}{\sigma}$ for $t \in (0,\delta)$ with δ small enough.
- If $z_0 = -1+\frac{1}{\sigma}$ and $w_0 < 0$, we fall into *Case VII* and we do not study this case here.
- If $z_0 = -1+\frac{1}{\sigma}$ and $w_0 = 0$, this is analogous to the case $z_0 = 1-\frac{1}{\sigma}$ and $w_0 = 0$.
- If $z_0 = -1+\frac{1}{\sigma}$ and $w_0 > 0$, this is analogous to the case $z_0 = 1-\frac{1}{\sigma}$ and $w_0 < 0$.

In all cases, $-1+\frac{1}{\sigma} < z(t) < z_0 = 1-\frac{1}{\sigma}$ for $t \in (0,\delta)$, thus $\Phi'(z(t)) = 2z(t)$ and $z(t)$ solves the ODE

$$\begin{cases} \ddot{z}(t) + \dot{z}(t) + 2z(t) = 0\,, & t \in (0,\delta)\,, \\ z(0) = z_0\,, \\ \dot{z}(0) = w_0\,. \end{cases}$$

The unique solution to this problem is given by

$$z(t) = z_0 e^{-\frac{1}{2}t}\cos(\omega t) + \frac{2w_0+z_0}{2\omega}e^{-\frac{1}{2}t}\sin(\omega t)\,, \quad \text{for every } t \in [0,\delta)\,. \tag{A.8}$$

where $\omega = \frac{\sqrt{7}}{2}$. We distinguish several possibilities, depending on the extreme values attained by the function $z_0 e^{-\frac{1}{2}t}\cos(\omega t) + \frac{2w_0+z_0}{2\omega}e^{-\frac{1}{2}t}\sin(\omega t)$ for $t \in [0,+\infty)$. The explicit values of the extreme values attained by this function are not relevant for the discussion. However, it is possible to compute them by considering the first time where the derivative vanishes. The first critical point is $\bar{t} = \frac{1}{\omega}\arctan\left(\frac{4\omega w_0}{(4\omega^2+1)z_0+2w_0}\right)$ or $\bar{t} = \frac{1}{\omega}\arctan\left(\frac{4\omega w_0}{(4\omega^2+1)z_0+2w_0}\right) + \frac{\pi}{\omega}$ (depending on the sign of w_0). All critical points are of the form $t = \bar{t} + \frac{n}{\omega}\pi$, $n \in \mathbb{N}$. The values attained at these critical points are

$$e^{-\frac{1}{2}\frac{n}{\omega}}\left(z_0 e^{-\frac{1}{2}\bar{t}}\cos(\omega\bar{t}) + \frac{2w_0+z_0}{2\omega}e^{-\frac{1}{2}\bar{t}}\sin(\omega\bar{t})\right).$$

Note that their absolute value is therefore damped as n increases. For this reason, the extreme values are attained either at $t=0$ and $t=\bar{t}$ or at $t=\bar{t}$ and $t=\bar{t}+\frac{\pi}{\omega}$.

- The extreme value at time $t=\bar{t}$ lies in the interval $(-1+\frac{1}{\sigma}, 1-\frac{1}{\sigma})$. See Figure 7. Since at time $t=0$ we already have that $-1+\frac{1}{\sigma} < z_0 < 1-\frac{1}{\sigma}$ and all future extreme values are damped, this implies that the function $z_0 e^{-\frac{1}{2}t}\cos(\omega t)+\frac{2w_0+z_0}{2\omega}e^{-\frac{1}{2}t}\sin(\omega t)$ lies in the interval $(-1+\frac{1}{\sigma}, 1-\frac{1}{\sigma})$ for every $t \in [0,+\infty)$. In this case, formula (A.8) can be extended up to $\delta = +\infty$. The solution satisfies $z(t) \to 0$ as $t \to +\infty$. The rate of convergence is exponential with constants independent of k and depending on the size of the initial data.

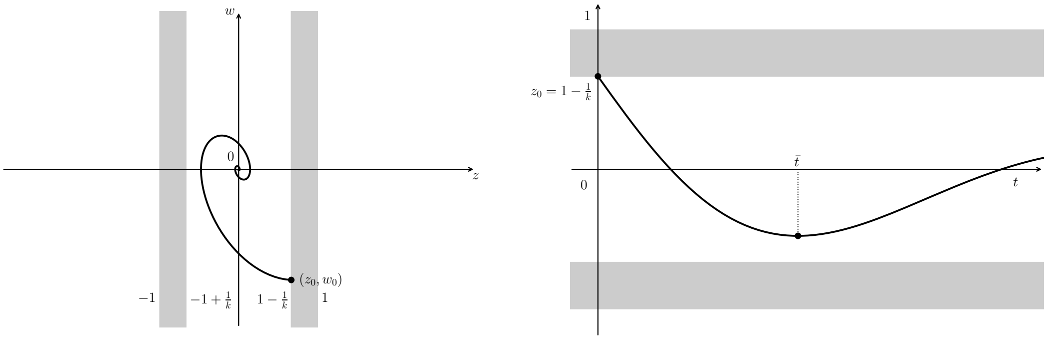

Figure 7. Example of an evolution described in *Case VI* when the extreme value at time $t = \bar{t}$ lies in the interval $\left(-1+\frac{1}{\sigma}, 1-\frac{1}{\sigma}\right)$. On the left, the trajectory is represented in the phase space. On the right, the trajectory is described as a function of time.

- The extreme value at time $t = \bar{t}$ lies outside the interval $\left(-1+\frac{1}{\sigma}, 1-\frac{1}{\sigma}\right)$. See Figure 8. This means that either the global minimum is less than or equal to $-1+\frac{1}{\sigma}$ or the global maximum is greater that or equal to $1-\frac{1}{\sigma}$. In either case, we extend (A.8) up to $\delta = T^\sigma_{VI}$, where T^σ_{VI} is the first exiting time such that the function $|z_0 e^{-\frac{1}{2}t}\cos(\omega t) + \frac{2w_0+z_0}{2\omega} e^{-\frac{1}{2}t}\sin(\omega t)|$ reaches the value $1-\frac{1}{\sigma}$. Note that T^σ_{VI} is bounded by $\bar{t}$. At time T^σ_{VI}, the sign of the exiting speed $\dot{z}(T^\sigma_{VI})$ depends on whether $z(T^\sigma_{VI}) = 1-\frac{1}{\sigma}$ (positive, in this case) or $z(T^\sigma_{VI}) = -1+\frac{1}{\sigma}$ (negative, in this case). After time T^σ_{VI}, the future of the evolution is obtained by restarting the evolution at time T^σ_{VI} and by considering *Case VII.*

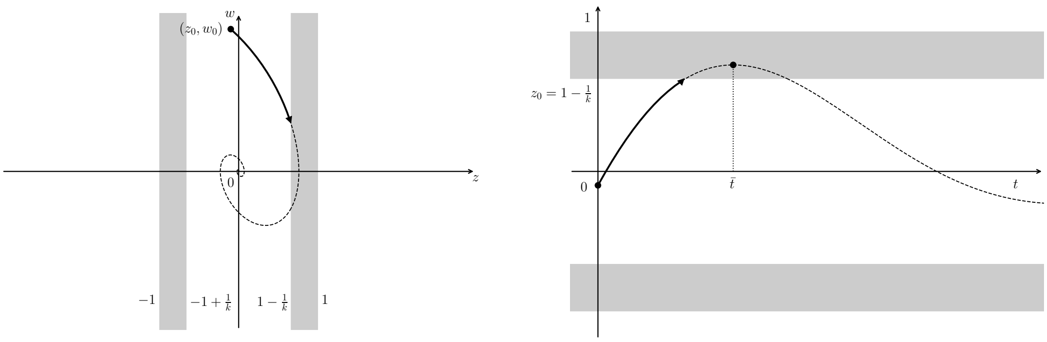

Figure 8. Example of an evolution described in *Case VI* when the extreme value at time $t = \bar{t}$ lies outside the interval $\left(-1+\frac{1}{\sigma}, 1-\frac{1}{\sigma}\right)$. On the left, the trajectory is represented in the phase space. On the right, the trajectory is described as a function of time. The solid line depicts $z(t)$ while the dashed line represents the function $z_0 e^{-\frac{1}{2}t}\cos(\omega t) + \frac{2w_0+z_0}{2\omega} e^{-\frac{1}{2}t}\sin(\omega t)$ beyond time T^σ_{VI}.

Case VII: Initial data $z_0 = 1-\frac{1}{\sigma}$ and $w_0 > 0$. (Arguing by symmetry, this is analogous to the case $z_0 = -1+\frac{1}{\sigma}$ and $w_0 < 0$.) Since $w_0 > 0$, by continuity there exists $\delta > 0$ such that $\dot{z}(t) > 0$ for $t \in (0,\delta)$. Hence, $z(t)$ is strictly increasing and, in particular, $1-\frac{1}{\sigma} = z_0 < z(t) < 1$ for $t \in (0,\delta)$ if δ is small enough. This implies $\Phi'(z(t)) = 2(\sigma-1)(1-z(t))$ for $t \in (0,\delta)$ and the trajectory $z(t)$ solves the ODE

$$\begin{cases} \ddot{z}(t) + \dot{z}(t) + 2(\sigma-1)(1-z(t)) = 0\,, & t \in (0,\delta)\,, \\ z(0) = z_0 = 1-\frac{1}{\sigma}\,, \\ \dot{z}(0) = w_0\,. \end{cases}$$

The explicit solution to this problem is

$$z(t) = 1 - \frac{w_0 + \frac{\mu}{\sigma}}{\lambda_\sigma + \mu} e^{-\lambda_\sigma t} + \frac{w_0 - \frac{\lambda_\sigma}{\sigma}}{\lambda_\sigma + \mu} e^{\mu t}\,, \quad \text{for } t \in [0,\delta)\,,$$

where λ_σ and μ are defined as in (A.7). Let us estimate the time T^σ_{VII} needed by the function $1 - \frac{w_0+\frac{\mu}{\sigma}}{\lambda_\sigma+\mu} e^{-\lambda_\sigma t} + \frac{w_0-\frac{\lambda_\sigma}{\sigma}}{\lambda_\sigma+\mu} e^{\mu t}$ to reach the value 1 by solving

$$1 - \frac{w_0 + \frac{\mu}{\sigma}}{\lambda_\sigma + \mu} e^{-\lambda_\sigma T^\sigma_{VII}} + \frac{w_0 - \frac{\lambda_\sigma}{\sigma}}{\lambda_\sigma + \mu} e^{\mu T^\sigma_{VII}} = 1 \iff T^\sigma_{VII} = \frac{1}{\lambda_\sigma + \mu} \log\Big(\frac{w_0 + \frac{\mu}{\sigma}}{w_0 - \frac{\lambda_\sigma}{\sigma}} \Big).$$

Note that $w_0 - \frac{\lambda_\sigma}{\sigma} > 0$ for k large enough depending on the size of the initial data. We remark that $T^\sigma_{VII} \to 0$ as $k \to +\infty$, hence $T^\sigma_{VII} < 1$ for k large enough, depending only on the size of the initial data. The exiting speed is $\dot{z}(T^\sigma_{VII}) = \lambda_\sigma \frac{w_0+\frac{\mu}{\sigma}}{\lambda_\sigma+\mu} e^{-\lambda_\sigma T^\sigma_{VII}} + \mu \frac{w_0-\frac{\lambda_\sigma}{\sigma}}{\lambda_\sigma+\mu} e^{\mu T^\sigma_{VII}} > 0$. Thus the future evolution is obtained by restarting the evolution at time T^σ_{VII} and by considering *Case I*.

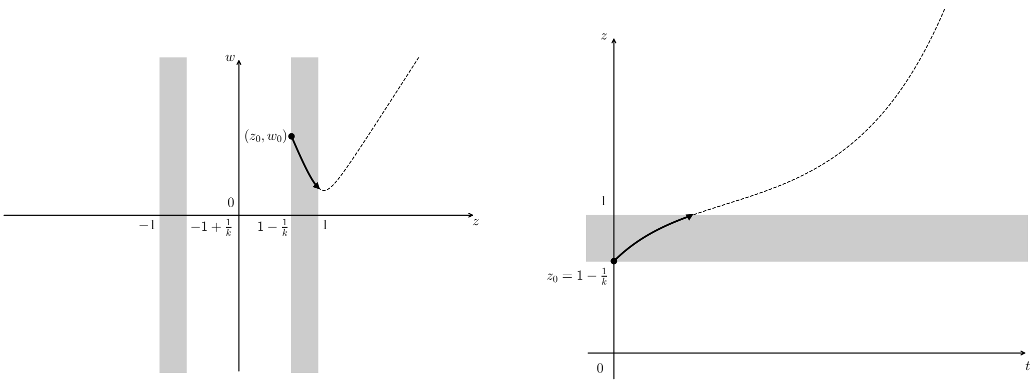

Figure 9. Example of an evolution described in *Case VII*. On the left, the trajectory is represented in the phase space. On the right, the trajectory is described as a function of time. The solid line depicts $z(t)$ while the dashed line represents the function $1 - \frac{w_0+\frac{\mu}{\sigma}}{\lambda_\sigma+\mu} e^{-\lambda_\sigma t} + \frac{w_0-\frac{\lambda_\sigma}{\sigma}}{\lambda_\sigma+\mu} e^{\mu t}$ beyond time T^σ_{VII}.

A.2. **Proof of the decay rate.** Combining all the cases discussed above, we can prove Theorem A.1

Proof of Theorem A.1. By the previous discussion, we deduce that $z(t)$ transits in the intervals $(1 - \frac{1}{\sigma}, 1)$ and $(-1, -1 + \frac{1}{\sigma})$ at most twice. The time spent in these intervals is either T^σ_V or T^σ_{VII}. In either case, the time spent in these intervals is bounded by a constant depending only on the size of the initial data. The time spent to do a transition from $(1 - \frac{1}{\sigma}, 1)$ to $(-1, -1 + \frac{1}{\sigma})$ (or viceversa) is T^σ_{VI}, which is bounded by the size of the initial data. This transition may occur only once. Then the solution $z(t)$ lies in only one of the intervals $[-1 + \frac{1}{\sigma}, 1 - \frac{1}{\sigma}]$ $[1, +\infty)$, $(-\infty, 1]$ for the rest of the evolution, converging to equilibrium exponentially fast. The worst exponential rate is $\frac{1}{2}$. The constant M depends on the bounds of the times T^σ_V, T^σ_{VI}, and T^σ_{VII}. □

Acknowledgments

G. M. Coclite, N. De Nitti, F. Maddalena, and G. Orlando are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

G. M. Coclite, F. Maddalena, and G. Orlando were supported by the Italian Ministry of University and Research under the Programme "Department of Excellence" Legge 232/2016 (Grant No. CUP - D93C23000100001).

G. M. Coclite, F. Maddalena, and G. Orlando have received funding from the INdAM - GNAMPA 2023 Project Codice CUP E53C22001930001, “Modelli variazionali ed evolutivi per problemi di adesione e di contatto”.

G. M. Coclite has been partially supported by the Project funded under the National Recovery and Resilience Plan (NRRP), Mission 4 Component 2 Investment 1.4 -Call for tender No. 3138 of 16/12/2021 of Italian Ministry of University and Research funded by the European Union -NextGenerationEUoAward Number: CN000023, Concession Decree No. 1033 of 17/06/2022 adopted by the Italian Ministry of University and Research, CUP: D93C22000410001, Centro Nazionale per la Mobilità Sostenibile and the Italian Ministry of Education, University and Research under the Programme Department of Excellence Legge 232/2016 (Grant No. CUP - D93C23000100001).

G. Orlando has been supported by the project “Approccio integrato e predittivo per il controllo della criminalità marittima” in the program “Research for Innovation” (REFIN) - POR Puglia FESR FSE 2014-2020, Codice CUP: D94I20001410008.

E. Zuazua has been funded by the Alexander von Humboldt-Professorship program, the ModConFlex Marie Curie Action, HORIZON-MSCA-2021-DN-01, the COST Action MAT-DYN-NET, the Transregio 154 Project “Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks” of the DFG, grants PID2020-112617GB-C22 and TED2021-131390B-I00 of MINECO (Spain), and by the Madrid Goverment – UAM Agreement for the Excellence of the University Research Staff in the context of the V PRICIT (Regional Programme of Research and Technological Innovation).

References

[1] J. M. Ball. On the asymptotic behavior of generalized processes, with applications to nonlinear evolution equations. *J. Differ. Equations*, 27:224–265, 1978.

[2] J. M. Ball. Global attractors for damped semilinear wave equations. *Discrete Contin. Dyn. Syst.*, 10(1-2):31–52, 2004.

[3] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[4] T. Cazenave and A. Haraux. *An introduction to semilinear evolution equations*, volume 13 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998.

[5] G. M. Coclite, G. Devillanova, G. Florio, M. Ligabò, and F. Maddalena. Thermo-elastic waves in a model with nonlinear adhesion. *Nonlinear Anal.*, 232:Paper No. 113265, 16, 2023.

[6] G. M. Coclite, G. Florio, M. Ligabò, and F. Maddalena. Nonlinear waves in adhesive strings. *SIAM J. Appl. Math.*, 77(2):347–360, 2017.

[7] G. M. Coclite, G. Florio, M. Ligabò, and F. Maddalena. Adhesion and debonding in a model of elastic string. *Comput. Math. Appl.*, 78(6):1897–1909, 2019.

[8] C. M. Dafermos. Asymptotic behavior of solutions of evolution equations. In *Nonlinear evolution equations (Proc. Sympos., Univ. Wisconsin, Madison, Wis., 1977)*, volume 40 of *Publication of the Mathematics Research Center, University of Wisconsin*, pages 103–123. Academic Press, New York-London, 1978.

[9] R. W. Dickey. Stability theory for the damped sine-Gordon equation. *SIAM J. Appl. Math.*, 30:248–262, 1976.

[10] M. Ghisi, M. Gobbino, and A. Haraux. Finding the exact decay rate of all solutions to some second order evolution equations with dissipation. *J. Funct. Anal.*, 271(9):2359–2395, 2016.

[11] M. Ghisi, M. Gobbino, and A. Haraux. Optimal decay estimates for the general solution to a class of semi-linear dissipative hyperbolic equations. *J. Eur. Math. Soc. (JEMS)*, 18(9):1961–1982, 2016.

[12] J. K. Hale. Dynamical systems and stability. *Journal of Mathematical Analysis and Applications*, 26(1):39–59, 1969.

[13] A. Haraux. Asymptotics for some nonlinear hyperbolic equations with a one-dimensional set of rest points. *Bol. Soc. Bras. Mat.*, 17(1):51–65, 1986.

[14] A. Haraux. Semi-linear hyperbolic problems in bounded domains. *Math. Rep.*, 3(1):i–xxiv and 1–281, 1987.

[15] A. Haraux. Slow and fast decay of solutions to some second order evolution equations. *J. Anal. Math.*, 95:297–321, 2005.

[16] A. Haraux and E. Zuazua. Decay estimates for some semilinear damped hyperbolic problems. *Arch. Ration. Mech. Anal.*, 100(2):191–206, 1988.

[17] J. P. LaSalle. An invariance principle in the theory of stability. Differ. Equations dynam. Systems, Proc. Int. Sympos. Puerto Rico 1965, 277-286 (1967)., 1967.
[18] F. Maddalena, D. Percivale, G. Puglisi, and L. Truskinovsky. Mechanics of reversible unzipping. *Contin. Mech. Thermodyn.*, 21(4):251–268, 2009.
[19] P. Martinez. Stabilization for the wave equation with Neumann boundary condition by a locally distributed damping. In *Contrôle des systèmes gouvernés par des équations aux dérivées partielles (Nancy, 1999)*, volume 8 of *ESAIM Proc.*, pages 119–136. Soc. Math. Appl. Indust., Paris, 2000.
[20] G. F. Webb. A bifurcation problem for a nonlinear hyperbolic partial differential equation. *SIAM J. Math. Anal.*, 10:922–932, 1979.
[21] E. Zuazua. Stability and decay for a class of nonlinear hyperbolic problems. *Asymptotic Anal.*, 1(2):161–185, 1988.
[22] E. Zuazua. Exponential decay for the semilinear wave equation with locally distributed damping. *Comm. Partial Differential Equations*, 15(2):205–235, 1990.
[23] E. Zuazua. Exponential decay for the semilinear wave equation with localized damping in unbounded domains. *J. Math. Pures Appl. (9)*, 70(4):513–529, 1991.

(G. M. Coclite) Politecnico di Bari, Dipartimento di Meccanica, Matematica e Management, Via E. Orabona 4, 70125 Bari, Italy.
Email address: giuseppemaria.coclite@poliba.it

(N. De Nitti) EPFL, Institut de Mathématiques, Station 8, 1015 Lausanne, Switzerland.
Email address: nicola.denitti@epfl.ch

(F. Maddalena) Politecnico di Bari, Dipartimento di Meccanica, Matematica e Management, Via E. Orabona 4, 70125 Bari, Italy.
Email address: francesco.maddalena@poliba.it

(G. Orlando) Politecnico di Bari, Dipartimento di Meccanica, Matematica e Management, Via E. Orabona 4, 70125 Bari, Italy.
Email address: gianluca.orlando@poliba.it

(E. Zuazua) Friedrich-Alexander-Universität Erlangen-Nürnberg, Department of Data Science, Chair for Dynamics, Control and Numerics (Alexander von Humboldt Professorship), Cauerstr. 11, 91058 Erlangen, Germany.

Chair of Computational Mathematics, Fundación Deusto, Avenida de las Universidades, 24, 48007 Bilbao, Basque Country, Spain.

Universidad Autónoma de Madrid, Departamento de Matemáticas, Ciudad Universitaria de Cantoblanco, 28049 Madrid, Spain.
Email address: enrique.zuazua@fau.de