Wave propagation and control in $1-d$ flexible multi-structures

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Abstract. This monograph is devoted to analyze the vibrations of a simplified $1-d$ model of a multi-body structure consisting of a finite number of flexible strings distributed along a planar graph.

We first discuss issues on existence and uniqueness of solutions that can be solved by standard methods. Then we analyze how solutions propagate along the graph as the time evolves. The problem of the observation of waves is a natural framework to analyze this issue. Roughly, the question can be formulated as follows: Can we obtain complete information on the vibrations by making measurements in one single extreme of the network? This formulation is relevant both in the context of control and inverse problems.

Using the Fourier development of solutions and techniques of Nonharmonic Fourier Analysis, we give spectral conditions that guarantee the observability property to hold in any time larger than twice the total lengths of the network in a suitable Hilbert that can be characterized in terms of Fourier series by means of properly chosen weights. When the network graph is a tree, we characterize these weights in terms of the eigenvalues of the corresponding elliptic problem. The resulting weighted observability inequality allows to identify the observable energy in Sobolev terms in some particular cases. That is the case, for instance, when the network is star-shaped and the ratios of the lengths of its strings are algebraic irrational numbers.

The techniques developed to handle this problem and the results we have obtained, allow us to solve also other similar problems. In particular, the simultaneous observability problem for strings or membranes from an interior region and the Schrödinger, heat or beam-type equations on networks controlled from one exterior node are also studied.

We also describe systematically the control theoretical consequences of the observability properties we have obtained here, in terms of the approximate, spectral and exact controllability of networks.

Contents

Introduction

In last years a considerable effort has been devoted to the study of mechanical systems constituted by multiple coupled elements as beams, plates, strings or membranes. Those systems are known as multi-structures. Their practical relevance is enormous. However, the complexity of the mathematical models describing their evolution is generally is considerable. In [**48**] and [**51**] wide information on this topic may be found.

The difficulty mentioned above makes it necessary to study the most simple versions of those models to know which results can be expected in more complex situations or when the model is better adapted to practical needs.

This monograph is devoted to analyze the vibrations of a simplified $1-d$ model of a multi-body structure consisting of a finite number of flexible strings distributed along a planar graph.

The model under consideration is, to some extend, the simplest one in the context of multi-body or multi-link continuous structures. However, as we shall see along the monograph, a fine analysis of the nature of the possible vibrations of these planar networks of flexible strings is far from trivial. The main goal of this book is to present is a self-contained way the state of the art of the problem of propagation, observation and control of wave on planar networks. As we shall see, this requires important developments related with non-harmonic Fourier series, Diophantine approximation, graph theory and propagation techniques.

The main tool for analyzing the propagation of waves along the graph will be the d'Alembert formula which allows solving the wave equation both in the space and the time directions. In the model under consideration the wave equation holds along each of the strings of the network. The d'Alembert formula allows then representing the solutions on each string explicitly. However, the overall dynamics turns out to be rather complex. This is due to the interaction of the various strings at the junction points. How the energy of waves is transferred from one string to another turns out to be a global problem in which several ingredients arise:

– the lengths of the various strings constituting the graph;
– the topology of the graph;
– the boundary conditions imposed at the extremes of the graph.

The problem of observation or observability concerns, roughly speaking, the issue of determining whether one can determine the total energy vibrations by partial measurements made for instance, in one or several interior or external nodes of the network. In other words, the property of observability is related with the distribution or propagation of vibrations along the various components of the multi-structure. This problem is relevant, not only because it is a way of analyzing deeply the nature of vibrations, but because it is also of immediate application in the context of inverse and control problems. Part of the book will also devoted to present systematically the consequences of our analysis in what concerns control problems. In particular, we shall analyze the properties of approximate, spectral and exact controllability of the networks.

As we mentioned above graph theory and Diophantine approximations issues enter in a crucial way on the analysis of the property of observability. The topology of the graph does play indeed a fundamental role. For instance, when the graph contains closed circuits there may exist vibrations of the network that do remain concentrated in that circuit, without being propagated to the rest of the network. In those cases, obviously, it is impossible to achieve the observation and/or control property if the observer or controlled is not located on the circuit where the solution is trapped. But whether a circuit may support a localized vibration depends also strongly on the mutual lengths of the strings composing the circuit. When all the ratios of the lengths of these strings are rational numbers, such a localized vibration does exist. However, if some of these ratios is irrational, then necessarily, part of the energy of the vibration will be transferred to some other components of the network. But, in order to determine the amount of energy that is actually transferred one needs to know further properties of that irrational ratio (being algebraic or not, a Liouville number....) and then apply the existing results on Diophantine approximation.

As we shall see, the overall picture is quite complex, but we hope that monograph will succeed on describing the main phenomena one may encounter. We shall mainly focus on three cases with different degrees of complexity and such that the corresponding results are also of quite different nature:

The star. it concerns the case where a finite number of strings are connected on a single point by one of their extremes. In this case, using d'Alembert formula, one can give sharp results characterizing the space of observation and/or control in Fourier series. We mainly discuss the most difficult case in which observation and/or control are localized in a single extreme of the network. The weights in the corresponding norms depend on the ratios of the lengths of the strings. The time needed for observation turns out to be twice the sum of all lengths of the networks.

The tree. It is well known that when all but one external node of the network are observed in a tree-like configuration, the whole energy of solutions may be observed (see [**51**]). This can be easily seen by an energy argument. Indeed, using sidewise energy estimate one can show that the observation inequality holds in the sharp energy space in a time which is twice the length of the longest path joining the points of the network with some of the observed ends.

Here we analyze the opposite case in which the observation is made at one single extreme of the tree-like network. The observation time turns out to be again, as in the case of one star, twice the sum of the lengths of the strings forming the network. At this point, it is important to note that the case of a tree is a generalization of the previous case of a star. Thus, one has to also generalize the condition on the irrationality of the ratios of the lengths of the strings arising in the case of the stars. To do that it is important to observe that the fact of two string having mutually irrational lengths can also be interpreted in spectral terms. Indeed, it means that the spectra of the two strings have empty intersection. The latter notion turns out to be the appropriate one to be extended to the general case of trees. The tree turns out to be observable from one end if and only if the spectra of all pairs of subtrees of the tree that match on a nodal point are disjoint. Obviously, this property is also related to the values of the lengths of the strings composing the tree, but does not have an easy interpretation as in the case of the star. Nevertheless, generically, trees also satisfy this property.

General networks. The propagation techniques we have employed in the analysis of star and trees are hard to apply in the case of a network supported by a general graph. Indeed, in the general case we lack of a natural ordering on the graph to analyze the propagation of waves. Actually, as we mentioned above, the presence of close circuits may trap the waves. Thus, we proceed in a different way by applying a consequence of the celebrated Beurling-Malliavin's Theorem on the completeness of families of real exponentials obtained by Haraux and Jaffard in [**34**] when analyzing the control of plates. Using the min-max principle, one can show that the spectral density of a general graph is the same as that of a single string whose length is the sum of the lengths of all the strings entering in the network. Then, when the time is greater than twice the total length, as a consequence of Beurling-Malliavin's Theorem, we deduce that there exist some Fourier weights so that the observation property holds in the corresponding weighted norm if and only if all the eigenfunctions of the network are observable. So far we do not know of any necessary and sufficient condition guaranteeing that all the eigenfunctions are observable in the general case. We know however, what that property is in the particular case of stars and trees discussed above: the lengths of the strings are mutually irrational in the case of stars or the spectra of all pair of subtrees with a common end-point are mutually disjoint in the more general case of the trees.

In view of this last result on general networks, we could have presented the material in this monograph in a completely different order. Indeed, we could have started from the most general results on the case of general networks using Beurling-Malliavin's Theorem to later discuss the particular cases of stars and trees using d'Alembert formula and Diophantine approximation. However, we have preferred to do all the way around. This corresponds actually to the order and chronology in which the progress was done in the field, starting from the work [**57**] on the case of a star composed of three strings and continuing to the series of notes [**26, 27, 28, 29**].

We became interested on this subject along several discussions with Günter Leugering on this subject and his book in collaboration with John Lagnese and George Schmidt [**51**] was a great help to start. As we said before, the model we consider in this monograph is the simplest one in the context of vibrations of networks. The interested reader is referred to [**51**] where many other models can be found with a description of the state of the art in what concerns the well-posedness of the initial boundary problems and the observation and/or control problems for networks of strings, beams, membranes and plates.

Before getting into the analysis of the star we discuss a simpler issue that, nevertheless, allows presenting some of the main difficulties of the theory. It concerns the simultaneous control of two strings connected at one end-point (which is in fact completely equivalent to the problem of controlling one single string from one interior point). In this case we already we see the necessity that both stings have mutually irrational lengths. Moreover, we also see that the time needed to control the strings is twice the sum of the lengths of both strings. This seems to contradict a first intuition that would suggest that the time needed to control both strings simultaneously should be twice the maximum of the lengths of the strings, i.e., $2\max(\ell_1, \ell_2)$, instead of $2(\ell_1 + \ell_2)$. But intuition fails and, in fact, the time

$2(\ell_1 + \ell_2)$ turns out to be sharp under the assumption that the ratio ℓ_1/ℓ_2 is irrational. In other words, even when ℓ_1/ℓ_2 is irrational, the time needed to control simultaneously the two strings together by means of the same control, is $2(\ell_1 + \ell_2)$, which is strictly greater than the time needed to control each string respectively with two different controls that would be $2\max(\ell_1, \ell_2)$.

It is interesting to analyze the relation of this result with the so-called Geometric Control Conditions (GCC) introduced by Bardos, Lebeau and Rauch [**11**] in the context of the boundary observation and/or control of the wave equation in bounded domains of $\mathbb{R}^n$. The GCC requires that all the rays of Geometric Optics enter the observation region in a finite, uniform time which turns out to be the minimal one for observation/control. In the case of two strings observed from one common end or the equivalent problem of the string controlled at an interior point, in view of GCC, one could expect the sharp time needed for observation/control to be equal to $2\max(\ell_1, \ell_2)$. But this is not the case, the fact that the rays pass once by the point of observation does not guarantee that the energy concentrated on that ray will be conveniently observed[1]. In fact, we need the ray to pass once more through the point of observation to be able to make a full measurement of the energy along the ray. This yields the control observation time $2(\ell_1 + \ell_2)$. But, in fact, passing twice by the observation point is not sufficient either. The irrationality of the ratio ℓ_1/ℓ_2 is needed to guarantee that, when passing through the observation point the second time, the solution is not exactly at the configuration as in the first one, which, of course, would make the second observation to be insufficient too. Finally, even when ℓ_1/ℓ_2 is irrational, we cannot get a uniform bound of the energy of the solution but rather a weaker measurement in a weaker norm. The nature of this norm, which is represented in Fourier series by means of some weights depending on ℓ_1/ℓ_2, depends very strongly on the irrationality class to which the number ℓ_1/ℓ_2 belongs. In fact, in the most favorable case, i.e., when ℓ_1/ℓ_2 is an algebraic number of degree two, one looses one derivative of the solution which, in Sobolev terms means that, for instance, an H^1 observation in time yields only control of the L^2-norm of the solution. In other more pathological cases, like when ℓ_1/ℓ_2 is, for instance, a Liouville number, one may loose an infinite number of derivatives in the sense that the weights entering in the Fourier representation of the observed norm may have an exponential decay.

We have so far described the content of the main body of the monograph: the propagation, observation and control of waves on stars, trees and general networks. But these are only a few of the problems arising in this context. We have complemented this material with the discussion of two important closely related problems:

– The simultaneous observation/control of two strings from a common subinterval. In this case one obtains better results than in the case when the observer/controller was located at a single point. Indeed, this time the results do hold in the sharp energy space without any loss of derivatives.

– The observation/control of general networks through all the nodal points. This is a problem of relevance in applications. From a technological point of view, putting observer/controllers at all the nodal points is feasible. However, one would

[1]The wave equation is a second order problem and therefore, even in $1-d$, for a pointwise observation mechanism to be efficient we need to measure not only the position, but also the space derivative.

like to know, for instance, if one can reduce the number of applied control forces by identifying a priori the nodes on which the same force will be applied. This is needed necessarily in order to diminish the complexity of the control mechanism. Thus, we would to know how many different control forces are needed to control the whole structure and to identify the nodes on which each control should be applied. We shall see that the total number of controls needed is four and this is a consequence of our previous analysis and the celebrated Four Colors Theorem.

So far, we have only discussed the wave equation on planar networks of strings. But of course, the same issues arise for all other models like beams, Schrödinger or heat equations. The theory of observation and control of Partial Differential Equations in open domains of the euclidean space is by now quite well-developed (We refer to the survey article [**82**] for an updated account of the developments in this field). However, very little is known in the context of PDE's on networks. In particular, as far as we know, nothing is known on the three models mentioned above.

The last part of this monograph is devoted to discuss those three models. Roughly speaking, we show that the result proved in the previous sections on the wave equation do yield similar results for those three models. To do that we employ two different results. In the case of the heat equation on the network, we use a classical result by Russel [**73**] guaranteeing that, whenever the wave equation is controllable in some time, then the heat equation is controllable in an arbitrarily small time. The results of this monograph on the observation and/or control of the wave equation on the network then immediately imply similar results on the corresponding heat model. In what concerns the Schödinger and beams models we use the fact that the time frequencies of the complex exponentials involved in the Fourier representation of solutions of these two models are the squares of those entering in the solutions of the wave equations. Thus, the gap between consecutive eigenfrequencies increases, This allows obtaining observability inequalities for Schödinger and beam equation. But, this time, as expected, the observability inequalities hold in an arbitrarily small time.

As we have already mentioned this monograph collects the existing results on simple $1-d$ models on networks. Much remains to be done in this field. At the end of this book we include a list of open problems and possible subjects of future research. We hope this book to attract the attention to this amazing field of research.

Finally, some comments on the notations used along this book. The numbering of objects is made locally in each chapter. The theorems, lemmas, etc., have a first number to indicate the chapter in which they appear. Thus, Proposition III.4, is the fourth proposition of Chapter III. For sections, subsections and formulas, the explicit reference to the chapter is omitted. That is why, when they are cited in a chapter different from that where they appear, we use an additional number to indicate the chapter where they were defined. For instance, formula (5) of Chapter IV is cited in that chapter as (5), but in others chapters as (IV.5). Concerning the constants, they all have been denoted by the letter C. Thus, C may stand for numbers that are different from line to line of the text. Only when we intend to explicitly indicate the dependence of C on some parameter, or to avoid confusions, we have used some other notations for the constants.

CHAPTER I

Preliminaries

1. An elastic string

Let start with a simple example. Consider an elastic string of length one which is fixed at its ends. The deformation of the string is given by the function $\phi(t,x) : \mathbb{R} \times (0,1) \to \mathbb{R}$. The function ϕ is the unique solution of the wave equation

$$(1) \qquad \begin{array}{ll} \phi_{tt} - \phi_{xx} = 0 & \text{in } \mathbb{R} \times (0,1), \\ \phi(t,0) = \phi(1,0) = 0 & \text{in } \mathbb{R}, \\ \phi(0,x) = \phi_0(x), \quad \phi_x(0,x) = \phi_1(x) & \text{in } (0,1), \end{array}$$

where ϕ_0 and ϕ_1 are the initial deformation and velocity of the string, respectively.

The solution of system (1) is expressed by the Fourier formula

$$(2) \qquad \phi(t,x) = \sum_{n=1}^{\infty} (a_n \cos n\pi t + \frac{b_n}{n\pi} \sin n\pi t) \sin n\pi x,$$

where (a_n) and (b_n) are the sequences of Fourier coefficients in the orthogonal basis of $L^2(0,1)$:

$$\theta_n(x) = \sin n\pi x, \quad n = 1, 2, ...$$

The energy of the solution ϕ is defined as

$$E_\phi(\phi_0, \phi_1, t) = \frac{1}{2} \int_0^! \left(|\phi_x(t,x)|^2 + |\phi_t(t,x)|^2 \right).$$

It is easy to prove that the energy of a solution is constant, that is $E_\phi(t) = E_\phi(0)$. The energy is a norm in the space $H_0^1(0,1) \times L^2(0,1)$ of initial states of (1) and may be expressed in terms of the Fourier coefficients (a_n) and (b_n) as

$$(3) \qquad E_\phi(\phi_0, \phi_1) = \frac{1}{4} \sum_{n=1}^{\infty} (n^2 \pi^2 a_n^2 + b_n^2).$$

Assume now that we observe the motion of the string at one of its points. To fix ideas, suppose we know the values of the velocity ϕ_t and the tension ϕ_x at some point $x = \xi$ in a time interval $(0,T)$. Let us define the *observation function*

$$\Phi(\phi_0, \phi_1, \xi, T) = \frac{1}{4} \int_0^T |\phi_t(t,\xi)|^2 dt + \frac{1}{4} \int_0^T |\phi_x(t,\xi)|^2 dt.$$

Let us note that for $T = 2M$ with $M \in \mathbb{N}$ it holds

$$(4) \qquad \Phi(\phi_0, \phi_1, \xi, T) = M E_\phi(\phi_0, \phi_1).$$

Indeed, from the formula (2) we have

$$\phi_t(t,\xi) = \sum_{n=1}^{\infty}(-n\pi a_n \sin n\pi t + b_n \cos n\pi t)\sin n\pi\xi,$$

$$\phi_x(t,\xi) = \sum_{n=1}^{\infty}(n\pi a_n \cos n\pi t + b_n \sin n\pi t)\cos n\pi\xi$$

and then, in view of the 2-periodicity of the functions $\sin n\pi t$ and $\cos n\pi t$

$$\int_0^{2M} |\phi_t(t,\xi)|^2 dt = M\int_0^2 |\phi_t(t,\xi)|^2 dt = M\sum_{n=1}^{\infty}(n^2\pi^2 a_n^2 + b_n^2)\sin^2 n\pi\xi, \tag{5}$$

$$\int_0^{2M} |\phi_x(t,\xi)|^2 dt = M\int_0^2 |\phi_x(t,\xi)|^2 dt = M\sum_{n=1}^{\infty}(n^2\pi^2 a_n^2 + b_n^2)\cos^2 n\pi\xi. \tag{6}$$

Therefore, in view of (3), (5) and (6) we obtain (4).

Clearly, the function $\Phi(\phi_0,\phi_1,\xi,T)$ is increasing in T, so, if $2 \le T \le 2M$ with $M \in \mathbb{N}$ we obtain

$$\Phi(\phi_0,\phi_1,\xi,2) \le \Phi(\phi_0,\phi_1,\xi,T) \le \Phi(\phi_0,\phi_1,\xi,2M),$$

or equivalently,

$$E_\phi(\phi_0,\phi_1) \le \Phi(\phi_0,\phi_1,\xi,T) \le ME_\phi(\phi_0,\phi_1).$$

That means, that for all $\xi \in [0,1]$ and $T \ge 2$ the norms defined by E_ϕ and $\Phi(\cdot,\xi,T)$ are equivalent. That is, it is possible to estimate the energy of the solution ϕ from the measurements of ϕ_t, ϕ_x made at point ξ during a time equal of length at least two. In particular, when $T = 2$ those two norms coincide:

$$E_\phi(\phi_0,\phi_1) = \Phi(\phi_0,\phi_1,\xi,2).$$

In other words, the energy of the solution can be measured at a point of the string. However, to do this, we should observe the velocity and the tension of the string at that point during a time at least equal to two.

When $\xi = 0$ or $\xi = 1$, the observation function Φ becomes simpler:

$$\Phi(\phi_0,\phi_1,0,T) = \frac{1}{4}\int_0^T |\phi_x(t,0)|^2 dt.$$

This suggest to consider a weaker observation function for the interior points of the strings:

$$\Psi(\phi_0,\phi_1,\xi,T) = \frac{1}{4}\int_0^T |\phi_x(t,\xi)|^2 dt.$$

We already know that, when $\xi = 0$ or $\xi = 1$ this function defines a norm in the space of initial data, which is equivalent to that defined by the energy. The following question naturally arises: does the function Ψ define a norm in $H_0^1(0,1)\times L^2(0,1)$? If so, is that norm equivalent to the energy?

Assume $T = 2$, then in view of (6) it holds

$$\Psi(\phi_0,\phi_1,\xi,2) = \frac{1}{4}\sum_{n=1}^{\infty}(n^2\pi^2 a_n^2 + b_n^2)\cos^2 n\pi\xi. \tag{7}$$

Formula (7) is very similar to (3) and clearly

$$\sum_{n=1}^{\infty}(n^2\pi^2a_n^2+b_n^2)\cos^2 n\pi\xi \le \sum_{n=1}^{\infty}(n^2\pi^2a_n^2+b_n^2),$$

and then

$$\Psi(\phi_0,\phi_1,\xi,2) \le E_\phi(\phi_0,\phi_1).$$

However, the converse inequality

$$E_\phi(\phi_0,\phi_1) \le C\Psi(\phi_0,\phi_1,\xi,2),$$

or equivalently,

$$C\sum_{n=1}^{\infty}(n^2\pi^2a_n^2+b_n^2) \le \sum_{n=1}^{\infty}(n^2\pi^2a_n^2+b_n^2)\cos^2 n\pi\xi, \tag{8}$$

is not true for any $\xi \in (0,1)$ and any constant $C>0$. Obviously, inequality (8) is equivalent to the existence of a constant $C>0$ such that, for every $n\in\mathbb{N}$,

$$|\cos n\pi\xi| \ge C. \tag{9}$$

But this inequality is false in general. Indeed, if ξ is a rational number that can be expressed as

$$\xi = \frac{2p+1}{2q}, \qquad p,q\in\mathbb{Z}, \tag{10}$$

then, when $n=qk$ with k odd

$$\cos n\pi\xi = \cos\frac{(2p+1)\,k}{2}\pi = 0.$$

Thus, in this case, $\cos n\pi\xi = 0$ for an infinite number of values of n and consequently, inequality (9) cannot be true. Let us note that in the right hand term of inequality (8) may vanish for sequences (a_n) and (b_n) with are not identically zero. That means that the function $\Psi(\cdot,\xi,2)$ is not a norm in $H_0^1(0,1)\times L^2(0,1)$.

On the other hand, when the number ξ cannot be expressed in the form (10) all the numbers $\cos n\pi\xi$ are different from zero. This implies that the function $\Psi(\cdot,\xi,2)$ does define a norm in $H_0^1(0,1)\times L^2(0,1)$. But in general this norm is weaker than the energy.

In fact, inequality (9) is equivalent to the existence of a positive number α such that, for all $k,n\in\mathbb{Z}$,

$$\left| n\pi\xi - \frac{2k+1}{2}\pi \right| \ge \alpha$$

that is

$$|(2\xi)\,n-(2k+1)| \ge \alpha_0 := \frac{2\alpha}{\pi}.$$

This is a rational approximation property of the number 2ξ and is false in general. We will discuss this issue in detail in Chapter III. For certain values of ξ we obtain weaker inequalities. For instance, if 2ξ may be expanded in continuous fraction $[0,c_1,c_2,....]$ with bounded sequence (c_n) then there exists a constant C_ξ such that

$$|(2\xi)\,n-(2k+1)| \ge \frac{C_\xi}{n}.$$

This implies that

$$|\cos n\pi\xi| \ge \frac{C_\xi}{n}$$

and therefore

$$\Psi(\phi_0, \phi_1, \xi, 2) \geq C_\xi \sum_{n=1}^{\infty} (a_n^2 + \frac{b_n^2}{n^2\pi^2}) = C_\xi ||\phi_0||^2_{L^2(0,1)} + ||\phi_1||^2_{H^{-1}(0,1)}.$$

Summarizing, for the values of ξ indicated above it holds

$$C_\xi \left(||\phi_0||^2_{L^2(0,1)} + ||\phi_1||^2_{H^{-1}(0,1)} \right) \leq \Psi(\phi_0, \phi_1, \xi, 2) \leq ||\phi_0||^2_{H^1_0(0,1)} + ||\phi_1||^2_{L^2(0,1)}.$$

This is the best result we may obtain. The information contained in $\Psi(\phi_0, \phi_1, \xi, 2)$ is actually a partial information on the energy of the solution whenever ξ is not one of the extremes of the string. This is also the case when we consider other kind of observation functions, e.g.,

$$\int_0^T |\phi(t, \xi)|^2 dt.$$

As we shall see in the following chapters, this is always the situation when we observe the vibrations of a network of strings. We can recover only a weaker information of the energy from measurements made at some points of the strings, even if at those points we measure both the velocity and the tension.

It should be pointed out that when the observation of a string is made on a larger set, say on some interval $\omega \subset (0, 1)$, then we can recover the energy of the solution from the observation function

$$\int_\omega \int_0^T |\phi_x(t, x)|^2 dt.$$

Indeed, assume $T = 2$ then

$$\int_\omega \int_0^2 |\phi_x(t, x)|^2 dt \geq \sum_{n=1}^{\infty} (n^2\pi^2 a_n^2 + b_n^2) \int_\omega \sin^2 n\pi x \; dx.$$

But, for any $\omega \subset (0, 1)$ there exists a constant $C_\omega > 0$ such that

$$\int_\omega \sin^2 n\pi x \; dx \geq C_\omega$$

for every $n \in \mathbb{N}$. Therefore,

$$C_\omega \sum_{n=1}^{\infty} (n^2\pi^2 a_n^2 + b_n^2) \leq \int_\omega \int_0^2 |\phi_x(t, x)|^2 dt \leq |\omega| \sum_{n=1}^{\infty} (n^2\pi^2 a_n^2 + b_n^2),$$

that is

$$C_\omega E_\phi \leq \int_\omega \int_0^2 |\phi_x(t, x)|^2 dt \leq |\omega| E_\phi.$$

Moreover, using the d'Alembert formula for the representation of the solutions of the wave equation, it is possible to prove that the property

$$C_1 E_\phi \leq \int_\omega \int_0^T |\phi_x(t, x)|^2 dt \leq C_2 E_\phi,$$

for some positive constants C_1 and C_2 is still true for any $T > 2$ dist $\{\omega, \{0, 1\}\}$.

Once again, for networks of strings, observing on an interval of one of the strings will not help. We can recover information only on the string where the observation is made.

2. Networks of strings

2.1. Elements on graphs. A graph G is a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set, whose elements are called *vertices* of G, and $\mathcal{E}$ is a family of non-ordered pairs $\mathbf{v}, \mathbf{w}$ of vertices, which we will denote by $\widehat{\mathbf{vw}}$. The elements of $\mathcal{E}$ are called *edges* of G with vertices $\mathbf{v}, \mathbf{w}$. When the graph G does not contain edges of the form $\widehat{\mathbf{vv}}$ it is said that the graph is *simple*[1].

A *path* between the vertices $\mathbf{v}$ and $\mathbf{w}$ of a graph G is a set of edges of the form

$$\widehat{\mathbf{v}\mathbf{v}_1}, \widehat{\mathbf{v}_1\mathbf{v}_2}, ..., \widehat{\mathbf{v}_{m-1}\mathbf{v}_m}, \widehat{\mathbf{v}_m\mathbf{w}}.$$

If all the edges forming a path are different, it is said that the path is *simple*; if all the vertices $\mathbf{v}_1, ..., \mathbf{v}_m$ are different, the path is called *elementary*.

A *closed path* is a path between a vertex and itself. An elementary closed path is called a *cycle*. When the graph G does not contain cycles it is said that G is a *tree*.

Graphs with a finite number of vertices are called *finite*. In this book we shall be concerned only with finite graphs.

Let us suppose that G is a finite graph with N vertices and M edges:

$$\mathcal{V} = \{\mathbf{v}_1, ..., \mathbf{v}_N\}, \qquad \mathcal{E} = \{\mathbf{e}_1, ..., \mathbf{e}_M\}.$$

The multiplicity $m(\mathbf{v})$ of the vertex $\mathbf{v}$ is the number of edges that meet at $\mathbf{v}$:

$$m(\mathbf{v}) := \operatorname{card}\{\mathbf{e} \in \mathcal{E}: \quad \mathbf{v} \in \mathbf{e}\}.$$

We also define the sets

$$\mathcal{V}_{\mathcal{S}} := \{\mathbf{v} \in \mathcal{V}: \quad m(\mathbf{v}) = 1\}, \quad \mathcal{V}_{\mathcal{M}} := \mathcal{V} \setminus \mathcal{V}_{\mathcal{S}}.$$

Let us observe that $\mathcal{V}_{\mathcal{S}}$ is the set of those vertices that belong to a single edge. These vertices are called *exterior*. The set $\mathcal{V}_{\mathcal{M}}$ contains the remaining vertices, i.e., those that belong to more than one edge; those vertices are called *interior*.

For a vertex $\mathbf{v}$ we denote

$$I_{\mathbf{v}} := \{i : \mathbf{v} \in \mathbf{e}_i\},$$

which is the set of the indices of all those edges of G, which are incident to $\mathbf{v}$. If the vertex $\mathbf{v}_j$ is exterior, $I_{\mathbf{v}_j}$ contains a single index; it will be denoted by $i(j)$ and, if this does not lead to misunderstanding, simply by i.

In this book we consider only simple finite graphs whose vertices are points of a plane. The edges of the graph are viewed as the rectilinear segments joining some of those points. The length of the segment corresponding to the edge $\mathbf{e}_i$ is called length of $\mathbf{e}_i$ and is denoted by ℓ_i.

We will also assume that the edges of the graphs may meet only at the vertices of G. Such graphs are known as *planar graphs*.

On every edge of G we choose an orientation (that is, one of the vertices has been chosen as the initial one). Then $\mathbf{e}_i$ may be parametrized as a function of its arc length by means of the functions $x_i : [0, \ell_i] \to \mathbf{e}_i$.

We define the incidence matrix of G

$$\varepsilon_{ij} = \begin{cases} 1 & \text{if} \quad x_i(0) = \mathbf{v}_j, \\ \text{-1} & \text{if} \quad x_i(\ell_i) = \mathbf{v}_j. \end{cases}$$

[1]Sometimes the term graph is used only for simple graphs, that is, for those that do not have edges with equal vertices. Non-simple graphs are then called *pseudo-graphs*.

Let us denote by L the sum of the length of all the edges of the graphs, the *length of the graph*. To indicate to which graph it corresponds, we shall write, if necessary, L_G.

Given functions $u^i : [0, \ell_i] \to \mathbb{R}$, $i = 1, ..., M$, we will denote by $\bar{u} : G \to \mathbb{R}$ the function defined for $\mathbf{x} \in \mathbf{e}_i$ by

$$\bar{u}(\mathbf{x}) = u^i(x_i^{-1}(\mathbf{x})).$$

In this case, we will say that $\bar{u}$ is a function defined on the graph G with components u^i. Frequently, we will indicate this fact just by writing $\bar{u} = (u^1, ..., u^M)$. In particular, the vector with equal to zero components will be denoted by $\bar{0}$.

2.2. Equations of the motion of the network. Now we consider a network of elastic strings that undergo small vibrations, transversal to some plane. At rest, the network coincides with a planar graph G contained in that plane.

Let us suppose that the function $u^i = u^i(t, x) : \mathbb{R} \times [0, \ell_i] \to \mathbb{R}$, describes the transversal displacement in time t of the string that coincides at rest with the edge $\mathbf{e}_i$. Then, for every $t \in \mathbb{R}$, the functions u^i, $i = 1, ..., M$, define a function $\bar{u}(t)$ on G with components $u^i : \mathbb{R} \times [0, \ell_i] \to \mathbb{R}$ given by $u^i(t, x) = u^i(t, x_i(x))$. This function allows to identify the network with its rest graph; in this sense, the vertices of G will be called nodes and the vertices, strings.

As a model of the motion of the network we assume that the displacements u^i satisfy the following non-homogeneous system

$$
\begin{aligned}
&(11) \qquad u^i_{xx} - u^i_{tt} = 0 && \text{in } \mathbb{R} \times [0, \ell_i], \quad i = 1, ..., M,\\
&(12) \qquad u^{i(j)}(t, \mathbf{v}_j) = h_j(t) && t \in \mathbb{R}, \quad j = 1, ..., r,\\
&(13) \qquad u^{i(j)}(t, \mathbf{v}_j) = 0 && t \in \mathbb{R}, \quad j = r + 1, ..., N,\\
&(14) \qquad u^i(t, \mathbf{v}) = u^j(t, \mathbf{v}) && t \in \mathbb{R}, \quad \mathbf{v} \in \mathcal{V}_{\mathcal{M}}, \ i, j \in I_{\mathbf{v}},\\
&(15) \qquad \textstyle\sum_{i \in I_{\mathbf{v}}} \partial_n u^i(t, \mathbf{v}) = 0 && t \in \mathbb{R}, \quad \mathbf{v} \in \mathcal{V}_{\mathcal{M}},\\
&(16) \qquad u^i(0, x) = u^i_0(x), \quad u^i_t(0, x) = u^i_1(x) && x \in [0, \ell_i], \quad i = 1, ..., M,
\end{aligned}
$$

where $\mathcal{C}$ is a non-empty subset of $\mathcal{V}_{\mathcal{S}}$ (the set of controlled nodes) and $\partial_n u^i(t, \mathbf{v}) := \varepsilon_{ij} u^i_x(t, x_i^{-1}(\mathbf{v}))$ is the exterior normal derivative of u_i at the node $\mathbf{v}$. Here we have assumed that the numbering of nodes has been chosen such that

$$\mathcal{C} = \{\mathbf{v}_1, ..., \mathbf{v}_r\}.$$

Thus, (11)-(16) corresponds to a network with r controlled exterior nodes.

Equation (11) is the classical 1-d wave equation, which is verified by the deformations of the strings of the network. The equalities (12), (13) reflect the condition that over some of the exterior nodes, precisely over those corresponding to the vertices contained in $\mathcal{C}$, some controls act to regulate their displacements, while the remaining nodes are fixed. The relations (14) and (15) express the continuity of the network and the balance of forces at the interior nodes. Finally, (16) indicates that the initial deformation and velocity of the strings (i.e., at time $t = 0$) given. The pair $(\bar{u}_0, \bar{u}_1)$ is called *initial state* of the network.

In general, we will suppose that the graph G does not contain vertices of multiplicity two, since those would be irrelevant in our model. Indeed, they may be considered as interior points of an edge whose length coincides with the sum of the lengths of the edges coupled at that vertex.

In order to study of the system (11)-(16), we need a proper functional setting . We define the Hilbert spaces

$$V = \{\bar{u} \in \prod_{i=1}^{M} H^1(0,\ell_i) : u^i(\mathbf{v}) = u^j(\mathbf{v}) \text{ if } \mathbf{v} \in \mathcal{V}_{\mathcal{M}} \text{ and } u^i(\mathbf{v}) = 0 \text{ if } \mathbf{v} \in \mathcal{V}_{\mathcal{S}}\},$$
$$H = \prod_{i=1}^{M} L^2(0,\ell_i),$$

provided with the Hilbert structures

$$< \bar{u}, \bar{w} >_V := \sum_{i=1}^{M} < u^i, w^i >_{H^1[0,\ell_i]} = \sum_{i=1}^{M} \int_0^{\ell_i} u_x^i w_x^i dx,$$
$$< \bar{u}, \bar{w} >_H := \sum_{i=1}^{M} < u^i, w^i >_{L^2[0,\ell_i]} = \sum_{i=1}^{M} \int_0^{\ell_i} u^i w^i dx,$$

respectively. Besides, we will denote

$$U = \left(L^2(0,T)\right)^r.$$

The study of the solvability of system (11)-(16) may be done in the standard way for non-homogeneous systems followings the classic *transposition method* (see [**61**]): first we study the homogeneous problem ($h_j \equiv 0$ for all $j = 1, ..., r$)

$$\phi^i_{xx} - \phi^i_{tt} = 0 \qquad \text{in } \mathbb{R} \times [0,\ell_i], \quad i = 1, ..., M, \tag{17}$$
$$\phi^{i(j)}(t, \mathbf{v}_j) = 0 \qquad t \in \mathbb{R}, \quad j = 1, ..., N, \tag{18}$$
$$\phi^i(t, \mathbf{v}) = \phi^j(t, \mathbf{v}) \qquad t \in \mathbb{R}, \quad \mathbf{v} \in \mathcal{V}_{\mathcal{M}}, \ i,j \in I_{\mathbf{v}}, \tag{19}$$
$$\textstyle\sum_{i \in I_{\mathbf{v}}} \partial_n \phi^i(t, \mathbf{v}) = 0 \qquad t \in \mathbb{R}, \quad \mathbf{v} \in \mathcal{V}_{\mathcal{M}}, \tag{20}$$
$$\phi^i(0,x) = \phi^i_0(x), \quad \phi^i_t(0,x) = \phi^i_1(x) \qquad x \in [0,\ell_i], \quad i = 1, ..., M; \tag{21}$$

Further, the solution of (11)-(16) in the general non-homogeneous case is defined by transposition. Let us describe the main steps, since some of its elements are widely used in through this book. The details of this procedure may be found in [**61**] or [**43**]. The application of this technique to the concrete problem of string networks may be found in [**51**].

Since the injection $V \subset H$ is dense and compact, when H is identified with its dual H' by means of the Riesz-Fréchet isomorphism, we can define the operator $-\Delta_G : V \to V'$ by

$$\langle -\Delta_G \bar{u}, \bar{v} \rangle_{V' \times V} = \langle \bar{u}, \bar{v} \rangle_H.$$

The operator $-\Delta_G$ is an isometry from V to V'. The notation La $-\Delta_G$ is justified by the fact that, for smooth functions $\bar{u} \in V$, the operator $-\Delta_G$ coincides with the Laplace operator.

It may be shown that the spectrum of the operator $-\Delta_G$ is formed by an increasing positive sequence $(\mu_n)_{n \in \mathbb{N}}$ of eigenvalues. The corresponding eigenfunctions $(\bar{\theta}_n)_{n \in \mathbb{N}}$ may be chosen to form an orthonormal basis of H.

The spaces V and H may be characterized as

$$\begin{aligned} V &= \left\{ \bar{u} = \sum_{n\in\mathbb{N}} u_n \bar{\theta}_n : \quad |||\bar{u}|||_V^2 := \sum_{n\in\mathbb{N}} \mu_n u_n^2 < \infty \right\}, \\ H &= \left\{ \bar{u} = \sum_{n\in\mathbb{N}} u_n \bar{\theta}_n : \quad |||\bar{u}|||_H^2 := \sum_{n\in\mathbb{N}} u_n^2 < \infty \right\}, \end{aligned}$$

and the norms of V and H are equivalent to $|||.|||_V$ and $|||.|||_H$, respectively. The spaces V and H are Hilbert spaces with respect to the scalar products that generate the corresponding norms.

The solution of the homogeneous system (17)-(21) with initial data

$$\bar{\phi}_0 = \sum_{n\in\mathbb{N}} \phi_{0,n} \bar{\theta}_n, \qquad \bar{\phi}_1 = \sum_{n\in\mathbb{N}} \phi_{1,n} \bar{\theta}_n, \tag{22}$$

is then defined by the formula

$$\bar{\phi}(t,x) := \sum_{n\in\mathbb{N}} (\phi_{0,n} \cos \lambda_n t + \frac{\phi_{1,n}}{\lambda_n} \sin \lambda_n t) \bar{\theta}_n(x). \tag{23}$$

Once again, this definition is justified by the fact that, for sufficiently smooth initial data $\bar{\phi}_0, \bar{\phi}_1$, the function determined by (23) is the unique solution of (17)-(21).

For a solution $\bar{u}$ of (11)-(16) in the classic sense, the energy is defined as the sum of the energies of its components, that is,

$$\mathbf{E}_{\bar{u}}(t) := \sum_{i=1}^{M} \mathbf{E}_{u_i}(t) \quad \text{with} \quad \mathbf{E}_{u_i}(t) := \frac{1}{2} \int_0^{\ell_i} \left(\left| u_t^i(t,x) \right|^2 + \left| u_x^i(t,x) \right|^2 \right) dx.$$

From the equations (11)-(15), it is easily proved that

$$\frac{d}{dt} \mathbf{E}_{\bar{u}}(t) = \sum_{i=1}^{r} u_t^i(t, \mathbf{v}_j) \partial_n u^i(t, \mathbf{v}_j). \tag{24}$$

In particular, in the homogeneous case the energy is conserved: $\mathbf{E}_{\bar{\phi}}(t) = \mathbf{E}_{\bar{\phi}}(0)$, for every $t \in \mathbb{R}$. Besides, if the initial data are expressed by (22) then

$$\mathbf{E}_{\bar{\phi}} = \frac{1}{2} \sum_{n\in\mathbb{N}} (\mu_n \phi_{0,n}^2 + \phi_{1,n}^2) = \frac{1}{2} (|||\bar{\phi}_0|||_V^2 + |||\bar{\phi}_1|||_H^2). \tag{25}$$

Since the sum in (25) is convergent for every $(\bar{\phi}_0, \bar{\phi}_1) \in V \times H$, this formula may be taken as the definition of the energy of the solution with initial state in $V \times H$.

From the definition (23) it holds that, for all $T \in \mathbb{R}$ and $(\bar{\phi}_0, \bar{\phi}_1) \in V \times H$, the solution $\bar{\phi}$ satisfies

$$\bar{\phi} \in C([0,T] : V) \bigcap C^1([0,T] : H). \tag{26}$$

In addition, $\bar{\phi}$ is the unique solution of the system (17)-(21) in the sense of distributions, which has the property (26).

For every $r \in \mathbb{R}$ we consider the Hilbert spaces

$$V^r := \left\{ \bar{u} = \sum_{n\in\mathbb{N}} u_n \bar{\theta}_n : \quad \|\bar{u}\|_r^2 := \sum_{n\in\mathbb{N}} \mu_n^r u_n^2 < \infty \right\}, \tag{27}$$

$$h^r := \left\{ (u_n) : \quad \|(u_n)\|_r^2 := \sum_{n\in\mathbb{N}} \mu_n^r |u_n|^2 < \infty \right\},$$

provided with the norms $\|\cdot\|_r$, where (u_n) denotes a sequence of real numbers u_n. The canonical isomorphism $\sum_{n\in\mathbb{N}} u_n\bar{\theta}_n \to (u_n)$ is an isometry between V^r and h^r.

Let us observe that V^r is the domain of $(-\Delta_G)^{\frac{r}{2}}$ considered as an operator from H to H. Besides, $V = V^1$ and $H = V^0$.

Further, we introduce the Hilbert spaces

$$\mathcal{W}^r := V^r \times V^{r-1},$$

endowed with the natural product structures. We then have

$$\mathcal{W}^1 = V \times H, \qquad \mathcal{W}^0 = H \times V'.$$

Therefore, it is possible to define for the initial state $(\bar{\phi}_0, \bar{\phi}_1) \in \mathcal{W}^r$ the solution of the homogeneous problem (17)-(21) by (26). In this case,

$$\bar{\phi} \in C([0,T] : V^r) \bigcap C^1([0,T] : V^{r-1}).$$

The following step in the study of the solvability of the system (11)-(16) consists in proving that, for every $T > 0$, there exists a constant $C > 0$ such that, at every exterior node $\mathbf{v} \in \mathcal{V}_{\mathcal{S}}$, the smooth solutions of (17)-(21) satisfy the inequality

$$\int_0^T |\partial_n \phi^i(t,\mathbf{v})|^2 dt \leq \mathbf{E}_{\bar{\phi}}. \tag{28}$$

This property is know as *hidden regularity*, since it is not a consequence of (26); it is an specific property of the solutions of (17)-(21) and in general, for the solutions of the Dirichlet problems for wave equations. The inequality (28) may be proved using D'Alembert formula for the representation of the solutions of (17). In [**51**], this inequality is proved by means of the multipliers technique, which is also useful in the more wide context of equations in several dimensions and having variable coefficients.

In what follows, in order to simplify the notations, we will suppose in the rest of this subsection that $r = 1$, that is, only one node of the network is controlled. We also assume that the index $i = 1$ corresponds to the strings that contains the controlled node.

Fix $T > 0$ and define for every $t \in (0,T]$ the operator $\mathbf{A}_t : H \times V \to L^2(0,T)$, which associates to every pair $(\bar{\phi}_1, -\bar{\phi}_0) \in H \times V$ the normal derivative $\partial_n \phi^1(.,\mathbf{v}_1)$ in the controlled node of the solution (23) of the system (17)-(21).

In view of (28) and (25), $\mathbf{A}_t$ is continuous. Then, the operator $\mathbf{A}_t^* : L^2(0,T) \to V' \times H$, adjoint of $\mathbf{A}_t$, will be also continuous (we have identified $L^2(0,T)$ and H with their duals).

Further, for every $h \in L^2[0,T]$ we define the solution of the system (11)-(16) with initial state $(\bar{u}_0, \bar{u}_1) \in H \times V'$ as

$$\bar{u} = \mathbf{A}_t^* h + \mathbf{S}_t(\bar{u}_0, \bar{u}_1), \tag{29}$$

where $\mathbf{S}_t(\bar{u}_0, \bar{u}_1)$ is the solution of (17)-(21) given by (23) in time t.

To clarify the meaning of this formula, let us calculate the operator$\mathbf{A}_t^*$. We consider the operator $\mathbf{B}$ defined for $h \in C^1([0,t])$ by

$$\mathbf{B}h = \langle \bar{u}(t), \bar{u}_t(t) \rangle_H,$$

where $\bar{u}$ is the solution in the classical sense of the problem (11)-(16) with initial data $\bar{u}_0 = \bar{u}_1 = 0$.

If we multiply the equation (11) by u_i and integrate over $[0,t]\times[0,\ell_i]$ it holds, after integration by parts,

$$\int_0^t \int_0^{\ell_i} (u^i_{tt} - u^i_{xx})\phi^i dtdx = \int_0^{\ell_i} \left(u^i\phi^i_t - u^i_t\phi^i\right)\Big|_0^t dx + \int_0^t \left(u^i_x\phi^i - u^i\phi^i_x\right)\Big|_0^{\ell_i} d\tau.$$

If we add these equalities we get, in view of the boundary conditions (12)-(15),

$$\int_0^t h\partial_n\phi^1(\tau,\mathbf{v}_1)d\tau = \sum_{i=1}^M \int_0^{\ell_i} \left(u^i(t,x)\phi^i_t(t,x) - u^i_t(t,x)\phi^i(t,x)\right) dx,$$

and this equality means that

$$\langle \partial_n\phi^1(t,\mathbf{v}_1), h\rangle_{L^2(0,t)} = \langle \bar{u}(t), \bar{\phi}_t(t)\rangle_{H\times H} - \langle \bar{u}_t(t), \bar{\phi}(t)\rangle_{V'\times V}.$$

Consequently we have

$$\langle \mathbf{A}\bar{\phi}, h\rangle_{L^2(0,t)} = \langle \mathbf{B}h, \bar{\phi}\rangle_{(H\times V')\times(H\times V)}.$$

That is, for $h \in C^1([0,t])$ it holds $\mathbf{B}h = \mathbf{A}^*_t h$. Taking into account that the operator $\mathbf{A}^*_t$ is continuous and that $C^1([0,t])$ is dense in $L^2(0,t)$, we can ensure that $\mathbf{A}^*_t$ coincides with the extension of $\mathbf{B}$ to $L^2(0,t)$.

This fact gives sense to the equality (29). In the classical case $h \in C^1([0,t])$, $(\bar{u}_0,\bar{u}_1) \in (H\times V')$, $u^i_0, u^i_1 \in C^1([0,\ell_i])$, formula (29) simply expresses the fact that the solution of the non-homogeneous problem with initial state $(\bar{u}_0,\bar{u}_1)$ can be represented as the sum of the solution of the homogeneous problem with initial state $(\bar{u}_0,\bar{u}_1)$ and the solution of the non-homogeneous problem with initial state $(\bar{0},\bar{0})$. This fact is an immediate consequence of the lineal character of the system (11)-(16).

Finally, for every $h \in L^2[0,T]$ the solution $\bar{u}$ of (11)-(16) defined by (29) has the property

$$\bar{u} \in C([0,T]:H)\bigcap C^1([0,T]:V').$$

3. The control problem

The control problem in time T consists in determining for which initial states of the networks it is possible to choose the controls $h_j \in L^2(0,T)$, $j = 1,...,r$, such that the systems reaches the equilibrium position after a time T. More precisely,

Definition I.1. *Let $T > 0$. We say that the initial state $(\bar{u}_0,\bar{u}_1) \in H\times V'$, is* ***controllable in time*** *T, if there exist functions $h_j \in L^2(0,T)$, $j = 1,...,r$, such that the solution of (11)-(16) with initial state $(\bar{u}_0,\bar{u}_1)$ satisfies*

$$\bar{u}|_{t=T} = \bar{u}_t|_{t=T} = \bar{0}.$$

Remark I.1. *Sometimes, under the conditions of the Definition I.1 it is also said that $(\bar{u}_0,\bar{u}_1)$ is exactly controllable. When for every $\varepsilon > 0$ there exist controls h^ε_i such that the corresponding solutions $\bar{u}^\varepsilon$ verify $\|(\bar{u}^\varepsilon|_T, \bar{u}^\varepsilon_t|_T)\|_{H\times V'} < \varepsilon$, it is said that $(\bar{u}_0,\bar{u}_1)$ is* ***approximately controllable in time*** *T.*

Then, the control problem in time T consists in characterizing the set of controllable initial states in time T. The following definition classifies the system (11)-(16) according to the answer to the control problem.

DEFINITION I.2. *Let $T > 0$. We say that the set $K \subset H \times V'$ is controllable in time T, if all the initial states $(\bar{u}_0, \bar{u}_1) \in K$ are controllable in time T. Then, we shall say that the system (11)-(16) is*

1) ***approximately** controllable in time T if there exists a set K, which is controllable in time T and is dense[2] en $H \times V'$;*
2) ***spectrally** controllable in time T if the subspace $Z \times Z$ is controllable in time T, where Z is the set of all the finite linear combinations of the eigenfunctions of the operator $-\Delta_G$;*
3) ***exactly** controllable in time T if the whole space $H \times V'$ is controllable in time T.*

Let us note that, due to the linear character of the system (11)-(16), if the set K is controllable, so is the subspace span K of all the finite linear combinations of elements of K. That is why it is natural to talk of controllable subspaces instead of controllable sets.

3.1. An equivalent formulation of the control problem. Let us observe first, that the control problem admits an equivalent formulation in terms of operators. Let $\mathbf{P}_T : U \to H \times V'$ be the operator defined by

$$\mathbf{P}_T \bar{h} := (\bar{u}(T), \bar{u}_t(T)),$$

where $\bar{u}$ is the solution of the system (11)-(16) with initial state $(\bar{0}, \bar{0})$.

Let us denote by $\mathcal{W}_T$ the rang of $\mathbf{P}_T$; that is, $\mathcal{W}_T$ is the set of those states that can be reached after a time T starting from the rest state.

Let us note that the initial state $(\bar{u}_0, \bar{u}_1) \in H \times V'$ is controllable in time T if, and only if, $(\bar{u}_0, \bar{u}_1) \in \mathcal{W}_T$. This fact is due to the invariance of the system (11)-(16) under the change of variable $t \to T - t$: if $\bar{u}$ is a solution of (11)-(16) then, $\bar{w}(t) = \bar{u}(T-t)$ is also a solution. Thus, given $(\bar{u}_0, \bar{u}_1) \in \mathcal{W}_T$, if $\bar{u}$ is a solution satisfying

$$(\bar{u}(0), \bar{u}_t(0)) = (\bar{0}, \bar{0}), \qquad (\bar{u}(T), \bar{u}_t(T)) = (\bar{u}_0, \bar{u}_1)$$

with control $\hat{h}$ then, $\bar{w}(t) = \bar{u}(T - t)$ satisfies

$$(\bar{w}(0), \bar{w}_t(0)) = (\bar{u}_0, \bar{u}_1), \qquad (\bar{w}(T), \bar{w}(T)) = (\bar{0}, \bar{0}).$$

Consequently, to drive $(\bar{u}_0, \bar{u}_1)$ to $(\bar{0}, \bar{0})$ it is sufficient to choose the control $\hat{h}(T-t)$.

As a consequence, if the initial states $(\bar{u}_0, \bar{u}_1)$ and $(\bar{v}_0, \bar{v}_1)$ are controllable in time T then it is possible to find a control $\hat{h} \in U$ driving $(\bar{u}_0, \bar{u}_1)$ to $(\bar{v}_0, \bar{v}_1)$. Indeed, it suffices to take $\hat{h} = \hat{h}_1 + \hat{h}_2$, where $\hat{h}_1$, $\hat{h}_2$ are, the controls that drive $(\bar{u}_0, \bar{u}_1)$ to $(\bar{0}, \bar{0})$ and $(\bar{0}, \bar{0})$ to $(\bar{v}_0, \bar{v}_1)$, respectively.

Thus, the control problem in time T is reduced to study the rang $\mathcal{W}_T$ of the operator $\mathbf{P}_T$. On the other hand, on the basis of general results of Functional Analysis (see Theorem II.1), the space $\mathcal{W}_T$ may be described in terms of the operator adjoint to $\mathbf{P}_T$. This is essentially the HUM.

Let us observe now that, according to the definition (29) of the solution of (11)-(16), the adjoint of the operator $\mathbf{P}_T$ coincides with $\mathbf{A}_T$, that is, the adjoint of $\mathbf{P}_T$ is the operator that associates to $(\bar{\phi}_1, -\bar{\phi}_0) \in H \times V$ the vector $\partial_n \bar{u}|_{\mathcal{C}} \in U$, whose components are the normal derivatives $\partial_n u^i(., \mathbf{v}_j)$, $j = 1, ..., r$, of the solution of the homogeneous system (17)-(21) with initial state $(\bar{\phi}_0, \bar{\phi}_1)$. That is why the control

[2] In other words, the system (11)-(16) is approximately controllable if all the initial states $(\bar{u}_0, \bar{u}_1) \in H \times V'$ are approximately controllable.

problem is reduced to the study of properties of the solutions of the homogeneous system (17)-(21).

On the other hand, $(\bar{u}_0, \bar{u}_1) \in \mathcal{W}_T$, that is, $(\bar{u}_0, \bar{u}_1) = \mathbf{P}_T \bar{h}$ for some $\bar{h} \in U$, if, and only if, for all $(\bar{\phi}_1, -\bar{\phi}_0) \in Z \times Z$ the following equality is satisfied

$$\langle(\bar{u}_0, \bar{u}_1), (\bar{\phi}_1, -\bar{\phi}_0)\rangle_{(H\times V')\times(H\times V)} = \langle \mathbf{P}_T \bar{h}, (\bar{\phi}_1, -\bar{\phi}_0)\rangle_{(H\times V')\times(H\times V)}.$$

Then, $(\bar{u}_0, \bar{u}_1) \in \mathcal{W}_T$ if, and only if,

$$\langle(\bar{u}_0, \bar{u}_1), (\bar{\phi}_1, -\bar{\phi}_0)\rangle_{(H\times V')\times(H\times V)} = \langle \bar{h}, \mathbf{P}_T^*(\bar{\phi}_1, -\bar{\phi}_0)\rangle_U = \langle \bar{h}, \partial_n \bar{\phi}|_{\mathcal{C}}\rangle_U.$$

Let us write this result in usual terms:

PROPOSITION I.1. *The initial state $(\bar{u}_0, \bar{u}_1) \in H \times V'$ is controllable in time T with control $\bar{h} = (h_1, ..., h_r) \in U$ if, and only if, for every $(\bar{\phi}_0, \bar{\phi}_1) \in Z \times Z$ the following equality holds*

$$\langle \bar{u}_0, \bar{\phi}_1\rangle_H - \langle \bar{u}_1, \bar{\phi}_0\rangle_{V'\times V} = \sum_{j=1}^{r} \int_0^T h_j(t) \partial_n \phi^i(t, \mathbf{v}_j) dt, \tag{30}$$

where $\bar{\phi}$ is the solution of the system (17)-(21) with initial state $(\bar{\phi}_0, \bar{\phi}_1)$.

REMARK I.2. *The relation (30) suggests an algorithm for the construction of the control $\bar{h}$. If we look for the control in the form $\bar{h} = -\partial_n \bar{\psi}|_{\mathcal{C}}$, where $\bar{\psi}$ is a solution of the homogeneous system (17)-(21), then the equality (30) is the Euler equation $I'(\bar{\psi}_0, \bar{\psi}_1) = 0$ corresponding to the quadratic functional $I : V \times H \to \mathbb{R}$ defined by*

$$I(\bar{\phi}_0, \bar{\phi}_1) = \frac{1}{2} \int_0^T \sum_{j=1}^{r} |\partial_n \phi^i(t, \mathbf{v}_j)|^2 dt + \langle \bar{u}_0, \bar{\phi}_1\rangle - \langle \bar{u}_1, \bar{\phi}_0\rangle.$$

Therefore, if $(\bar{\psi}_0, \bar{\psi}_1)$ is a minimizer of I, the relation (30) will be verified. The functional is continuous and convex. So, in order to guarantee the controllability of an initial state $(\bar{u}_0, \bar{u}_1) \in H \times V'$ it is sufficient that I be coercive. This is the central idea of the Hilbert Uniqueness Method (HUM) due to [**59**]. *In Chapter II we will describe in detail this technique.*

4. A controllability theorem and its limitations

A natural starting point for the study of the control problem for a network of strings is the following theorem due to J. Schmidt.

THEOREM I.1 (Schmidt, [**76**]). *If G is a tree (does not contain closed paths) and the set $\mathcal{C}$ contains all the exterior nodes, except at most one, then the system (11)-(16) is exactly controllable in any time $T \geq T^*$, where T^* is equal to twice the length of the largest simple path connecting the uncontrolled node with the controlled ones.*

The proof of this theorem is rather simple. The main ingredient is the possibility of representing the solutions of the 1-d wave equation at every string by means of the D'Alembert formula. In the Section 3.1 of the Chapter 3 we describe the proof for the case of a network formed by three strings with two controlled nodes. There it is explained also, how to proceed in the case of arbitrary trees. Both facts, the tree structure and that all the exterior nodes, except at most one, are controlled, play an essential role in the proof.

However, the conditions of Theorem I.1 seems to be very strong: a high number of controls and a simple topological configuration of the graph. The question on whether these conditions may be weakened naturally arises. Could be the system (11)-(16) exactly controllable when there are more than two uncontrolled exterior nodes or when G contains circuits, at least for some values of the values of the lengths of the strings? It turns out that in both cases the answer is negative. In Chapter 5 (Section 3) we will prove the following result:

THEOREM I.2. *If G is a tree and in the system (11)-(16) there are at least two uncontrolled nodes then, there exist initial states controllable in any finite time T.*

This fact adds a particular interest to Theorem I.1, as it serves as a criterion on the exact controllability of the system (11)-(16).

In these notes we will mainly study networks of strings, which are controlled from their exterior nodes and which do not verify the conditions of Theorem I.1. That is why, we will be able to expect the controllability of the system in subspaces of $H \times V'$, which are strictly smaller.

CHAPTER II

Some useful tools

1. D'Alembert formula and observability from the boundary of the 1-d wave equation

In this section we shall write the D'Alembert formula for the solutions of the $1-d$ wave equation in a way that allows to use certain formal calculations for the study of the propagation of the solutions along the network. This allows, in particular, to prove observability properties of the solutions from one end of the string.

1.1. D'Alembert formula. Let us assume that the function $u(t,x)$ satisfies the $1-d$ wave equation in $\mathbb{R}\times\mathbb{R}$. Then, for every $t_*\in\mathbb{R}$ the function u may be expressed by means of the D'Alembert formula

$$u(t,x)=\frac{1}{2}\left(u(t_*,x+t-t_*)+u(t_*,x-t+t_*)\right)+\frac{1}{2}\int_{x-t+t_*}^{x+t-t_*}u_t(t_*,\xi)d\xi. \tag{1}$$

In account of the symmetry of the wave equation with respect to the variables x,t, the formula (1) is also valid if we change the role of these variables. Thus, if $u(t,x)$ satisfies the $1-d$ wave equation in $\mathbb{R}\times[0,\ell]$ then, for every $a\in[0,\ell]$, $u(t,x)$ may be expressed by the formula

$$u(t,x)=\frac{1}{2}\left(u(t+x-a,a)+u(t-x+a,a)\right)+\frac{1}{2}\int_{t-x+a}^{t+x-a}u_x(\tau,a)d\tau. \tag{2}$$

From (2), after derivation, we obtain the equalities

$$\begin{aligned}
u_x(t,x) &= \frac{1}{2}\left(u_t(t+x-a,a)-u_t(t-x+a,a)\right)+ && (3)\\
&\quad +\frac{1}{2}\left(u_x(t+x-a,a)+u_x(t-x+a,a)\right), && (4)\\
u_t(t,x) &= \frac{1}{2}\left(u_t(t+x-a,a)+u_t(t-x+a,a)\right)+\\
&\quad +\frac{1}{2}\left(u_x(t+x-a,a)-u_x(t-x+a,a)\right).
\end{aligned}$$

If we denote

$$G(t):=u_t(t,0),\quad F(t):=u_x(t,0),\qquad \widehat{G}(t):=u_t(t,\ell),\quad \widehat{F}:=u_x(t,\ell),$$

then the formulas (3)-(4) for $x=\ell$, $a=0$ may be written as

$$\widehat{F}=\ell^+F+\ell^-G,\qquad \widehat{G}=\ell^-F+\ell^+G, \tag{5}$$

and, for $x=0$, $a=\ell$

$$F=\ell^+\widehat{F}-\ell^-\widehat{G},\qquad G=-\ell^-\widehat{F}+\ell^+\widehat{G}, \tag{6}$$

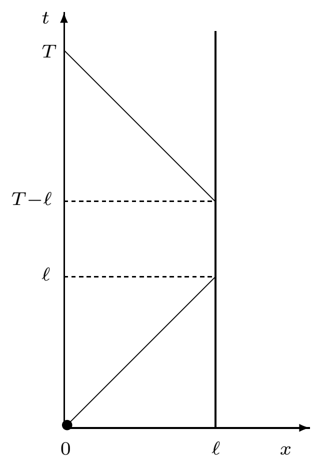

FIGURE 1. Region of application of the D'Alembert formula

where ℓ^+, ℓ^- are the linear operator that act over a function f depending on time t according to

$$\ell^{\pm} f(t) := \frac{f(t+\ell) \pm f(t-\ell)}{2}. \tag{7}$$

Let us remark that the formulas (5) and (6) express the relation between the traces of u_t and u_x in the extremes of the interval $[0,\ell]$. Obviously, (6) is the inverse relation to (5).

1.2. Observability from the boundary of the $1-d$ wave equation. The following proposition contains a very useful result on the observability of $1-d$ waves from the boundary. It will be frequently used in what follows.

PROPOSITION II.1. *If $u(t,x)$ satisfies the wave equation $u_{tt} = u_{xx}$ in $\mathbb{R} \times [0,\ell]$ then*

$$\mathbf{E}_u(t) \le \frac{1}{4} \int_{t-\ell}^{t+\ell} \left(|u_x(\tau,0)|^2 + |u_t(\tau,0)|^2 \right) d\tau.$$

PROOF. In view of (3)-(4), it holds

$$\begin{aligned}
\mathbf{E}_u(t) &= \frac{1}{8} \int_0^\ell \big\{ |u_t(t+x,0) - u_t(t-x,0) + u_x(t+x,0) + u_x(t-x,0)|^2 + \\
&+ |u_t(t+x,0) + u_t(t-x,0) + u_x(t+x,0) - u_x(t-x,0)|^2 \big\} \, dx \\
&\le \frac{1}{4} \int_0^\ell \big\{ |u_t(t+x,0)|^2 + |u_t(t-x,0)|^2 + |u_x(t+x,0)|^2 + |u_x(t-x,0)|^2 \big\} \, dx \\
&= \frac{1}{4} \int_{t-\ell}^{t} \big\{ |u_t(\tau,0)|^2 + |u_x(\tau,0)|^2 \big\} \, dx + \frac{1}{4} \int_t^{t+\ell} \big\{ |u_t(\tau,0)|^2 + |u_x(\tau,0)|^2 \big\} \, dx \\
&= \frac{1}{4} \int_{t-\ell}^{t+\ell} \left(|u_x(\tau,0)|^2 + |u_t(\tau,0)|^2 \right) dt.
\end{aligned}$$

□

PROPOSITION II.2. *For all $\ell > 0$, a, $b \in \mathbb{R}$ the operators ℓ^+, ℓ^- are continuous from $L^2[a-\ell, b+\ell]$ to $L^2[a,b]$.*

PROOF. We will prove that in addition the norm of the operators $\ell^{\pm}$, considered as elements of $\mathcal{L}(L^2[a-\ell, b+\ell], L^2[a,b])$, is not greater than one. In fact,

$$\begin{aligned}
\int_a^b |\ell^{\pm} f(t)|^2 dt &= \frac{1}{4}\int_a^b |f(t+\ell) \pm f(t-\ell)|^2 \, dt \\
&\leq \frac{1}{2}\int_a^b |f(t+\ell)|^2 \, dt + \frac{1}{2}\int_a^b |f(t-\ell)|^2 \, dt \\
&\leq \frac{1}{2}\int_{a+\ell}^{b+\ell} |f(t)|^2 dt + \frac{1}{2}\int_{a-\ell}^{b-\ell} |f(t)|^2 dt \leq \int_{a-\ell}^{b+\ell} |f(t)|^2 dt.
\end{aligned}$$

□

2. The Hilbert Uniqueness Method (HUM): reduction to an observability problem.

2.1. Description of the method. In this section we describe the main tool used along these notes for the study of control problems: The *Hilbert Uniqueness Method* (HUM)[1], which allows to reduce the control problem to the study of observability properties of the solutions of a homogeneous system.

We illustrate the application of HUM for the system (11)-(16), but we use, in general, an abstract setting that allows to avoid the difficulties related to the notations. Besides, it allows to use the results in other situations in which the method is also applied for the study of control problems: when the equation (11) is replaced by the Schrödinger or heat equations, or when the boundary conditions or the choice of the controls are different.

The starting point of HUM consists in reducing the control problem to the identification of the image of a continuous linear operator as it has been described in Section 3. Having this, the description of controllable initial states is based on the following general result of Functional Analysis: if E and F are Hilbert spaces and $\mathbf{A} : F \to E$ is a continuous linear operator with adjoint $\mathbf{A}^* : E' \to F$ (we have identified F and F' through the Riesz-Fréchet isometry) then

THEOREM II.1. *If $\mathbf{A}^*$ is injective then the image of $\mathbf{A}$ coincides with the set*

$$M = \{u \in E : \quad \exists C_u > 0 \text{ such that } |\langle \phi, u\rangle_{E'\times E}| \leq C_u \|\mathbf{A}^*\phi\|_F \ \ \forall\, \phi \in E'\}\,.$$

PROOF. We will show first that Im $\mathbf{A} \subset M$. If $u \in$ Im $\mathbf{A}$, that is, $u = \mathbf{A}p$ for $p \in F$ then, for all $\phi \in E'$

$$|\langle \phi, u\rangle_{E'\times E}| = |\langle \phi, \mathbf{A}p\rangle_{E'\times E}| = |\langle \mathbf{A}^*\phi, p\rangle_F| \leq \|\mathbf{A}^*\phi\|_F \|p\|_F,$$

and thus $u \in M$ with $C_u = \|p\|_F$.

The inclusion $M \subset$ Im $\mathbf{A}$ is more delicate. Since $\mathbf{A}^*$ is injective, $\mathbf{A}^*\phi = 0$ if, and only if, $\phi = 0$. Consequently, the function $\|\phi\|_{\mathbf{A}} = \|\mathbf{A}^*\phi\|_F$ is a norm in E'. Let $H_{\mathbf{A}}$ be the completion of E' with respect to that norm. This means that there exists an isometry $\kappa : (E', \|.\|_{\mathbf{A}}) \to H_{\mathbf{A}}$ such that $\kappa(E')$ is dense in $H_{\mathbf{A}}$. If we

[1] The name of this method is due to its author J.-L. Lions (see [**59**], [**60**]).

identify E' and $\kappa(E')$ through κ, it holds $E' \subset H_{\mathbf{A}}$. This imbedding is dense and continuous. Indeed, since $\mathbf{A}^*$ is bounded,

$$\|\phi\|_{\mathbf{A}} = \|\mathbf{A}^*\phi\|_F \leq C\|\phi\|_{E'}.$$

For $u \in M$ and $\phi \in E'$ we will denote by $\langle \phi, u \rangle$ the imagine by ϕ of the linear and continuous functional obtained by extending ϕ to M by continuity: if the sequence $(\phi_n) \subset E'$ converges to ϕ en $H_{\mathbf{A}}$ then

$$|\langle \phi_n, u \rangle_{E' \times E} - \langle \phi_m, u \rangle_{E' \times E}| = |\langle \phi_n - \phi_m, u \rangle_{E' \times E}| \leq C_u \|\mathbf{A}^*(\phi_n - \phi_m)\|_F = C_u \|\phi_n - \phi_m\|_{\mathbf{A}},$$

and then $\langle \phi_n, u \rangle$ is a Cauchy sequence in, $\mathbb{R}$ ((ϕ_n) is convergent), and thus is convergent. Now define $\langle \phi, u \rangle = \lim_{n \to \infty} \langle \phi_n, u \rangle$. The mapping $\langle ., u \rangle : H_{\mathbf{A}} \to \mathbb{R}$ is then linear and continuous, since when passing to the limit in the relations $|\langle \phi_n, u \rangle| \leq C_u \|\phi_n\|_{\mathbf{A}}$ it holds

$$|\langle \phi, u \rangle| \leq C_u \|\phi\|_{\mathbf{A}}. \tag{8}$$

Let us consider now the functional $I : H_{\mathbf{A}} \to \mathbb{R}$ defined by

$$I(\phi) = \frac{1}{2}\|\phi\|_{\mathbf{A}}^2 - \langle \phi, u \rangle,$$

which is clearly continuous and convex. Once again, in view of (8),

$$|I(\phi)| \geq \frac{1}{2}\|\phi\|_{\mathbf{A}}^2 - |\langle \phi, u \rangle| \geq \frac{1}{2}\|\phi\|_{\mathbf{A}}^2 - C_u\|\phi\|_{\mathbf{A}} \to \infty$$

as $\|\phi\|_{\mathbf{A}} \to \infty$. Then there exists a minimizer $\hat{\phi} \in H_{\mathbf{A}}$ that, taking into account that I is differentiable, satisfies the Euler equation $I'\hat{\phi} = 0$ and that is

$$\langle \phi, \hat{\phi} \rangle_{\mathbf{A}} = \langle \mathbf{A}^*\phi, \mathbf{A}^*\hat{\phi} \rangle_F = \langle \phi, u \rangle \qquad \text{for all } \phi \in H_{\mathbf{A}}.$$

In particular, for $\phi \in E'$,

$$\langle \phi, \mathbf{A}\mathbf{A}^*\hat{\phi} \rangle_{E' \times E} = \langle \mathbf{A}^*\phi, \mathbf{A}^*\hat{\phi} \rangle_F = \langle \phi, u \rangle_{E' \times E}.$$

This means that

$$u = \mathbf{A}\mathbf{A}^*\hat{\phi} \in \text{Im } \mathbf{A}. \tag{9}$$

□

Remark II.1. *Proceeding in a similar way as in the proof of the previous theorem it may be shown that is possible to identify "by continuity" $H'_{\mathbf{A}}$ with a subspace of E. In such case,*

$$H'_{\mathbf{A}} = \{u \in E : \quad \exists C_u > 0 \textit{ such that } | < \phi, u >_{E' \times E} | \leq C_u \|\mathbf{A}^*\phi\|_F \quad \textit{for every } \phi \in E'\}.$$

Then, from the theorem it follows that $\text{Im } \mathbf{A} = H'_{\mathbf{A}}$.

Remark II.2. *In general, it takes place the equality* $\overline{\text{Im } \mathbf{A}} = (\ker \mathbf{A}^*)^{\perp}$. *From this fact , it holds that* Im $\mathbf{A}$ *is dense in* E *if, and only if* $\mathbf{A}^*$ *is injective. Consequently, Theorem II.1 provides a description of* Im $\mathbf{A}$ *whenever it is dense in* E. *On the other hand, the injectiveness of* $\mathbf{A}^*$ *is equivalent to the fact that the equation* $\mathbf{A}^*\phi = v$ *has at most one solution. This is a uniqueness property to which refer the name of HUM.*

Let us assume now that W is a Hilbert space such that $W \subset E$ with continuous and dense embedding. This allows to extend by continuity the linear and continuous functionals in W to E, such that we can consider $E' \subset W'$.

The following result is very useful in order to characterize subspaces of Im $\mathbf{A}$.

COROLLARY II.1. *The subspace W is contained in the image of the operator $\mathbf{A}$ if, and only if, there exists a constant $C > 0$ such that*

$$\|\phi\|_{W'} \leq C\|\mathbf{A}^*\phi\|_F, \tag{10}$$

for every $\phi \in E'$. In such case, for every $u \in W$ there exists $p \in F$ such that

$$\|p\|_F \leq 2C\|u\|_W. \tag{11}$$

PROOF. Consider the set

$$\Gamma = \{\phi \in E' : \|\mathbf{A}^*\phi\|_F = 1\} \subset E'.$$

Let us observe, that the existence of a constant $C > 0$ such that

$$\|\phi\|_{W'} \leq C\|\mathbf{A}^*\phi\|_F, \tag{12}$$

for all $\phi \in E'$ means that Γ is bounded in W'.

On the other hand, the fact $W \subset$ Im $\mathbf{A}$ is equivalent to the fact that Γ is weakly bounded in W'. Indeed, according to Theorem II.1, $W \subset$ Im $\mathbf{A}$ if an only if, for every $u \in W$ there exists a constant C_u such that

$$| < \phi, u >_{W' \times W} | = | < \phi, u >_{E' \times E} | \leq C_u\|\mathbf{A}^*\phi\|_F, \tag{13}$$

for every $\phi \in E'$. Consequently, if $W \subset$ Im $\mathbf{A}$ then, for all $\phi \in \Gamma$, $u \in W'$,

$$| < \phi, u >_{W' \times W} | \leq C_u, \tag{14}$$

that is, Γ is weakly bounded. Conversely, if the inequality (14) is verified and $\psi \in E'$ then, choosing $\phi = \dfrac{\psi}{\|\mathbf{A}^*\psi\|_F} \in \Gamma$ ($\|\mathbf{A}^*\psi\|_F \neq 0$ as $\mathbf{A}^*$ is injective) it holds

$$| < \psi, u >_{W' \times W} | = \|\mathbf{A}^*\psi\|_F | < \phi, u >_{W' \times W} | \leq C_u\|\mathbf{A}^*\psi\|_F, \tag{15}$$

and then $u \in$ Im $\mathbf{A}^*$.

Finally, it suffices to recall the fact that the properties of being bounded and weakly bounded coincide in Hilbert spaces [2].

In order to prove (11) it suffices to choose for $u \in W$, the element $p \in F$ obtained in the proof of Theorem II.1, that is, $p = \mathbf{A}^*\hat{\phi}$, where $\hat{\phi}$ is a minimizer of the functional

$$I(\phi) = \frac{1}{2}\|\phi\|_{\mathbf{A}}^2 - \langle \phi, u \rangle.$$

Then we will have,

$$0 = I(0) \geq I(\hat{\phi}) = \frac{1}{2}\|\hat{\phi}\|_{\mathbf{A}}^2 - \langle \hat{\phi}, u \rangle.$$

Then,

$$\|\hat{\phi}\|_{\mathbf{A}}^2 \leq 2\langle \hat{\phi}, u \rangle \leq 2\|\phi\|_{W'}\|u\|_W \leq 2C\|\hat{\phi}\|_{\mathbf{A}}\|u\|_W.$$

Finally, since $\|p\|_F = \|\hat{\phi}\|_{\mathbf{A}}$, it holds

$$\|p\|_F \leq 2C\|u\|_W.$$

$\square$

[2] This is an immediate consequence of the Banach-Steinhaus theorem and the reflexiveness of the Hilbert spaces

REMARK II.3. *In particular,* Im $\mathbf{A} = E$*, that is, the operator* $\mathbf{A}$ *is surjective if, and only if there exists a constant* $C > 0$ *such that*

$$\|\phi\|_{E'} \leq C\|\mathbf{A}^*\phi\|_F, \tag{16}$$

for every $\phi \in E'$*. This condition is equivalent to the continuity of* $(\mathbf{A}^*)^{-1}$*.*

REMARK II.4. *Due to the continuity of* $\mathbf{A}^*$*, it is sufficient to prove the inequality* (10) *for a dense subspace of* E'*.*

REMARK II.5. *Inequality (16) is known as* ***observability inequality****. All along this book, any inequality of the form (10) will be called generically observability inequality.*

Let us see now another possible way of constructing subspaces of Im $\mathbf{A}$, which will be frequently used in what follows. Assume that $\mathbf{B} : E' \to E'$ is a continuous operator, whose image is dense in E' and verifies the properties

1) There exists a constant $C > 0$ such that, for every $\phi \in E'$,
$$\|\mathbf{B}\phi\|_{E'} \leq C \|\mathbf{A}^*\phi\|_F \,,$$
for all $\phi \in E'$.
2) If $\mathbf{B}$ is not injective then, neither is $\mathbf{A}^*$; that is, if there exists $\phi \in E' \setminus \{0\}$ such that $\mathbf{B}\phi = 0$ then there exists $\psi \in E' \setminus \{0\}$ such that $\mathbf{A}^*\psi = 0$.

Let us note that an operator $\mathbf{B}$ with the properties indicated above is injective if, and only if, $\mathbf{A}^*$ is injective. Consequently, it holds that the subspace Im $\mathbf{A}$ is dense in E if, and only if, $\mathbf{B}$ is injective.

In such case, property 1 would correspond to the fact that $\mathbf{B} \circ (\mathbf{A}^*)^{-1}$ be continuous. Moreover, if in addition $\mathbf{B}$ would be surjective, then, according to Banach theorem on the open mapping, its inverse $\mathbf{B}^{-1}$ would be also continuous and then the same is true of $\mathbf{A}^*$; so we would have Im $\mathbf{A} = E$.

This cannot be asserted if $\mathbf{B}$ is not surjective. However, it is true for some smaller subspace:

PROPOSITION II.3. *If* $\mathbf{B}$ *is a continuous operator with dense image having the property 1 then* Im $\mathbf{B}^* \subset$ Im $\mathbf{A}$*, where* $\mathbf{B}^*$ *is the adjoint operator to* $\mathbf{A}$*.*

PROOF. If $u \in$ Im $\mathbf{B}^*$, that is, $u = \mathbf{B}^*v$ then

$$\langle u, \phi\rangle_{E\times E'} = \langle \mathbf{B}^*v, \phi\rangle_{E\times E'} = \langle v, \mathbf{B}\phi\rangle_{E\times E'} \leq \|v\|_E \|\mathbf{B}\phi\|_{E'} \leq C \|v\|_E \|\mathbf{A}^*\phi\|_F \,,$$

and so the assertion follows from Theorem II.1. □

Property 2 guarantees that the previous result is exact in the sense that it provides a subspace dense in Im $\mathbf{A}$ whenever such a subspace exists. Unfortunately that subspace may not coincide with the image of $\mathbf{A}$, it may be smaller.

In this book we will use the results described above in the following concrete situation. Let H be a separable Hilbert space and $\{\theta_n\}_{n\in\mathbb{N}}$ an orthonormal basis of H. Let us denote by Φ the set of all the formal linear combinations $\bar{X} = \sum_{n\in\mathbb{N}} x_n\theta_n$, $x_n \in \mathbb{R}$, and Z the set of the finite linear combinations.

Let (α_n), (β_n) be sequences of real numbers different from zero and define the Hilbert space

$$E := \left\{ (\bar{X}, \bar{Y}) \in \Phi \times \Phi : \quad \|(\bar{X}, \bar{Y})\|_E^2 := \sum_{n \in \mathbb{N}} (\alpha_n^2 x_n^2 + \beta_n^2 y_n^2) < \infty \right\}$$

provided with the norm $\|.\|_E$. Then, the dual of E may be identified with the space

$$E' = \left\{ (\bar{X}, \bar{Y}) \in \Phi \times \Phi : \quad \|(\bar{X}, \bar{Y})\|_{E'}^2 := \sum_{n \in \mathbb{N}} (\alpha_n^{-2} x_n^2 + \beta_n^{-2} y_n^2) < \infty \right\}$$

endowed with the norm $\|.\|_{E'}$.

Let us consider as before the linear and continuous operator $\mathbf{A} : F \to E$ with injective adjoint $\mathbf{A}^*$. Let now (c_n) be another sequence verifying $c_n \geq c\alpha_n$, $c_n \geq d\beta_n$ for some $c, d > 0$ and define the space

$$W := \left\{ (\bar{X}, \bar{Y}) \in \Phi \times \Phi : \quad \|(\bar{X}, \bar{Y})\|_W^2 := \sum_{n \in \mathbb{N}} c_n^2 (\alpha_n^2 x_n^2 + \beta_n^2 y_n^2) < \infty \right\} \subset E.$$

Then the results of Corollary II.1 allows us to assert that

Proposition II.4. *$W \subset \mathrm{Im}\ \mathbf{A}$ if, and only if, there exists a constant $C > 0$ such that*

$$\|(\bar{X}, \bar{Y})\|_{W'}^2 := \sum_{n \in \mathbb{N}} c_n^{-2} (\alpha_n^{-2} x_n^2 + \beta_n^{-2} y_n^2) \leq C \|\mathbf{A}^*(\bar{X}, \bar{Y})\|_F^2, \tag{17}$$

for all $\bar{X}, \bar{Y} \in Z$, that is, for all finite linear combinations (x_n), (y_n).

Clearly, if the inequality (17) holds, then $\mathrm{Im}\ \mathbf{A}$ contains the subspace $Z \times Z$ of all the finite linear combinations, but this condition is not necessary in general. To clarify when this happens let us note that, due to the linearity of $\mathbf{A}$, $Z \times Z \subset \mathrm{Im}\ \mathbf{A}$ if, and only if, $(\bar{\theta}_n, \bar{0})$ and $(\bar{0}, \bar{\theta}_n)$ belong to $\mathrm{Im}\ \mathbf{A}$ for every $n \in \mathbb{N}$. According to Theorem II.1, the latter fact is equivalent to the existence, for every $n \in \mathbb{N}$, of constants $C_n^1,\ C_n^2 > 0$ such that

$$| < (\bar{X}, \bar{Y}), (\bar{\theta}_n, \bar{0}) >_{E' \times E} | \leq C_n^1 \|\mathbf{A}^*(\bar{X}, \bar{Y})\|_F,$$

$$| < (\bar{X}, \bar{Y}), (\bar{0}, \bar{\theta}_n) >_{E' \times E} | \leq C_n^2 \|\mathbf{A}^*(\bar{X}, \bar{Y})\|_F.$$

It suffices now to note that

$$< (\bar{X}, \bar{Y}), (\bar{\theta}_n, \bar{0}) >_{E' \times E} = x_n, \qquad < (\bar{X}, \bar{Y}), (\bar{0}, \bar{\theta}_n) >_{E' \times E} = y_n$$

to conclude:

Proposition II.5. *$Z \times Z \subset \mathrm{Im}\ \mathbf{A}$ if, and only if, for every $n \in \mathbb{N}$ there exists a constant $C_n > 0$ such that*

$$|x_n| + |y_n| \leq C_n \|\mathbf{A}^*(\bar{X}, \bar{Y})\|_F,$$

for all $\bar{X},\ \bar{Y} \in Z$.

2.2. Application to the control of the network. Let us apply now the previous results to the control problem of the network. From Theorem II.1 it holds immediately

COROLLARY II.2. *The initial state* $(\bar{u}_0, \bar{u}_1) \in V' \times H$ *is controllable in time* T *if, and only if, there exists a constant* $C > 0$ *such that*

$$C \int_0^T \sum_{j=1}^r |\partial_n \phi^i(t, \mathbf{v}_j)|^2 dt \geq \left| \langle \bar{u}_0, \bar{\phi}_1 \rangle_H - \langle \bar{u}_1, \bar{\phi}_0 \rangle_{V' \times V} \right|^2$$

for every solution $\bar{\phi}$ *of the system (17)-(21) with initial state* $(\bar{\phi}_0, \bar{\phi}_1) \in Z \times Z$.

It is interesting to point out how the formula (9), obtained in the proof of Theorem II.1, provides an algorithm for the construction of the control $\bar{h}$ that drives the controllable state $(\bar{u}_0, \bar{u}_1) \in H \times V'$ to $(\bar{0}, \bar{0})$ in time T: we should solve the extremal problem

$$I(\Psi^*) = \min_W I(\Psi) \tag{18}$$

for the functional

$$I(\Psi) = \frac{1}{2} \int_0^T \sum_{j=1}^r |\partial_n \phi^i(t, \mathbf{v}_j)|^2 dt + \langle \bar{u}_0, \bar{\phi}_1 \rangle - \langle \bar{u}_1, \bar{\phi}_0 \rangle$$

over the space W, which is the completion of $Z \times Z$ with the norm

$$\|(\bar{\phi}_0, \bar{\phi}_1)\|_W = \left[\int_0^T \sum_{j=1}^r |\partial_n \phi^i(t, \mathbf{v}_j)|^2 dt \right]^{\frac{1}{2}}$$

and ϕ is the solution of (17)-(21) with initial state $\Psi = (\bar{\phi}_0, \bar{\phi}_1)$.

Let $\Psi^* = (\bar{\varphi}_0^*, \bar{\varphi}_1^*)$ be the solution of the problem (18). Next, we solve the homogeneous system (17)-(21) with initial data $(\bar{\varphi}_1^*, -\bar{\varphi}_0^*)$. Let $\bar{\phi}$ be the corresponding solution. The control will be the trace $\partial_n \bar{\phi}|_{\mathcal{C}}$ of this solution.

Besides, from remarks II.2 and II.3 it follows

COROLLARY II.3. *The system (I.11)-(I.16) is approximately controllable in time* T *if, and only if, the following unique continuation property is verified*

$$\partial_n \phi^i(t, \mathbf{v}_j) = 0, \ j = 1, ..., r, \quad \text{for almost every } t \in [0, T] \quad \text{implies} \quad (\bar{\phi}_0, \bar{\phi}_1) = (\bar{0}, \bar{0}).$$

Moreover, all the initial states $(\bar{u}_0, \bar{u}_1) \in V' \times H$ *are exactly controllable in time* T *if, and only if, there exists a constant* $C > 0$ *such that*

$$C \int_0^T \sum_{j=1}^r |\partial_n \phi^i(t, \mathbf{v}_j)|^2 dt \geq \|(\bar{\phi}_0, \bar{\phi}_1)\|_{V \times H}^2 \tag{19}$$

for all $(\bar{\phi}_0, \bar{\phi}_1) \in V \times H$.

The inequality (19) may be expressed in terms of the Fourier coefficients $(\phi_{0,n})$, $(\phi_{1,n})$ of the initial data $\bar{\phi}_0, \bar{\phi}_1$ as

$$C \int_0^T \sum_{j=1}^r |\partial_n \phi^i(t, \mathbf{v}_j)|^2 dt \geq \sum_{n \in \mathbb{N}} \left(\mu_n \phi_{0,n}^2 + \phi_{1,n}^2 \right). \tag{20}$$

Unfortunately, this inequality is not valid for the system (11)-(16), except under the very restrictive conditions on the graph G and the location of the controlled nodes indicated in Theorem I.1. All along this book, we shall deal with situations when the inequality (19) is not true, that is, there exists initial states $(\bar{u}_0, \bar{u}_1) \in V' \times H$, which are not controllable in time T. We will be only able to prove weaker inequalities of the type

$$\int_0^T \sum_{j=1}^r |\partial_n \phi^i(t, \mathbf{v}_j)|^2 dt \geq \sum_{n\in\mathbb{N}} c_n^2 \left(\mu_n u_{0,n}^2 + u_{1,n}^2\right), \tag{21}$$

with coefficients c_n different from zero. This will allow to ensure, according to the Proposition II.4, that the space of initial states $(\bar{u}_0, \bar{u}_1) \in V' \times H$ defined by

$$\sum_{n\in\mathbb{N}} \frac{1}{c_n^2} u_{0,n}^2 < \infty, \quad \sum_{n\in\mathbb{N}} \frac{1}{c_n^2 \mu_n} u_{1,n}^2 < \infty, \tag{22}$$

is controllable in time T.

From that fact, it would hold, in particular, that the system is spectrally controllable (and then approximately controllable) in time T.

Let us remark that, if we would be able proof in addition that the coefficients c_n en (21) verify a uniform inequality of the form

$$c_n \mu_n^\varepsilon \geq C > 0,$$

for some $\varepsilon \in \mathbb{R}$, then, the sequences $(u_{0,n}), (u_{1,n})$ such that

$$\sum_{n\in\mathbb{N}} \mu_n^\varepsilon u_{0,n}^2 < \infty, \quad \sum_{n\in\mathbb{N}} \mu_n^{\varepsilon-1} u_{1,n}^2 < \infty,$$

would satisfy the inequalities (22). This would imply that the space $\mathcal{W}^\varepsilon$ is controllable in time T.

Remark II.6. *Let us assume that $r = 1$ and the inequality (21) is verified. If we replace $\bar{\phi}$ by its explicit expression (I.23), we obtain*

$$\int_0^T |\sum_{k\in\mathbb{N}} \varkappa_k (\phi_{0,k} \cos \lambda_k t + \frac{\phi_{1,k}}{\lambda_k} \sin \lambda_k t)|^2 dt \geq \sum_{k\in\mathbb{N}} c_k^2 (\mu_k \phi_{0,k}^2 + \phi_{1,k}^2), \tag{23}$$

where $\varkappa_k = \partial_n \theta_k^1(\mathbf{v}_1)$.

If we define for $k < 0$, $\lambda_k := -\lambda_{|k|}$ and denote $a_k = \frac{1}{2}\left(u_{0,|k|} - i\frac{u_{1,|k|}}{\lambda_k}\right)$ for $k \in \mathbb{Z}_*$, the inequality (23) becomes

$$\int_0^T |\sum_{k\in\mathbb{Z}_*} \varkappa_{|k|} a_k e^{i\lambda_k t}|^2 dt \geq \sum_{k\in\mathbb{N}} c_k^2 \mu_k |a_k|^2.$$

Consequently, we can assert that the latter inequality would be verified for every finite sequence (a_k), in general of complex numbers, satisfying $a_{-k} = \overline{a_k}$.

Let us note, however, that,

$$\frac{1}{2}|\sum_{k\in\mathbb{Z}_*} \varkappa_{|k|} a_k e^{i\lambda_k t}|^2 \leq |\sum_{k>0} \varkappa_{|k|} a_k e^{i\lambda_k t}|^2 + |\sum_{k<0} \varkappa_{|k|} a_k e^{i\lambda_k t}|^2$$

and since

$$\sum_{k<0} \varkappa_{|k|} a_k e^{i\lambda_k t} = \sum_{k>0} \varkappa_k \overline{a_k} e^{-i\lambda_k t} = \overline{\sum_{k>0} \varkappa_k a_k e^{i\lambda_k t}},$$

we obtain that the following inequalities hold

$$\int_0^T |\sum_{k\in\daleth} \varkappa_k a_k e^{i\lambda_k t}|^2 dt \geq C \sum_{k\in\daleth} c_k^2 \mu_k \left|a_k\right|^2,$$

$$\int_0^T |\sum_{k\in\mathbb{N}} \varkappa_k a_k e^{-i\lambda_k t}|^2 dt \geq C \sum_{k\in\mathbb{N}} c_k^2 \mu_k \left|a_k\right|^2,$$

for every finite complex sequence (a_k).

3. The moments method

In this section we describe an alternative method for the study of the control problem: the *method of moments*. These methods turns out to be useful not only for networks of strings, but also in the study of systems obtained by replacing in (11)-(16) the wave equation by the heat equation and, in general, by equations, whose solutions may be computed using the method of separation of variables.

3.1. Description of the method. Let H be a Hilbert space and $(\mathbf{a}_n)$ a sequence of elements of H. Given a sequence $(m_n) \in l^2$, the following problem is known as *problem of moments*: find an element $v \in H$ such that

$$\langle v, \mathbf{a}_n \rangle_H = m_n, \qquad n \in \mathbb{Z}. \tag{24}$$

A problem of moments appears in a natural way in the study of control problems when we try to find the control v that drives an initial state to rest in time T directly from Proposition I.1. In this case, the space H is $L^2(0,T)$ and the sequence $(\mathbf{a}_n)$ is formed by the complex exponentials $\mathbf{a}_n = e^{i\lambda_n t}$. This leads to the problem of moments

$$\int_0^T v(t) e^{i\lambda_n t} dt = m_n, \qquad n \in \mathbb{Z}, \tag{25}$$

where the sequence (m_n) depends on the Fourier coefficients of the initial state to be controlled.

Historically, this approach was the first giving important results on the controllability of systems described by partial differential equations. For more details, the reader may consult the papers [**31**], [**73**], [**33**], [**74**], [**32**].

A natural way to search for a solution of (24) is to solve first the problem for the sequences of the canonical basis $\bar{\mathbf{e}}^k = \left(\delta_n^k\right)$ of l^2. Here, the symbol δ_n^k is the Kronecker δ (δ_n^k is one if $n = k$ and zero otherwise). If we denote by v_k the corresponding solutions (assuming that such solutions exists), we will have

$$\langle v_k, \mathbf{a}_n \rangle = \delta_n^k \qquad n, k \in \mathbb{Z}.$$

A sequence with this property is called *biorthogonal sequence to* the sequence $(\mathbf{a}_n)$ in H. The usefulness of a biorthogonal sequence is immediate: if we choose

$$v = \sum_{k\in\mathbb{N}} m_k v_k, \tag{26}$$

we have, at least formally, that, for every n, it holds

$$\langle v, \mathbf{a}_n \rangle = \sum_{k\in\mathbb{N}} m_k \langle v_k, \mathbf{a}_n \rangle = \sum_{k\in\mathbb{N}} m_k \delta_n^k = m_n.$$

Under additional summability conditions on the sequence (m_n), formula (26) provides a solution of (24):

PROPOSITION II.6. *If* $(v_n) \subset H$ *is a biorthogonal sequence to* $(\mathbf{a}_n)$ *in* H *then, for every sequence* (m_n) *such that*

$$\sum_{n\in\mathbb{N}} |m_n| \, \|v_n\|_H < \infty, \tag{27}$$

there exists a solution $v \in H$ *of (24). That solution is given by (26).*

PROOF. It suffices to note that the function v defined by (26) belongs to H:

$$\|v\|_H \leq \sum_{n\in\mathbb{N}} |m_n| \, \|v_n\|_H < \infty.$$

□

Thus, solving a problem of moments with this technique involves to fundamental steps: to determine a biorthogonal sequence and to estimate the norms of its elements. According to Proposition II.6, if there exists a biorthogonal sequence, we will be able to indicate a dense in l^2 subspace of sequences, defined by (27), for which the problem of moments has a solution. In particular, *the existence of a biorthogonal sequence guarantees the solvability of the problem of moments for every finite sequence* (m_n).

As it has been pointed out above, in the study of the control problems the problems of moments (25) are relevant, where (λ_n) is a sequence of complex numbers such that $(\Re\lambda_n)$ is increasing[3]. In this case, a biorthogonal sequence may be constructed in a relatively easy way thanks to the developments of Paley and Wiener [**67**].

After performing the change of variables $t \to t + A$ with $A = \frac{T}{2}$, the problem (25) may be written in the symmetric form

$$\int_{-A}^{A} \tilde{v}(t) e^{i\lambda_n t} dt = \tilde{m}_n, \tag{28}$$

which is a problem of moments in $L^2(-A, A)$.

Let us assume that F is an entire function satisfying:

1) $F \in L^\infty(\mathbb{R})$;
2) F is of exponential type not greater than A: there exist constants $M, A > 0$ such that $|F(z)| \leq M e^{A|z|}$ for every $z \in \mathbb{C}$.
3) all the numbers λ_n are simple zeros of F:

$$F(\lambda_n) = 0, \quad F'(\lambda_n) \neq 0.$$

Then, it is easy to see that the functions

$$F_k(z) := \frac{F(z)}{(z - \lambda_k) F'(\lambda_k)} \tag{29}$$

[3] $\Re z$ denotes the real part of the complex number z.

satisfy the property 2. Besides, it may be shown, using the Phragmén-Lindelöf theorem (see, e.g., Theorem 11, p. 82 in [**81**]), that there exists a constant $C > 0$ such that for every $k \in \mathbb{N}$,

$$\|F_k\|_{L^2(\mathbb{R})} \le \frac{C}{|F'(\lambda_k)|} \|F\|_{L^\infty(\mathbb{R})}\,; \tag{30}$$

in particular, the functions F_k belong to $L^2(\mathbb{R})$.

Finally, let us observe that $F_k(\lambda_n) = \delta_k^n$.

Now we are ready to apply the fundamental tool of this technique:

THEOREM II.2 (Paley and Wiener, [**67**]). *The function F is the Fourier transform of a function $\varphi \in L^2(\mathbb{R})$ with support contained in the interval $[-A, A]$, that is,*

$$F(z) = \int_{-A}^{A} e^{izt}\varphi(t)dt,$$

if, and only if, F is an entire function of exponential type at most A and $F \in L^2(\mathbb{R})$.

If we apply Theorem II.2 to the functions F_k defined by (29) it holds that there exist functions $v_k \in L^2(-A, A)$ such that

$$F_k(z) = \int_{-A}^{A} e^{izt}v_k(t)dt, \qquad k \in \mathbb{N}.$$

From these inequalities we obtain

$$\int_{-A}^{A} e^{i\lambda_n t}v_k(t)dt = F_k(\lambda_n) = \delta_k^n,$$

and thus, the sequence (v_k) would be biorthogonal to $\left(e^{i\lambda_n t}\right)$ in $L^2(-A, A)$. By this reason, the function F is called *generating function* for the sequence $\left(e^{i\lambda_n t}\right)$.

On the other hand, from Plancherel's identity

$$\|v_k\|_{L^2(-A,A)} = \|F_k\|_{L^2(\mathbb{R})}\,.$$

Consequently, in view of (30) there exists a constant $C > 0$ such that for every $k \in \mathbb{N}$

$$\|v_k\|_{L^2(-A,A)} \le \frac{C}{|F'(\lambda_k)|}. \tag{31}$$

Then, if we succeed in constructing a generating function F of the sequence (λ_n), the problem of identifying subspaces of sequences (m_n) for which the problem of moments (28) has a solution is reduced to estimate the sequence of norms $|F'(\lambda_k)|$.

REMARK II.7. *If it would be possible to establish uniform estimates of the form*

$$|F'(\lambda_k)| \ge C\,|\lambda_k|^{-\alpha}\,,$$

then it would hold

$$\|v_k\|_{L^2(-A,A)} \le C\,|\lambda_k|^{\alpha}$$

and, according to Proposition II.6 the problem of moments (28) would have a solution for every sequence (m_n) satisfying

$$\sum_{n\in\mathbb{N}} |m_n|\,|\lambda_k|^{\alpha} < \infty.$$

It may be useful to characterize subspaces of sequences of the type h^r for which the problem of moments has a solution. Let us observe that, if there exists $\gamma \in \mathbb{R}$ such that

$$\sum_{n\in\mathbb{N}} |\lambda_k|^{\gamma} < \infty,$$

then, from the Cauchy-Schwarz inequality holds

$$\sum_{n\in\mathbb{N}} |m_n|\,|\lambda_k|^{\alpha} < \sum_{n\in\mathbb{N}} |m_n|^2\,|\lambda_k|^{2(\alpha-\frac{\gamma}{2})} \sum_{n\in\mathbb{N}} |\lambda_k|^{\gamma}\,.$$

Thus, the problem of moments (28) would have a solution for every sequence $(m_n) \in h^{\alpha-\frac{\gamma}{2}}$.

REMARK II.8. *It is relatively easy to construct an entire function F vanishing at the elements of the sequence (λ_n) if we have additional information on the numbers λ_n. If there exists $p \in \mathbb{N}$ such that*

$$\sum_{n\in\mathbb{Z}} \frac{1}{|\lambda_n|^p} < \infty,$$

we could take, e.g.,

$$F(z) = \prod_{n\in\mathbb{Z}} \left(\frac{\sin\left(\pi z/\lambda_n\right)}{\pi z/\lambda_n}\right)^p,$$

which is a bounded function for $z \in \mathbb{R}$. To guarantee that the zeros of F are all simple is less easy. However, the truly difficult problem consists in estimating $F'(\lambda_n)$. In [**70**] *and* [**58**] *it may be found wide information on this theme. Good examples of the difficulties involved in the application of this technique are the works* [**32**], [**31**], [**33**].

The following result due to Russell is very useful since it allows to obtain a biorthogonal sequence to the exponential family that appears in connection with the heat equation from a biorthogonal sequence of the family of exponentials of the wave equation. Essentially, this result is contained in [**73**], though we state it in a form similar to that of [**2**, Teorema II.5.20].

Let $(\lambda_n)_{n\in\mathbb{Z}_*}$ be a sequences of real numbers such that $\lambda_{-n} = -\lambda_n$ and $(\varkappa_n)_{n\in\mathbb{Z}_*}$ a symmetric sequence of complex numbers: $\varkappa_{-n} = \varkappa_n$.

THEOREM II.3 (Russell, [**73**]). *If there exists a sequence (v_n) biorthogonal to $\left(\varkappa_n e^{i\lambda_n t}\right)_{n\in\mathbb{Z}_*}$ in $L^2(-A,A)$ then, for every $\varepsilon > 0$ there will exist a sequence (w_n) biorthogonal to $\left(\varkappa_n e^{-\lambda_n^2 t}\right)_{n\in\mathbb{N}}$ in $L^2(-\varepsilon,\varepsilon)$. Besides, there exist positive constants C_ε and γ such that*

$$\|w_n\|_{L^2(-\varepsilon,\varepsilon)} \le C_\varepsilon \|v_n\|_{L^2(-A,A)}\, e^{\gamma|\lambda_n|},$$

for all $n \in \mathbb{N}$.

3.2. Application of the method of moments to the control of the network. Now we will see how a problem of moments arises in a natural way associated to the control problem for the system (11)-(16). This will provide an alternative approach for the study of the controllability of a network. In what follows we will consider for simplicity $r = 1$, that is, the network is controlled from one exterior node.

According to Proposition I.1, the initial state $(\bar{u}_0, \bar{u}_1) \in H \times V'$ is controllable in time T if, and only if, there exists $h \in L^2(0,T)$ such that, for every $(\bar{\phi}_0, \bar{\phi}_1) \in Z \times Z$ the following equality holds

$$\int_0^T h(t)\partial_n \phi^1(t, \mathbf{v}_1)dt = \langle \bar{u}_1, \bar{\phi}_0 \rangle_{V' \times V} - \langle \bar{u}_0, \bar{\phi}_1 \rangle_{H \times H}, \tag{32}$$

where $\bar{\phi}$ is the solution of the homogeneous system (17)-(21) with initial state $(\bar{\phi}_0, \bar{\phi}_1)$.

Let us observe that, if

$$\bar{\phi}_0 = \sum_{n \in \mathbb{N}} \phi_{0,n} \bar{\theta}_n, \qquad \bar{\phi}_1 = \sum_{n \in \mathbb{N}} \phi_{1,n} \bar{\theta}_n$$

then, from formula (23) we have

$$\partial_n \phi^1(t, \mathbf{v}_1) = \sum_{k \in \mathbb{Z}_*} \varkappa_k \left(\phi_{0,k} \cos \lambda_k t + \frac{\phi_{1,k}}{\lambda_k} \sin \lambda_k t \right),$$

where $\varkappa_k = \partial_n \theta_k^1(\mathbf{v}_1)$ is the value of the normal derivative of the eigenfunction $\bar{\theta}_k$ in the controlled node. With this, the condition (32) says that the initial state $\bar{u}_0 = \sum_{k \in \mathbb{N}} u_{0,k} \bar{\theta}_k$, $\bar{u}_1 = \sum_{k \in \mathbb{N}} u_{1,k} \bar{\theta}_k$ is controllable in time T with control h if, and only if, for all the finite sequences $(\phi_{0,k})$, $(\phi_{1,k})$ the following equality is satisfied

$$\int_0^T \varkappa_k \sum_{k \in \mathbb{N}} \left(\phi_{0,k} \cos \lambda_k t + \frac{\phi_{1,k}}{\lambda_k} \sin \lambda_k t \right) h(t)dt = \sum_{k \in \mathbb{N}} \left(u_{1,k} \phi_{0,k} - u_{0,k} \phi_{1,k} \right). \tag{33}$$

By choosing (33) $\phi_{0,k} = 1$, $\phi_{0,k} = 0$ for $k \neq k$ and $\phi_{1,k} = 0$ for every k, what corresponds to the initial data $\bar{\phi}_0 = \bar{\theta}_k$, $\bar{\phi}_1 = \bar{0}$, we will obtain

$$\int_0^T \varkappa_k \cos \lambda_k t \; h(t)dt = u_{1,k}. \tag{34}$$

In an analogous way, with $\phi_{1,k} = \lambda_k$, $\phi_{1,k} = 0$ for $k \neq k$ and $\phi_{0,k} = 0$ for all k,

$$\int_0^T \varkappa_k \sin \lambda_k t \; h(t)dt = -\lambda_k u_{0,k}. \tag{35}$$

Naturally, the relations (34), (35) are necessary for (33) to be satisfied. Besides, they are sufficient. Indeed, if we multiply (34) by $\phi_{0,k}$, (35) by $\phi_{1,k}$ and add over a finite set $I \subset \mathbb{N}$ we obtain

$$\int_0^T \varkappa_k \sum_{k \in I} \left(\phi_{0,k} \cos \lambda_k t + \frac{\phi_{1,k}}{\lambda_k} \sin \lambda_k t \right) h(t)dt = \sum_{k \in I} \left(u_{1,k} \phi_{0,k} - u_{0,k} \phi_{1,k} \right),$$

and this is the equality (33).

Now, combining the equalities (34), (35) it holds

$$\int_0^T \varkappa_k e^{i\lambda_k t} \; h(t)dt = u_{1,k} - i\lambda_k u_{0,k}, \tag{36}$$

$$\int_0^T \varkappa_k e^{-i\lambda_k t} \; h(t)dt = u_{1,k} + i\lambda_k u_{0,k} \quad k \in \mathbb{N}. \tag{37}$$

If we define for $k < 0$, $\lambda_k = -\lambda_{-k}$ then the previous results and (36)-(37) may be unified in

PROPOSITION II.7. *The initial state* $(\bar{u}_0, \bar{u}_1)$ *is controllable in time* T *with control* h *if, and only if, the following equalities are verified*

$$\int_0^T \varkappa_{|k|} e^{i\lambda_k t}\ h(t)dt = u_{1,|k|} - i\lambda_k u_{0,|k|} \quad \text{for every} k \in \mathbb{Z}_*, \tag{38}$$

The equalities (38) constitute a problem of moments for the sequence $\left(\varkappa_{|k|} e^{i\lambda_k t}\right)_{k\in\mathbb{Z}_*}$.

Let us observe that, if h is a real function (what is natural for the system (11)-(16)) any of the relations (36)-(37) implies (34) and (35). The reason to write two equalities consists in the fact that the method, which we will use to solve the problem of moments does not guarantees a priori that the function h is real. However, if we are able to construct a complex function v satisfying (38) then, the real part of v would satisfy (34), (35). Indeed, it suffices to note that (37) may be written as

$$\int_0^T \varkappa_k e^{i\lambda_k t}\ \overline{h(t)} dt = u_{1,k} - i\lambda_k u_{0,k} \quad \text{para } k > 0,$$

from which we obtain, after adding this equality to the first one,

$$\int_0^T \varkappa_k e^{i\lambda_k t} \frac{h(t) + \overline{h(t)}}{2} dt = u_{1,k} - i\lambda_k u_{0,k} \quad \text{para } k > 0.$$

This means, that the real function

$$\hat{h}(t) = \frac{h(t) + \overline{h(t)}}{2}$$

satisfies (34) and (35).

As a consequence of the Proposition II.7 the following characterization of the spectral controllability of the system (11)-(16) is obtained:

PROPOSITION II.8. *The system (I.11)-(I.16) is spectrally controllable in time* T *if, and only if, there exists a sequence* $(v_k)_{k\in\mathbb{Z}_*}$ *biorthogonal to* $\left(\varkappa_{|k|} e^{i\lambda_k t}\right)_{k\in\mathbb{Z}_*}$ *en* $L^2(0,T)$.

PROOF. The fact that the existence of a sequence biorthogonal to $\left(\varkappa_{|k|} e^{i\lambda_k t}\right)_{k\in\mathbb{Z}_*}$ in $L^2(0,T)$ implies the spectral controllability is immediate: the problem of moments (38) would have a solution for any finite sequence $(u_{0,n})$, $(u_{1.n})$ and then, in view of Proposition II.7, all the initial states from $Z \times Z$ would be controllable in time T.

To see that this condition is also necessary, we assume that the system (I.11)-(I.16) is spectrally controllable and construct a sequence biorthogonal to $\left(\varkappa_{|k|} e^{i\lambda_k t}\right)_{k\in\mathbb{Z}_*}$ in $L^2(0,T)$.

For every $m \in \mathbb{N}$, let $g_m, h_m \in L^2(0,T)$ be the controls that correspond to the initial states $(\bar{\theta}_m, \bar{0})$ and $(\bar{0}, \bar{\theta}_m)$, respectively. In such case, according to Proposition II.7, we have the equalities

$$\int_0^T \varkappa_k e^{i\lambda_k t}\ h_m(t)dt = \delta_{|k|}^m, \qquad \int_0^T \varkappa_k e^{i\lambda_k t}\ g_m(t)dt = -i\lambda_k \delta_{|k|}^m,$$

for $m \in \mathbb{N}$, $k \in \mathbb{Z}_*$.

Let us define the functions

$$v_m = \frac{1}{2}\left(h_{|m|} + \frac{i}{\lambda_{|m|}} g_{|m|}\right), \quad m \in \mathbb{Z}_*. \tag{39}$$

We will have

$$\begin{aligned}\int_0^T \varkappa_k e^{i\lambda_k t}\ v_m(t)dt &= \frac{1}{2}\int_0^T \varkappa_k e^{i\lambda_k t}\ h_{|m|}(t)dt + \frac{i}{2\lambda_m}\int_0^T \varkappa_k e^{i\lambda_k t}\ g_{|m|}(t)dt \\ &= \frac{1}{2}\delta_{|k|}^{|m|} + \frac{\lambda_k}{2\lambda_{|m|}}\delta_{|k|}^{|m|} = \delta_k^m.\end{aligned}$$

This means that the sequence $(v_m)_{m\in\mathbb{Z}_*}$ is biorthogonal to $\left(\varkappa_{|k|}e^{i\lambda_k t}\right)_{k\in\mathbb{Z}_*}$. □

If we know subspaces of controllable initial states for system (11)-(16), then it is possible to give more precise information on the biorthogonal sequence constructed in Proposition II.8:

Proposition II.9. *If the subspace $\mathcal{W}^r$ of initial states for the system (I.11)-(I.16) is controllable in time T then there exists a sequence $(v_k)_{k\in\mathbb{Z}_*}$ biorthogonal to $\left(\varkappa_{|k|}e^{i\lambda_k t}\right)_{k\in\mathbb{Z}_*}$ in $L^2(0,T)$, which satisfies*

$$||v_k||_{L^2(0,T)} \le C\lambda_k^{r-1}, \qquad k\in\mathbb{Z}_*,$$

where C is a positive constant independent of k.

Proof. If the subspace $\mathcal{W}^r$ is controllable in time T, there exists a constant $C>0$ such that

$$\int_0^T |\partial_n \phi^1(t,\mathbf{v}_1)|^2 dt \ge C||(\bar\phi_0,\bar\phi_1)||_{V^{1-r}\times V^{-r}}.$$

Then, in view of Corollary II.1, for every $(\bar u_0,\bar u_1)$ there exists $h\in L^2(0,T)$ such that

$$||h||_{L^2(0,T)} \le C||\,(\bar u_0,\bar u_1)\,||_{W^r}.$$

Thus, the functions g_m, h_m constructed in Proposition II.8 satisfy

$$||g_m||_{L^2(0,T)} \le C\lambda_m^r, \qquad ||h_m||_{L^2(0,T)} \le C\lambda_m^{r-1}.$$

Then, from (39) it holds

$$||v_m||_{L^2(0,T)} \le C\lambda_m^{r-1}.$$

Now it suffices to recall that the sequence $(v_m)_{m\in\mathbb{Z}_*}$ is biorthogonal to $\left(\varkappa_{|k|}e^{i\lambda_k t}\right)_{k\in\mathbb{Z}_*}$. □

Remark II.9. *If we perform the change of variable $t\to t-\frac{T}{2}$ we obtain that the assertions of the propositions II.8 and II.9 remain to be valid if we replace the space $L^2(0,T)$ by $L^2(-\frac{T}{2},\frac{T}{2})$.*

Remark II.10. *The numbers $\varkappa_k = \partial_n\theta_k^1(\mathbf{v}_1)$ have a direct incidence in the spectral controllability of the system (I.11)-(I.16). If $\varkappa_k = 0$ for some k then, from (34), (35) it follows that the initial state $(\bar u_0,\bar u_1)$ is controllable only if $u_{0,k}=u_{1,k}=0$, that is, if $\bar u_0$ and $\bar u_1$ are orthogonal to $\bar\theta_k$. In this case, the space of controllable initial states is not dense in $H\times V'$. Consequently, the condition $\varkappa_k\neq 0$ for every $k\in\mathbb{N}$ is necessary for the approximate controllability (and in particular for the spectral) of the system (I.11)-(I.16).*

For the sequence $(|\varkappa_k|)$ an upper bound is easily obtained. If we consider the solutions

$$\bar\phi(t,x) = \cos\lambda_k t\ \bar\theta_k(x), \quad k\in\mathbb{N},$$

of the homogeneous system (I.17)-(I.21) and apply the inequality (I.28) it holds

$$|\varkappa_k|^2 \int_0^T |\cos \lambda_k t|^2 \, dt = \int_0^T \left|\phi^1_{k,x}(t, \mathbf{v}_1)\right|^2 dt \leq C\mathbf{E}_{\bar{\phi}} = C\lambda_k^2.$$

In an analogous way, taking $\bar{\phi}(t,x) = \sin \lambda_k t \; \bar{\theta}_k(x)$ we will have

$$|\varkappa_k|^2 \int_0^T |\sin \lambda_k t|^2 \, dt = \int_0^T \left|\phi^1_{k,x}(t, \mathbf{v}_1)\right|^2 dt \leq C\mathbf{E}_{\bar{\phi}} = C\lambda_k^2.$$

From these two inequalities we see that the sequence $\varkappa_k$ satisfies

$$|\varkappa_k| \leq C\lambda_k, \quad k \in \mathbb{N}. \tag{40}$$

4. Riesz bases and Ingham-type inequalities

In this section we describe the technique for the proof of observability inequalities based on a theorem recently proved by Baiocchi, Komornik and Loreti in [**10**] and Avdonin and Moran in [**5**], which provides a Riesz basis of $L^2(0,T)$ formed by finite linear combinations of complex exponentials $(e^{i\lambda_n t})$. From this result, we obtain in II.7 a useful consequence: if we prove an Ingham-type inequality for the sequence (λ_n), then a similar inequality is true for the sequence (λ_n^s) with $s > 1$.

4.1. Riesz bases. In general, if $\mathbf{H}$ is a separable Hilbert space, the sequence $(\mathbf{a}_n) \subset \mathbf{H}$ is called *Riesz basis of the closure of its linear span* if there exists constants $c_1, c_2 > 0$ such that the following inequality is verified

$$c_1 ||\bar{\gamma}||_{l^2} \leq || \sum_{n \in \mathbb{Z}} \gamma_n \mathbf{a}_n ||_H \leq c_2 ||\bar{\gamma}||_{l^2},$$

for every finite sequence of complex numbers such that $\bar{\gamma} = (\gamma_n)$. In particular, if the sequence $(\mathbf{a}_n)$ is complete in $\mathbf{H}$ it is called *Riesz basis*[4] *of* $\mathbf{H}$.

Thus, to prove observability inequalities (20) it would be very useful to have the information on the fact that the sequence $(e^{i\lambda_n t})$ forms a Riesz basis of $L^2(0,T)$.

Let us observe that, essentially, the technique derived from the use of Riesz bases coincides with the method of moments, since a theorem due to Bari [**12**] asserts that the inequality

$$c_1 ||\bar{\gamma}||_{l^2} \leq || \sum_{n \in \mathbb{Z}} \gamma_n \mathbf{a}_n ||_H$$

is equivalent to the fact that the problem of moments (24) has a solution for any $(m_n) \in l^2$.

4.2. Generalized Ingham theorems. An important theorem due to Ingham [**38**] asserts that the sequence $(e^{i\lambda_n t})$ forms a Riesz basis of the closure of its linear span in $L^2(0,T)$ if the sequence (λ_n) satisfies the separation condition

$$\lambda_{n+1} - \lambda_n \geq \gamma > 0, \tag{41}$$

with $\gamma > \frac{2\pi}{T}$.

A stronger version of this result was given by Beurling in [**16**]: if the sequence (λ_n) satisfies the condition (41), then $(e^{i\lambda_n t})$ forms a Riesz basis in the closure of its linear span in $L^2(0,T)$ for every T satisfying

$$T > 2\pi D^+(\lambda_n),$$

[4]An equivalent definition is that $(\mathbf{a}_n)$ is the image of an orthonormal basis of $\mathbf{H}$ by a continuous bijection. In [**81**] the reader may find more information on this topic.

where $D^+(\lambda_n)$ is the upper density of the sequence (λ_n):

$$D^+(\lambda_n) := \lim_{n\to\infty} \frac{n^+(r,(\lambda_n))}{r},$$

with $n^+(r,(\lambda_n))$ being the maximum number of elements of (λ_n) contained in an interval of length r.

The inequality corresponding to this assertion

$$\text{(I)} \qquad C_1 \|\bar{c}\|_{l^2}^2 \leq \int_0^T \left| \sum_{n\in\mathbb{Z}} c_n e^{i\lambda_n t} \right|^2 dt \leq C_2 \|\bar{c}\|_{l^2}^2 ,$$

is known as *Ingham inequality.* This inequality has been an extremely useful tool in the study of the control problems.

In several concrete problems, however, the separation condition (41) is not verified. This is the case, for example, of the networks of strings (see Proposition III.7). That is why a lot of work has been devoted to obtaining inequalities similar to (I), more precisely, of the type

$$(\mathrm{I}_{\mathbf{B}}) \qquad \int_0^T \left| \sum_{n\in\mathbb{Z}} c_n e^{i\lambda_n t} \right|^2 dt \geq C \|\mathbf{B}\bar{c}\|_{l^2}^2 ,$$

where $\mathbf{B} : l^2 \to l^2$ is a continuous operator, usually with a simple structure, when the elements of the sequence (λ_n) may get close. We refer to the works [**20**], [**21**], [**41**], [**40**], [**5**], [**8**], [**9**], [**10**] for further information.

Let us observe that an inequality of type $(\mathrm{I}_{\mathbf{B}})$ guarantees that the problem of moments has a solution for every $\bar{c}$, which belongs to the image of the adjoint of $\mathbf{B}$. This subspace is necessarily smaller than l^2 if the sequence (λ_n) does not satisfy the separation condition (41), since otherwise $\mathbf{B}$ would have a bounded inverse, what would lead to the Ingham inequality, which is not true in the case of lack of separation.

The most complete result in this direction was simultaneously obtained by Baiocchi, Komornik and Loreti in [**8**], [**9**], [**10**] and Avdonin and Moran in [**5**]. In their papers an inequality like $(\mathrm{I}_{\mathbf{B}})$ is proved for increasing sequences of real numbers (λ_n) with the following generalized separation property:

There exist $\delta > 0$ and a natural number M such that

$$(42) \qquad \lambda_{n+M} - \lambda_n \geq M\delta$$

for every $n \in Z$.

This means that there may be at most M consecutive elements of the sequence (λ_n) that are close; in a larger number there must be some separation.

Remark II.11. *The separation property (42) may be described in an equivalent way in terms of the upper density of the sequence (λ_n). It turns out that, if $T > 2\pi D^+(\lambda_n)$ then there exist $\delta > \frac{2\pi}{T}$ and $M \in \mathbb{N}$ such that (λ_n) satisfies the separation condition (42). The details of the proof may be found in* [**10**].

In order to state the main result of the papers mentioned above and to describe how the operator $\mathbf{B}$ corresponding to this result is constructed, we need some preliminary elements.

Let us fix a sequence (λ_n) satisfying the separation condition (42). We will say that two integer numbers n, m are equivalent if $|\lambda_n - \lambda_m| < |n-m|\,\delta$. This is an equivalence relation in $\mathbb{Z}$. Let us denote by Λ_k, $k \in \mathbb{Z}$, the equivalence classes of $\mathbb{Z}$ with respect to the defined relation. Obviously, every Λ_k is formed by consecutive numbers and contains $d(k) \le M$ elements. We denote by $n(k)$ the smaller of the elements of Λ_k. Besides, we assume that the numbering of the classes has been chosen so that $n(k+1) - 1 \in \Lambda_k$, that is, $n(k+1) = n(k) + d(k)$.

For every $m \in \mathbb{N}$ we pick k such that $m \in \Lambda_k$ and define the function

$$f_m(t) = \sum_{j=n(k)}^{m} \frac{e^{i\lambda_j t}}{\pi_{j,m}},$$

where $\pi_{j,m}$ is the product of all the differences $\lambda_m - \lambda_j$ with $n(k) \le j < m$ if $n(k) < m$ and $\pi_{n(k),n(k)} = 1$. These functions are *called divided differences of the family* $(e^{i\lambda_n t})$, see [**39**], p. 246 or [**79**].

Theorem II.4 (Baiocchi *et al.* [**10**], Avdonin-Moran [**5**]). *For all the values of $\delta > 0$, $M \in \mathbb{N}$ and $T > \frac{2\pi}{\delta}$ if (λ_n) satisfies the separation condition (42), then the sequence (f_n) forms a Riesz basis in the closure of its linear span in $L^2(0,T)$.*

The following result is also proved in [**5**]. It allows to clarify what happens when the value of T is not sufficiently large.

Theorem II.5. *If the sequence (λ_n) satisfies the separation condition (42) and $T < 2\pi D^+$ then there exists a proper subsequence $(\hat{n}) \subset \mathbb{Z}$ such that $(f_{\hat{n}})$ forms a Riesz basis in $L^2(0,T)$.*

As a consequence, after applying Theorem III.3.10(e) from [**2**], it holds

Corollary II.4. *If the sequence (λ_n) satisfies the condition $D^+(\lambda_n) < \infty$ then, for every $T < 2\pi D^+(\lambda_n)$, there exist complex numbers c_n, not all of then equal to zero , such that*

$$\sum_{n\in\mathbb{Z}} |c_n|^2 < \infty, \qquad \sum_{n\in\mathbb{Z}} c_n e^{i\lambda_n t} = 0 \qquad \text{en } L^2(0,T).$$

In what follows, we write the result of Theorem II.4 as an equivalent inequality of type $(\mathrm{I}_{\mathbf{B}})$.

Let $\bar{\mathbf{e}}^n$, $n \in \mathbb{Z}$, be the canonical basis of l^2 and consider the subspaces

$$\mathbf{L}_k = \operatorname{span}_{n\in\Lambda_k} (\bar{\mathbf{e}}^n).$$

Each subspace $\mathbf{L}_k$ has finite dimension $d(k) \le M$. Then, l^2 is decomposed as

$$l^2 = \bigoplus_{k\in\mathbb{Z}} \mathbf{L}_k.$$

Let $m \in \mathbb{N}$. For every $\bar{h} = (h_1, ..., h_m) \in \mathbb{R}^m$, we define the operators $\mathbf{A}_m(\bar{h}) : \mathbb{R}^m \to \mathbb{R}^m$ by $\mathbf{A}_m(\bar{h})\bar{x} = A_m(\bar{h})\bar{x}$, where $A_m(\bar{h})$ is the matrix with components

$$\mathbf{A}_{m,ij}(\bar{h}) = \begin{cases} \prod_{k=1}^{\prime j} (h_i - h_k)^{-1} & \text{if } i \le j, \\ 1 & \text{if } i = j = 1, \\ 0 & \text{if } i > j. \end{cases} \tag{43}$$

where the symbol $'$ in the product sign indicates that the factor corresponding to $k = i$ has been excluded.

These matrices are invertible if all the numbers h_j are pairwise distinct. Now we take

$$\mathbf{B}_k = \Big(\mathbf{A}_{d(k)}(\lambda_{n(k)}, ..., \lambda_{n(k)+d(k)-1})\Big)^{-1}.$$

Finally, the operator $\mathbf{B}$ is defined for $\bar{\mathbf{v}} = \sum_{k\in\mathbb{Z}} \bar{\mathbf{v}}_k$ as

$$\mathbf{B}\bar{\mathbf{v}} = \sum_{k\in\mathbb{Z}} \mathbf{B}_k \bar{\mathbf{v}}_k,$$

where $\bar{\mathbf{v}}_k$ is the projection of $\bar{\mathbf{v}}$ over $\mathbf{L}_k$. This is the operator that appears in the inequality ($\mathrm{I}_\mathbf{B}$) corresponding to the assertion of Theorem II.4.

THEOREM II.6. *For all $\delta > 0$, $M \in \mathbb{N}$ and $T > \frac{2\pi}{\delta}$, there exist constants $C_1, C_2 > 0$ such that, if the sequence (λ_n) satisfies the separation condition (42) then*

$$C_1 \left\|\mathbf{B}\bar{c}\right\|_{l^2}^2 \geq \int_0^T \left|\sum_{n\in\mathbb{Z}} c_n e^{i\lambda_n t}\right|^2 dt \geq C_2 \left\|\mathbf{B}\bar{c}\right\|_{l^2}^2, \tag{$\mathrm{I}_\mathbf{B}$}$$

for every finite sequence $\bar{c}$.

We should remark that the operator $\mathbf{B}$ has a structure that makes it easy to obtain information from the inequality ($\mathrm{I}_\mathbf{B}$). According to its definition we have

$$\begin{aligned}
\left\|\mathbf{B}\bar{c}\right\|_{l^2}^2 &= \sum_{k\in\mathbb{Z}} \left\|\mathbf{B}_k \bar{c}_k\right\|_{l^2}^2 = \sum_{k\in\mathbb{Z}} \left\|\Big(\mathbf{A}_{d(k)}(\lambda_{n(k)}, ..., \lambda_{n(k)+d(k)-1})\Big)^{-1} \bar{c}_k\right\|_{l^2}^2 \\
&\geq \sum_{k\in\mathbb{Z}} \gamma_k^2 \left\|\bar{c}_k\right\|_{l^2}^2,
\end{aligned}$$

where

$$\gamma_k = \left\|\Big(\mathbf{A}_{d(k)}(\lambda_{n(k)}, ..., \lambda_{n(k)+d(k)-1})\Big)\right\|^{-1}.$$

Taking into account that

$$\left\|\bar{c}_k\right\|_{l^2}^2 = \sum_{n=n(k)}^{n(k)+d(k)-1} \left|c_n\right|^2,$$

from the inequality (42) it holds

$$\int_0^T \left|\sum_{n\in\mathbb{Z}} c_n e^{i\lambda_n t}\right|^2 dt \geq C_2 \sum_{k\in\mathbb{Z}} \gamma_k^2 \sum_{n=n(k)}^{n(k)+d(k)-1} \left|c_n\right|^2.$$

It simply says that it is sufficient to choose weights γ_k^2 in the coefficients corresponding to $n \in \Lambda_k$. Thus, we have obtained

COROLLARY II.5. *If the strictly increasing sequence (λ_n) satisfies the separation condition (42) or equivalently, $D^+(\lambda_n) < \infty$ then, for every $T > 2\pi D^+(\lambda_n)$ there exists positive numbers γ_n, such that*

$$\int_0^T \left|\sum_{n\in\mathbb{Z}} c_n e^{i\lambda_n t}\right|^2 dt \geq \sum_{n\in\mathbb{Z}} \gamma_n^2 \left|c_n\right|^2,$$

for every finite sequence (c_n).

It is possible to make more precise estimations of $\left\|\mathbf{A}_{d(k)}^{-1}\bar{c}_k\right\|_{l^2}^2$ leading to inequalities with weights, which vary inside of every group Λ_k. It depends on the particular structure of the operator $\mathbf{A}_m$, and of course, of the sequence (λ_n).

4.3. A new inequality. The following result turns to be very useful for the identification of subspaces of controllable initial states for the Schrödinger and beams equations if we know subspaces of controllable initial states for the system (11)-(16). This result will be used in Chapter VII.

THEOREM II.7. *Let (λ_n) be an increasing sequence of positive numbers with upper density $D^+(\lambda_n) < \infty$. Assume that there exist constants $C > 0$ and $\alpha < 0$ such that the inequality*

$$\int_0^T \left|\sum_{n\in\mathbb{Z}} c_n e^{i\lambda_n t}\right|^2 dt \geq C \sum_{n\in\mathbb{Z}} \lambda_n^{2\alpha} c_n^2, \tag{44}$$

is verified for every finite sequence $\bar{c}$. Then, for all $\tau > 0$ and $s > 1$, there exists a constant $C_1 > 0$ such that

$$\int_0^\tau \left|\sum_{n\in\mathbb{Z}} c_n e^{i\lambda_n^s t}\right|^2 dt \geq C_1 \sum_{n\in\mathbb{Z}} \lambda_n^{2\alpha} c_n^2,$$

for every finite sequence $\bar{c}$.

The proof of this assertion is based on Theorem II.6. We will need the following technical results.

PROPOSITION II.10. *Let $K : \mathbb{R}^m \to \mathbb{R}^m$ be a linear operator defined by the matrix $A=(a_{ij})$. If $\|A\|$ is the norm of A considered as a linear operator from $l^2(\mathbb{R}^m)$ to $l^2(\mathbb{R}^m)$ then*

$$\max_{i,j=1,\ldots,m} |a_{ij}| \leq \|A\| \leq \sqrt{m} \max_{i,j=1,\ldots,m} |a_{ij}|.$$

This fact is easily proved with the aid of Schur's Lemma or directly using the Cauchy-Schwarz inequality (see, e.g., [**36**], Theorem 3.4.7).

PROPOSITION II.11. *Let $\mathbf{A}_m(\bar{h}) : \mathbb{R}^m \to \mathbb{R}^m$ be defined by (43) and assume that $1 < h_1 \leq h_2 \leq \cdots \leq h_m$. If $\bar{h}^s = (h_1^s, ..., h_m^s)$ ($\bar{h}^s$ is formed by the s-powers of the components of $\bar{h}$). Then, for every $s > 1$,*

$$\left\|\mathbf{A}_m(\bar{h}^s)\right\| \leq \sqrt{m}\left\|\mathbf{A}_m(\bar{h})\right\|. \tag{45}$$

PROOF. Let us observe that

$$\left|h_i^s - h_j^s\right| \geq s\left|h_i - h_j\right| h_1^{s-1}$$

and thus, from the definition of $\mathbf{A}_{m,ij}(\bar{h}^s)$ we obtain

$$\left|\mathbf{A}_{m,ij}(\bar{h}^s)\right| \leq \left|\mathbf{A}_{m,ij}(\bar{h})\right| h_1^{(s-1)(j-1)} s^{(j-1)}.$$

Now, using Proposition II.10,

$$\begin{aligned}\left\|\mathbf{A}_m(\bar{h}^s)\right\| &\leq \sqrt{m}\max_{i,j=1,\ldots,m}\left|\mathbf{A}_{m,ij}(\bar{h}^s)\right| \leq \sqrt{m}\max_{i,j=1,\ldots,m}\left|\mathbf{A}_{m,ij}(\bar{h})\right| h_1^{(s-1)(j-1)} s^{(j-1)} \\ &\leq \sqrt{m}\max_{i,j=1,\ldots,m}\left|\mathbf{A}_{m,ij}(\bar{h})\right| \max_{j=1,\ldots,m} h_1^{(s-1)(1-j)} s^{(1-j)}.\end{aligned}$$

Taking into account that $h_1 > 1$ it follows

$$\max_{j=1,\dots,m} h_1^{(s-1)(1-j)} s^{(1-j)} = 1$$

and then

$$\left\| \mathbf{A}_m(\bar{h}^s) \right\| \leq \sqrt{m} \max_{i,j=1,\dots,m} \left| \mathbf{A}_{m,ij}(\bar{h}) \right|.$$

Applying once again Proposition II.10, inequality (45) is obtained. $\square$

PROOF OF THEOREM II.7. Let us choose $\delta < \frac{1}{D^+}$ and $T > \frac{2\pi}{\delta}$. According to II.6, there exists a constant $C_1 > 0$ such that

$$C_1 \left\| \mathbf{B}\bar{c} \right\|_{l^2}^2 \geq \int_0^T \left| \sum_{n\in\mathbb{Z}} c_n e^{i\lambda_n t} \right|^2.$$

In account of the hypothesis (44) of the lemma and the fact

$$\left\| \mathbf{B}\bar{c} \right\|_{l^2}^2 = \sum_{k\in\mathbb{Z}} \left\| \left(\mathbf{A}_{d(k)}(\lambda_{n(k)}, \dots, \lambda_{n(k)+d(k)-1}) \right)^{-1} \bar{c}_k \right\|_{l^2}^2,$$

we obtain

$$\left\| \left(\mathbf{A}_{d(k)}(\lambda_{n(k)}, \dots, \lambda_{n(k)+d(k)-1}) \right)^{-1} \bar{c}_k \right\|_{l^2}^2 \geq C \sum_{n\in\Lambda_k} \lambda_n^{2\alpha} c_n^2,$$

for all $\bar{c}_k \in \mathbb{R}^{d(k)}$. Since the sequence (λ_n) is increasing, this inequality implies

$$\left\| \left(\mathbf{A}_{d(k)}(\lambda_{n(k)}, \dots, \lambda_{n(k)+d(k)-1}) \right)^{-1} \bar{c}_k \right\|_{l^2}^2 \geq C \lambda_{n(k)+d(k)-1}^{2\alpha} \sum_{n\in\Lambda_k} c_n^2.$$

Thus, we can conclude that

$$\left\| \mathbf{A}_{d(k)}(\lambda_{n(k)}, \dots, \lambda_{n(k)+d(k)-1}) \right\| \leq C \lambda_{n(k)+d(k)-1}^{-\alpha}.$$

Now, Proposition II.11 allows us to ensure that

$$\left\| \mathbf{A}_{d(k)}(\lambda_{n(k)}^s, \dots, \lambda_{n(k)+d(k)-1}^s) \right\| \leq C \lambda_{n(k)+d(k)-1}^{-\alpha}. \tag{46}$$

Let $\tau > 0$ and choose n_0 such that $\delta' := \delta s \lambda_{n_0}^{s-1} > \frac{2\pi}{\tau}$. Then, for every $n \geq n_0$ it holds

$$\lambda_{n+M}^s - \lambda_n^s > s\left(\lambda_{n+M} - \lambda_n\right) \lambda_n^{s-1} \geq M\delta s \lambda_n^{s-1} = M\delta'.$$

In particular, every set of the partition (Λ_k^s) of $\mathbb{Z}$, defined for the sequence (λ_n^s) for the value δ', which contains and element $n \geq n_0$, will be contained in one of the sets Λ_k.

Once again from Theorem II.6 we obtain that for every finite sequence $\bar{c}$,

$$\int_0^T \left| \sum_{n\in\mathbb{Z}} c_n e^{i\lambda_n^s t} \right|^2 dt \geq C_2 \left\| \mathbf{B}^s \bar{c} \right\|_{l^2}^2. \tag{47}$$

Here, the operator $\mathbf{B}^s$ corresponds to the sequence (λ_n^s) and to δ', that is, to the partition (Λ_k^s).

It is possible to prove (see Lemma 3.1 in [**10**]) that, if $\mathbf{B}_\delta$ and $\mathbf{B}_{\delta'}$ are the operators defined by (43) for the partitions generated by δ and δ', respectively, then there exist constants $d_1, d_2 > 0$, depending only on δ and δ', such that

$$d_1 \left\| \mathbf{B}_\delta \bar{c} \right\|_{l^2}^2 \leq \left\| \mathbf{B}_{\delta'} \bar{c} \right\|_{l^2}^2 \leq d_2 \left\| \mathbf{B}_\delta \bar{c} \right\|_{l^2}^2,$$

for every finite sequence $\bar{c}$.

Thus, we may assume that the operator $\mathbf{B}^s$ has been constructed for the partition (Λ_k).

Now it suffices to note that

$$\|\mathbf{B}^s \bar{c}\|_{l^2}^2 \geq \sum_{k\in\mathbb{Z}} \left(\left\| \mathbf{A}_{d(k)}(\lambda_{n(k)}, ..., \lambda_{n(k)+d(k)-1}) \right\|_{l^2}^{-1} \right)^2 \|\bar{c}_k\|_{l^2}^2$$

and to use the inequality (46) to conclude that

$$\int_0^T \left| \sum_{n\in\mathbb{Z}} c_n e^{i\lambda_n^s t} \right|^2 dt \geq C_2 \sum_{k\in\mathbb{Z}} \lambda_{n(k)+d(k)-1}^{2\alpha} \|\bar{c}_k\|_{l^2}^2 .$$

Finally, let us observe that there exists a constant $C > 0$ such that for every n satisfying $n(k) \leq n \leq n(k) + d(k) - 1$, it holds

$$\lambda_{n(k)+d(k)-1} \leq C\lambda_n.$$

This concludes the proof. $\square$

CHAPTER III

The three string network

This chapter is devoted to the study of the control problem for the simplest of non trivial networks of strings[1]: the three string network. This chapter has mainly a didactic intension. The most of the results presented here will be generalized later in Chapter IV for the case of networks supported by tree-shaped graphs. However, the generality of the problem in that case, involves complex notations, unavoidable if the take into account the need of referring in a precise sense to each of the multiple elements, which form a network. This cause the methods we use, which are essentially simple, to appear hidden behind the notations. That is why we have try to describe the fundamental ideas in a simple context, paying attention in those aspects that will allow to extend the technique to the general framework of tree-shaped networks.

1. The three string network with two controlled nodes

1.1. Equations of the motion of the network. Let T, ℓ_0, ℓ_1, ℓ_2 be positive numbers. We consider the following non-homogeneous system

$$(1)\qquad \begin{cases} u^i_{tt} - u^i_{xx} = 0 & \text{in } \mathbb{R}\times[0,\ell_i] \quad i=0,1,2,\\ u^0(t,0) = u^1(t,0) = u^2(t,0) & t \in \mathbb{R}\\ u^0_x(t,0) + u^1_x(t,0) + u^2_x(t,0) = 0 & t\in\mathbb{R}\\ u^i(t,\ell_0) = v^i(t), \quad u^2(t,\ell_i) = 0 & t\in\mathbb{R} \qquad i=0,1,\\ u^i(0,x) = u^i_0(x), \quad u^i_t(0,x) = u^i_1(x) & x\in[0,\ell_i] \qquad i=0,1,2. \end{cases}$$

which models the vibrations of a network formed by three elastic strings $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2$ with lengths ℓ_0, ℓ_1, ℓ_2 coupled at one of their extremes. The functions $u^i = u^i(t,x) : [0,\ell_i] \to \mathbb{R}$, $i = 0,1,2$, represent the transversal displacement of the strings. On the free nodes of the strings $\mathbf{e}_0$ and $\mathbf{e}_1$ some external controls v^0 and v^1 act to regulate the motion of those points. Let us observe that in (1), the parametrization of the strings has been chosen so that the points $x = 0$ correspond to the common node, while $x = \ell_i$, to the exterior node of the string $\mathbf{e}_i$.

Sea $T > 0$. According to the general results described in Chapter I, the homogeneous system (1) ($v^0 = v^1 = 0$)

$$(2)\qquad \begin{cases} \phi^i_{tt} - \phi^i_{xx} = 0 & \text{in } \mathbb{R}\times[0,\ell_i] \quad i=0,1,2,\\ \phi^0(t,0) = \phi^1(t,0) = \phi^2(t,0) & t \in \mathbb{R}\\ \phi^0_x(t,0) + \phi^1_x(t,0) + \phi^2_x(t,0) = 0 & t\in\mathbb{R}\\ \phi^i(t,\ell_0) = 0 & t\in\mathbb{R} \qquad i=0,1,2,\\ \phi^i(0,x) = \phi^i_0(x), \quad \phi^i_t(0,x) = \phi^i_1(x) & x\in[0,\ell_i] \qquad i=0,1,2, \end{cases}$$

[1]With this, we want to indicate that it is the simplest of those networks, which are not reduced to a single string.

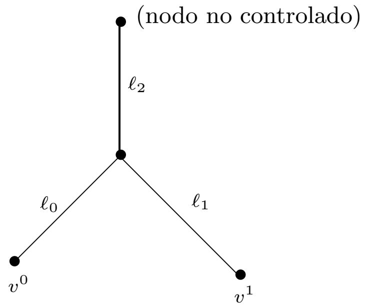

FIGURE 1. The three string network with two controlled nodes

has a unique solution $\bar{\phi}$ with initial state $(\bar{\phi}_0, \bar{\phi}_1) \in V \times H$ satisfying

$$\bar{\phi} \in C([0,T] : V) \bigcap C^1([0,T] : H). \tag{3}$$

This solution is expressed in terms of the Fourier coefficients $(\phi_{0,n})$, $(\phi_{1,n})$ of the initial data in the orthonormal basis $(\bar{\theta}_n)$ formed by the eigenfunctions of the elliptic operator $-\Delta_G$ associated to (1) by the formula

$$\bar{\phi}(t,x) = \sum_{n\in\mathbb{N}} (\phi_{0,n} \cos \lambda_n t + \frac{\phi_{1,n}}{\lambda_n} \sin \lambda_n t) \bar{\theta}_n(x). \tag{4}$$

The energy of $\bar{\phi}$ is constant in time; it may be computed by the relation

$$\mathbf{E}_{\bar{\phi}} = \sum_{n\in\mathbb{N}} (\mu_n \phi_{0,n}^2 + \phi_{1,n}^2). \tag{5}$$

For the non-homogeneous system, for every $v^0, v^1 \in L^2(0,T)$, there exists a solution of (31), defined by transposition that satisfies

$$\bar{u} \in C([0,T] : H) \bigcap C^1([0,T] : V'),$$

which will be the unique solution of (1) having the latter property.

1.2. The control problem for the three string network. The control problem in time T for the three string network defined by system (1) consists in *characterizing the initial states $(\bar{u}_0, \bar{u}_1) \in H \times V'$ of the network for which there exist controls $v^0, v^1 \in L^2(0,T)$ such that the corresponding solution of* (1) *satisfies*

$$\bar{u}(T) = \bar{u}_t(T) = \bar{0}.$$

The control of a three string network from two exterior nodes satisfies the hypotheses of Theorem I.1. In this case it holds

THEOREM III.1. *The system (1) is exactly controllable in time $T^* = 2(\ell_2 + \max\{\ell_0, \ell_1\})$.*

PROOF. Let us assume that $\ell_0 > \ell_1$, such that $T^* = 2(\ell_0 + \ell_2)$. In view of Proposition I.1, the initial state $(\bar{u}_0, \bar{u}_1) \in H \times V'$ is controllable in time T with controls $v^0, v^1 \in L^2(0,T)$ if, and only if,

$$\int_0^{T^*} \phi_x^0(t, \ell_0) v^0(t) dt + \int_0^{T^*} \phi_x^1(t, \ell_1) v^1(t) dt = \langle \bar{u}_0, \bar{\phi}_1 \rangle_H - \langle \bar{u}_1, \bar{\phi}_0 \rangle_{V' \times V},$$

for every solution $\bar{\phi}$ of the system (2) with initial state $(\bar{\phi}_0, \bar{\phi}_1) \in Z \times Z$. Corollary II.3 of Theorem II.1 allows us to ensure that the system (1) is exactly controllable in time T if, and only if, there exists a constant $C > 0$ such that

$$\int_0^{T^*} |\phi_x^0(t, \ell_0)|^2 dt + \int_0^{T^*} |\phi_x^1(t, \ell_1)|^2 dt \geq C\mathbf{E}_{\bar{\phi}}, \tag{6}$$

for every solution $\bar{\phi}$ of the homogeneous system (2) with initial state in $Z \times Z$.

In order to prove the inequality (6), it suffices to find $\hat{t} \in \mathbb{R}$ such that

$$\int_0^{2(\ell_0+\ell_2)} \left(\left|\phi_x^0(t, \ell_1)\right|^2 + \left|\phi_x^1(t, \ell_2)\right|^2 \right) dt \geq C\mathbf{E}_{\phi^i}(\hat{t}), \qquad i = 0, 1, 2. \tag{7}$$

Thanks to Proposition II.1 it is immediate for $i = 0, 1$ (that is, for the components of the solution corresponding to the controlled strings) if $\hat{t} \in [\ell_0, 2\ell_2 + \ell_0]$. For $i = 2$ the idea is simple: the D'Alembert formula allows to express

$$\phi_x^0(t, 0) = \ell_0^+ \phi_x^0(t, \ell_0), \qquad \phi_t^0(t, 0) = \ell_0^- \phi_x^0(t, \ell_0), \tag{8}$$

$$\phi_x^1(t, 0) = \ell_1^+ \phi_x^1(t, \ell_1), \qquad \phi_t^1(t, 0) = \ell_1^- \phi_x^1(t, \ell_1). \tag{9}$$

In account of the transmission conditions in the common node

$$\begin{aligned} \phi_t^2(t, 0) &= \phi_t^1(t, 0) = \ell_1^- \phi_x^1(t, \ell_1), \\ \phi_x^2(t, 0) &= -\left(\phi_x^0(t, 0) + \phi_x^1(t, 0)\right) = \left(\ell_0^+ \phi_x^0(t, \ell_0) + \ell_1^+ \phi_x^1(t, \ell_1)\right). \end{aligned}$$

Then, according to Proposition II.1,

$$\begin{aligned} \mathbf{E}_{\phi^2}(\hat{t}) &\leq \int_{\hat{t}-\ell_2}^{\hat{t}+\ell_2} \left(\left|\phi_t^2(t, 0)\right|^2 + \left|\phi_x^2(t, 0)\right|^2 \right) dt \\ &= \int_{\hat{t}-\ell_2}^{\hat{t}+\ell_2} \left(\left|\ell_1^- \phi_x^1(t, \ell_1)\right|^2 + \left|\ell_0^+ \phi_x^0(t, \ell_0) + \ell_1^+ \phi_x^1(t, \ell_1)\right|^2 \right) dt. \end{aligned}$$

From this inequality and applying Proposition II.2 we obtain

$$C\mathbf{E}_{\phi^2}(\hat{t}) \leq \int_{\hat{t}-\ell_2-\ell_1}^{\hat{t}+\ell_2+\ell_1} \left|\phi_x^1(t, \ell_1)\right|^2 dt + \int_{\hat{t}-\ell_0-\ell_2}^{\hat{t}+\ell_0+\ell_2} \left|\phi_x^0(t, \ell_0)\right|^2 dt,$$

and thus, choosing $\hat{t} = \ell_0 + \ell_2$, the inequalities (7) will be verified. □

Remark III.1. *It is clear that the same procedure would work in the case of a general tree-shaped network controlled from all of its exterior nodes, except one: it suffices to apply an induction argument. The application of the D'Alembert formula and Proposition II.1 allows to estimate the norms*

$$\int_{-\alpha-\ell}^{\alpha+\ell} \left|\phi_x(t, \ell)\right|^2 dt, \qquad \int_{-\alpha-\ell}^{\alpha+\ell} \left|\phi_t(t, \ell)\right|^2 dt$$

of the traces ϕ_x and ϕ_t in the extreme $x = \ell$ from the norms

$$\int_{-\alpha}^{\alpha} \left|\phi_x(t, 0)\right|^2 dt, \qquad \int_{-\alpha}^{\alpha} \left|\phi_t(t, 0)\right|^2 dt$$

of the traces ϕ_x and ϕ_t in the extreme $x = 0$. Thus, if we start from the controlled nodes, when we arrive to an interior node we will have estimates of the traces ϕ_x and ϕ_t of all the components that are coupled in that node, except of one of them. The coupling conditions allows then to obtain estimations for the traces of those *components, and this would make possible to continue until we reach the string containing the uncontrolled node.*

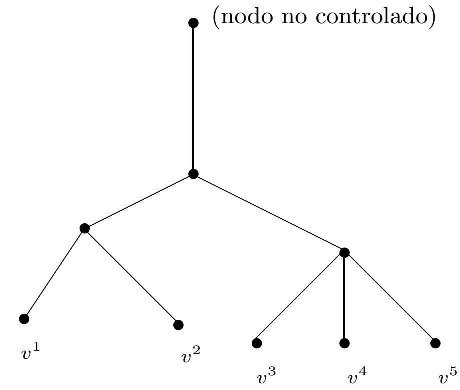

FIGURE 2. Tree-shaped network with one uncontrolled node

2. A simpler problem: simultaneous control of two strings

A control problem similar to that of the three string network is the one of the two strings $\mathbf{e}_1$ and $\mathbf{e}_2$ of lengths ℓ_1 and ℓ_2, which are simultaneously controlled from one of their ends. This problem was implicitly studied in [**41**]. Later, in [**78**] and [**7**] an essentially complete solution was obtained. The results of [**78**] are based on a generalization of the Ingham inequality proved in [**41**]. This technique, however, allowed only to guarantee the controllability of the system in a time larger than the one, which is really necessary. In [**7**] the method of moments was used; this methods provided the optimal control time. Here we describe a different solution, based on completely elementary arguments, which in addition provides more information than the other mentioned techniques.

The system corresponding to the simultaneous control of two strings is

$$\begin{cases} u^i_{tt} - u^i_{xx} = 0 & (t,x) \in \mathbb{R}\times[0,\ell_i], \\ u^i(t,\ell_i) = 0, \quad u^i(t,0) = v(t) & t \in \mathbb{R}, \\ u^i(0,x) = u^i_0(x), \quad u^i_t(0,x) = u^i_1(x) & x \in [0,\ell_i], \end{cases} \tag{10}$$

for $i = 1, 2$. In this case *simultaneous* refers to the fact that the control v applied to both strings is the same. To this sort of problems, considered for the first time by Russell in [**75**], is devoted the chapter 5 of [**60**].

For every $T > 0$ the system (10) is well posed for initial states from $(u^i_0, u^i_1) \in L^2(0,\ell_i) \times H^{-1}(0,\ell_i)$, $i = 1, 2$ and $v \in L^2(0,T)$: there exists a unique solution that satisfies

$$u^i \in C([0,T] : L^2(0,\ell_i)) \cap C^1([0,T] : H^{-1}(0,\ell_i)), \quad i = 1, 2.$$

When $v \equiv 0$ the system (10) becomes

$$\begin{cases} \phi^i_{tt} - \phi^i_{xx} = 0 & (t,x) \in \mathbb{R}\times[0,\ell_i], \\ \phi^i(t,\ell_i) = \phi^i(t,0) = 0 & t \in \mathbb{R}, \\ \phi^i(0,x) = \phi^i_0(x), \quad \phi^i_t(0,x) = \phi^i_1(x) & x \in [0,\ell_i], \end{cases} \tag{11}$$

with $i = 1,2$. Let us observe that the system (11) is formed by two wave equations with homogeneous Dirichlet boundary conditions, which are *uncoupled*. Both equations are also well posed for $(\phi^i_0, \phi^i_1) \in H^1_0(0,\ell_i) \times L^2(0,\ell_i)$ and the corresponding solutions are expressed by the formula

$$\phi^i(t,x) = \sum_{n\in\mathbb{N}} (\phi^i_{0,n} \cos\sigma^i_n t + \frac{\phi^1_{1,n}}{\sigma^i_n} \sin\sigma^i_n t) \sin\sigma^i_n x, \quad i = 1,2, \tag{12}$$

where (σ^i_n) is the sequence formed by the square roots of the eigenvalues of the string $\mathbf{e}_i$:

$$\sigma^i_n = \frac{n\pi}{\ell_i}, \quad n \in \mathbb{N},$$

and $(\phi^i_{0,n})$, $(\phi^i_{1,n})$ are the sequences of the Fourier coefficients of ϕ^i_0, ϕ^i_1, respectively, in the orthonormal basis $(\sin\sigma^i_n x)$ of $L^2(0,\ell_i)$:

$$\phi^i_0(x) = \sum_{n\in\mathbb{N}} \phi^i_{0,n} \sin\sigma^i_n x, \qquad \phi^i_1(x) = \sum_{n\in\mathbb{N}} \phi^i_{1,n} \sin\sigma^i_n x, \qquad i = 1,2.$$

The control problem in time T consists in *characterizing the initial states*(u^i_0, u^i_1), $i = 1,2$, *of the system* (10) *such that there exists* $v \in L^2(0,T)$ *with the property that the solutions* u^1, u^2 *of* (10) *satisfy*

$$u^i(T,x) = u^i_t(T,x) = 0, \qquad i = 1,2,$$

for $x \in [0,\ell_i]$.

Let us observe that, though the system (11) is formed by two *uncoupled* equations, the fact that the same control is used generates coupling conditions, similar to those in the three string network. In fact, if we apply HUM, it turns out that the observability inequality associated to (10) is

$$\int_0^T |\phi^1_x(t,0) + \phi^2_x(t,0)|^2 dt \geq \sum_{i=1,2} \sum_{n\in\mathbb{N}} (c^i_n)^2 ((\sigma^i_n \phi^i_{0,n})^2 + (\phi^i_{1,n})^2). \tag{13}$$

If there exists sequences of positive numbers (c^i_n), $i = 1,2$ such that (13) is verified by all the solutions ϕ^1, ϕ^2 of (11) with initial states $(\phi^1_0, \phi^1_1) \in Z^1 \times Z^1$, $(\phi^2_0, \phi^2_1) \in Z^2 \times Z^2$, respectively, then, all the initial states (u^i_0, u^i_1), $i = 1,2$, satisfying

$$\sum_{n\in\mathbb{N}} \frac{1}{(c^i_n)^2} (u^i_{0,n})^2 + \sum_{n\in\mathbb{N}} \frac{1}{(\sigma^i_n c^i_n)^2} (u^i_{1,n})^2 < \infty$$

are controllable in time T.

Our main observability result for the solutions of (11) is

THEOREM III.2. *Let $T^* = 2(\ell_1 + \ell_2)$. The following inequalities take place*

$$\int_0^{T^*} |\phi_x^1(t,0) + \phi_x^2(t,0)|^2 dt \quad \geq \quad \ell_1 \sum_{n\in\mathbb{N}} \left(\sin \sigma_n^1 \ell_2\right)^2 \left((\sigma_n^1 \phi_{0,n}^1)^2 + (\phi_{1,n}^1)^2\right),$$

$$\int_0^{T^*} |\phi_x^1(t,0) + \phi_x^2(t,0)|^2 dt \quad \geq \quad \ell_2 \sum_{n\in\mathbb{N}} \left(\sin \sigma_n^2 \ell_1\right)^2 \left((\sigma_n^2 \phi_{0,n}^2)^2 + (\phi_{1,n}^2)^2\right),$$

for any solution of (11) with initial states $(\phi_0^i, \phi_1^i) \in Z^i \times Z^i$.

PROOF. We will proved the second inequality; the first one is proved in a similar way.

Let us observe that, due to the $2\ell_1$-periodicity in time of the solutions of (11) it follows $\ell_1^- \phi_x^1(t,0) = 0$, where ℓ_1^- is the operator defined by (II.7) corresponding to the number ℓ_1. Then, if we apply Proposition II.2 we obtain

$$\begin{aligned}
\int_0^{T^*} |\phi_x^1(t,0) + \phi_x^2(t,0)|^2 dt \quad &\geq \quad \int_{\ell_1}^{T^*-\ell_1} |\ell_1^- \phi_x^1(t,0) + \ell_1^- \phi_x^2(t,0)|^2 dt \\
&= \quad \int_{\ell_1}^{T^*-\ell_1} |\ell_1^- \phi_x^2(t,0)|^2 dt. \qquad (14)
\end{aligned}$$

On the other hand, $\psi = \ell_1^- \phi^2$ is a solution of the equation

$$\psi_{tt} - \psi_{xx} = 0$$

in $\mathbb{R} \times [0, \ell_1]$ and thus, from Proposition II.1 it results

$$\int_{\ell_1}^{T^*-\ell_1} |\psi_x(t,0)|^2 dt \geq 4\mathbf{E}_\psi. \tag{15}$$

Taking into account that $\psi_x(t,0) = \ell_1^- \phi_x^2(t,0)$, from (14) and (15) we obtain

$$\int_0^{T^*} |\phi_x^1(t,0) + \phi_x^2(t,0)|^2 dt \geq 4\mathbf{E}_{\ell_1^- \phi^2}. \tag{16}$$

It remains just to calculate the energy $\mathbf{E}_{\ell_1^- \phi^2}$. From the formula (12) we obtain that

$$\ell_1^- \phi^2(t,x) = \sum_{n\in\mathbb{N}} (\phi_{0,n}^2 \ell_1^- \cos \sigma_n^2 t + \frac{\phi_{1,n}^2}{\sigma_n^2} \ell_1^- \sin \sigma_n^2 t) \sin \sigma_n^2 x. \tag{17}$$

In view of the relations

$$\ell_1^- \cos \sigma_n^2 t = \frac{1}{2}\left(\cos \sigma_n^2 (t + \ell_1) - \cos \sigma_n^2 (t - \ell_1)\right) = -\sin \sigma_n^2 \ell_1 \sin \sigma_n^2 t,$$

$$\ell_1^- \sin \sigma_n^2 t = \frac{1}{2}\left(\sin \sigma_n^2 (t + \ell_1) - \sin \sigma_n^2 (t - \ell_1)\right) = \sin \sigma_n^2 \ell_1 \cos \sigma_n^2 t,$$

the equality (17) becomes

$$\ell_1^- \phi^2(t,x) = \sum_{n\in\mathbb{N}} \sin \sigma_n^2 \ell_1 (\frac{\phi_{1,n}^2}{\sigma_n^2} \ell_1^- \cos \sigma_n^2 t - \phi_{0,n}^2 \ell_1^- \sin \sigma_n^2 t) \sin \sigma_n^2 x.$$

If we apply formula (I.25) for the energy it follows

$$\mathbf{E}_{\ell_1^- \phi^2} = \frac{\ell_2}{4} \sum_{n\in\mathbb{N}} \left(\sin \sigma_n^2 \ell_1\right)^2 \left((\sigma_n^2 \phi_{0,n}^1)^2 + (\phi_{1,n}^2)^2\right).$$

Thus, it suffices to replace the latter expression in (16) to obtain the inequality of the theorem. $\square$

2.1. Identification of controllable subspaces. The aim of this subsection is to identify subspaces of controllable initial data of the system (10) in time $T \geq 2(\ell_1 + \ell_2)$ with the aid of Theorem III.2.

An easily identifiable subspace is that of the finite linear combinations of the eigenfunctions. It takes place

PROPOSITION III.1. *The system (10) is spectrally controllable in some time $T \geq 2(\ell_1 + \ell_2)$ if, and only if, the quotient $\dfrac{\ell_1}{\ell_2}$ is an irrational number.*

PROOF. If $\dfrac{\ell_1}{\ell_2}$ is irrational then the coefficients $\sin \sigma_n^1 \ell_2$, $\sin \sigma_n^2 \ell_1$, $n \in \mathbb{N}$, appearing in the inequalities in Proposition III.2 are all different from zero. Indeed, if $\sin \sigma_n^1 \ell_2 = 0$ for some n, then there would exist $k \in \mathbb{N}$ such that

$$\frac{n\pi}{\ell_1}\ell_2 = k\pi,$$

that is, $\dfrac{\ell_1}{\ell_2} = \dfrac{n}{k} \in \mathbb{Q}$. Then, the initial states (u_0^i, u_1^i), $i = 1, 2$, satisfying

$$\sum_{n\in\mathbb{N}} \frac{1}{(\sin \sigma_n^1 \ell_2)^2} (u_{0,n}^1)^2 + \sum_{n\in\mathbb{N}} \frac{1}{(\sigma_n^1 \sin \sigma_n^1 \ell_2)^2} (u_{1,n}^1)^2 \quad < \quad \infty, \tag{18}$$

$$\sum_{n\in\mathbb{N}} \frac{1}{(\sin \sigma_n^2 \ell_1)^2} (u_{0,n}^2)^2 + \sum_{n\in\mathbb{N}} \frac{1}{(\sigma_n^2 \sin \sigma_n^2 \ell_1)^2} (u_{1,n}^2)^2 \quad < \quad \infty, \tag{19}$$

are controllable in time $T \geq 2(\ell_1 + \ell_2)$.

In particular, the initial states $(\phi_0^1, \phi_1^1) \in Z^1 \times Z^1$, $(\phi_0^2, \phi_1^2) \in Z^2 \times Z^2$ would be controllable.

Let us see that the condition $\dfrac{\ell_1}{\ell_2} \notin \mathbb{Q}$ is also necessary for the approximate controllability. If $\dfrac{\ell_1}{\ell_2} = \dfrac{n}{k}$ with $n, k \in \mathbb{N}$ then, for every $p \in \mathbb{N}$ the functions

$$\phi^1(t,x) = \sin \frac{pn\pi t}{\ell_1} \sin \frac{pn\pi x}{\ell_1}, \qquad \phi^2(t,x) = -\sin \frac{pk\pi t}{\ell_2} \sin \frac{pk\pi x}{\ell_2},$$

are solutions of (11) and satisfy

$$\phi_x^1(t,0) + \phi_x^2(t,0) \equiv 0.$$

Consequently, the system (10) is not approximately controllable and, in particular, is not spectrally controllable. $\square$

For the further identification of controllable initial states of the system (10) with the aid of Theorem III.2 we will need some definitions from Number Theory. For $\eta \in \mathbb{R}$ we denote by $|||\eta|||$ the distance from η to the set $\mathbb{Z}$:

$$|||\eta||| = |\min\{x \in \mathbb{R} : \ \eta - x \in \mathbb{Z}\}| \,.$$

PROPOSITION III.2. *If* $\dfrac{\ell_1}{\ell_2}$ *is irrational, then all the initial states* (u_0^1, u_1^1), (u_0^2, u_1^2) *satisfying*

$$\sum_{n\in\mathbb{N}} \frac{1}{|||n\frac{\ell_2}{\ell_1}|||^2}(u_{0,n}^1)^2 + \sum_{n\in\mathbb{N}} \frac{1}{n^2|||n\frac{\ell_2}{\ell_1}|||^2}(u_{1,n}^1)^2 \quad < \quad \infty, \tag{20}$$

$$\sum_{n\in\mathbb{N}} \frac{1}{|||n\frac{\ell_1}{\ell_2}|||^2}(u_{0,n}^2)^2 + \sum_{n\in\mathbb{N}} \frac{1}{n^2|||n\frac{\ell_1}{\ell_2}|||^2}(u_{1,n}^2)^2 \quad < \quad \infty, \tag{21}$$

are controllable in time $T \geq 2(\ell_1+\ell_2)$.

PROOF. Let us observe that for each $x \in \mathbb{R}$

$$2|||\frac{x}{\pi}||| \leq |\sin x| \leq \pi|||\frac{x}{\pi}||| \tag{22}$$

(the proof of this fact may be found in Proposition A.1 in Appendix A).

Then,

$$2|||n\frac{\ell_2}{\ell_1}||| \leq \left|\sin\sigma_n^1\ell_2\right| \leq \pi|||n\frac{\ell_2}{\ell_1}|||, \qquad 2|||n\frac{\ell_1}{\ell_2}||| \leq \left|\sin\sigma_n^2\ell_1\right| \leq \pi|||n\frac{\ell_1}{\ell_2}|||.$$

Thus, the relations (20)-(21) are equivalent to (18)-(19). □

Therefore, in order to characterize subspaces of controllable initial states (10) it suffices to estimate the norms of the sequences $|||n\frac{\ell_2}{\ell_1}|||$, $|||n\frac{\ell_1}{\ell_2}|||$, $n \in \mathbb{N}$.

A natural way of getting additional information is the following: let $\rho : \mathbb{R} \to \mathbb{R}_+$ be an increasing function and define

$$\mathbf{\Psi}_\rho = \left\{x \in \mathbb{R}_+ : \liminf_{n\to\infty} |||nx|||\rho(n) > 0\right\}.$$

Then, if $\dfrac{\ell_1}{\ell_2}, \dfrac{\ell_2}{\ell_1} \in \Psi_\rho$ the inequalities

$$\sum_{n\in\mathbb{N}} \rho(n)(u_{0,n}^1)^2 + \sum_{n\in\mathbb{N}} \frac{\rho(n)}{n^2}(u_{1,n}^1)^2 \quad < \quad \infty, \tag{23}$$

$$\sum_{n\in\mathbb{N}} \rho(n)(u_{0,n}^2)^2 + \sum_{n\in\mathbb{N}} \rho(n)(u_{1,n}^2)^2 \quad < \quad \infty, \tag{24}$$

guarantees the controllability of the initial state (u_0^1, u_1^1), (u_0^2, u_1^2).

However, in such a general setting the problem turns out to be extremely difficult. That is why we restrict ourselves to the case $\rho(x) = x^\alpha$ with $\alpha > 0$. This choice is motivated by two reason. The first one is that this choice of ρ leads to the identification of subspaces of controllable initial states of the form

$$(u_0^i, u_1^i) \in \hat{H}^\alpha(0,\ell_i) \times \hat{H}^{\alpha-1}(0,\ell_i),$$

where

$$\hat{H}^\varepsilon(0,\ell_i) = \left\{u(x) = \sum_{n\in\mathbb{N}} u_n \sin\sigma_n^i x : \sum_{n\in\mathbb{N}} n^{2\varepsilon}(u_n)^2 < \infty\right\}.$$

Let us note that $\hat{H}^\varepsilon(0,\ell_i)$ is the Sobolev space $H^\varepsilon(0,\ell_i)$ with certain additional boundary conditions. In particular, $\hat{H}^1(0,\ell_i) = H_0^1(0,\ell_i)$ and $\hat{H}^0(0,\ell_i) = L^2(0,\ell_i)$.

The second reason for this choice of the function ρ is that the problem of describing the sets

$$\Psi_\alpha := \mathbf{\Psi}_{x^\alpha} = \left\{x \in \mathbb{R}_+ : \liminf_{n\to\infty} |||nx|||n^\alpha > 0\right\},$$

is a classical and difficult problem in Number Theory. In [**52**] and [**19**] the reader may find information on this theme. See also Appendix A where we have gathered the most relevant facts, which will be used in what follows.

Positive results

The following results are known

1) For every $\alpha > 0$ the sets Ψ_α have the property: if $\xi \in \Psi_\alpha$ then $\dfrac{1}{\xi} \in \Psi_\alpha$
2) Ψ_1 coincides with the set of irrational numbers $\eta \in \mathbb{R}$ having a continuous fraction expansion $[a_0, a_1, ..., a_n, ...]$ (see, e.g., [**52**], p. 6) with bounded (a_n). The set Ψ_1 is not denumerable and has Lebesgue measure equal to zero.
3) For every $\varepsilon > 0$ the complementary of the set $\Psi_{1+\varepsilon}$ is of measure zero (see Proposition A.2 in Appendix A). This set is usually denote in the literature by $\mathbf{B}_\varepsilon \subset \mathbb{R}$. As a consequence of the Roth theorem (Theorem A.2), the set $\mathbf{B}_\varepsilon$ contains all the algebraic irrational numbers, that is, all those numbers, which are root of polynomials of degree greater than one with integer coefficients.

As a consequence we obtain

Corollary III.1. *a) If $\frac{\ell_1}{\ell_2} \in \mathbf{B}_\varepsilon$ then the subspaces of initial states*

$$(u_0^i, u_1^i) \in \hat{H}^{1+\varepsilon}(0, \ell_i) \times \hat{H}^{\varepsilon}(0, \ell_i),$$

is controllable in any time $T \geq 2(\ell_1 + \ell_2)$. In particular, if $\frac{\ell_1}{\ell_2}$ is an algebraic irrational number, this subspace is controllable for any $\varepsilon > 0$.

b) If $\frac{\ell_1}{\ell_2}$ admits a bounded expansion in continuous fraction then, the subspace of initial states

$$(u_0^i, u_1^i) \in H_0^1(0, \ell_i) \times L^2(0, \ell_i),$$

is controllable in any time $T \geq 2(\ell_1 + \ell_2)$.

Negative results

We now describe some results that may be obtained as a counterpart to those provided by Proposition III.2.

Proposition III.3. *If there exists a sequence $(n_k) \subset \mathbb{N}$ such that*

$$|||n_k \frac{\ell_1}{\ell_2}|||\rho(n_k) \to 0 \quad or \quad |||n_k \frac{\ell_2}{\ell_1}|||\rho(n_k) \to 0, \qquad k \to \infty,$$

then, there exist initial states (u_0^1, u_1^1), (u_0^2, u_1^2) satisfying (23)-(24) which are not controllable in any finite time T.

Proof. Recall that the fact that all the initial states satisfying (23)-(24) are controllable in time T is equivalent to the fact that the following inequalities are verified:

$$\int_0^T |\phi_x^1(t,0) + \phi_x^2(t,0)|^2 dt \quad \geq \quad C_1 \sum_{n\in\mathbb{N}} \frac{1}{\rho^2(n)} \left((\frac{n\pi}{\ell_1}\phi_{0,n}^1)^2 + (\phi_{1,n}^1)^2 \right), \tag{25}$$

$$\int_0^T |\phi_x^1(t,0) + \phi_x^2(t,0)|^2 dt \quad \geq \quad C_2 \sum_{n\in\mathbb{N}} \frac{1}{\rho^2(n)} \left((\frac{n\pi}{\ell_2}\phi_{0,n}^2)^2 + (\phi_{1,n}^2)^2 \right), \tag{26}$$

for any solution of (11) with initial states $(\phi_0^i, \phi_1^i) \in Z^i \times Z^i$, $i = 1, 2$.

We assume that

$$|||n_k \frac{\ell_1}{\ell_2}|||\rho(n_k) \to 0 \tag{27}$$

and we will proved that under this assumption the inequality (26) is impossible.

Indeed, from (27) it holds that, for every $k \in \mathbb{N}$, there exists $m_k \in \mathbb{N}$ such that

$$\left| n_k \frac{\ell_1}{\ell_2} - m_k \right| \rho(n_k) \to 0.$$

Then,

$$\left| \sigma_{n_k}^2 - \sigma_{m_k}^1 \right| \rho(n_k) = \left| \frac{\pi n_k}{\ell_2} - \frac{\pi m_k}{\ell_1} \right| \rho(n_k) \to 0. \tag{28}$$

On the other hand, after replacing in (26) the solutions

$$\phi_k^1(t, x) = \cos \sigma_{m_k}^1 t \sin \sigma_{m_k}^1 x, \qquad \phi_k^2(t, x) = -\cos \sigma_{n_k}^2 t \sin \sigma_{n_k}^2 x,$$

it holds

$$\int_0^T |\sigma_{m_k}^1 \cos \sigma_{m_k}^1 t - \sigma_{n_k}^2 \cos \sigma_{n_k}^2 t|^2 dt \geq C_2 \rho^{-2}(n_k)(\sigma_{n_k}^2)^2$$

and then

$$\left| \sigma_{n_k}^2 - \sigma_{m_k}^1 \right|^2 \geq C \rho^{-2}(n_k) \tag{29}$$

(we have used the inequality

$$\int_0^T |x \cos xt - y \cos yt|^2 dt \leq 4|x - y|^2 x^2 T,$$

for $y \geq x \geq 1$, which is easily obtained using, for example, the main value theorem).

Thus, from (29) we obtain

$$\left| \sigma_{n_k}^2 - \sigma_{m_k}^1 \right| \rho(n_k) \geq C,$$

what contradicts the property (28) of the sequences (n_k) and (m_k). $\square$

The first important consequence of Proposition III.3 is based on the Dirichlet theorem: *for all* $\alpha < 1$, $\xi \in \mathbb{R}$ *and* $\varepsilon > 0$ *there exist an infinite number of values of* n *such that* $|||n\xi|||n^\alpha < \varepsilon$ (see [**19**], Section I.5).

COROLLARY III.2. *For all the values* ℓ_1, ℓ_2 *of the lengths of the strings and every* $\alpha < 1$ *there exist initial states*

$$(u_0^i, u_1^i) \in \hat{H}^\alpha(0, \ell_i) \times \hat{H}^{\alpha - 1}(0, \ell_i), \qquad i = 1, 2,$$

which are not controllable in any finite time T*. In particular, there exist non-controllable initial states in* $L^2(0, \ell_i) \times H^{-1}(0, \ell_i)$*, that means that the system (10) is not exactly controllable in any finite time.*

The following result of negative character is based on a construction due to Liouville. Let us consider the series

$$\xi = \sum_{k \in \mathbb{N}} 10^{-a_k}, \tag{30}$$

where (a_k) is an increasing sequence of natural numbers. Then, for each $p \in \mathbb{N}$,

$$|\xi 10^{a_p} - m| = 10^{a_p} \sum_{k>p} 10^{-a_k} < 10^{a_p - a_{p+1}}.$$

Let us assume that $\rho : \mathbb{R} \to \mathbb{R}_+$ is an increasing function. Fix $\varepsilon > 0$ and choose a sequence (a_k) that verifies

$$10^{a_k - a_{k+1}} < \frac{\varepsilon}{\rho(10^{a_k})},$$

or equivalently,

$$a_{k+1} > a_k + \lg \frac{\rho(10^{a_k})}{\varepsilon}.$$

Then, for the natural numbers $n_p = 10^{a_p}$, $p \in \mathbb{N}$, it will be true

$$|||n_p \xi|||\rho(n_p) < \varepsilon.$$

Summarizing, it is possible to construct real numbers, which are approximated by rationals faster than any given order ρ.

From Proposition III.3 it follows

Corollary III.3. *For any increasing function $\rho : \mathbb{R} \to \mathbb{R}_+$, it may be found values of the lengths ℓ_1, ℓ_2 of the strings such that there exist initial data in the subspace defined by (23)-(24), which are not controllable in any finite time T. In other words, the subspace of controllable initial states may be arbitrarily small.*

Remark III.2. *The numbers of the form (30) are called Liouville's numbers. The discovery of such numbers had a transcendental importance in the history of mathematics: Liouville had been able to prove that, if ξ is an algebraic irrational number of order p (that is, ξ is a root of a polynomial of degree p with rational coefficients and there is no polynomials of smaller degree having that property) then, the inequality*

$$|\xi n - m| < \frac{1}{n^{p-1}}$$

has no solutions $m, n \in \mathbb{N}$. Therefore, the numbers defined by (30) are not algebraic. This fact allowed to show for the first time the existence of non algebraic numbers.

3. The three string network with one controlled node

The rest of this chapter will be devoted to the study of the control problem for the three string network with a single exterior controlled node. The scheme we will follow will allow us to deal in Chapter IV the case of general tree-shaped networks. That is the main reason of the detail study we present here, though it is possible to reduce the of identifying subspaces of controllable initial states to the analogous problem for the system of two simultaneously controlled strings already solved in Section 2 (see also Section 8).

The motion of the network is described by the system

$$(31)\qquad \begin{cases} u^i_{tt} - u^i_{xx} = 0 & \text{in } \mathbb{R}\times[0,\ell_i] \quad i = 0,1,2, \\ u^0(t,0) = u^1(t,0) = u^2(t,0) & t \in \mathbb{R} \\ u^0_x(t,0) + u^1_x(t,0) + u^2_x(t,0) = 0 & t \in \mathbb{R} \\ u^0(t,\ell_0) = v(t), \quad u^i(t,\ell_i) = 0 & t \in \mathbb{R} \quad i = 1,2, \\ u^i(0,x) = u^i_0(x), \quad u^i_t(0,x) = u^i_1(x) & x \in [0,\ell_i] \quad i = 0,1,2, \end{cases}$$

which coincides with (1), except by the fact that now $v^1 = 0$.

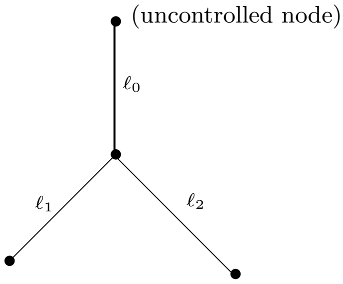

FIGURE 3. The three string network

Let us observe that the homogeneous version of the system (31), that is, when $v = 0$, coincides with (2).

From the general results of Chapter I we know that there exists a sequence (c_n) of positive numbers such that

$$\int_0^T |\phi_x^0(t,\ell_0)|^2 dt \geq \sum_{n\in\mathbb{N}} c_n^2(\mu_n \phi_{0,n}^2 + \phi_{1,n}^2), \tag{32}$$

for every solution $\bar{\phi}$ de (2) with initial state $(\bar{\phi}_0, \bar{\phi}_1) \in Z \times Z$, if, and only if, the space of initial states $(\bar{u}_0, \bar{u}_1)$ verifying

$$\sum_{n\in\mathbb{N}} \frac{u_{0,n}^2}{c_n^2} < \infty, \qquad \sum_{n\in\mathbb{N}} \frac{u_{1,n}^2}{c_n^2\mu_n} < \infty, \tag{33}$$

is controllable in time T.

Let us observe that the exact controllability of the system (31) is then equivalent to the existence of a subsequence (c_n) with the property (32) and having and positive lower bound:

$$c_n \geq c > 0, \qquad n \in \mathbb{N}. \tag{34}$$

Unfortunately, that is impossible for the system (2), independently of the lengths ℓ_0, ℓ_1, ℓ_2 of the strings. In Proposition III.7 we will prove that, for any ℓ_0, ℓ_1, ℓ_2, there exists a subsequence $(n_k) \subset \mathbb{N}$ such that

$$\lim_{k\to\infty} (\lambda_{n_k+1} - \lambda_{n_k}) = 0.$$

Then, if the inequality (32) is verified, if we replace the solutions of (2)

$$\bar{\phi}^k(t,x) = \frac{1}{\varkappa_{n_k+1}} \cos\lambda_{n_k+1}t\ \bar{\theta}_{n_k+1}(x) - \frac{1}{\varkappa_{n_k}} \cos\lambda_{n_k}t\ \bar{\theta}_{n_k}(x),$$

where

$$\varkappa_n = \theta_{n,x}^0(\ell_0),$$

we would obtain

$$C\left(\frac{\lambda_{n_k+1}^2}{\varkappa_{n_k+1}^2} + \frac{\lambda_{n_k}^2}{\varkappa_{n_k}^2}\right) \leq \int_0^T |\cos\lambda_{n_k+1}t - \cos\lambda_{n_k}t|^2 dt \leq \frac{T^3}{3}|\lambda_{n_k+1} - \lambda_{n_k}|^2.$$

But $|\varkappa_n| \leq C\lambda_n$ (see formula (70) in Remark III.4). Consequently, if

$$c_n \geq c > 0, \qquad n \in \mathbb{N},$$

it would exist a constant $C > 0$ such that

$$C \leq |\lambda_{n_k+1} - \lambda_{n_k}|^2,$$

for every k. This contradicts the choice of the subsequence (λ_{n_k}).

As a consequence, if a sequence (c_n) has the property (32) then, necessarily,

$$\liminf_{n\to\infty} c_n = 0.$$

Thus, the system (31) is not exactly controllable for any choice of the values of ℓ_0, ℓ_1, ℓ_2 and T. That is why we concentrate on proving inequalities of type (32) with coefficients c_n (they will be called weights), which do not satisfy the condition (34).

4. An observability inequality

In this section we prove the following property of the solutions of (2)

THEOREM III.3. *There exists a positive constant C such that every solution $\bar{\phi}$ of (2) with initial data $(\bar{\phi}_0, \bar{\phi}_1) \in V \times H$ satisfies the inequalities*

$$\int_0^{T^*} |\phi_x^0(t,\ell_0)|^2 dt \geq C\mathbf{E}_{\ell_j^- \bar{\phi}}, \quad j = 1, 2,$$

where $T^* = 2(\ell_0 + \ell_1 + \ell_2)$.

In Theorem III.3, $\ell_j^- \bar{\phi}$ is the function obtained by applying the operator ℓ^- defined (II.7) for $\ell = \ell_j$ to the solution $\bar{\phi}$ of (2). Due to the linearity of ℓ_j^-, the function $\ell_j^- \bar{\phi}$ is also a solution of (2). In particular, its energy $\mathbf{E}_{\ell_j^- \bar{\phi}}(t)$ is conserved in time.

If we intend to prove an inequality of the type

$$\int_0^T |\phi_x^0(t,\ell_0)|^2 dt \geq C\mathbf{E}_{\bar{\phi}}, \tag{35}$$

for the solutions of (2), what is equivalent to prove that for some $\hat{t} \in \mathbb{R}$ and $i = 0, 1, 2$,

$$\int_0^T |\phi_x^0(t,\ell_0)|^2 dt \geq C\mathbf{E}_{\phi_i}(\hat{t}),$$

it is natural to try to proceed as in Section 1, that is, to apply Proposition II.1 to estimate the energy of each string. Thanks to Proposition II.1, we will have for every $\hat{t} \in \mathbb{R}$ and $i = 1, 2$,

$$\mathbf{E}_{\phi^i}(\hat{t}) \leq C\left(\int |\phi_x^i(t,0)|^2 dt + \int |\phi_t^i(t,0)|^2 dt\right),$$

$$\mathbf{E}_{\phi^0}(\hat{t}) \leq C\left(\int |\phi_x^0(t,\ell_0)|^2 dt + \int |\phi_t^0(t,\ell_0)|^2 dt\right) = \int |\phi_x^0(t,\ell_0)|^2 dt$$

(we have not written the limit in the integrals, we will come back later to that issue in detail).

Thus, if we are able to prove that there exists $C > 0$ such that, for $i = 1, 2$,

$$\int |\phi_x^i(t,0)|^2 dt \quad \leq \quad C \int_0^T |\phi_x^0(t,\ell_0)|^2 dt, \tag{36}$$

$$\int |\phi_t^i(t,0)|^2 dt \quad \leq \quad C \int_0^T |\phi_x^0(t,\ell_0)|^2 dt, \tag{37}$$

we obtain the inequality (35).

The inequality (37) is proved without difficulty for $i = 1, 2$: in view of the coupling conditions and the formulas (8) it results

$$\phi_t^1(t,0) = \phi_t^2(t,0) = \phi_t^0(t,0) = \ell_0^- \phi_x^0(t,\ell_0) \tag{38}$$

and then

$$\int |\phi_t^i(t,0)|^2 dt = \int |\ell_0^- \phi_x^0(t,\ell_0)|^2 dt \leq \int |\phi_x^0(t,\ell_0)|^2 dt, \qquad i = 1, 2.$$

However, the inequalities (36) are more delicate; moreover, they are not true[2]. In spite of what happens with $\phi_t^1(.,0)$, $\phi_t^2(.,0)$, the traces $\phi_x^1(.,0)$, $\phi_x^2(.,0)$ can not be expressed in a direct way as a function of $\phi_x^0(.,\ell_0)$; for them the coupling condition in this node allows just to guarantee that

$$\phi_x^1(t,0) + \phi_x^2(t,0) = -\ell_0^+ \phi_x^0(t,\ell_0).$$

Nevertheless, the boundary conditions $\phi_x^1(t,\ell_1) = \phi_x^2(t,\ell_2) = 0$ provide additional information:

$$0 = \phi_x^i(t,\ell_1) = \ell_i^+ \phi_t^i(t,0) + \ell_i^- \phi_x^i(t,0), \qquad i = 1, 2,$$

from where it holds

$$\ell_i^- \phi_x^i(t,0) = -\ell_i^+ \phi_t^i(t,0) = \ell_i^+ \ell_0^- \phi_x^0(t,\ell_0), \qquad i = 1, 2. \tag{39}$$

In this way, we arrive to the system of equations

$$\left\{ \begin{array}{c} \phi_x^1(t,0) + \phi_x^2(t,0) = f(t), \\ \ell_1^- \phi_x^1(t,0) = g_1(t), \qquad \ell_2^- \phi_x^2(t,0) = g_2(t), \end{array} \right. \tag{40}$$

which is satisfied by the traces $\phi_x^1(.,0)$, $\phi_x^2(.,0)$, where

$$f(t) = -\ell_0^+ \phi_x^0(t,\ell_0), \qquad g_i(t) = \ell_i^+ \ell_0^- \phi_x^0(t,\ell_0), \qquad i = 1, 2.$$

Let us observe that f, g_1, g_2 are functions such that their norms in L^2 may be estimated in terms of the norm L^2 of $\phi_x^0(.,\ell_0)$ with the help of Proposition II.2.

Then the following question arises: is the information (40) sufficient to prove the inequality (35), at least with a weaker energy? The answer is yes; the idea is the following: if we apply, for example, the operator ℓ_1^- to the first of the equations (40) we obtain uncoupled conditions

$$\ell_1^- \phi_x^1(.,0) = g_1, \qquad \ell_1^- \phi_x^2(.,0) = \ell_1^- f - g_1. \tag{41}$$

Due to the linearity of the system (2) and the operators ℓ_1^- and ℓ_2^-, if $\bar\phi$ is a solution of (2) then the functions $\ell_1^- \bar\phi$ and $\ell_2^- \bar\phi$ (the operators act in this case over

[2]If this inequalities were true, it would result that the whole energy space $H \times V'$ be controllable. This, as we will see later, never happens, independently of the values of the lengths ℓ_0, ℓ_1, ℓ_2 of the strings.

the variable t of $\bar{u}$) are also solutions of (2). Besides, the following inequalities take place

$$(\ell_j^- \phi^i)_x = \ell_j^- \phi_x^i, \quad (\ell_j^- \phi^i)_t = \ell_1^- \phi_t^i \ , \quad \text{for} \quad i = 0, 1, 2, \ \text{ and } \ j = 1, 2.$$

Then, if we choose the solution $\bar{w} = \ell_1^- \bar{u}$ of (2) we will have

$$(w^i)_x = \ell_1^- \phi_x^i, \quad (w^i)_t = \ell_1^- \phi_t^i \ , \quad \text{for} \quad i = 0, 1, 2,$$

and the relations (41) become

$$w_x^1(.,0) = g_1, \qquad w_x^2(.,0) = \ell_1^- f - g_1.$$

This implies that $w_x^1(.,0)$, $w_x^2(.,0)$, may be estimated in terms of the L^2-norm of $\phi_x^0(.,\ell_0)$. Of course, the same happens with the traces $w_t^1(.,0)$, $w_t^2(.,0)$ and $w_x^0(.,\ell_0)$ due to the continuity of ℓ_1^- (Proposition II.2). With this it may be proved that

$$\int |\phi_x^0(t,\ell_0)|^2 dt \geq C\mathbf{E}_{\bar{w}}.$$

Following this simple argument we will prove that

Proposition III.4. *There exists a positive constant C such that every solution $\bar{\phi}$ of (2) with initial data $(\bar{\phi}_0, \bar{\phi}_1) \in V \times H$ satisfies the inequalities*

$$\int_0^{T_j} |\phi_x^0(t,\ell_0)|^2 dt \geq C\mathbf{E}_{\ell_j^- \bar{\phi}}, \quad j = 1, 2,$$

where[3] $T_j = 2(\ell_0 + \ell_j + \max\{\ell_1, \ell_2\})$.

Proof. We have almost already prove this result; we just need to follow carefully the integration intervals to obtain the indicated observation time. We will prove the assertion for $i = 1$; for $i = 2$ the proof is, obviously, similar.

Let us observe first that, since $w_t^0(t,\ell_0) = 0$, $w_x^0(t,\ell_0) = \ell_1^- \phi_x^0(t,\ell_0)$, then, in view of Proposition II.1, for any $\hat{t} \in \mathbb{R}$, the energy $\mathbf{E}_{w^0}$ of the solution $\bar{w}$ on the string $\mathbf{e}_0$ satisfies

$$\mathbf{E}_{w^0}(\hat{t}) \leq C \int_{\hat{t}-\ell_0}^{\hat{t}+\ell_0} |w_x^0(t,\ell_0)|^2 dt = C \int_{\hat{t}-\ell_0}^{\hat{t}+\ell_0} |\ell_1^- \phi_x^0(t,\ell_0)|^2 dt,$$

and from Proposition II.2 it follows that, for $\hat{t} \in [\ell_0 + \ell_1, T_1 - \ell_0 - \ell_1]$,

$$\mathbf{E}_{w^0}(\hat{t}) \leq C \int_{\hat{t}-\ell_0-\ell_1}^{\hat{t}+\ell_0+\ell_1} |\phi_x^0(t,\ell_0)|^2 dt \leq C \int_0^{T_1} |\phi_x^0(t,\ell_0)|^2 dt. \tag{42}$$

If we can prove that there exist $C > 0$ and $\hat{t} \in \mathbb{R}$ such that

$$\int_{\hat{t}-\ell_i}^{\hat{t}-\ell_i} |w_x^i(t,0)|^2 dt \leq C \int_0^{T_1} |\phi_x^0(t,\ell_0)|^2 dt, \tag{43}$$

$$\int_{\hat{t}-\ell_i}^{\hat{t}-\ell_i} |w_t^i(t,0)|^2 dt \leq C \int_0^{T_1} |\phi_x^0(t,\ell_0)|^2 dt, \tag{44}$$

[3]This result is very close to Theorem III.3 but in a larger observation time. If $\ell_1 < \ell_2$ the inequality of the theorem is immediately obtained for $j = 1$, since $T_1 = T^*$.

$i = 1, 2$, for every solution of (31) with $v \equiv 0$, then, based on Proposition II.1 we would obtain

$$\mathbf{E}_{w^i}(\hat{t}) \leq C \int_0^{T_1} |\phi_x^0(t, \ell_0)|^2 dt, \quad i = 0, 1, 2$$

and then, in account of (42),

$$\mathbf{E}_{\bar{w}}(\hat{t}) = \mathbf{E}_{w^0}(\hat{t}) + \mathbf{E}_{w^1}(\hat{t}) + \mathbf{E}_{w^2}(\hat{t}) \leq C \int_0^{T_1} |\phi_x^0(t, \ell_0)|^2 dt.$$

So, we concentrate ourselves in proving the inequalities (43). As it has been pointed out above (in the formulas (38)-(39)) we have the equalities

$$w_t^i(t, 0) = \ell_i^- \phi_t^i(t, 0) = \ell_i^- \ell_0^- \phi_x^0(t, \ell_0), \qquad i = 1, 2, \tag{45}$$

$$w_x^1(t, 0) = \ell_1^- \phi_x^1(t, 0) = -\ell_1^+ \phi_t^1(t, 0), \qquad w_t^2(t, 0) = w_t^1(t, 0) = \ell_1^- \ell_0^- \phi_x^0(t, \ell_0). \tag{46}$$

Then, combining (45) with Proposition II.2, we can ensure that, for any $\hat{t} \in \mathbb{R}$,

$$\int_{\hat{t}-\ell_1}^{\hat{t}+\ell_1} |w_t^1(t, 0)|^2 dt = \int_{\hat{t}-\ell_1}^{\hat{t}+\ell_1} |\ell_1^- \ell_0^- \phi_x^0(t, 0)|^2 dt \leq C \int_{\hat{t}-\ell_0-2\ell_1}^{\hat{t}+\ell_0+2\ell_1} |\phi_x^0(t, \ell_0)|^2 dt.$$

In a similar way it is possible to prove the inequalities

$$\begin{aligned} \int_{\hat{t}-\ell_1}^{\hat{t}+\ell_1} |w_x^1(t, 0)|^2 dt &\leq C \int_{\hat{t}-\ell_0-2\ell_1}^{\hat{t}+\ell_0+2\ell_1} |\phi_x^0(t, \ell_0)|^2 dt, \\ \int_{\hat{t}-\ell_2}^{\hat{t}+\ell_2} |w_t^2(t, 0)|^2 dt &\leq C \int_{\hat{t}-\ell_0-\ell_1-\ell_2}^{\hat{t}+\ell_0+\ell_1+\ell_2} |\phi_x^0(t, \ell_0)|^2 dt, \\ \int_{\hat{t}-\ell_2}^{\hat{t}+\ell_2} |w_x^2(t, 0)|^2 dt &\leq C \int_{\hat{t}-\ell_0-\ell_1-\ell_2}^{\hat{t}+\ell_0+\ell_1+\ell_2} |\phi_x^0(t, \ell_0)|^2 dt. \end{aligned} \tag{47}$$

Now it is easy to see that if we choose $\hat{t} = \ell_0 + \ell_1 + \max\{\ell_1, \ell_2\}$ in (47) we obtain[4] the inequalities (42), (43). With this the proposition is proved □

Let us assume now that $\ell_1 \leq \ell_2$. Then $T_1 = 2(\ell_0 + \ell_1 + \ell_2)$ and $T_2 = 2(\ell_0 + \ell_1 + \ell_2) \geq T_1$. In fact, the value of the observation time T_2 may be reduced.

The possibility of choosing an observation time smaller than T_2 (T_1 already coincides with T^*), which allows to obtain the assertion of the theorem from Proposition III.4, is based on a property of *generalized periodicity* in time of the solutions of the homogeneous system (2) (see Proposition III.5), which allows that all the L^2-information on the traces $\phi_x^0(t, \ell_0)$ of these solutions may be obtained in a time interval of length T^*. This makes the observation in a larger time superfluous.

We define the operator

$$\mathcal{Q} := \ell_0^+ \ell_1^- \ell_2^- + \ell_0^- \ell_1^+ \ell_2^- + \ell_0^- \ell_1^- \ell_2^+.$$

Then,

Proposition III.5. *For every solution $\bar{\phi}$ of (2) with initial data $(\bar{\phi}_0, \bar{\phi}_1) \in V \times H$ it holds*

$$\mathcal{Q}\phi_x^0(t, \ell_0) = 0.$$

[4]We have that $\hat{t} \in [\ell_0 + \ell_1, T_1 - \ell_0 - \ell_1]$, what is necessary for the inequality (42). This value of $\hat{t}$ has been chosen so that the numbers $\hat{t} - \ell_0 - 2\ell_1$, $\hat{t} + \ell_0 + 2\ell_1$, $\hat{t} - \ell_0 - \ell_1 - \ell_2$, $\hat{t} + \ell_0 + \ell_1 + \ell_2$ belong to the interval $[0, T_1]$.

PROOF. From the relations $\phi_t^0(t,0) = -\ell_0^- \phi_x^0(t,\ell_0)$, $\phi_x^0(t,0) = \ell_0^+ \phi_x^0(t,\ell_0)$, we have that

$$\begin{aligned} \mathcal{Q}\phi_x^0(t,\ell_0) &= \ell_1^-\ell_2^-(\ell_0^+\phi_x^0(t,\ell_0)) + (\ell_1^+\ell_2^- + \ell_1^-\ell_2^+)\ell_0^-\phi_x^0(t,\ell_0) \\ &= \ell_1^-\ell_2^-\phi_x^0(t,0) - (\ell_1^+\ell_2^- + \ell_1^-\ell_2^+)\phi_t^0(t,0). \end{aligned}$$

We recall now that $\phi_t^0(t,0) = \phi_t^1(t,0) = \phi_t^2(t,0)$

$$\mathcal{Q}\phi_x^0(t,\ell_0) = \ell_1^-\ell_2^-\phi_x^0(t,0) + \ell_2^-(-\ell_1^+\phi_t^1(t,0)) + \ell_1^-(-\ell_2^+\phi_t^2(t,0)), \tag{48}$$

and from the equalities $\ell_1^+\phi_t^1(t,0) + \ell_1^-\phi_x^1(t,0) = 0$, $\ell_2^+\phi_t^2(t,0) + \ell_2^-\phi_x^2(t,0) = 0$ (obtained as in the proof of Proposition III.4 from the coupling and boundary conditions) from (48) it holds

$$\begin{aligned} \mathcal{Q}\phi_x^0(t,\ell_0) &= \ell_1^-\ell_2^-\phi_x^0(t,0) + \ell_2^-\ell_1^-\phi_t^1(t,0) + \ell_1^-\ell_2^-\phi_t^2(t,0) \\ &= \ell_1^-\ell_2^-(\phi_x^0(t,0) + \phi_t^1(t,0) + \phi_t^2(t,0)) = 0. \end{aligned}$$

□

The usefulness of Proposition III.5 in our context relies on the following property:

For every $T > 0$ there exists a constant $C_T > 0$ such that every continuous function ψ, which is a solution of $Q\psi \equiv 0$, satisfies the inequality

$$\int_0^T |\psi(t)|^2 dt \le C_T \int_0^{T^*} |\psi(t)|^2 dt, \tag{49}$$

where, as before, $T^* = 2(\ell_0 + \ell_1 + \ell_2)$.

This fact, when applied to $u_x^0(t,\ell_0)$, give the assertion of Theorem III.3 from Proposition III.4.

The proof of this property will be given en Chapter IV in more general conditions. Now we restrict ourselves to the particular version corresponding to the operator $\mathcal{Q}$, which allows to illustrate clearly the idea of the proof in the general case.

Let us assume that $\ell_* = \min\{\ell_0,\ell_1,\ell_2\}$. If we are able to prove that, for arbitrary $T > 0$ and ψ satisfying $Q\psi \equiv 0$,

$$\int_0^T |\psi(t)|^2 dt \le C \int_0^{T-2\ell_*} |\psi(t)|^2 dt, \tag{50}$$

then we can get the proof of the assertion (49), iterating this inequality as many times as necessary to obtain $T - 2n\ell_* \le T^*$.

Let us observe that, according to the definition of $\mathcal{Q}$, the equality $\mathcal{Q}\psi \equiv 0$ may be written as

$$(\ell_0^+\ell_1^-\ell_2^- + \ell_0^-\ell_1^+\ell_2^- + \ell_0^-\ell_1^-\ell_2^+)\psi \equiv 0$$

and then, from the definition of the operators $\ell_i^\pm$ results[5]

$$\begin{aligned} &3\psi(t+\ell_0+\ell_1+\ell_2) - \psi(t+\ell_0+\ell_1-\ell_2) - \psi(t+\ell_0-\ell_1+\ell_2) - \\ &-\psi(t-\ell_0-\ell_1-\ell_2) - -\psi(t-\ell_0+\ell_1+\ell_2) - \\ &-\psi(t-\ell_0+\ell_1-\ell_2) - \psi(t-\ell_0-\ell_1+\ell_2) + 3\psi(t+\ell_0+\ell_1+\ell_2) = 0. \end{aligned}$$

[5]It is a lengthly, but completely elementary computation.

Replacing the variable t by $t-(\ell_0+\ell_1+\ell_2)$ the previous inequality may be written as

$$\psi(t)=\sum_{\tau\in\Gamma}c_\tau\psi(t-\tau),$$

where

$$\Gamma:=\{2\ell_0,2\ell_1,2\ell_2,2(\ell_0+\ell_1),2(\ell_0+\ell_2),2(\ell_1+\ell_2),2(\ell_0+\ell_1+\ell_2)\}$$

and the coefficients c_τ are equal to 1 or $-\frac{1}{3}$. From this, using the Cauchy-Schwarz inequality,

$$\int_{T^*}^{T}|\psi(t)|^2dt\le C\sum_{\tau\in\Gamma}\int_{T^*}^{T}|\psi(t-\tau)|^2dt=C\sum_{\tau\in\Gamma}\int_{T^*-\tau}^{T-\tau}|\psi(t)|^2dt, \tag{51}$$

for some constant C independent of ψ (in fact, it may be taken $C=\frac{55}{9}$).

But every $\tau\in\Gamma$ satisfies $2\ell^*\le\tau\le T^*$, so we have $T^*-\tau\ge 0$ and $T-\tau\le T-2\ell_*$; and then the following inequality

$$\int_{T^*-\tau}^{T-\tau}|\psi(t)|^2dt\le\int_0^{T-2\ell_*}|\psi(t)|^2dt,$$

which, after being replaced in (51) gives

$$\int_{T^*}^{T}|\psi(t)|^2dt\le C\sum_{\tau\in\Gamma}\int_0^{T-2\ell_*}|\psi(t)|^2dt=7C\int_0^{T-2\ell_*}|\psi(t)|^2dt.$$

Finally

$$\int_0^{T}|\psi(t)|^2dt=\int_0^{T^*}|\psi(t)|^2dt+\int_{T^*}^{T}|\psi(t)|^2dt\le C\int_0^{T-2\ell_*}|\psi(t)|^2dt,$$

which is the inequality (50).

Thus, we have proved the inequality (49) and with this, Theorem III.3 is also proved.

5. Properties of the sequence of eigenvalues

The relation of the operator $\mathcal{Q}$ with the system (2) is not purely technical. This operator is closely related to the boundary conditions (2), as Proposition III.5 shows. Besides, the operator $\mathcal{Q}$ is linked in a direct way to the eigenvalues of $-\Delta_G$. Indeed, if we apply $\mathcal{Q}$ to the function $e^{i\lambda t}$, where λ is an arbitrary real number, since $\ell^+e^{i\lambda t}=\cos\lambda\ell\ e^{i\lambda t}$ and $\ell^-e^{i\lambda t}=i\sin\lambda\ell\ e^{i\lambda t}$, we obtain

$$\begin{aligned}\mathcal{Q}e^{i\lambda t}&=\left(\ell_0^+\ell_1^-\ell_2^-+\ell_0^-\ell_1^+\ell_2^-+\ell_0^-\ell_1^-\ell_2^+\right)e^{i\lambda t}\\&=-(\cos\lambda\ell_0\sin\lambda\ell_1\sin\lambda\ell_2+\sin\lambda\ell_0\cos\lambda\ell_1\sin\lambda\ell_2+\\&\quad+\sin\lambda\ell_0\sin\lambda\ell_1\cos\lambda\ell_2)\,e^{i\lambda t}.\end{aligned}$$

Thus, if we define the function

$$q(\lambda):=-(\cos\lambda\ell_0\sin\lambda\ell_1\sin\lambda\ell_2+\sin\lambda\ell_0\cos\lambda\ell_1\sin\lambda\ell_2+\sin\lambda\ell_0\sin\lambda\ell_1\cos\lambda\ell_2), \tag{52}$$

we arrive to the relation

$$\mathcal{Q}e^{i\lambda t}=q(\lambda)e^{i\lambda t}.$$

It takes place

PROPOSITION III.6. *Let $\lambda \neq 0$. Then λ^2 is an eigenvalue of $-\Delta_G$ if, and only if, $q(\lambda) = 0$.*

PROOF. The necessity of this condition is immediate: if λ^2 is an eigenfunction with associated eigenfunction $\bar{\theta}$ then the function $\bar{\phi}(t,x) = e^{i\lambda t}\bar{\theta}(x)$ is a solution of (2). According to Proposition III.5 we have

$$0 = \mathcal{Q}\phi_x^0(t,\ell_0) = \mathcal{Q}e^{i\lambda t}\theta_x^0(\ell_0) = q(\lambda)e^{i\lambda t}\theta_x^0(\ell_0).$$

From this inequality it holds $q(\lambda) = 0$ if $\theta_x^0(\ell_0) \neq 0$. On the other hand, if $\theta_x^0(\ell_0) = 0$ then the function θ^0, which is a solution of a second order ordinary differential equation satisfies $\theta^0(\ell_0) = \theta_x^0\ (\ell_0) = 0$, and this implies $\theta^0 \equiv 0$; in particular, $\theta^0(0) = 0$. From the boundary conditions we have that $\theta^1(0) = \theta^2(0) = 0$. This means that λ^2 is also an eigenvalue of the strings $\mathbf{e}_1$ and $\mathbf{e}_2$ and therefore,

$$\sin\lambda\ell_1 = \sin\lambda\ell_2 = 0.$$

If we replace these equalities in (52), we obtain $q(\lambda) = 0$.

Now we will see that the condition $q(\lambda) = 0$ is also sufficient for λ^2 to be an eigenvalue. It suffices to construct a non-zero eigenfunction $\bar{\theta}$ associated to λ^2. We look for the components of $\bar{\theta}$ in the form:

$$\theta^i(x) = a_i \sin\lambda(x-\ell_i), \quad i = 0,1,2, \tag{53}$$

what guarantees that the boundary condition are satisfied $\theta^i(\ell_i) = 0$. The remaining boundary conditions lead to the linear system

$$\begin{aligned} &a_0\sin\lambda\ell_0 = a_1\sin\lambda\ell_1 = a_2\sin\lambda\ell_2,\\ &a_0\lambda\cos\lambda\ell_0 + a_1\lambda\cos\lambda\ell_1 + a_2\lambda\cos\lambda\ell_2 = 0,\end{aligned}$$

whose determinant coincides with $\lambda q(\lambda)$. Thus, if $q(\lambda) = 0$ we can find numbers a_0, a_1, a_2, not all of then equal to zero, such that the function $\bar{\theta}$ defined by (53) is an eigenfunction. □

REMARK III.3. *The proof of the fact that the condition given in Proposition III.6 is necessary may be done in a simpler way without using the operator $\mathcal{Q}$,since the boundary conditions imply that an eigenfunction is necessarily given by (53). If this eigenfunction does not vanish identically, the determinant of the liner system is equal to zero. Thus, $q(\lambda) = 0$. However, we have used the properties of $\mathcal{Q}$ because this is natural technique we will use in Chapter IV to prove properties related to more general networks.*

An important consequence of Proposition III.6 is that, if we denote by (σ_n) the increasing sequence formed by the elements of the set

$$\Sigma = \frac{\pi}{\ell_0}\mathbb{N} \cup \frac{\pi}{\ell_1}\mathbb{N} \cup \frac{\pi}{\ell_2}\mathbb{N},$$

which are the positive square roots of the eigenvalues of the strings with homogeneous Dirichlet boundary conditions and by (λ_n) the increasing sequence formed by the positive square roots of the eigenvalues of the network, then, for every $n \in \mathbb{N}$,

$$\lambda_n < \sigma_n < \lambda_{n+1} < \sigma_{n+1}. \tag{54}$$

Indeed, in every interval (σ_n, σ_{n+1}) the function $q(\lambda)$ may be expressed as

$$q(\lambda) = h_1(\lambda)h_2(\lambda),$$

where

$$h_1(\lambda) = \sin\lambda\ell_0 \sin\lambda\ell_1 \sin\lambda\ell_2, \qquad h_2(\lambda) = \cot\lambda\ell_0 + \cot\lambda\ell_1 + \cot\lambda\ell_2.$$

Let us observe that under the hypothesis that all the ratios $\frac{\ell_i}{\ell_j}$ $(i \neq j)$ are irrationals, the numbers λ_n are the positive zeros of $h_2(\lambda)$, while the numbers σ_n are the points where $h_2(\lambda) \to \pm\infty$. It suffices now to note that on every interval (σ_n, σ_{n+1}) the function $h_2(\lambda)$ is strictly increasing to conclude that necessarily the numbers σ_n and λ_n alternate, that is, (54) is verified.

The inequalities (54) allow to obtain information on the numbers λ_n from the properties of the sequence (σ_n).

Let us observe that from (54) we obtain for every $n \in \mathbb{N}$,

$$\lambda_{n+4} - \lambda_n > \sigma_{n+3} - \sigma_n.$$

But, for every $n \in \mathbb{N}$, among the four numbers $\sigma_n, \sigma_{n+1}, \sigma_{n+2}, \sigma_{n+3}$ there are at least two corresponding to the same string. This implies that for every n, there exists $i = 0, 1, 2$ such that

$$\sigma_{n+3} - \sigma_n > \frac{\pi}{\ell_i}.$$

Consequently, the following generalized separation property is valid

$$\lambda_{n+4} - \lambda_n > \pi \min\left(\frac{1}{\ell_0}, \frac{1}{\ell_1}, \frac{1}{\ell_2}\right),$$

for every $n \in \mathbb{N}$. This generalized separation property allows to apply the technique derived from Theorem II.4 in the proof of observability inequalities of the type (32).

On the other hand, if n_λ and n_σ denote respectively, the counting functions[6] of the sequences (λ_n) and (σ_n) then

$$n_\sigma(r) \le n_\lambda(r) \le n_\sigma(r) + 1.$$

The function $n_\sigma(r)$ may explicitly computed as the sum of the counting functions of the sequences $\left(\frac{n\pi}{\ell_0}\right)$, $\left(\frac{n\pi}{\ell_1}\right)$ and $\left(\frac{n\pi}{\ell_2}\right)$. It holds

$$n_\sigma(r) = [\frac{r\ell_0}{\pi}] + [\frac{r\ell_1}{\pi}] + [\frac{r\ell_2}{\pi}],$$

where $[\eta]$ denotes the integer part of the number η. Therefore, we obtain

$$r\frac{\ell_0 + \ell_1 + \ell_2}{\pi} - 1 \le n_\lambda(r) \le 1 + r\frac{\ell_0 + \ell_1 + \ell_2}{\pi} + 1. \tag{55}$$

Then, the sequence (λ_n) has density

$$D(\lambda_n) = \lim_{r\to\infty} \frac{n_\lambda(r)}{r} = \frac{\ell_0 + \ell_1 + \ell_2}{\pi},$$

which coincides with the density of the sequence (σ_n). It is essentially due to this reason that the time of observation in Theorem III.3 $T^* = 2(\ell_0+\ell_1+\ell_2) = 2\pi D(\lambda_n)$ is optimal.

Besides, the inequality (55) implies that

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{\pi}{\ell_0 + \ell_1 + \ell_2}. \tag{56}$$

[6]The counting function $n(r)$ of the positive sequence (λ_n) is the number of elements of the sequence contained on the interval $(0, r]$.

This shows that the eigenvalues of the network behave asymptotically as the eigenvalues of a single string of length $\ell_0 + \ell_1 + \ell_2$.

Another important consequence of the inequalities (54) is the following.

PROPOSITION III.7. *For any values ℓ_0, ℓ_1, ℓ_2 of the lengths of the strings there exists a subsequence $(n_k) \subset \mathbb{N}$ such that*

$$\lim_{k\to\infty} (\lambda_{n_k+1} - \lambda_{n_k}) = 0.$$

PROOF. According to Dirichlet's theorem on the simultaneous approximation of real numbers by rationals (see [**19**], Section I.5), for every $\varepsilon > 0$ there exists an infinite number of values of $k \in \mathbb{N}$ for which there exist natural numbers p_k, q_k such that

$$\left|k\frac{\ell_1}{\ell_0} - p_k\right| < \varepsilon, \qquad \left|k\frac{\ell_2}{\ell_0} - q_k\right| < \varepsilon.$$

Then

$$\left|\frac{\pi k}{\ell_0} - \frac{\pi p_k}{\ell_1}\right| < \varepsilon', \qquad \left|\frac{\pi k}{\ell_0} - \frac{\pi q_k}{\ell_2}\right| < \varepsilon', \tag{57}$$

where

$$\varepsilon' = \min\left\{\frac{\pi\varepsilon}{\ell_1}, \frac{\pi\varepsilon}{\ell_2}\right\}.$$

Let $n_k \in \mathbb{N}$ such that

$$\sigma_{n_k} = \min\left\{\frac{\pi k}{\ell_0}, \frac{\pi p_k}{\ell_1}, \frac{\pi q_k}{\ell_2}\right\}.$$

Then, from (57) we obtain the inequalities

$$\sigma_{n_k+2} - \sigma_{n_k} < 2\varepsilon'.$$

But, in view of (54), the latter inequality implies

$$\lambda_{n_k+1} - \lambda_{n_k} < 2\varepsilon',$$

for infinite values of de $k \in \mathbb{N}$.

Taking into account that ε' may be chosen arbitrarily small, the assertion of the proposition is obtained. ☐

6. Observability of the Fourier coefficients of the initial data

Our aim in this section is to express the inequalities of Theorem III.3 in terms of the Fourier coefficients of the initial data of the solution $\bar{\phi}$ of (2). To do this, we should express $\mathbf{E}_{\ell_j^-\bar{\phi}}$, $j = 1, 2$, in terms of those coefficients.

If $\bar{\phi}_0, \bar{\phi}_1 \in Z$, that is, if the sequences $(\phi_{0,n})$ and $(\phi_{1,n})$ are finite, then from the formula (4) it follows

$$\ell_j^-\bar{\phi}(x) = \sum_{n\in\mathbb{N}} (\phi_{0,n}\ell_j^- \cos\lambda_n t + \frac{\phi_{1,n}}{\lambda_n}\ell_j^- \sin\lambda_n t)\bar{\theta}_n(x). \tag{58}$$

But we have the relations

$$\ell_j^- \cos\lambda_n t = \frac{1}{2}\left(\cos\lambda_n(t+\ell_j) - \cos\lambda_n(t-\ell_j)\right) = -\sin\lambda_n\ell_j \sin\lambda_n t,$$
$$\ell_j^- \sin\lambda_n t = \frac{1}{2}\left(\sin\lambda_n(t+\ell_j) - \sin\lambda_n(t-\ell_j)\right) = \sin\lambda_n\ell_j \cos\lambda_n t;$$

if we replace them in (58) we obtain

$$\ell_j^- \bar\phi(x) = \sum_{n\in\mathbb{N}} \sin\lambda_n \ell_j \Big(\frac{\phi_{1,n}}{\lambda_n}\cos\lambda_n t - \phi_{0,n}\sin\lambda_n t\Big)\bar\theta_n(x). \tag{59}$$

Using the formula (5) for the energy of $\ell_j^- \bar\phi$ we arrive to

$$\mathbf{E}_{\ell_j^- \bar\phi} = \sum_{n\in\mathbb{N}} \sin^2 \lambda_n \ell_j (\mu_n \phi_{0,n}^2 + \phi_{1,n}^2) \tag{60}$$

and therefore, Theorem III.3 allows us to ensure that there exists a constant $C > 0$ such that the inequalities

$$\int_0^{T^*} |\phi_x^0(t,\ell_0)|^2 dt \geq C \sum_{n\in\mathbb{N}} \sin^2 \lambda_n \ell_j (\mu_n \phi_{0,n}^2 + \phi_{1,n}^2), \quad j = 1, 2,$$

are verified for every $\bar\phi_0, \bar\phi_1 \in Z$. Since $Z \times Z$ is dense in $V \times H$, this inequality is still valid for all $\bar\phi_0 \in V$, $\bar\phi_1 \in H$.

If we denote

$$c_n := \max\{|\sin\lambda_n \ell_1|, |\sin\lambda_n \ell_2|\} \tag{61}$$

we will have

Theorem III.4. *There exists a positive constant C such that every solution $\bar\phi$ of (2) with initial state $(\bar\phi_0, \bar\phi_1) \in V \times H$ satisfies*

$$\int_0^{T^*} |\phi_x^0(t,\ell_0)|^2 dt \geq C \sum_{n\in\mathbb{N}} c_n^2 (\mu_n \phi_{0,n}^2 + \phi_{1,n}^2).$$

7. Study of the weights c_n

Theorem III.4 provides a satisfactory result as it allows to ensure the controllability of certain subspace of initial data. However, that subspace depends on the coefficients c_n.

Let us observe first that we cannot obtain an inequality like the one given in Theorem III.4, with non-vanishing coefficients c_n not necessarily defined by (61), when the ratio ℓ_1/ℓ_2 is an irrational number. Indeed, if $\dfrac{\ell_1}{\ell_2} = \dfrac{p}{q}$ con $p, q \in \mathbb{Z}$ then, for every $k \in \mathbb{Z}$ the function $\bar\phi = (\phi^0, \phi^1, \phi^2)$ defined by

$$\phi^0 \equiv 0, \quad \phi^1 = \sin\frac{kp\pi t}{\ell_1}\sin\frac{kp\pi x}{\ell_1}, \quad \phi^2 = -\sin\frac{kq\pi t}{\ell_2}\sin\frac{kq\pi x}{\ell_2},$$

is a solution of (2) and satisfies

$$\phi_x^0(t,\ell_0) \equiv 0.$$

This implies that the system is not approximately controllable[7]. .

In fact this condition is also sufficient.

Proposition III.8. *If the ratio ℓ_1/ℓ_2 is an irrational number, then all the coefficients c_n, $n \in \mathbb{N}$, defined by (61), are different from zero.*

[7]Moreover, the unique continuation property fails on a subspace of infinite dimension

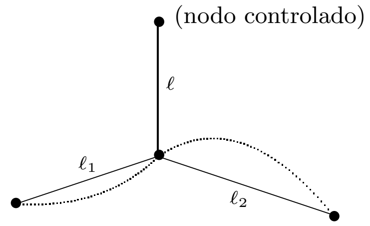

FIGURE 4. A localized vibration when $\frac{\ell_2}{\ell_1} \in \mathbb{Q}$.

PROOF. It suffices to observe that $c_n = 0$ implies $|\sin \lambda_n \ell_1| = |\sin \lambda_n \ell_2| = 0$, and then

$$\lambda_n \ell_1 = p\pi, \qquad \lambda_n \ell_2 = q\pi.$$

An that is

$$\frac{\ell_1}{\ell_2} = \frac{p}{q} \in \mathbb{Q}.$$

□

Summarizing we have obtained the following characterization of the lengths of the strings for which the system is approximately or spectrally controllable in some finite time.

COROLLARY III.4. *The following properties of the system of the three string network are equivalent:*

1) *The system is spectrally controllable in time* $T \geq T^*$;
2) *The system is approximately controllable in time* $T \geq T^*$;
3) *The ratio* ℓ_1/ℓ_2 *is an irrational number.*

In what follows we will try to find conditions over the values of ℓ_0, ℓ_1, ℓ_2 , which allows us to ensure that for some $\alpha \in \mathbb{R}$ all the initial states $(\bar{u}_0, \bar{u}_1) \in \mathcal{W}^\alpha$ are controllable in time $T^* = 2(\ell_0 + \ell_1 + \ell_2)$. More precisely, if we define

$$\Phi_\alpha = \left\{(\ell_0, \ell_1, \ell_2) \in \mathbb{R}^3_+ : \mathcal{W}^\alpha \subset \mathcal{W}_{T^*}\right\},$$

our aim is to indicate explicit conditions guaranteeing that $(\ell_0, \ell_1, \ell_2) \in \Phi_\alpha$.

If we can prove that for the values ℓ_0, ℓ_1, ℓ_2 of the lengths of the strings there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$,

$$c_n \geq C\lambda_n^{-\alpha}, \tag{62}$$

then, as it has been pointed out in Section 3.1, we will have $(\ell_0, \ell_1, \ell_2) \in \Phi_\alpha$.

Let us consider the function

$$\mathbf{a}^\alpha(\ell_1, \ell_2, \lambda) := (|\sin \lambda \ell_1| + |\sin \lambda \ell_2|)\lambda^\alpha.$$

It is clear that, if for some values ℓ_1, ℓ_2 the function $\mathbf{a}^\alpha(\ell_1, \ell_2, \lambda)$ has a positive lower bound:

$$\mathbf{a}^\alpha(\ell_1, \ell_2, \lambda) \geq a > 0 \text{ for all } \lambda \in \mathbb{R}_+,$$

then, for every $\ell_0 \in \mathbb{R}_+$, then the inequality (62) holds. Thus, we will be concerned with the study of the function $\mathbf{a}$.

It takes place

PROPOSITION III.9. *If there exists a constant $C > 0$ such that $|||n\frac{\ell_1}{\ell_2}||| \geq Cn^{-\alpha}$ for every $n \in \mathbb{N}$, then*

$$\mathbf{a}^\alpha(\ell_1, \ell_2, \lambda) \geq a > 0 \text{ for every } \lambda \geq 1. \tag{63}$$

PROOF. Let us assume that the inequality (63) is false. Then there exists a sequence (λ_k) such that

$$\mathbf{a}^\alpha(\ell_1, \ell_2, \lambda_k) = |\sin\lambda_k\ell_1|\lambda_k^\alpha + |\sin\lambda_k\ell_2|\lambda_k^\alpha \to 0 \ (k \to \infty). \tag{64}$$

Let us observe that in these conditions $\lambda_k \to \infty$. Indeed, if (λ_k) has a finite limit point $\lambda_* \geq 1$ then, from the continuity of $\mathbf{a}^\alpha$ follows

$$\mathbf{a}^\alpha(\ell_1, \ell_2, \lambda_*) = 0.$$

Consequently

$$\sin\lambda_*\ell_1 = \sin\lambda_*\ell_2 = 0,$$

and this may only happen if $\dfrac{\ell_1}{\ell_2}$ is an rational number. But in such case there exist values of $n \in \mathbb{N}$ such that

$$|||n\frac{\ell_1}{\ell_2}||| = 0,$$

and this contradicts the hypothesis of the proposition.

From (64) it holds

$$|\sin\lambda_k\ell_1|\lambda_k^\alpha \to 0, \qquad |\sin\lambda_k\ell_2|\lambda_k^\alpha \to 0. \tag{65}$$

Let us denote for every $k \in \mathbb{N}$,

$$\varepsilon_k := |||\frac{\ell_1\lambda_k}{\pi}|||, \qquad m_k := \frac{\ell_1\lambda_k}{\pi} - \varepsilon_k \in \mathbb{N},$$
$$\delta_k := |||\frac{\ell_2\lambda_k}{\pi}|||, \qquad n_k := \frac{\ell_2\lambda_k}{\pi} - \delta_k \in \mathbb{N}.$$

Since $0 \leq \varepsilon_k, \delta_k \leq \frac{1}{2}$,

$$\lim_{k\to\infty} \frac{m_k}{\lambda_k} = \frac{1}{\pi}, \qquad \lim_{k\to\infty} \frac{n_k}{\lambda_k} = \frac{1}{\pi}.$$

In particular, as $\lambda_k \to \infty$, the same happens with the sequences m_k and n_k. Besides,

$$\pi\varepsilon_k \leq 2|\sin\varepsilon_k\pi| = |\sin(m_k + \varepsilon_k)\pi| = \sin\lambda_k\ell_1,$$

and thus, from (65) we obtain $\varepsilon_k\lambda_k^\alpha \to 0$. Analogously, $\delta_k\lambda_k^\alpha \to 0$.

Then we have

$$\lim_{k\to\infty}\left(n_k\frac{\ell_1}{\ell_2} - m_k\right)n_k^\alpha = \frac{\pi}{\ell_2}\lim_{k\to\infty}\left(\varepsilon_k\frac{n_k}{\lambda_k} - \delta_k\frac{m_k}{\lambda_k}\right)n_k^\alpha = \frac{1}{\ell_2}\lim_{k\to\infty}(\varepsilon_k n_k^\alpha - \delta_k n_k^\alpha) = 0.$$

From this

$$|||n_k\frac{\ell_1}{\ell_2}|||n_k^\alpha \leq \left(n_k\frac{\ell_1}{\ell_2} - m_k\right)n_k^\alpha \to 0,$$

what contradicts the fact $|||n\frac{\ell_1}{\ell_2}||| \geq Cn^{-\alpha}$ for all $n \in \mathbb{N}$. $\square$

The condition provided by Proposition III.9, which implies the inequality (62), is sufficient for $\mathcal{W}^\alpha \subset \mathcal{W}_T$. Now we will see that this condition is also necessary in the following sense

PROPOSITION III.10. *If there exists a sequence of natural numbers n_k, $n_k \to \infty$, for which*

$$|||n_k \frac{\ell_1}{\ell_2}|||n_k^\alpha \to 0, \text{ as } k \to \infty,$$

then there exist values of $\ell_0 \in \mathbb{R}$ such that for every $T > 0$, the space $\mathcal{W}^\alpha$ is not contained in $\mathcal{W}_T$. That is, there exists initial states in $\mathcal{W}^\alpha$, which are not controllable.

PROOF. It suffices to choose ℓ_0 such that

$$|||n_k \frac{\ell_0}{\ell_2}|||n_k^\alpha \to 0, \text{ as } k \to \infty.$$

In fact, let m and $\tilde{m}$ be the closest to $n_k\frac{\ell_1}{\ell_2}$ and $n_k\frac{\ell_0}{\ell_2}$, respectively, natural numbers. Let (σ_p) be the sequence defined in page (54) and denote by p_k the index for which

$$\sigma_{p_k} = \min\left(n_k\frac{\pi}{\ell_2}, m\frac{\pi}{\ell_1}, \tilde{m}\frac{\pi}{\ell_0}\right).$$

Then we have

$$|\sigma_{p_k+2} - \sigma_{p_k}|\, \sigma_{p_k}^\alpha \to 0,$$

and this implies, in view of the inequalities (54),

$$|\lambda_{p_k+1} - \lambda_{p_k}|\, \lambda_{p_k}^\alpha \to 0. \tag{66}$$

On the other hand, the fact that all the initial states $(\bar{u}_0, \bar{u}_1) \in \mathcal{W}^\alpha$ are controllable in time T is equivalent to the following inequality being true

$$\int_0^T |\phi_x^0(t,\ell_0)|^2 dt \geq C \sum_{n\in\mathbb{N}} \lambda_n^{2\alpha}(\mu_n \phi_{0,n}^2 + \phi_{1,n}^2),$$

for all solution (2) with $(\bar{\phi}_0, \bar{\phi}_1) \in Z \times Z$. However, proceeding as in page 34, it can be easily proved that, due to (66) the latter inequality is impossible. $\square$

As in the case of the simultaneous control of two strings, Proposition III.9 reduces the problem of identifying subspaces of controllable initial states to the following diophantine approximation problem: given $\alpha < 0$, determine for which values of ξ there exists a constant $C > 0$ such that the inequality

$$|||n\xi||| \geq Cn^\alpha, \tag{67}$$

is true for each $n \in \mathbb{N}$.

In view of the results described in Section 2.1 we obtain:

COROLLARY III.5. *a) If $\frac{\ell_1}{\ell_2} \in B_\varepsilon$ then, the space $\mathcal{W}^{1+\varepsilon}$ is controllable in any time $T \geq T^*$. In particular, if $\frac{\ell_1}{\ell_2}$ is an algebraic irrational number then, $\mathcal{W}^{1+\varepsilon}$ is controllable for any $\varepsilon > 0$.*

b) If $\frac{\ell_1}{\ell_2}$ admits a bounded expansion in continuous fraction then the subspace $\mathcal{W}^1$ is controllable in any time $T \geq T^*$.

c) There exist values of the lengths ℓ_0, ℓ_1, ℓ_2 such that no subspace of the form $\mathcal{W}^\alpha$ is controllable in finite time T.

REMARK III.4. *As we will see later, the numbers* $\varkappa_n = \theta^0_{n,x}(\ell_0)$*, where* $\bar\theta_n$ *are the eigenfunctions of the elliptic problem associated to (2), are relevant for the control problem when we attempt to prove the observability inequalities in a direct way.*

The eigenfunctions $\bar\theta_n$ *may be explicitly expressed in terms of the eigenvalues*

$$\bar\theta_n = \begin{pmatrix} \theta^0_n \\ \theta^1_n \\ \theta^2_n \end{pmatrix} = \gamma_n \begin{pmatrix} \dfrac{\sin\lambda_n(\ell_0 - x)}{\sin\lambda_n\ell_0} \\ \dfrac{\sin\lambda_n(\ell_1 - x)}{\sin\lambda_n\ell_1} \\ \dfrac{\sin\lambda_n(\ell_2 - x)}{\sin\lambda_n\ell_2} \end{pmatrix},$$

where

$$\gamma_n = \sqrt{2}\left\{\frac{\ell_0}{\sin^2\lambda_n\ell_0} + \frac{\ell_1}{\sin^2\lambda_n\ell_1} + \frac{\ell_2}{\sin^2\lambda_n\ell_2}\right\}^{-\frac{1}{2}}.$$

Then

$$\varkappa_n = -\lambda_n\sqrt{2}\left\{\ell_0 + \ell_1\frac{\sin^2\lambda_n\ell_0}{\sin^2\lambda_n\ell_1} + \ell_2\frac{\sin^2\lambda_n\ell_0}{\sin^2\lambda_n\ell_2}\right\}^{-\frac{1}{2}}.$$

A rather rough estimation is

$$|\varkappa_n| \geq \lambda_n\sqrt{\frac{2}{\ell_0+\ell_1+\ell_2}}\,|\sin\lambda_n\ell_1\sin\lambda_n\ell_2|. \tag{68}$$

Is the lengths ℓ_0, ℓ_1, ℓ_2 *satisfy the condition (S)*[8] *then, according to Proposition A.4 from Appendix A, for* $\varepsilon > 0$ *there exists a constant* $C_\varepsilon > 0$ *such that*

$$|\sin\lambda_n\ell_1| \geq \frac{C_\varepsilon}{\lambda_n^{1+\varepsilon}}, \qquad |\sin\lambda_n\ell_2| \geq \frac{C_\varepsilon}{\lambda_n^{1+\varepsilon}}, \quad n\in\mathbb{N},$$

and with this, from (68) it follows

$$|\varkappa_n| \geq \frac{C_\varepsilon}{\lambda_n^{1+\varepsilon}}. \tag{69}$$

However, with the aid of Theorem III.4 it is possible to establish more precise estimations under weaker conditions imposed to the lengths. Indeed, it suffices to apply the inequality of the theorem to the solutions $\sin\lambda_n t\,\bar\theta_n$ *and* $\cos\lambda_n t\,\bar\theta_n$ *to get*

$$|\varkappa_n|^2\int_0^T|\cos\lambda_n t|^2\,dt \geq Cc_n^2\lambda_n^2, \qquad |\varkappa_n|^2\int_0^T|\sin\lambda_n t|^2\,dt \geq Cc_n^2\lambda_n^2.$$

From this we obtain

$$|\varkappa_n| \geq C\lambda_n\max\{|\sin\lambda_n\ell_1|, |\sin\lambda_n\ell_2|\}.$$

This inequality is obviously more exact than (68). From it we can obtain: If the ratio $\frac{\ell_1}{\ell_2}$ *belongs to* $\mathbf{B}_\varepsilon$ *then,*

$$|\varkappa_n| \geq \frac{C}{\lambda_n^\varepsilon}.$$

In spite of the estimation (68), the latter does not impose any restriction over the ratio ℓ_0*.*

[8] The numbers ℓ_0, ℓ_1, ℓ_2 are said to satisfy the condition (S) if they are linearly independent over $\mathbb{Q}$ and the ratios $\frac{\ell_i}{\ell_j}$ are algebraic numbers. For more details see Appendix A.

Proceeding a similar way, from the inequality (I.28) we obtain

$$C\lambda_n \geq |\varkappa_n|\,, \tag{70}$$

independently of the values of the lengths.

8. Relation between the simultaneous control of two strings and the control of the three string network from one exterior node

As the reader has already noted, the conditions on the lengths of the strings that allow to identify subspaces of controllable initial states, are the same for the simultaneous control of two strings and for the control of one exterior node of the three string network. Besides, when these conditions are satisfied, the corresponding subspaces of controllable initial states coincide, up to the boundary conditions, on the uncontrolled strings. This in not by chance. The reason is the following:

THEOREM III.5. *If $\mathcal{V}$ is a subspace of controllable initial states in time T for the system of the simultaneous control of two strings (10) then the subspace $(L^2(0,\ell_0)\times H^{-1}(0,\ell_0))\times\mathcal{V}$ of initial states for the system of the three string network (31) is controllable in time $T+2\ell_0$.*

We need some preliminary elements for the proof of this fact. We consider the spaces

$$\begin{aligned}
\mathcal{W}_0 &= \left\{(\bar\phi_0,\bar\phi_1)\in V\times H:\quad \phi_0^0(0)=\phi_0^1(0)=\phi_0^2(0)=0\right\},\\
\mathcal{V}_0 &= \left(H_0^1(0,\ell_1)\times L^2(0,\ell_1)\right)\times\left(H_0^1(0,\ell_2)\times L^2(0,\ell_2)\right).
\end{aligned}$$

For $(\bar\phi_0,\bar\phi_1)\in\mathcal{W}_0$ we denote by $\bar\phi$ the solution of the homogeneous system for the three string network (2) with initial state $(\bar\phi_0,\bar\phi_1)$, and by $\bar\psi=(0,\psi^1,\psi^2)$, where ψ^1,ψ^2 are the solutions of the homogeneous system (11) with initial states (ϕ_0^1,ϕ_1^1) and (ϕ_0^2,ϕ_1^2), respectively.

Let us choose $T\geq 2(\ell_0+\ell_1+\ell_2)$ and denote

$$||(\bar\phi_0,\bar\phi_1)||_E^2 := \int_0^T |\phi_x^0(t,\ell_0)|^2dt,$$

$$||(\bar\phi_0,\bar\phi_1)||_S^2 := \int_{\ell_0}^{T-\ell_0} |\psi_x^1(t,0)+\psi_x^2(t,0)|^2dt.$$

In view of the results of Proposition III.1 and Corollary III.4, the functions $||.||_E$ and $||.||_S$ define norms in $\mathcal{W}_0$ and $\mathcal{V}_0$ respectively, if, and only if, $\dfrac{\ell_1}{\ell_2}$ is an irrational number.

PROPOSITION III.11. *There exists a constant $C>0$ such that for every $(\bar\phi_0,\bar\phi_1)\in\mathcal{W}_0$,*

$$C||(\bar\phi_0,\bar\phi_1)||_E \geq ||(\phi_0^0,\phi_1^0)||_{H_0^1\times L^2} + ||(\bar\phi_0,\bar\phi_1)||_S.$$

PROOF. Let us observe that, if we apply D'Alembert formulas (II.5)to the component ϕ^0 we have, in account of the fact that $\phi_t^0(t,\ell_0)\equiv 0$,

$$\phi_x^0(t,0)=\ell_0^-\phi_x^0(t,\ell_0),\qquad \phi_t^0(t,0)=-\ell_0^+\phi_x^0(t,\ell_0).$$

Then, from Proposition II.2 we obtain the inequalities

(71)
$$\int_0^T |\phi_x^0(t,\ell_0)|^2dt \geq \int_{\ell_0}^{T-\ell_0} |\phi_x^0(t,0)|^2dt, \qquad \int_0^T |\phi_x^0(t,\ell_0)|^2dt \geq \int_{\ell_0}^{T-\ell_0} |\phi_t(t,0)|^2dt.$$

On the other hand, if $\mathbf{E}_{\phi^0}$ is the energy of the component ϕ^0 from Proposition II.1 it follows,

$$\mathbf{E}_{\phi^0}(0) \leq \int_{-\ell_0}^{\ell_0} |\phi_x^0(t,\ell_0)|^2dt.$$

And then, from the property (49) (see Proposition III.5)

$$||(\phi_0^0,\phi_1^0)||^2_{H_0^1\times L^2}{=}2\mathbf{E}_{\phi^0}(0) \leq C\int_0^T |\phi_x^0(t,\ell_0)|^2dt. \tag{72}$$

Let us observe now that the solution $\bar{\phi}$ may be decomposed as

$$\bar{\phi} = \bar{\psi} + \bar{\omega}, \tag{73}$$

were $\bar{\omega} = (\omega^0,\omega^1,\omega^2)$ is the unique solution of the problem

$$\begin{cases} \omega^i_{tt} - \omega^i_{xx} = 0 & \text{in } \mathbb{R}\times[0,\ell_i], & i=0,1,2,\\ \omega^i(t,\ell_i) = 0, \qquad \omega^i(t,0) = \phi^i(t,0) & \text{en } \mathbb{R}, & i=0,1,2,\\ \omega^0(0,x) = \phi^0_0(x), \qquad \omega^0_t(0,x) = \phi^0_1(x) & \text{in } [0,\ell_0], & \\ \omega^i(0,x) = \omega^i_t(0,x) = 0, & \text{in } [0,\ell_i], & i=1,2. \end{cases} \tag{74}$$

Indeed, for every $i=0,1,2$, the function

$$\bar{\eta} = \bar{\phi} - \bar{\psi} - \bar{\omega}$$

satisfies

$$\begin{cases} \eta^i_{tt} - \eta^i_{xx} = 0 & \text{in } \mathbb{R}\times[0,\ell_i],\\ \eta^i(t,0) = \eta^i(t,\ell_i) = 0 & \text{in } \mathbb{R},\\ \eta^i(0,x) = \eta^i_t(0,x) = 0 & \text{in } [0,\ell_i]. \end{cases}$$

Thus, $\bar{\eta} \equiv \bar{0}$,that is, (73) is verified. In particular,

$$\phi^i_x(t,0) = \psi^i_x(t,0) + \omega^i_x(t,0), \qquad i=0,1,2. \tag{75}$$

In view of the coupling conditions

$$\phi^0_x(t,0) + \phi^1_x(t,0) + \phi^2_x(t,0) = 0,$$

from (75) follows

$$-\phi^0_x(t,0) = \psi^1_x(t,0) + \psi^2_x(t,0) + \omega^1_x(t,0) + \omega^2_x(t,0). \tag{76}$$

Thus, we have

$$\begin{aligned} \int_{\ell_0}^{T-\ell_0} |\psi^1_x(t,0)+\psi^2_x(t,0)|^2dt \quad &\leq \quad \int_{\ell_0}^{T-\ell_0} |\psi^1_x(t,0)+\psi^2_x(t,0)+\omega^1_x(t,0)+\omega^2_x(t,0)|^2dt \\ &\quad + \int_{\ell_0}^{T-\ell_0} |\omega^1_x(t,0)+\omega^2_x(t,0)|^2dt \\ &\leq \quad \int_{\ell_0}^{T-\ell_0} |\phi^0_x(t,0)|^2dt + \int_{\ell_0}^{T-\ell_0} |\omega^1_x(t,0)+\omega^2_x(t,0)|^2dt. \end{aligned}$$

On the other hand, if we apply Lemma 4.2 from [**35**] to the system (74), we obtain that there exists a constant $C > 0$ such that

$$\int_{\ell_0}^{T-\ell_0} |\omega_x^1(t,0) + \omega_x^2(t,0)|^2 dt \le C \int_{\ell_0}^{T-\ell_0} |\omega_t^0(t,0)|^2 dt = C \int_{\ell_0}^{T-\ell_0} |\phi_t^0(t,0)|^2 dt.$$

So, we arrive to the inequality

$$\int_{\ell_0}^{T-\ell_0} |\psi_x^1(t,0) + \psi_x^2(t,0)|^2 dt \le \int_{\ell_0}^{T-\ell_0} |\phi_x^0(t,0)|^2 dt + C \int_{\ell_0}^{T-\ell_0} |\phi_t^0(t,0)|^2 dt,$$

and, in view of (71),

$$\int_{\ell_0}^{T-\ell_0} |\psi_x^1(t,0) + \psi_x^2(t,0)|^2 dt \le C \int_0^T |\phi_x^0(t,\ell_0)|^2 dt. \tag{77}$$

Finally, combining the inequalities (72) and (77) the assertion of the proposition is obtained. □

Proposition III.12. *Let $\bar g \in H$ be a continuous function such that $g^0(0) \neq 0$. Then, there exists a constant $C > 0$ such that for every $(\bar\phi_0, \bar\phi_1) \in \mathcal{W}_0$ and every $\lambda \in \mathbb{R}$,*

$$C||(\bar\phi_0 + \lambda \bar g, \bar\phi_1)||_E \ge ||(\phi_0^0, \phi_1^0)||_{H_0^1 \times L^2} + ||(\bar\phi_0, \bar\phi_1)||_S.$$

Proof. Let us denote by $\bar\varphi_\lambda$ the solution of the system (31) with initial state $(\bar\phi_0 + \lambda \bar g, \bar\phi_1)$. Let us observe that

$$|\lambda|^2 = |\phi_0^0(0) + \lambda g^0(0)|^2 \le C||\phi_0^0 + \lambda g^0||_{H^1}^2 \le C\mathbf{E}_{\varphi_\lambda^0}(0). \tag{78}$$

In a way analogous to that followed in the proof of Proposition III.11 it is proved that

$$\mathbf{E}_{\varphi_\lambda^0}(0) \le C||(\bar\phi_0 + \lambda \bar g, \bar\phi_1)||_E^2. \tag{79}$$

Then, from the relations (78), (79) we have

$$||(\bar\phi_0, \bar\phi_1)||_E \le ||(\bar\phi_0 + \lambda, \bar\phi_1)||_E + |\lambda| ||(\bar g, \bar 0)||_E \le C||(\bar\phi_0 + \lambda \bar g, \bar\phi_1)||_E$$

and the assertion holds from Proposition III.11 □

Proof of Theorem III.5. Let us denote by $\mathcal{F}_S$ and $\mathcal{F}_E$ the completions of H and $\mathcal{V}_0$ with the norms $||.||_E$ and $||.||_S$, respectively. In account of the fact that

$$H = \mathbb{R}\bar g + \mathcal{W}_0,$$

Proposition III.11 allows us to ensure that

$$\left(H_0^1(0,\ell_0) \times L^2(0,\ell_0)\right) \times \mathcal{F}_S \supset \mathcal{F}_E.$$

Then, the spaces of controllable initial states $\mathcal{C}_E = \mathcal{F}_E'$, $\mathcal{C}_S = \mathcal{F}_S'$ of the systems (31) and (10) given by HUM satisfy the relation

$$\mathcal{C}_E \subset (L^2(0,\ell_0) \times H^{-1}(0,\ell_0)) \times \mathcal{C}_S.$$

In particular, if $\mathcal{V} \subset \mathcal{C}_S$ then

$$\mathcal{V} \subset (L^2(0,\ell_0) \times H^{-1}(0,\ell_0)) \times \mathcal{V},$$

and this is the assertion of the theorem. □

Remark III.5. *The advantage of this approach is that it provides subspaces of controllable initial states of the system (31) in which no restriction is imposed on the regularity of the components ϕ_0^0, ϕ_1^0, in spite of what is needed in Corollary III.5.*

9. Lack of observability in small time

Due to the finite speed of propagation of the waves along the strings of the network (equal to one in this case), it is natural to expect that, when the control time T is small the system is not controllable and by that reason, that and observability inequality of the type (32) is impossible. This would imply that the system (31) is not spectrally controllable in that time. It turns out that such lack of controllability, even approximate, will occur whenever $T < T^* = 2(\ell_0 + \ell_1 + \ell_2)$.

For an arbitrary network, the lack of spectral controllability for values of T smaller than twice its lengths may be proved on the basis of results from the Theory on Non Harmonic Fourier Series (concretely, the Beurling-Malliavin theorem) and the asymptotic properties of the sequence of eigenvalues of the problem (see Chapter V). However, for the three string network it is possible to give a completely elementary proof of this fact based on the explicit construction (shown in Figure 5)of a solution $\bar\phi$ of (2), whose trace $\phi_x^0(.,\ell_0)$ in the observation point vanishes during a time $T < T^*$. This allows to ensure that the system (31) is not even approximately controllable. It holds

THEOREM III.6. *Let $T < T^*$. Then, there exist non-zero initial states*

$$(\bar\phi_0, \bar\phi_1) \in \bigcap_{\alpha\in\mathbb{R}} \mathcal{W}^\alpha,$$

for which the solution $\bar\phi$ of (2) satisfies

$$\phi_x^0(t,\ell_0) = 0 \qquad in \ \ [0,T]. \tag{80}$$

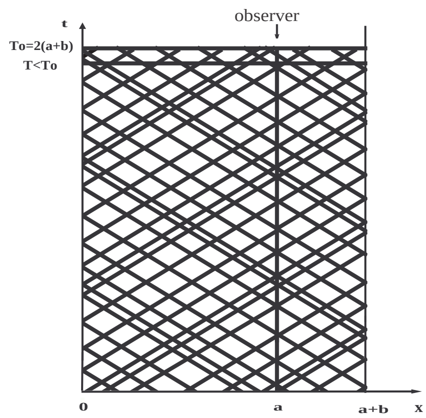

FIGURE 5. Support of a non-observable solution

In the proof of this theorem we use some technical results. Let $T > 0$ and $0 < \sigma < T$. We define the operator $I_\sigma : L^2(0,T) \to L^2(0,T-\sigma)$ by the formula

$$(I_\sigma f)(t) := \int_t^{t+\sigma} f(\tau)d\tau.$$

For arbitrary values of σ_1, $\sigma_2 \in (0,T)$ the system of functional equations

$$\left\{ \begin{array}{lll} I_{\sigma_i} f_i = 0 & \text{en } L^2(0,T-\sigma_i) & i=1,2, \\ f_1 + f_2 = 0 & \text{en } L^2(0,T), & \end{array} \right. \tag{81}$$

admits the trivial solution $f_1 = f_2 = 0$. Our aim is to study for which values of T this is the only solution of (81). The answer is given by the following

Lemma III.1. *Let $T_0 = \sigma_1 + \sigma_2$. Then, if $T < T_0$, the system (81) admits non-trivial solutions $f_i \in C^\infty([0,T])$, $i = 1,2$.*

Before proving this lemma let us see how Theorem III.6 may be obtained from it. It is clear that it is sufficient to prove Theorem III.6 for large values of T so that we assume $T \geq 2(\ell_0 + \hat{\ell})$, where $\hat{\ell}$ is the largest of the numbers ℓ_1, ℓ_2.

Let f_1, f_2 be non-zero solutions of(81) for $\sigma_1 = 2\ell_1$, $\sigma_2 = 2\ell_2$ and $\tilde{T} = T - 2\ell_0$. We define the functions

$$\phi^i(t,x) = \frac{1}{2}\int_{t-x}^{t+x} f_i(\tau - \ell_0)d\tau, \qquad i = 1,2,$$

for $x \in [0,\ell_i]$, $t \in [x + \ell_0, T - \ell_0 - x]$. These functions satisfy

$$(\mathrm{S}_i) \quad \left\{ \begin{array}{l} \phi^i_{xx}(t,x) = \phi^i_{tt}(t,x) \\ \phi^i(t,0) = \phi^i(t,\ell_i) = 0, \qquad \phi^i_x(t,0) = 0, \end{array} \right.$$

whenever $x \in [0,\ell_i]$ and $t \in [\ell_0 + x, T - \ell_0 - x]$.

Each of the functions ϕ^i may be extended to a solution of (S_i), which we will denote again by ϕ^i, defined in the region $[\ell_0, T-\ell_0] \times [0,\ell_0]$. Note that these functions have been chosen such that $\phi^i_x(t,0) = f_i(t)$ for $t \in [\ell_0, T-\ell_0]$. Besides,

$$\phi^1_x(t,0) + \phi^2_x(t,0) = f_1(t) + f_2(t) \quad \text{and} \quad \phi^1(t,0) = \phi^2(t,0) = 0,$$

and then, $\bar{\phi} = (\phi^0 = 0, \phi^1, \phi^2)$ is a solution of (2) defined in the time interval $[\ell_0, T-\ell_0]$. Consequently, the unique solution of (2) defined on $[0,T]$ that coincides with $\bar{u}$ on $[\ell_0, T-\ell_0]$ satisfies the inequality (80).

It just remains to prove that the initial data of $\bar{\phi}$ belong to $\mathcal{W}^{\ \alpha}$ for every real α.

As $\phi^0 \equiv 0$ and $f_1, f_2 \in C^\infty([0,T])$ this is equivalent to proving that for some $T^* \in [\ell_0, T-\ell_0]$ and every $k \in \mathbb{N}$ the following inequalities hold

$$\frac{\partial^{2k}}{\partial x^{2k}}\phi^i(T^*,0) = \frac{\partial^{2k}}{\partial x^{2k}}\phi^i(T^*,\ell_i) = \frac{\partial^{2k+1}}{\partial x^{2k+1}}\phi^1(T^*,0) + \frac{\partial^{2k+1}}{\partial x^{2k+1}}\phi^2(T^*,0) = 0, \tag{82}$$

and

$$\frac{\partial^{2k}}{\partial x^{2k}}\phi^i_t(T^*,0) = \frac{\partial^{2k}}{\partial x^{2k}}\phi^i_t(T^*,\ell_i) = \frac{\partial^{2k+1}}{\partial x^{2k+1}}\phi^1_t(T^*,0) + \frac{\partial^{2k+1}}{\partial x^{2k+1}}\phi^2_t(T^*,0) = 0. \tag{83}$$

Let us observe that, if f is a smooth function then

$$(I_\sigma f)^{(k)}(t) = (I_\sigma f^{(k)})(t).$$

This implies that, if f_1 and f_2 are smooth solutions of (81) then so are the functions $f_1^{(k)}$, $f_2^{(k)}$. That is why the functions $f_1^{(m)}$ and $f_2^{(m)}$ are also solutions of the equation (81) that defines f_1 and f_2. Then, choosing, e.g., $T^* = \ell_0 + \hat{\ell}$ we obtain the equalities (82) and (83).

The proof of Lemma III.1 is based on the following facts:

PROPOSITION III.13. *If* $\frac{\sigma_1}{\sigma_2} \in \mathbb{Q}$ *then, for every* $T > 0$ *there exists non-zero functions* $\varphi \in C^\infty([0,T])$ *such that*

$$\begin{aligned} I_{\sigma_1}\varphi = 0 \quad &in \quad [0, T-\sigma_1], \\ I_{\sigma_2}\varphi = 0 \quad &in \quad [0, T-\sigma_2]. \end{aligned}$$

PROOF. If $\frac{\sigma_1}{\sigma_2} \in \mathbb{Q}$ there exist numbers $p, q \in \mathbb{N}$, $\gamma \in \mathbb{R}$ such that

$$\frac{\sigma_1}{p} = \frac{\sigma_2}{q} = \gamma.$$

Let $\varphi \in C^\infty(\mathbb{R})$ a not identically vanishing, γ-periodic function such that

$$\int_0^\gamma \varphi(\tau)d\tau = 0.$$

Then,

$$I_{\sigma_1}\varphi = \int_t^{t+\sigma_1} \varphi(\tau)d\tau = \int_t^{t+\gamma p} \varphi(\tau)d\tau = p\int_t^{t+\gamma} \varphi(\tau)d\tau = 0.$$

In a similar way it may be proved that $I_{\sigma 2}\varphi = 0$. □

PROPOSITION III.14. *Let* $\varepsilon > 0$, $T = \sigma_1 + \sigma_2 - \varepsilon$ *and* $\frac{\sigma_1}{\sigma_2} \notin \mathbb{Q}$. *Then, there exists a non-zero function* $\varphi \in C^\infty([0,T])$, *such that*

$$\begin{aligned} I_{\sigma_1}\varphi = 0 \quad &in \quad [0, T-\sigma_1], \\ I_{\sigma_2}\varphi = 0 \quad &in \quad [0, T-\sigma_2]. \end{aligned} \tag{84}$$

PROOF. The real number σ_2 may be expressed as $\sigma_2 = n\sigma_1 + \omega$, $n \in \mathbb{N}$, $\omega \in (0, \sigma_1)$. Since $\frac{\sigma_1}{\sigma_2}$ is irrational, so is $\frac{\omega}{\sigma_2}$. Let us consider the sequence $\{\omega_k\}$ defined by

$$\omega_k \in (0, \sigma_1), \quad k - \omega_k \in \sigma_1\mathbb{Z}$$

(the values of $k\omega$ modulo σ_1). As a consequence of the irrationality of ω, we have $\omega_k \neq \omega_l$ if $k \neq l$ and that the sequence $\{\omega_k\}$ is dense in the interval $[0, \sigma_1]$. Then there exist $k_1 < 0$, $k_2 > 0$ such that $\omega_{k_1}, \omega_{k_2} \in [\sigma_1 - \varepsilon, \sigma_1)$ and $\omega_k \in [0, \sigma_1 - \varepsilon)$ for every k satisfying $k_1 < k < k_2$[9].

Now let us define the subsets of $[0, \sigma_1)$:

$$\Omega_k = (\omega_k, \omega_k + \gamma)$$

for $k_1 < k \leq k_2$, where $\gamma > 0$ is sufficiently small so that it holds

$$\overline{\Omega_k} \cap \overline{\Omega_l} = \emptyset \;\text{ if } k \neq l \;\text{ and }\; \Omega_k \subset (0, \sigma_1) \;\text{ for } k_1 < k, l \leq k_2.$$

It is not difficult to show that the sets Ω_k have the following properties:

(i) if $t \in \Omega_k$ with $k_1 < k < k_2$ and $t = \omega_k + \tau$ for some $\tau \in (0, \gamma)$ then,

$$\begin{aligned} t + \omega = \omega_{k+1} + \tau \quad &\text{if} \quad \omega_k < \omega_{k+1}, \\ t + \omega = \omega_{k+1} + \tau - \sigma_1 \quad &\text{if} \quad \omega_{k+1} < \omega_k. \end{aligned}$$

(ii) if $t \in [0, \sigma_1 - \varepsilon) \setminus \cup\Omega_k$, then,

$$\begin{aligned} t + \omega \notin \cup\Omega_k \quad &\text{if} \quad t < \sigma_1 - \omega, \\ t + \omega - \sigma_1 \notin \cup\Omega_k \quad &\text{if} \quad t \geq \sigma_1 - \omega. \end{aligned}$$

[9]In other words, $k_1 < 0$ and $k_2 > 0$ are the values of k with the smallest non-zero absolute value such that ω_k is in the interval $[\ell_1 - \varepsilon, \ell_1)$.

Let us choose now a function $\psi \in C^\infty([0,\sigma_1])$ with support contained in the interval $(0,\gamma)$ and satisfying $\int_0^\gamma \psi(\tau)d\tau = 0$ and define the function φ in $[0,\sigma_1]$ by

$$\varphi(t) = \begin{cases} \varphi(t-\omega_k) & \text{if} \quad t \in \Omega_k, \\ 0 & \text{if} \quad t \in [0,\sigma_1] \setminus \cup\Omega_k. \end{cases}$$

Then it follows $\varphi \in C^\infty([0,\sigma_1])$ and $\operatorname{supp}\varphi \subset \cup\Omega_k \subset (0,\sigma_1)$. In particular, the σ_1-periodic extension of φ to $\mathbb{R}$, which we still denote by φ, verifies $\varphi \in C^\infty(\mathbb{R})$. Let us check that φ is in addition one of the functions, whose existence is asserted in the lemma.

Let $t_1, t_2 \in [0,\sigma_1] \setminus \cup\Omega_k$, then

$$\begin{aligned} \int_{t_1}^{t_2} \varphi(\tau)d\tau &= \sum_{m:\Omega_m \subset (t_1,t_2)} \int_{\Omega_m} \varphi(\tau)d\tau = \sum_m \int_{\omega_m}^{\omega_m+\gamma} \varphi(\tau-\omega_m)d\tau \\ &= \sum_m \int_0^\gamma \varphi(\tau)d\tau = 0. \end{aligned}$$

In particular, if we choose $t_1 = 0$, $t_2 = \sigma_1$ we get $\int_0^{\sigma_1} \varphi(\tau)d\tau = 0$, and therefore, since φ is σ_1-periodic, $I_{\sigma_1}\varphi = 0$ in $\mathbb{R}$.

It remains to calculate $I_{\sigma_2}\varphi$ for the values of t in the interval $[0,\sigma_1-\varepsilon)$. Two cases are possible:

Case 1. $t \in \Omega_k$ for some k. Then, $t = \omega_k + \tau$ with $\tau \in (0,\gamma)$. We will assume that $\omega_k < \omega_{k+1}$, since when $\omega_{k+1} < \omega_k$ the result is obtained in a similar way.

In view of the property (i) of the sets Ω_k mentioned above, we obtain

$$\begin{aligned} I_{\sigma_2}\varphi(t) &= \int_t^{t+\omega} \varphi(s)ds = \int_{\omega_k+\tau}^{\omega_{k+1}+\tau} \varphi(s)ds \\ &= \int_{\omega_k+\tau}^{\omega_k+\gamma} \varphi(s)ds + \int_{\omega_{k+1}}^{\omega_{k+1}+\tau} \varphi(s)ds + \int_{\omega_k+\gamma}^{\omega_{k+1}} \varphi(s)ds. \end{aligned}$$

But $\omega_k+\gamma$ and ω_{k+1} do not belong to $\cup\Omega_k$, $\int_{\omega_k+\gamma}^{\omega_{k+1}} \varphi(s)ds = 0$, and thus

$$\begin{aligned} I_{\sigma_2}\varphi(t) &= \int_{\omega_k+\tau}^{\omega_k+\gamma} \varphi(s-\omega_k)ds + \int_{\omega_{k+1}}^{\omega_{k+1}+\tau} \varphi(s-\omega_{k+1})ds \\ &= \int_\tau^\gamma \psi(s)ds + \int_0^\tau \psi(s)ds = 0. \end{aligned}$$

Case 2. $t \notin \cup\Omega_k$. In view of the property (ii), if $t < \sigma_1 - \omega$, then $t+\omega$ does not belong to $\cup\Omega_k$ and we have

$$I_{\sigma_2}\varphi(t) = \int_t^{t+\omega} \varphi(s)ds = 0.$$

If $t \geq \sigma_1 - \omega$, then $t - \sigma_1 + \omega \notin \cup\Omega_k$ and it holds

$$I_{\sigma_2}\varphi(t) = \int_t^{t+\omega-\sigma_1} \varphi(s)ds = 0.$$

This proves the proposition □

Proof of Lemma III.1. It follows immediately from the previous propositions. It suffices to take $f_1 = \varphi$ and $f_2 = -\varphi$ according to Propositions III.13 or III.14, depending on whether $\frac{\sigma_1}{\sigma_2}$ is rational of irrational. □

10. Application of the method of moments to the control of the three string network

In this section we will study the problem of moments (II.38)

$$\int_0^T \varkappa_{|k|} e^{i\lambda_k t}\ h(t)dt = u_{1,|k|} - i\lambda_k u_{0,|k|} \quad \text{for every } k \in \mathbb{Z}_*, \tag{85}$$

for the three string network. Recall that, in view of the results of Section II.3, the existence of a solution $h \in L^2(0,T)$ of the problem of moments (85) is equivalent to the controllability in time T of the initial state $(\bar{u}_0, \bar{u}_1)$ with

$$\bar{u}_0 = \sum_{n\in\mathbb{N}} u_{0,n}\bar{\theta}_n, \qquad \bar{u}_1 = \sum_{n\in\mathbb{N}} u_{1,n}\bar{\theta}_n.$$

With this, our aim is to show an alternative way to the study of the control problem for the system (31).

If we perform in (85) the change of variable $t \to t - \frac{T}{2}$, we obtain

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i\lambda_n t}\ h(t - \frac{T}{2})dt = \frac{1}{\varkappa_n}\left(u_{1,n} - i\lambda_n u_{0,n}\right) e^{i\lambda_n \frac{t}{2}}.$$

Denoting $m_n := \frac{1}{\varkappa_n}\left(u_{1,n} - i\lambda_n u_{0,n}\right) e^{i\lambda_n \frac{T}{2}}$, $A := \frac{T}{2}$, the problem (85) will be written in the form (II.28). This implies, in account of Proposition II.6, that, if we can construct a sequence (v_n) biorthogonal to a $\left(e^{i\lambda_n t}\right)$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ then the initial states satisfying

$$\sum_{n\in\mathbb{Z}_*}\left|\frac{1}{\varkappa_n}\left(u_{1,n} - i\lambda_n u_{0,n}\right) e^{i\lambda_n \frac{T}{2}}\right| \|v_n\|_{L^2} < \infty \tag{86}$$

are controllable in time T with control

$$v = \sum_{n\in\mathbb{Z}_*} \frac{1}{\varkappa_n}\left(u_{1,n} - i\lambda_n u_{0,n}\right) e^{i\lambda_n \frac{T}{2}} v_n.$$

The inequality (86) is equivalent to

$$\sum_{n\in\mathbb{Z}_*} \frac{1}{|\varkappa_n|}\left(\left|u_{1,n}\right| + \lambda_n \left|u_{0,n}\right|\right) \|v_n\|_{L^2} < \infty.$$

In particular, if the biorthogonal sequence (v_n) has been obtained from a generating function F then all the initial states that satisfy

$$\sum_{n\in\mathbb{Z}_*} \frac{1}{|\varkappa_n|\,|F'(\lambda_n)|}\left(\left|u_{1,n}\right| + \lambda_n \left|u_{0,n}\right|\right) < \infty \tag{87}$$

are controllable in time T.

Let us remark that for the three string network it is easy to construct a generating function, since we already know a function that vanishes at the numbers λ_n: recall that, as it has been shown in Proposition III.6, $q(\lambda_n) = 0$, where

$$\begin{aligned} q(z) \quad &= \quad \cos z\ell_0 \sin z\ell_1 \sin z\ell_2 + \sin z\ell_0 \cos z\ell_1 \sin z\ell_2 \\ &\qquad + \sin z\ell_0 \cos z\ell_1 \sin z\ell_1 \cos z\ell_2, \end{aligned} \tag{88}$$

and this is an entire function bounded on the real axis: $|q(z)| \leq 3$.

On the other hand, if we replace in (88) $\cos z\ell_k$ and $\sin z\ell_k$ by their expressions in terms of $e^{iz\ell_k}$ and $e^{-iz\ell_k}$:

$$\cos z\ell_k = \frac{1}{2}\left(e^{iz\ell_k} + e^{-iz\ell_k}\right), \qquad \sin z\ell_k = -\frac{i}{2}\left(e^{iz\ell_k} - e^{-iz\ell_k}\right),$$

we can see that q may be written as the sum of eight terms of the form $c_h e^{izh}$, where c_h are constants and

$$|h| \leq \ell_0 + \ell_1 + \ell_2.$$

Then, there exists a constant $C > 0$ such that for every $z \in \mathbb{C}$

$$|q(z)| \leq Ce^{|z|(\ell_0+\ell_1+\ell_2)},$$

that is, the function q is of exponential type at most $\ell_0 + \ell_1 + \ell_2$.

Then, based on the results of the Subsection II.3.1, we can assert that there exists a sequence (v_n) biorthogonal to $\left(e^{i\lambda_n t}\right)$ in any interval $(-\frac{T}{2}, \frac{T}{2})$ with $T \geq 2\left(\ell_0 + \ell_1 + \ell_2\right)$ that satisfies

$$||v_n||_{L^2(-\frac{T}{2},\frac{T}{2})} \leq \frac{C}{|q'(\lambda_n)|}, \quad n \in \mathbb{N},$$

where the constant $C > 0$ does not depend on n.

This guarantees immediately that the spaces of sequences for which the problem of moments (85) has a solution is dense in l^2. Therefore, the space of controllable initial states in time $T \geq 2\left(\ell_0 + \ell_1 + \ell_2\right)$, is dense in $H \times V'$. Moreover, all the initial states from $Z \times Z$ are controllable.

Now we estimate $|q'(\lambda_n)|$ in order to identify larger subspaces of controllable initial states. Observe that the function q may be written in the form

$$q(z) = \sin z\ell_0 \sin z\ell_1 \sin z\ell_2 \left(\cot z\ell_0 + \cot z\ell_1 + \cot z\ell_2\right). \tag{89}$$

Then it follows

$$|q'(\lambda_n)| = |\sin \lambda_n\ell_0 \sin \lambda_n\ell_1 \sin \lambda_n\ell_2|\, \mathbf{A}_n, \tag{90}$$

where we have denoted

$$\mathbf{A}_n = \left(\frac{\ell_0}{\sin^2 \lambda_n\ell_0} + \frac{\ell_1}{\sin^2 \lambda_n\ell_1} + \frac{\ell_2}{\sin^2 \lambda_n\ell_2}\right).$$

In account of (87), we can ensure that the initial states satisfying

$$\sum_{n\in\mathbb{Z}_*} \frac{1}{|\varkappa_n|\,|q'(\lambda_n)|}\left(\left|u_{1,n}\right| + \lambda_n\left|u_{0,n}\right|\right) < \infty. \tag{91}$$

are controllable.

To make this information more precise, we need to estimate the product $|\varkappa_n|\,|q'(\lambda_n)|$. Recall that (see Remark III.4)

$$|\varkappa_n| = \frac{\sqrt{2}\lambda_n}{|\sin \lambda_n\ell_0|}\mathbf{A}_n^{-\frac{1}{2}},$$

and thus we have

$$|\varkappa_n|\,|q'(\lambda_n)| = \sqrt{2}\lambda_n\,|\sin \lambda_n\ell_1 \sin \lambda_n\ell_2|\,\mathbf{A}_n^{\frac{1}{2}}.$$

Then,

$$\begin{aligned}
|\varkappa_n|^2 |q'(\lambda_n)|^2 &= 2\lambda_n^2 |\sin\lambda_n\ell_1 \sin\lambda_n\ell_2|^2 \left(\frac{\ell_0}{\sin^2\lambda_n\ell_0} + \frac{\ell_1}{\sin^2\lambda_n\ell_1} + \frac{\ell_2}{\sin^2\lambda_n\ell_2}\right) \\
&\geq 2\lambda_n^2 \left(\ell_1 \sin^2\lambda_n\ell_2 + \ell_2 \sin^2\ell_1\right) \geq C\lambda_n^2 c_n^2.
\end{aligned}$$

Here $c_n = \max\left(|\sin\lambda_n\ell_1|, |\sin\lambda_n\ell_2|\right)$ are the coefficients defined by (61) in Section 6.

With this we can conclude that a sufficient condition for the initial state $(\bar{u}_0, \bar{u}_1)$ to be controllable is

$$\sum_{n\in\mathbb{Z}_*} \frac{1}{c_n}\left(|u_{1,n}| + \lambda_n |u_{0,n}|\right) < \infty. \tag{92}$$

Let us observe that this result is weaker than that given in Proposition 6, since, if the initial state $(\bar{u}_0, \bar{u}_1)$ satisfies (92) then it also satisfies

$$\sum_{n\in\mathbb{Z}_*} \frac{1}{c_n^2}\left(u_{1,n}^2 + \lambda_n u_{0,n}^2\right) < \infty.$$

Let us choose $\delta > 0$. The series

$$\sum_{n\in\mathbb{Z}_*} \frac{1}{\lambda_n^{1+\delta}}$$

converges for every $\delta > 0$, as $\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{\pi}{\ell_0+\ell_1+\ell_2}$. Then, with the help of the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
\sum_{n\in\mathbb{Z}_*} \frac{1}{c_n\lambda_n}\left(|u_{1,n}| + \lambda_n |u_{0,n}|\right) &< \sum_{n\in\mathbb{Z}_*} \frac{\lambda_n^{\delta-1}}{c_n^2}\left(u_{1,n}^2 + \lambda_n^2 u_{0,n}^2\right) \sum_{n\in\mathbb{Z}_*} \frac{1}{\lambda_n^{1+\delta}} \\
&\leq C \sum_{n\in\mathbb{Z}_*} \frac{\lambda_n^{\delta-1}}{c_n^2}\left(u_{1,n}^2 + \lambda_n^2 u_{0,n}^2\right).
\end{aligned}$$

Thus, for (92) to be verified and consequently the initial state $(\bar{u}_0, \bar{u}_1)$ to be controllable in time T, it is sufficient that

$$\sum_{n\in\mathbb{Z}_*} \frac{\lambda_n^{\delta-1}}{c_n^2}\left(u_{1,n}^2 + \lambda_n^2 u_{0,n}^2\right) < \infty.$$

In particular, if $\frac{\ell_1}{\ell_2} \in \mathbf{B}_\varepsilon$, the controllability condition (92) obtained with the method of moments guarantees that all the initial states from

$$(\bar{u}_0, \bar{u}_1) \in \mathcal{W}^{\frac{3}{2}+\varepsilon+\delta} = V^{\frac{3}{2}+\varepsilon+\delta} \times V^{\frac{1}{2}+\varepsilon+\delta},$$

with arbitrarily small $\delta > 0$ are controllable.

This difference between the result is due to the technique we have used, mainly to the possible inaccuracy in the estimation of the sequence $|\varkappa_n| |q'(\lambda_n)|$.

Remark III.6. *According to Proposition II.9, once we have identified subspaces of controllable initial states in time T of the form $\mathcal{W}^r$ it can be constructed,* a posteriori *a sequence $(\tilde{v}_n)$ biorthogonal to $\left(e^{i\lambda_n t}\right)$ in $L^2(0,T)$ satisfying*

$$\|\tilde{v}_n\|_{L^2(0,T)} \leq C\lambda_n^{r-1}.$$

Thus, in view of Corollary III.5, if $\frac{\ell_1}{\ell_2} \in \mathbf{B}_\varepsilon$ *then for the system of the three string network is can be constructed a sequence* $(\tilde{v}_n)$ *biorthogonal to* $\left(e^{i\lambda_n t}\right)$ *in* $L^2(0,T)$ *verifying*

$$\|\tilde{v}_n\|_{L^2(0,T)} \leq C\lambda_n^{\varepsilon}. \tag{93}$$

Let us remark that the biorthogonal sequence (v_n) *used in this section not necessarily coincides with* $(\tilde{v}_n)$. *Recall in addition, that we do not resort to that sequence, since to attempt to get information on controllable subspaces without using the information provided by Corollary III.5. The norms of the elements of the sequences* (v_n) *could be estimated in the following way. Since*

$$\|v_n\|_{L^2(-A,A)} \leq \frac{C}{|q'(\lambda_n)|},$$

it suffices to estimate $|q'(\lambda_n)|$. *From the equality (89) we obtain*

$$\begin{aligned} |q'(\lambda_n)| &\geq \ell_0 \left|\sin\lambda_n\ell_1 \sin\lambda_n\ell_2\right| + \ell_1 \left|\sin\lambda_n\ell_0 \sin\lambda_n\ell_2\right| + \ell_2 \left|\sin\lambda_n\ell_0 \sin\lambda_n\ell_1\right| \\ &\geq Cs(\lambda, \ell_0, \ell_1, \ell_2), \end{aligned}$$

where we have denoted

$$s(\lambda, \ell_0, \ell_1, \ell_2) := \left|\sin\lambda_n\ell_0\right|\left|\sin\lambda_n\ell_1\right| + \left|\sin\lambda_n\ell_0\right|\left|\sin\lambda_n\ell_2\right| + \left|\sin\lambda_n\ell_1\right|\left|\sin\lambda_n\ell_2\right|.$$

To obtain lower estimates of the function s *we need to impose additional restrictions on the lengths* ℓ_0, ℓ_1, ℓ_2. *Let us assume that those lengths satisfy the following rational approximation conditions, which we will call briefly* conditions *(S) (see also Definition A.1 in Appendix A):*

- ℓ_0, ℓ_1, ℓ_2 *are linearly independent over the field* $\mathbb{Q}$ *of rational numbers;*
- *all the ratios* $\frac{\ell_i}{\ell_j}$ *are algebraic numbers, that is, roots of polynomials with rational coefficients.*

Under these hypotheses it is proved in Proposition A.4 that for every $\varepsilon > 0$ *there exists a constant* $C_\varepsilon > 0$ *such that for every* $n = 1, 2, \ldots$, *the following inequality holds*

$$s(\lambda_n, \ell_0, \ell_1, \ell_2) \geq C_\varepsilon \left(\lambda_n\right)^{-1-e}.$$

This guarantees that

$$\|v_n\|_{L^2(-A,A)} \leq C\lambda_n^{1+\varepsilon}.$$

Unfortunately, we have imposed restrictive conditions on ℓ_0 *and we have been able to prove an estimate weaker than (93). This fact could be caused by two reason: that the norms of the elements of the sequence* (v_n) *are actually larger than those of the elements of the sequences* $(\tilde{v}_n)$ *or that the technique we have used to estimate* $|q'(\lambda_n)|$ *is not precise enough.*

CHAPTER IV

General trees

In this chapter we study the control problem from one exterior node for networks of strings, which are supported on a tree-shaped graph. We will follow the technique described in Chapter III for the three string network, which is the simplest example of a network supported by a tree-shaped graph, not reduced to a single string.

Let us briefly recall this technique. The key element is the construction of an operator $\mathbf{B} : V \times H \to V \times H$, which guarantees the existence of a constant $C > 0$ such that all the solutions of the homogeneous system (I.11)-(I.16) with initial states $(\bar{\phi}_0, \bar{\phi}_1) \in Z \times Z$ satisfy the observability inequality

$$C \int_0^T |\partial_n \phi^1(t, \mathbf{v}_1)|^2 dt \geq \|\mathbf{B}(\bar{\phi}_0, \bar{\phi}_1)\|^2_{V \times H},$$

where T is twice the total length of the network (here we used the notations introduced in Chapter I for general networks).

The operator $\mathbf{B}$ has the property of being essentially a diagonal operator: there exist real numbers b_n such that

$$\|\mathbf{B}(\bar{\phi}_0, \bar{\phi}_1)\|^2_{V \times H} = \sum_{n \in \mathbb{N}} b_n^2 \left(\mu_n \phi_{0,n}^2 + \phi_{1,n}^2\right).$$

This leads to the inequality

$$C \int_0^T |\partial_n \phi^1(t, \mathbf{v}_1)|^2 dt \geq \sum_{n \in \mathbb{N}} b_n^2 \left(\mu_n \phi_{0,n}^2 + \phi_{1,n}^2\right),$$

which allows to indicate subspaces $H \times V'$ of controllable initial states in time T.

1. Notations and statement of the problem

1.1. Notations for the elements of the graphs. In this section, we introduce precise notations for the elements of the rest configuration graph. This is needed to write the equations of the motion of the network in a way that takes into account the topological structure of the graph.

Let $\mathcal{A}$ be a planar, connected graph without closed paths. According to the usual terminology in Graph Theory, those graphs will be called *trees*. By the multiplicity of a vertex of $\mathcal{A}$ we mean the number of edges that branch out from that vertex. If the multiplicity is equal to one, the vertex is called exterior, otherwise, it is said to be interior. We assume that the graph $\mathcal{A}$ does not contain vertices of multiplicity two, since they are irrelevant for our model.

In what follows, we describe a procedure for indexing the edges and vertices of the graph. In Figure 1 an example is given of a tree with indices defined according to this rule. First, we choose an exterior vertex and denote it by $\mathcal{R}$. It is called the root of $\mathcal{A}$. The remaining edges and vertices will be denoted by $\mathbf{e}_{\bar{\alpha}}$ and $\mathcal{O}_{\bar{\alpha}}$, respectively, where $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$ is a multi-index (possibly empty) of variable length k defined by recurrence for every edge in the following way.

For the edge containing the root $\mathcal{R}$ we choose the empty index. Thus, that edge is denoted by $\mathbf{e}$ and its vertex different from $\mathcal{R}$ is denoted by $\mathcal{O}$.

Assume now that the interior vertex $\mathcal{O}_{\bar{\alpha}}$, contained in the edge $\mathbf{e}_{\bar{\alpha}}$, has multiplicity equal to $m_{\bar{\alpha}}+1$. This means that there are $m_{\bar{\alpha}}$ edges, different from $\mathbf{e}_{\bar{\alpha}}$, that branch out from $\mathcal{O}_{\bar{\alpha}}$. We denote these edges by $\mathbf{e}_{\bar{\alpha}\circ\beta}$, $\beta = 1, \ldots, m_{\bar{\alpha}}$ and the other vertex of the edge $\mathbf{e}_{\bar{\alpha}\circ\beta}$ by $\mathcal{O}_{\bar{\alpha}\circ\beta}$. Here, $\bar{\alpha}\circ\beta$ represents the index $(\alpha_1, \ldots, \alpha_k, \beta)$, obtained by adding a new component β to the index $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$. In general, if $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$ and $\bar{\beta} = (\beta_1, \ldots, \beta_m)$, then $\bar{\alpha}\circ\bar{\beta}$ will denote the multi-index of length $k+m$ defined by $\bar{\alpha}\circ\bar{\beta} = (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m)$.

Let now $\mathcal{M}$ be the set of the interior vertices of $\mathcal{A}$ and $\mathcal{S}$ the set of exterior vertices, except $\mathcal{R}$, and define

$$\mathcal{I}_{\mathcal{M}} = \{\bar{\alpha};\quad \mathcal{O}_{\bar{\alpha}} \in \mathcal{M}\}, \qquad \mathcal{I}_{\mathcal{S}} = \{\bar{\alpha};\quad \mathcal{O}_{\bar{\alpha}} \in \mathcal{S}\},$$

which are the sets of the indices of the interior and exterior vertices, except $\mathcal{R}$, respectively. Note that with these notations, we admit the empty multi-index, which corresponds to the vertex $\mathcal{O}$ and belongs to one of the sets $\mathcal{I}_{\mathcal{M}}$ or $\mathcal{I}_{\mathcal{S}}$. Finally, $\mathcal{I} = \mathcal{I}_{\mathcal{S}} \bigcup \mathcal{I}_{\mathcal{M}}$ is the set of the indices of all the vertices, except that of the root $\mathcal{R}$.

Further, for $\bar{\alpha} \in \mathcal{I}_{\mathcal{M}}$, the sets

$$\mathcal{A}_{\bar{\alpha}} = \left\{\mathbf{e}_{\bar{\alpha}\circ\bar{\beta}};\quad \bar{\alpha}\circ\bar{\beta} \in \mathcal{I}\right\}$$

are called sub-trees of $\mathcal{A}$. Note that $\mathcal{A}_{\bar{\alpha}}$ is formed by the edges having indices with a common initial part $\bar{\alpha}$. This means that $\mathcal{A}_{\bar{\alpha}}$ is also a tree branching out from the vertex of $\mathbf{e}_{\bar{\alpha}}$ different from $\mathcal{O}_{\bar{\alpha}}$. Then, if one chooses that vertex as the root $\mathcal{R}_{\bar{\alpha}}$ of $\mathcal{A}_{\bar{\alpha}}$ and denotes by $\mathbf{e}^{\bar{\alpha}}_{\bar{\beta}}$ the edge with index $\bar{\beta}$ in $\mathcal{A}_{\bar{\alpha}}$ according to the numbering rule defined above for trees, it holds that

$$\mathbf{e}_{\bar{\alpha}\circ\bar{\beta}} = \mathbf{e}^{\bar{\alpha}}_{\bar{\beta}} \qquad \mathcal{O}_{\bar{\alpha}\circ\bar{\beta}} = \mathcal{O}^{\bar{\alpha}}_{\bar{\beta}}.$$

In order to prove properties of trees, we shall often proceed by induction with respect to the largest length of the indices $\bar{\alpha}$ used to number the edges according to the procedure described above. To do this we should prove that:

(1) The property is true for the simplest case of a one-edged tree (i.e., the corresponding network is formed by a single string).
(2) If the property is true for all the sub-trees $\mathcal{A}_1, \ldots, \mathcal{A}_m$ branching out from $\mathcal{O}$, then it is also true for the whole tree $\mathcal{A}$.

In what follows such process will be called simply induction.

Besides, the length of the edge $\mathbf{e}_{\bar{\alpha}}$ will be denoted by $\ell_{\bar{\alpha}}$. Then, $\mathbf{e}_{\bar{\alpha}}$ may be parameterized by its arc length by means of the functions $\pi_{\bar{\alpha}}$, defined in $[0, \ell_{\bar{\alpha}}]$ such that $\pi_{\bar{\alpha}}(\ell_{\bar{\alpha}}) = \mathcal{O}_{\bar{\alpha}}$ and $\pi_{\bar{\alpha}}(0)$ is the other vertex of this edge.

Finally, we denote by $L_{\mathcal{A}}$ and $L_{\bar{\alpha}}$, $\bar{\alpha} \in \mathcal{I}$, the sum of the lengths of all the edges of the tree $\mathcal{A}$ (i.e., the total length of $\mathcal{A}$) and of its sub-trees $\mathcal{A}_{\bar{\alpha}}$, respectively.

1.2. Equations of the motion of the network. In this subsection we write the equations of the motion of the tree-shaped network with a controlled node (I.11)-(I.16) with the specific notations introduced for trees in this chapter. The vertex of

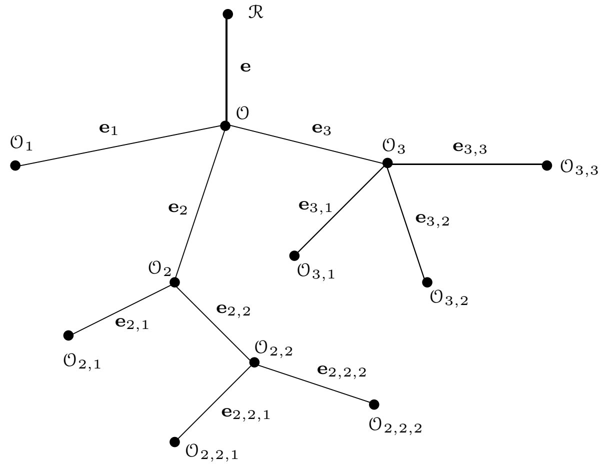

FIGURE 1. A tree with indices for its vertices and edges

the graph $\mathcal{A}$ corresponding to the controlled node of the network has been chosen as the root of the tree.

$$
\begin{array}{lll}
(1) & u^{\bar\alpha}_{tt}(t,x) = u^{\bar\alpha}_{xx}(t,x) & \text{in } \mathbb{R}\times[0,\ell_{\bar\alpha}], \quad \bar\alpha\in\mathcal{I},\\
(2) & u(t,0) = v(t), & \text{in } \ \mathbb{R},\\
(3) & u^{\bar\alpha}(t,\ell_{\bar\alpha}) = 0 & \text{in } \mathbb{R}, \ \ \bar\alpha\in\mathcal{I}_{\mathcal{S}},\\
(4) & u^{\bar\alpha\circ\beta}(t,0) = u^{\bar\alpha}(t,\ell_{\bar\alpha}) & \text{in } \mathbb{R}, \quad \beta=1, \ \ldots, m_{\bar\alpha}, \ \ \bar\alpha\in\mathcal{I}_{\mathcal{M}},\\
(5) & \sum_{\beta=1}^{m_{\bar\alpha}} u^{\bar\alpha\circ\beta}_x(t,0) = u^{\bar\alpha}_x(t,\ell_{\bar\alpha}) & \text{in } \mathbb{R}, \ \ \bar\alpha\in\mathcal{I}_{\mathcal{M}},\\
(6) & u^{\bar\alpha}(0,x) = u^{\bar\alpha}_0(x), \quad u^{\bar\alpha}_t(0,x) = u^{\bar\alpha}_1(x), & \text{in } [0,\ell_{\bar\alpha}], \ \bar\alpha\in\mathcal{I}.
\end{array}
$$

For every $\bar\alpha\in\mathcal{I}$, the function $u^{\bar\alpha}(t,x): \ \mathbb{R}\times[0,\ell_{\bar\alpha}]\to\mathbb{R}$ denotes the transversal displacement of the string with index $\bar\alpha$. We will denote by $\bar u$ the set whose elements are $u^{\bar\alpha}, \ \bar\alpha\in\mathcal{I}$. In particular, the sets of initial states $(u^{\bar\alpha}_0)_{\bar\alpha\in\mathcal{I}}, (u^{\bar\alpha}_1)_{\bar\alpha\in\mathcal{I}}$ of the strings are denoted by $\bar u^0$ and $\bar u^1$. With these notations, the remaining elements relative to the system (1)-(6) are defined exactly as in Subsection 2.2 of Chapter I.

We also consider the homogeneous version of the system (1)-(6)

$$
\begin{array}{lll}
(7) & \phi^{\bar\alpha}_{tt}(t,x) = \phi^{\bar\alpha}_{xx}(t,x) & \text{in } \mathbb{R}\times[0,\ell_{\bar\alpha}], \quad \bar\alpha\in\mathcal{I},\\
(8) & \phi(t,0) = 0, & \text{in } \ \mathbb{R},\\
(9) & \phi^{\bar\alpha}(t,\ell_{\bar\alpha}) = 0 & \text{in } \mathbb{R}, \ \ \bar\alpha\in\mathcal{I}_{\mathcal{S}},\\
(10) & \phi^{\bar\alpha\circ\beta}(t,0) = \phi^{\bar\alpha}(t,\ell_{\bar\alpha}) & \text{in } \mathbb{R}, \quad \beta=1, \ \ldots, m_{\bar\alpha}, \ \ \bar\alpha\in\mathcal{I}_{\mathcal{M}},\\
(11) & \sum_{\beta=1}^{m_{\bar\alpha}} \phi^{\bar\alpha\circ\beta}_x(t,0) = \phi^{\bar\alpha}_x(t,\ell_{\bar\alpha}) & \text{in } \mathbb{R}, \ \ \bar\alpha\in\mathcal{I}_{\mathcal{M}},\\
(12) & \phi^{\bar\alpha}(0,x) = \phi^{\bar\alpha}_0(x), \quad \phi^{\bar\alpha}_t(0,x) = \phi^{\bar\alpha}_1(x), & \text{in } [0,\ell_{\bar\alpha}], \ \bar\alpha\in\mathcal{I}.
\end{array}
$$

The solution of problem (7)-(12) is given by

$$
\bar\phi(t) = \sum_{k\in\mathbb{N}} \left(\phi_{0,k}\cos\lambda_k t + \frac{\phi_{1,k}}{\lambda_k}\sin\lambda_k t \right) \bar\theta_k, \tag{13}
$$

where $(\phi_{0,k})_{k\in\mathbb{N}}, (\phi_{1,k})_{k\in\mathbb{N}}$ are the sequences of Fourier coefficients of the initial state $(\bar{\phi}^0, \bar{\phi}^1)$ in the orthonormal basis $(\bar{\theta}_k)_{k\in\mathbb{N}}$ formed by the eigenfunctions of the elliptic operator $-\Delta_{\mathcal{A}}$ corresponding to (1)-(5). Recall that $(\mu_k)_{k\in\mathbb{N}}$ is the increasing sequence of eigenvalues and $\lambda_k := \sqrt{\mu_k}$.

For technical reasons, we will also consider solutions $\bar{\phi}$ of (7) such that $\phi^{\bar{\alpha}} \in C^2(\mathbb{R}\times[0,\ell_{\bar{\alpha}}])$, satisfying (9), (10) and (11), but not necessarily (8). That is, $\bar{\phi}$ is a smooth solution that satisfy the boundary conditions given in (7)-(11) at all the nodes, except at the root $\mathcal{R}$. These solutions will be briefly referred as *solutions of* (N). In the same way we define a *solution of* (N) *on the sub-tree* $\mathcal{A}_{\bar{\alpha}}$.

For a solution $\bar{\phi}$ of (N) we define the functions

$$G_{\bar{\alpha}}(t) := \phi^{\bar{\alpha}}_t(t,0), \qquad F_{\bar{\alpha}}(t) := \phi^{\bar{\alpha}}_x(t,0), \tag{14}$$

$$\widehat{G}_{\bar{\alpha}}(t) := \phi^{\bar{\alpha}}_t(t,\ell_{\bar{\alpha}}), \qquad \widehat{F}_{\bar{\alpha}}(t) := \phi^{\bar{\alpha}}_x(t,\ell_{\bar{\alpha}}), \tag{15}$$

for every $\bar{\alpha}\in\mathcal{I}$. These functions are the velocity and the tension at the extremes of the string $\mathbf{e}_{\bar{\alpha}}$.

According to the coupling conditions (10)-(11), we will have the formulas

$$G_{\bar{\alpha}\circ\beta}(t) = \widehat{G}_{\bar{\alpha}}(t), \qquad \sum_{\beta=1}^{m_{\bar{\alpha}}} F_{\bar{\alpha}\circ\beta}(t) = \widehat{F}_{\bar{\alpha}}(t), \tag{16}$$

for every $t\in\mathbb{R}$, $\bar{\alpha}\in\mathcal{I}_{\mathcal{M}}$, $\beta = 1, ..., m_{\bar{\alpha}}$.

On the other hand, from the D'Alembert formulas (5) we obtained the equalities

$$\widehat{F}_{\bar{\alpha}} = \ell^+_{\bar{\alpha}} F_{\bar{\alpha}} + \ell^-_{\bar{\alpha}} G_{\bar{\alpha}}, \qquad \widehat{G}_{\bar{\alpha}} = \ell^-_{\bar{\alpha}} F_{\bar{\alpha}} + \ell^+_{\bar{\alpha}} G_{\bar{\alpha}},$$

for all $\bar{\alpha}\in\mathcal{I}$. In view of them, the coupling conditions (16) at the interior nodes may be expressed as

$$G_{\bar{\alpha}\circ\beta}(t) = \ell^-_{\bar{\alpha}} F_{\bar{\alpha}} + \ell^+_{\bar{\alpha}} G_{\bar{\alpha}}, \tag{17}$$

$$\sum_{\beta=1}^{m_{\bar{\alpha}}} F_{\bar{\alpha}\circ\beta}(t) = \ell^+_{\bar{\alpha}} F_{\bar{\alpha}} + \ell^-_{\bar{\alpha}} G_{\bar{\alpha}}. \tag{18}$$

For a function $\bar{w}(t)$ defined on the tree $\mathcal{A}$ the energy of $\bar{w}$ on the string $\mathbf{e}_{\bar{\alpha}}$ is defined by

$$E^{\bar{\alpha}}_{\bar{w}}(t) := \frac{1}{2}\int_0^{\ell_{\bar{\alpha}}} \left(|w^{\bar{\alpha}}_t(t,x)|^2 + |w^{\bar{\alpha}}_x(t,x)|^2\right) dx.$$

For a sub-tree $\mathcal{A}_{\bar{\alpha}}$, we denote by $\mathbf{E}^{\bar{\alpha}}_{\bar{w}}$ the total energy of $\bar{w}$ on the sub-tree:

$$\mathbf{E}^{\bar{\alpha}}_{\bar{\omega}}(t) := \sum_{\bar{\beta}:\bar{\alpha}\circ\bar{\beta}\in\mathcal{I}} E^{\bar{\alpha}\circ\bar{\beta}}_{\bar{w}}(t).$$

In particular, the total energy of $\bar{w}$ on the network is

$$\mathbf{E}_{\bar{w}}(t) := \sum_{\bar{\alpha}\in\mathcal{I}} E^{\bar{\alpha}}_{\bar{w}}(t).$$

2. The operators $\mathcal{P}$ and $\mathcal{Q}$

In this section we define two linear operators $\mathcal{P}$ and $\mathcal{Q}$ that allow to express the relation

$$\mathcal{P}G + \mathcal{Q}F = 0 \tag{19}$$

between the velocity and the tension of the solutions of (N) at the root of the tree. These operators will play an essential role in the proof of the main observability results, so we study them in detail. In particular, we need information on how they act on the traces $F_{\bar{\alpha}}$ and $G_{\bar{\alpha}}$ of the other components of the solution at the interior nodes.

First, $\mathcal{P}$ and $\mathcal{Q}$ are constructed for a string. Then, using a recursive argument, they are obtained for general trees.

2.1. A tree formed by a single string. Assume that $\phi \in C^2(\mathbb{R} \times [0, \ell])$ satisfies the wave equation

$$\phi_{tt} - \phi_{xx} = 0$$

in $\mathbb{R} \times [0, \ell]$ and that $\phi(t, \ell) \equiv 0$. Thus, ϕ is a solution of (N) for the network formed by a single string of length ℓ. Let us note that in this case, with the notations (14)-(15), we get $\widehat{G}(t) := \phi_t(t, \ell) = 0$.

From the D'Alembert formula (II.5) it holds

$$0 = \ell^+ G + \ell^- F, \tag{20}$$

for every $t \in \mathbb{R}$. This is a relation of the type (19) with $\mathcal{P} = \ell^+, \quad \mathcal{Q} = \ell^-$.

2.2. Operators of type S. As stated above, we are interested not only in the existence of the operators $\mathcal{P}$ and $\mathcal{Q}$ satisfying (19), but also in their structure. That is why we consider a class of linear operators constituted by linear combinations of certain shift operators. This class of operators allows to describe the main properties of the operators $\mathcal{P}$ and $\mathcal{Q}$ we use in this chapter.

For the real number h we denote by τ_h the shift operator defined by

$$\tau_h f(t) := f(t + h).$$

As we shall be concerned only with algebraic properties of those operators, we may assume τ_h to act on the vector spaces of mappings $f = f(t) : \mathbb{R} \to \mathbf{W}$, where $\mathbf{W}$ is a vector space.

Let $\Lambda = \{\ell_1, \ldots, \ell_n\}$ be a set of positive numbers, not necessarily different. In what follows, whenever a set is denoted by Λ we tacitly assume that it may contain repeated elements. If $\tilde{\Lambda} = \left\{\tilde{\ell}_1, \ldots, \tilde{\ell}_{n'}\right\}$ is another such set, we use the notation $\Lambda \sqcup \tilde{\Lambda}$ for the set $\{\ell_1, \ldots, \ell_n, \tilde{\ell}_1, \ldots, \tilde{\ell}_{n'}\}$, which once again may contain repeated elements. Observe that this operation differs from the usual union of sets in the fact that the multiplicity of the elements is taken into account.

We set

$$S(\Lambda) := span\,\{\tau_h : \quad h \in \mathcal{H}_\Lambda\},$$

(the set all linear combinations of shift operators τ_h with $h \in \mathcal{H}_\Lambda$) where

$$\mathcal{H}_\Lambda = \left\{ h = \sum_{i=1}^{n} \varepsilon_i \ell_i, \ \varepsilon_i = \pm 1 \right\}.$$

Observe that the set $\mathcal{H}_\Lambda$ contains at most 2^n elements, so $S(\Lambda)$ is of finite dimension.

For an operator $\mathcal{B} \in S(\Lambda)$ we shall write $s(\mathcal{B}) := s(\Lambda) := \sum_{i=1}^n \ell_i$. We say that $\mathcal{B}$ *is of type* S if $\mathcal{B} \in S(\Lambda)$ for some set Λ.

The operators ℓ^+ and ℓ^-, defined in Chapter III by (7) for a string are of type S. They belong to $S(\{\ell\})$, since they may be expressed as

$$\ell^{\pm} = \frac{\tau_\ell \pm \tau_{-\ell}}{2}.$$

We write these formulas in a unified way as

$$\ell^{\varepsilon} = \frac{\tau_\ell + \varepsilon\tau_{-\ell}}{2}, \tag{21}$$

where $\varepsilon = \pm 1$.

In the following proposition we gather two elementary properties of the operators of type S. The operator $\mathcal{P}$ and $\mathcal{Q}$, which we will construct for the network with the property (19), will be products of the operators $\ell_i^{\pm}$ constructed for the lengths of the strings. The proposition shows why it is natural to consider the class of operators $S(\Lambda)$ to characterize the operators $\mathcal{P}$ and $\mathcal{Q}$.

Proposition IV.1. *(i)* $\mathcal{B} \in S(\Lambda)$ *if, and only if, it may be written as a linear combination of operators of the form* $\ell_1^{\varepsilon_1}\ell_2^{\varepsilon_2}\cdots\ell_n^{\varepsilon_n}$*, where each* ε_i *is* -1 *or* 1.

(ii) If $\mathcal{B}_1 \in S(\Lambda_1)$ *and* $\mathcal{B}_2 \in S(\Lambda_2)$ *then,* $\mathcal{B}_1\mathcal{B}_2 = \mathcal{B}_2\mathcal{B}_1 \in S(\Lambda_1 \sqcup \Lambda_2)$ *and* $s(\mathcal{B}_1\mathcal{B}_2) = s(\mathcal{B}_1) + s(\mathcal{B}_2)$.

Proof. These properties are based on the fact that, if $\alpha, \beta \in \mathbb{R}$ then,

$$\tau_\alpha\tau_\beta = \tau_{\alpha+\beta} = \tau_\beta\tau_\alpha.$$

This implies, in particular, that the operators $\ell_i^{\varepsilon_i}$ and $\ell_j^{\varepsilon_j}$ commute.

In account of (21), a product of the form $\ell_1^{\varepsilon_1}\ell_2^{\varepsilon_2}\cdots\ell_n^{\varepsilon_n}$ may be expressed as

$$\ell_1^{\varepsilon_1}\ell_2^{\varepsilon_2}\cdots\ell_n^{\varepsilon_n} = \prod_{i=1}^n \left(\frac{\tau_{\ell_i} + \varepsilon_i\tau_{-\ell_i}}{2}\right);$$

then we get

$$\ell_1^{\varepsilon_1}\ell_2^{\varepsilon_2}\cdots\ell_n^{\varepsilon_n} = \sum_{h\in\mathcal{H}_\Lambda} c_h\tau_h \in S(\Lambda).$$

Thus, any linear combination of the operators $\ell_1^{\varepsilon_1}\ell_2^{\varepsilon_2}\cdots\ell_n^{\varepsilon_n}$ is an operator of type $S(\Lambda)$.

Conversely, if $h \in \mathcal{H}_\Lambda$, then

$$h = \sum_{i=1}^n \varepsilon_i\ell_i, \qquad \varepsilon_i = \pm 1,$$

and in view of the fact that, as it follows of (21), for any ℓ, $\tau_{\varepsilon\ell} = \varepsilon\ell^{\varepsilon} + \ell^{-\varepsilon}$,

$$\tau_h = \prod_{i=1}^n \tau_{\varepsilon_i\ell_i} = \prod_{i=1}^n \left[\varepsilon_i\ell_i^{\varepsilon_i} + \ell_i^{-\varepsilon_i}\right].$$

From here, τ_h may be expressed as a sum of products of the form $\ell_1^{\varepsilon_1'}\ell_2^{\varepsilon_2}\cdots\ell_n^{\varepsilon_n'}$ and the same will be true for each $\mathcal{B} = \sum_{h\in\mathcal{H}_\Lambda} c_h\tau_h$.

The assertion (ii) follows immediately. If

$$\Lambda_1 = \{\ell_1, \dots, \ell_n\}$$

and

$$\Lambda_2 = \{\ell_{n+1}, \dots, \ell_N\}$$

then,

$$\Lambda_1 \sqcup \Lambda_2 = \{\ell_1, \dots, \ell_N\}.$$

If $\mathcal{B}_1 = \ell_1^{\varepsilon_1} \cdots \ell_n^{\varepsilon_n} \in S(\Lambda_1)$, $\mathcal{B}_2 = \ell_{n+1}^{\varepsilon_{n+1}} \cdots \ell_N^{\varepsilon_N} \in S(\Lambda_2)$,

$$\mathcal{B}_1\mathcal{B}_2 = \mathcal{B}_2\mathcal{B}_1 = \ell_1^{\varepsilon_1} \cdots \ell_n^{\varepsilon_n} \ell_{n+1}^{\varepsilon_1} \cdots \ell_N^{\varepsilon_N} \in S(\Lambda_1 \sqcup \Lambda_2).$$

Taking now into account that any $\mathcal{B}_1 \in S(\Lambda_1)$ and $\mathcal{B}_2 \in S(\Lambda_2)$ may be expressed, respectively, by means of linear combinations of the operators $\ell_1^{\varepsilon_1} \cdots \ell_n^{\varepsilon_n}$ and $\ell_{n+1}^{\varepsilon_{n+1}} \cdots \ell_N^{\varepsilon_N}$, (ii) is obtained in the general case. □

In the rest of this chapter, when an operator $\mathcal{B}$ of type S is applied to a function w depending on the real variable t (and, possibly, on other variables), we will assume that $\mathcal{B}$ acts on that variable. In particular, if $w(t,x)$ is a function defined on $\mathbb{R} \times [a,b]$ then

$$\ell^{\pm} w(t,x) = \frac{1}{2}(w(t+\ell,x) \pm w(t-\ell,x)).$$

The following facts are widely used in the proof of our main results.

Proposition IV.2. *Let $w(t,x)$ be a function defined on $\mathbb{R} \times [0,\ell]$. Then,*

$$\mathbf{E}_{\ell^{\pm} w}(t) \le \ell^{+} \mathbf{E}_w(t).$$

Proof. For every $t \in \mathbb{R}$

$$\begin{aligned}
\mathbf{E}_{\ell^{\pm} w}(t) &= \frac{1}{8}\int_0^{\ell} \left\{ |w_x(t+\ell,x) \pm w_x(t-\ell,x)|^2 + |w_t(t+\ell,x) \pm w_t(t-\ell,x)|^2 \right\} dx \\
&\le \frac{1}{4}\int_0^{\ell} \left\{ |w_x(t+\ell,x)|^2 + |w_x(t-\ell,x)|^2 + |w_t(t+\ell,x)|^2 + |w_t(t-\ell,x)|^2 \right\} dx \\
&= \frac{1}{2}(\mathbf{E}_w(t+\ell) + \mathbf{E}_w(t-\ell)) = \ell^{+}\mathbf{E}_w(t).
\end{aligned}$$

□

Proposition IV.3. *If $\mathcal{B}$ is an operator of type S with $s(\mathcal{B}) = s$ then there exist positive constants C_1, C_2, depending only on the coefficients of $\mathcal{B}$, such that*

(i) $$\int_a^b |\mathcal{B}f(t)|^2 dt \le C_1 \int_{a-s}^{b+s} |f(t)|^2 dt,$$

for all the functions f for which both integrals are defined[1].

(ii) If the function $w(t,x)$ is defined on $\mathbb{R} \times [0,\ell]$ and there exists a constant $M > 0$ such that $\mathbf{E}_w(t) \le M$ for every $t \in [a,b]$ then $\mathbf{E}_{\mathcal{B}w}(t) \le C_2 M$ for $t \in [a+s, b-s]$.

[1]In other words, $\mathcal{B}$ is continuous from $L^2[a-s,b+s]$ to $L^2[a,b]$.

PROOF. (i) When the set Λ is formed by a single element: $\Lambda = \{\ell\}$, we have $\mathcal{B} = c_1\ell^+ + c_2\ell^-$ and $s(\mathcal{B}) = \ell$. Then,

$$\begin{aligned}
\int_a^b |\mathcal{B}f(t)|^2 dt &= \int_a^b \left|c_1\ell^+ f(t) + c_2\ell^- f(t)\right|^2 dt \\
&= \int_a^b \left|\frac{c_1+c_2}{2} f(t+\ell) + \frac{c_1-c_2}{2} f(t-\ell)\right|^2 dt \\
&\le \left(\frac{c_1+c_2}{2}\right)^2 \int_a^b |f(t+\ell)|^2 dt + \left(\frac{c_1-c_2}{2}\right)^2 \int_a^b |f(t-\ell)|^2 dt \\
&\le \left(\frac{c_1+c_2}{2}\right)^2 \int_{a+\ell}^{b+\ell} |f(t)|^2 dt + \left(\frac{c_1-c_2}{2}\right)^2 \int_{a-\ell}^{b-\ell} |f(t)|^2 dt \\
&\le \left(c_1^2 + c_2^2\right) \int_{a-\ell}^{b+\ell} |f(t)|^2 dt.
\end{aligned}$$

When $n \ge 2$, it suffices to iterate this inequality taking under consideration Proposition IV.1(i). Let us note that C_1 may be chosen as the maximum of the squares of the coefficients of $\mathcal{B}$ in its representation given by Proposition IV.1(i) and then, C_1 depends only on $\mathcal{B}$.

(ii) Is an immediate consequence of Proposition IV.2. $\square$

The next proposition plays a crucial role in obtaining the optimal time in the observability inequalities that we prove for the solutions of the homogeneous system (1)-(6). Let us note that this fact was already proved in Section III.4 of Chapter III for the operator $\mathcal{Q}$ corresponding to the three string network.

PROPOSITION IV.4. *Let $\Lambda = \{\ell_1, ..., \ell_m\}$ with $\ell_1 \le ... \le \ell_m$ and denote $T_\Lambda = 2s(\Lambda) = 2\sum_{i=1}^m \ell_i$. Assume that $\mathcal{B} = \sum_{h\in\mathcal{H}_\Lambda} c_h\tau_h \in S(\Lambda)$ and that the coefficient $c_{\ell_1+\cdots+\ell_m}$ is different from zero. Then, for any $T > 0$ there exists a constant $C_T > 0$ such that*

$$\int_0^T |u(t)|^2 dt \le C_T \int_0^{T_\Lambda} |u(t)|^2 dt$$

for any continuous function u satisfying $\mathcal{B}u \equiv 0$.

PROOF. We shall prove that, for any natural number n and any function u satisfying $\mathcal{B}u \equiv 0$, it holds that

$$\int_0^{T_\Lambda + 2n\ell_1} |u(t)|^2 dt \le \gamma^n \int_0^{T_\Lambda} |u(t)|^2 dt, \tag{22}$$

where γ is a positive constant depending only on $\mathcal{B}$. Clearly, the assertion of the proposition immediately follows from inequality (22).

If $\mathcal{B}u \equiv 0$, i.e., $0 = \sum_{h\in\mathcal{H}_\Lambda} c_h\tau_h u(t) = \sum_{h\in\mathcal{H}_\Lambda} c_h u(t+h)$, then, replacing the variable t by $t - (\ell_1 + \cdots \ell_m)$ and taking into account that $c_{\ell_1+\cdots\ell_m} \ne 0$ we get

$$u(t) = \sum_{h'\in\mathcal{H}_\Lambda^*} \delta_{h'} u(t-h'), \tag{23}$$

where $\mathcal{H}_\Lambda^* = \{h' = h - (\ell_1 + \cdots \ell_m) : h \in \mathcal{H}_\Lambda,\ h \ne (\ell_1 + \cdots \ell_m)\}$ and $\delta_{h'} = -\frac{c_{h'+(\ell_1+\cdots\ell_m)}}{c_{\ell_1+\cdots\ell_m}}$.

From (23) and the Cauchy-Schwartz inequality it follows

$$|u(t)|^2 \le \delta \sum_{h' \in \mathcal{H}^*_\Lambda} |u(t-h')|^2, \tag{24}$$

where $\delta = \sum_{h' \in \mathcal{H}^*_\Lambda} \delta^2_{h'}$.

Note that, for every $h' \in \mathcal{H}^*_\Lambda$ we have $2\ell_1 \le h' \le 2(\ell_2 + \cdots + \ell_m)$, and therefore,

$$T_\Lambda + 2(n+1)\ell_1 - h' \le T_\Lambda + 2n\ell_1 \quad \text{and} \quad T_\Lambda + 2n\ell_1 - h' \ge 2(n+1)\ell_1 \ge 0.$$

This fact implies that

$$\int_{T_\Lambda + 2n\ell_1 - h'}^{T_\Lambda + 2(n+1)\ell_1 - h'} |u(t)|^2 dt \le \int_0^{T_\Lambda + 2n\ell_1} |u(t)|^2 dt. \tag{25}$$

On the other hand, from (24) it follows that

$$\begin{aligned} \int_{T_\Lambda + 2n\ell_1}^{T_\Lambda + 2(n+1)\ell_1} |u(t)|^2 dt \quad &\le \quad \delta \sum_{h' \in \mathcal{H}^*_\Lambda} \int_{T_\Lambda + 2n\ell_1}^{T_\Lambda + 2(n+1)\ell_1} |u(t-h')|^2 dt = \\ &= \quad \delta \sum_{h' \in \mathcal{H}^*_\Lambda} \int_{T_\Lambda + 2n\ell_1 - h'}^{T_\Lambda + 2(n+1)\ell_1 - h'} |u(t)|^2 dt. \end{aligned}$$

Now, taking into account (25), the previous inequality becomes

$$\int_{T_\Lambda + 2n\ell_1}^{T_\Lambda + 2(n+1)\ell_1} |u(t)|^2 dt \le (2^m - 1)\delta \int_0^{T_\Lambda + 2n\ell_1} |u(t)|^2 dt.$$

From this latter inequality we obtain

$$\begin{aligned} \int_0^{T_\Lambda + 2(n+1)\ell_1} |u(t)|^2 dt \quad &= \quad \int_0^{T_\Lambda + 2n\ell_1} |u(t)|^2 dt + \int_{T_\Lambda + 2n\ell_1}^{T_\Lambda + 2(n+1)\ell_1} |u(t)|^2 dt \le \\ &\le \quad \int_0^{T_\Lambda + 2n\ell_1} |u(t)|^2 dt + (2^m - 1)\delta \int_0^{T_\Lambda + 2n\ell_1} |u(t)|^2 dt \le \\ &\le \quad (1 + (2^m - 1)\delta) \int_0^{T_\Lambda + 2n\ell_1} |u(t)|^2 dt, \end{aligned}$$

which proves inequality (22) with $\gamma = 1 + (2^m - 1)\delta$. □

REMARK IV.1. *If $\mathcal{B}$ is an operator of type S there exists a unique function $b(\lambda)$ such that $\mathcal{B}e^{i\lambda t} = b(\lambda)e^{i\lambda t}$. Indeed, it suffices to express $\mathcal{B}$ in the form*

$$\mathcal{B} \;=\; \sum_{m \in \{0,1\}^n} d_m \ell_1^{\varepsilon_1} \cdots \ell_n^{\varepsilon_n}, \tag{26}$$

given by Proposition IV.1(i), to see that

$$\mathcal{B}e^{i\lambda t} = \sum d_m \ell_1^{\varepsilon_1} \cdots \ell_n^{\varepsilon_n} e^{i\lambda t}.$$

Taking into account that

$$\ell^+ e^{i\lambda t} = \cos \ell\lambda \; e^{i\lambda t}, \qquad \ell^- e^{i\lambda t} = i \sin \ell\lambda \; e^{i\lambda t},$$

it holds

$$\mathcal{B}e^{i\lambda t} = \sum d_m L_1^{\varepsilon_1}(\lambda) \cdots L_n^{\varepsilon_n}(\lambda) e^{i\lambda t},$$

where $L_i^{\varepsilon_i}(\lambda) = \cos \ell_i \lambda$ if $\varepsilon_i = 1$ and $L_i^{\varepsilon_i}(\lambda) = i \sin \ell_i \lambda$ if $\varepsilon_i = -1$. This means that the function $b(\lambda)$ may be constructed by replacing the operators ℓ_i^+ and ℓ_i^- in the decomposition (26) of $\mathcal{B}$ by $\cos \lambda t$ and $i \sin \lambda t$, respectively.

The uniqueness of $b(\lambda)$ is immediate: if $\mathcal{B} e^{i\lambda t} = b(\lambda) e^{i\lambda t} = c(\lambda) e^{i\lambda t}$, then $b(\lambda) = c(\lambda)$.

2.3. Construction of $\mathcal{P}$ and $\mathcal{Q}$ in the general case. The construction of $\mathcal{P}$ and $\mathcal{Q}$ will be done by induction. We remind that such operators have already been constructed for a network consisting of a single string.

We shall denote by Λ_i the set of all the lengths of the strings of the sub-tree $\mathcal{A}_i$ and by $\Lambda_{\mathcal{A}}$ that of all the lengths of the tree $\mathcal{A}$. Suppose that for the sub-trees $\mathcal{A}_i$, $i = 1, \ldots, m$, we have already constructed the operators $\mathcal{P}_i$, $\mathcal{Q}_i$ that belong to $S(\Lambda_i)$ and verify

$$\mathcal{P}_i G_i + \mathcal{Q}_i F_i = 0, \tag{27}$$

where G_i and F_i are the velocity and the tension at the root of the sub-tree $\mathcal{A}_i$, i.e., at the vertex $\mathcal{O}$ of $\mathcal{A}$.

We define the operators

$$\mathcal{P} = \ell^+ \sum_{i=1}^m \mathcal{P}_i \prod_{j \neq i} \mathcal{Q}_j + \ell^- \prod_{j=1}^m \mathcal{Q}_j, \tag{28}$$

$$\mathcal{Q} = \ell^- \sum_{i=1}^m \mathcal{P}_i \prod_{j \neq i} \mathcal{Q}_j + \ell^+ \prod_{j=1}^m \mathcal{Q}_j \tag{29}$$

(here the products denote the composition of operators).

Those are precisely the operators we are looking for.

Proposition IV.5. *The operators $\mathcal{P}$ and $\mathcal{Q}$ defined by (28)-(29) belong to $S(\Lambda_{\mathcal{A}})$. If $\bar{u}$ is a solution of* (N) *then*

$$\mathcal{P} G + \mathcal{Q} F = 0.$$

Proof. To prove that $\mathcal{P}, \mathcal{Q} \in S(\Lambda_{\mathcal{A}})$, it suffices to observe that, according to Proposition IV.1, all the terms of the sums in (28) and (29) belong to $S(\{\ell\} \sqcup \Lambda_1 \sqcup \ldots \sqcup \Lambda_m) = S(\Lambda_{\mathcal{A}})$. Using (17)-(18), the coupling conditions (16) between the strings may be expressed as

$$\sum_{i=1}^m F_i = \ell^- G + \ell^+ F, \qquad G_i = \ell^+ G + \ell^- F, \quad i = 1, \ldots, m. \tag{30}$$

From (28)-(29) we have

$$\begin{aligned} \mathcal{P} G + \mathcal{Q} F &= \sum_{i=1}^m (\mathcal{P}_i \prod_{j \neq i} \mathcal{Q}_j) \ell^+ G + \prod_{j=1}^m \mathcal{Q}_j \ell^- G + \sum_{i=1}^m (\mathcal{P}_i \prod_{j \neq i} \mathcal{Q}_j) \ell^- F + \prod_{j=1}^m \mathcal{Q}_j \ell^+ F \\ &= \sum_{i=1}^m (\mathcal{P}_i \prod_{j \neq i} \mathcal{Q}_j)(\ell^+ G + \ell^- F) + \prod_{j=1}^m \mathcal{Q}_j (\ell^- G + \ell^+ F). \end{aligned}$$

Then, using formulas (30),

$$\mathcal{P}G + \mathcal{Q}F = \sum_{i=1}^{m}(\mathcal{P}_i \prod_{j\neq i} \mathcal{Q}_j)G_i + \sum_{i=1}^{m}(\prod_{j=1}^{m} \mathcal{Q}_j)F_i = \sum_{i=1}^{m}(\prod_{j\neq i} \mathcal{Q}_j)(\mathcal{P}_i G_i + \mathcal{Q}_i F_i) = 0,$$

where the last equality follows from the hypotheses (27). Thus, $\mathcal{P}$ and $\mathcal{Q}$, defined by (28)-(29), satisfy the relation (19). □

REMARK IV.2. *From the definition, an $S(\Lambda)$-operator $\mathcal{B}$ may be written in the form*

$$\mathcal{B} = \sum_{h\in\mathcal{H}_\Lambda} c_h \tau_h. \tag{31}$$

In general, this representation is not unique, since some elements of $\mathcal{H}_\Lambda$ may coincide. However, the coefficient $c_{s(\mathcal{B})} = c_{\ell_1+\cdots+\ell_m}$, corresponding to the largest value of h, is determined in a unique way, as $\ell_1 + \cdots + \ell_m$ cannot be equal to another element of $\mathcal{H}_\Lambda$. Besides, it is easy to see that $c_{s(\mathcal{B})}$ is a multiplicative function, i.e., if $\mathcal{B}_1$ and $\mathcal{B}_2$ are S-operators with $s(\mathcal{B}_1) = s_1$ and $s(\mathcal{B}_2) = s_2$ then

$$c_{s_1+s_2}(\mathcal{B}_1\mathcal{B}_2) = c_{s_1}(\mathcal{B}_1)c_{s_2}(\mathcal{B}_2).$$

In the next proposition we study this coefficient for the operators $\mathcal{P}$ and $\mathcal{Q}$.

PROPOSITION IV.6. *Let $c_{L_\mathcal{A}}(\mathcal{B})$ denote the coefficient, corresponding to $h = s(\Lambda_\mathcal{A}) = L_\mathcal{A} \in \mathcal{H}_{\Lambda_\mathcal{A}}$ in the expansion (31) of an $S(\Lambda_\mathcal{A})$-operator $\mathcal{B}$. Then $c_{L_\mathcal{A}}(\mathcal{P}) = c_{L_\mathcal{A}}(\mathcal{Q}) > 0$.*

PROOF. We proceed by induction. For a string,

$$\mathcal{P} = \ell^+ = \frac{\tau_h + \tau_{-h}}{2}, \qquad \mathcal{Q} = \ell^+ = \frac{\tau_h - \tau_{-h}}{2}.$$

This implies $c_\ell(\mathcal{P}) = c_\ell(\mathcal{Q}) = \frac{1}{2}$.

Now assume the assertion is true for the sub-trees $\mathcal{A}_1, ..., \mathcal{A}_m$. It means that

$$c_{L_i}(\mathcal{P}) = c_{L_i}(\mathcal{Q}) > 0, \quad i = 1, ..., m, \tag{32}$$

where, as above, L_i is the sum of the lengths of all the strings of the sub-tree $\mathcal{A}_i$.

Then, from formula (28) and the assumption (32)

$$\begin{aligned}
c_{L_\mathcal{A}}(\mathcal{P}) &= c_{L_\mathcal{A}}(\ell^+ \sum_{i=1}^{m} \mathcal{P}_i \prod_{j\neq i} \mathcal{Q}_j + \ell^- \prod_{j=1}^{m} \mathcal{Q}_j) \\
&= c_{L_\mathcal{A}}(\ell^+ \sum_{i=1}^{m} \mathcal{P}_i \prod_{j\neq i} \mathcal{Q}_j) + c_{L_\mathcal{A}}(\ell^- \prod_{j=1}^{m} \mathcal{Q}_j) \\
&= c_\ell(\ell^+) \sum_{i=1}^{m} c_{L_i}(\mathcal{P}_i) \prod_{j\neq i} c_{L_j}(\mathcal{Q}_j) + c_\ell(\ell^-) \prod_{j=1}^{m} c_{L_j}(\mathcal{Q}_j) \\
&= \frac{1}{2}(m+1) \prod_{j=1}^{m} c_{L_j}(\mathcal{Q}_j) > 0.
\end{aligned}$$

In the same way it may be proved that

$$c_{L_\mathcal{A}}(\mathcal{Q}) = \frac{1}{2}(m+1) \prod_{j=1}^{m} c_{L_j}(\mathcal{Q}_j),$$

what completes the proof. $\square$

2.4. The action of $\mathcal{P}$ and $\mathcal{Q}$ on the tensions and velocities at the interior nodes. For the index $\bar{\alpha} = (\alpha_1, ..., \alpha_k) \in \mathcal{I}$ we denote

$$\tilde{\Lambda}_{\bar{\alpha}} := \{\ell, \ell_{\alpha_1}, \ell_{\alpha_1,\alpha_2}, ..., \ell_{\alpha_1,\alpha_2,...,\alpha_{k-1}}\}.$$

Observe that $\tilde{\Lambda}_{\bar{\alpha}}$ is the set of the lengths of the strings forming the unique simple path that connects the root $\mathcal{R}$ with the sub-tree $\mathcal{A}_{\bar{\alpha}}$. For completeness we take for the empty index $\tilde{\Lambda} = \emptyset$.

The following proposition gives information on how the operators $\mathcal{P}$ and $\mathcal{Q}$ act on traces of the components of a solution at the interior nodes of the network.

PROPOSITION IV.7. *For any $\bar{\alpha} \in \mathcal{I}$ there exist operators $\mathcal{L}_{\bar{\alpha}} \in S(\Lambda_{\mathcal{A}} \sqcup \tilde{\Lambda}_{\bar{\alpha}})$ such that, for any solution of* (N)

$$\mathcal{Q}F_{\bar{\alpha}} = \mathcal{L}_{\bar{\alpha}}G, \qquad \mathcal{P}F_{\bar{\alpha}} = -\mathcal{L}_{\bar{\alpha}}F.$$

PROOF. We proceed by induction. Note that from the relation $\mathcal{P}G + \mathcal{Q}F = 0$, it follows that when $\bar{\alpha}$ is the empty multi-index the property is true with $\mathcal{L} = -\mathcal{P} \in S(\Lambda_{\mathcal{A}}) = S(\Lambda_{\mathcal{A}} \sqcup \tilde{\Lambda})$. In particular, for a single string the assertion of the proposition holds.

Suppose now that the operators $\mathcal{L}_{\bar{\alpha}}$ have been already constructed for the sub-trees $\mathcal{A}_i$, $i = 1, ..., m$, of $\mathcal{A}$. This means that we have for $i = 1, ..., m$, the operators $\mathcal{L}^i_{\bar{\alpha}} \in S(\Lambda_i \sqcup \tilde{\Lambda}^i_{\bar{\alpha}})$ such that

$$\mathcal{P}_i F_{i\circ\bar{\alpha}} = -\mathcal{L}^i_{\bar{\alpha}} F_i, \qquad \mathcal{Q}_i F_{i\circ\bar{\alpha}} = \mathcal{L}^i_{\bar{\alpha}} G_i,$$

where $\tilde{\Lambda}^i_{\bar{\alpha}}$ is the set defined as $\tilde{\Lambda}_{\bar{\alpha}}$ for the sub-tree $\mathcal{A}_i$ and $\mathcal{P}_i$, $\mathcal{Q}_i$ are the operators $\mathcal{P}$, $\mathcal{Q}$ corresponding to that sub-tree.

Then, using relation (29),

$$\begin{aligned}
\mathcal{Q}F_{i\circ\bar{\alpha}} &= \ell^-(\sum_{j=1}^m \mathcal{P}_j \prod_{k\neq j} \mathcal{Q}_k)F_{i\circ\bar{\alpha}} + \ell^+(\prod_{k=1}^m \mathcal{Q}_k)F_{i\circ\bar{\alpha}} \\
&= \ell^-(\sum_{\substack{j=1\\ j\neq i}}^m \mathcal{P}_j \prod_{\substack{k\neq j\\ k\neq i}} \mathcal{Q}_k)\mathcal{Q}_i F_{i\circ\bar{\alpha}} + \ell^-(\mathcal{P}_i \prod_{k\neq i} \mathcal{Q}_k)F_{i\circ\bar{\alpha}} + \ell^+(\prod_{k=1}^m \mathcal{Q}_k)F_{i\circ\bar{\alpha}} \\
&= \mathcal{L}^i_{\bar{\alpha}} \left(\ell^-(\sum_{\substack{i=1\\ j\neq i}}^m \mathcal{P}_j \prod_{\substack{k\neq j\\ k\neq i}} \mathcal{Q}_k)G_i - \ell^-(\prod_{k\neq i} \mathcal{Q}_k)F_i + \ell^+(\prod_{k=1}^m \mathcal{Q}_k)G_i \right) \\
&= \mathcal{L}^i_{\bar{\alpha}} \left(\ell^- \sum_{j\neq i} (\prod_{\substack{k\neq j\\ k\neq i}} \mathcal{Q}_k)(\mathcal{P}_j G_i + \mathcal{Q}_j \widehat{F}) - \ell^-(\prod_{k\neq i} \mathcal{Q}_k)\widehat{F} + \ell^+(\prod_{k\neq i} \mathcal{Q}_k)G_i \right) \\
&= \mathcal{L}^i_{\bar{\alpha}}(\prod_{k\neq i} \mathcal{Q}_k)(\ell^+\widehat{G} - \ell^-\widehat{F}) = \mathcal{L}^i_{\bar{\alpha}}(\prod_{k\neq i} \mathcal{Q}_k)\left(\ell^+(\ell^-F + \ell^+G) - \ell^-(\ell^+F + \ell^-G)\right) \\
&= \mathcal{L}^i_{\bar{\alpha}}(\prod_{k\neq i} \mathcal{Q}_k)\left((\ell^+)^2 - (\ell^-)^2\right)G.
\end{aligned}$$

In a similar way, it may be obtained that

$$\mathcal{P}F_{i\circ\bar{\alpha}} = -\mathcal{L}_{\bar{\alpha}}^{i} \prod_{k\neq i} \mathcal{Q}_k \left((\ell^+)^2 - (\ell^-)^2\right) F.$$

Thus, we arrive to the recursive formula

$$\mathcal{L}_{i\circ\bar{\alpha}} = \mathcal{L}_{\bar{\alpha}}^{i} \prod_{k\neq i} \mathcal{Q}_k \left((\ell^+)^2 - (\ell^-)^2\right),$$

from which, in particular, according to Proposition IV.1, it holds that the operators $\mathcal{L}_{i\circ\bar{\alpha}}$ belong to $S(\Lambda_i \sqcup \tilde{\Lambda}_{\bar{\alpha}}^{i} \sqcup \{\ell,\ell\}) = S(\Lambda_i \sqcup \{\ell\} \sqcup \tilde{\Lambda}_{\bar{\alpha}}^{i} \sqcup \{\ell\}) = S(\Lambda_{\mathcal{A}} \sqcup \tilde{\Lambda}_{i\circ\bar{\alpha}})$. This proves the proposition. $\square$

The action of $\mathcal{P}$ and $\mathcal{Q}$ on the velocities $G_{\bar{\alpha}}$ may be described in a similar way:

Proposition IV.8. *For any $\bar{\alpha} \in \mathcal{I}$ there exist operators $\mathcal{K}_{\bar{\alpha}}$, $\widehat{\mathcal{K}}_{\bar{\alpha}} \in S(\Lambda_{\mathcal{A}} \sqcup \tilde{\Lambda}_{\bar{\alpha}})$ such that, for any solution of* (N)

$$\mathcal{Q}G_{\bar{\alpha}} = \mathcal{K}_{\bar{\alpha}}G, \qquad \mathcal{P}G_{\bar{\alpha}} = \widehat{\mathcal{K}}_{\bar{\alpha}}F.$$

Proof. From the relation $\mathcal{P}G + \mathcal{Q}F = 0$ it follows that for the empty multi-index $\mathcal{K} = \mathcal{Q}$ and $\widehat{\mathcal{K}} = -\mathcal{Q}$. For the remaining indices the operators $\mathcal{K}_{\bar{\alpha}}$ and $\widehat{\mathcal{K}}_{\bar{\alpha}}$ are constructed by recurrence. Assume that for the index $\bar{\alpha}$ the operators $\mathcal{K}_{\bar{\alpha}}$ and $\widehat{\mathcal{K}}_{\bar{\alpha}}$, verifying the conditions of the proposition, have been already constructed.

Then, for the indices $\bar{\alpha} \circ i$ with $i = 1, \ldots, m_{\bar{\alpha}}$, we have that

$$\mathcal{Q}G_{\bar{\alpha}\circ i} = \mathcal{Q}\widehat{G}_{\bar{\alpha}} = \ell_{\bar{\alpha}}^{+}\mathcal{Q}G_{\bar{\alpha}} + \ell_{\bar{\alpha}}^{-}\mathcal{Q}F_{\bar{\alpha}} = \left(\ell_{\bar{\alpha}}^{+}\mathcal{K}_{\bar{\alpha}} + \ell_{\bar{\alpha}}^{-}\mathcal{L}_{\bar{\alpha}}\right)G,$$

where $\mathcal{L}_{\bar{\alpha}}$ is the operator constructed in the previous proposition.

In an analogous way it may be obtained that

$$\mathcal{P}G_{\bar{\alpha}\circ i} = (\ell_{\bar{\alpha}}^{+}\widehat{\mathcal{K}}_{\bar{\alpha}} - \ell_{\bar{\alpha}}^{-}\mathcal{L}_{\bar{\alpha}})F.$$

Then, the needed operators may be constructed by the rules

$$\mathcal{K}_{\bar{\alpha}\circ i} = \ell_{\bar{\alpha}}^{+}\mathcal{K}_{\bar{\alpha}} + \ell_{\bar{\alpha}}^{-}\mathcal{L}_{\bar{\alpha}}. \tag{33}$$

$$\widehat{\mathcal{K}}_{\bar{\alpha}\circ i} = \ell_{\bar{\alpha}}^{+}\widehat{\mathcal{K}}_{\bar{\alpha}} - \ell_{\bar{\alpha}}^{-}\mathcal{L}_{\bar{\alpha}}. \tag{34}$$

As in the proof of the Proposition IV.7, from the relations (33)-(34) it holds, in particular, that the operators $\mathcal{K}_{\bar{\alpha}\circ i}$ and $\widehat{\mathcal{K}}_{\bar{\alpha}\circ i}$ belong to $S(\Lambda_{\mathcal{A}} \sqcup \tilde{\Lambda}_{\bar{\alpha}\circ i})$. $\square$

2.5. Action of $\mathcal{P}$ and $\mathcal{Q}$ on the solution. If $\bar{\phi}$ is a solution of (N) and $\mathcal{B}$ is an operator of type S, then, due to the linearity of $\mathcal{B}$ and (N), $\mathcal{B}\bar{\phi}$ is also a solution of (N). Moreover, if $G_{\bar{\alpha}}^{\mathcal{B}\bar{\phi}}$ and $F_{\bar{\alpha}}^{\mathcal{B}\bar{\phi}}$, $\bar{\alpha} \in \mathcal{I}$, denote the velocity and strength traces of the strings at the vertices of the network for the solution $\mathcal{B}\bar{\phi}$, then

$$G_{\bar{\alpha}}^{\mathcal{B}\bar{\phi}} = \mathcal{B}G_{\bar{\alpha}}, \qquad F_{\bar{\alpha}}^{\mathcal{B}\bar{\phi}} = \mathcal{B}F_{\bar{\alpha}}.$$

That is true, in particular, when $\mathcal{B}$ is one of the operators $\mathcal{P}$ or $\mathcal{Q}$. The following lemma contains a fundamental technical step in our construction

Lemma IV.1. *There exists a constant C, independent of $\bar{\phi}$, such that*

$$\mathbf{E}_{\mathcal{P}\bar{\phi}}(t) \leq C \int_{T^*-2L_{\mathcal{A}}}^{T^*+2L_{\mathcal{A}}} |F(t)|^2 dt, \qquad \mathbf{E}_{\mathcal{Q}\bar{\phi}}(t) \leq C \int_{T^*-2L_{\mathcal{A}}}^{T^*+2L_{\mathcal{A}}} |G(t)|^2 dt \tag{35}$$

for every $T^* \in \mathbb{R}$ and $t \in [T^* - L_{\mathcal{A}}, T^* + L_{\mathcal{A}}]$.

PROOF. (i) Fix $T^* \in \mathbb{R}$. We shall prove first that

$$\mathbf{E}_{\mathcal{P}\bar{\phi}}(T^*) \leq C \int_{T^*-2L_{\mathcal{A}}}^{T^*+2L_{\mathcal{A}}} |F(t)|^2 dt, \qquad \mathbf{E}_{\mathcal{Q}\bar{\phi}}(T^*) \leq C \int_{T^*-2L_{\mathcal{A}}}^{T^*+2L_{\mathcal{A}}} |G(t)|^2 dt. \tag{36}$$

As a consequence of the Propositions IV.7 and IV.8 we have

$$\begin{aligned} \mathcal{Q}F_{\bar{\alpha}} &= \mathcal{L}_{\bar{\alpha}}G, & \mathcal{Q}G_{\bar{\alpha}} &= \mathcal{K}_{\bar{\alpha}}G, \\ \mathcal{P}F_{\bar{\alpha}} &= -\mathcal{L}_{\bar{\alpha}}F, & \mathcal{P}G_{\bar{\alpha}} &= \widehat{\mathcal{K}}_{\bar{\alpha}}F \end{aligned}$$

for $\bar{\alpha} \in \mathcal{I}$. Then, from Propositions II.1 and IV.3(i) it follows that

$$E^{\bar{\alpha}}_{\mathcal{Q}\bar{\phi}}(T^*) \leq C \int_{T^*-\ell_{\bar{\alpha}}}^{T^*+\ell_{\bar{\alpha}}} \left(|\mathcal{L}_{\bar{\alpha}}G(t)|^2 + |K_{\bar{\alpha}}G(t)|^2\right) dt \leq C \int_{T^*-2L_{\mathcal{A}}}^{T^*+2L_{\mathcal{A}}} |G(t)|^2 dt,$$

$$E^{\bar{\alpha}}_{\mathcal{P}\bar{\phi}}(T^*) \leq C \int_{T^*-\ell_{\bar{\alpha}}}^{T^*+\ell_{\bar{\alpha}}} \left(|\mathcal{L}_{\bar{\alpha}}F(t)|^2 + |\widehat{\mathcal{K}}_{\bar{\alpha}}F(t)|^2\right) dt \leq C \int_{T^*-2L_{\mathcal{A}}}^{T^*+2L_{\mathcal{A}}} |F(t)|^2 dt,$$

where, as above, $E^{\bar{\alpha}}$ is the energy of the solution in the string $\mathbf{e}_{\bar{\alpha}}$.
It suffices to note that $\mathbf{E} = \sum_{\bar{\alpha}\in\mathcal{I}} E^{\bar{\alpha}}$ to obtain the inequalities (36).

(ii) Now we prove that these inequalities remain true for all $t \in [T^* - L_{\mathcal{A}}, L^* + T_{\mathcal{A}}]$. Indeed, if $t \in [T^* - L_{\mathcal{A}}, T^* + L_{\mathcal{A}}]$, from the formula (I.24) for the energy we have

$$\begin{aligned} \mathbf{E}_{\mathcal{P}\bar{\phi}}(t) &= \mathbf{E}_{\mathcal{P}\bar{\phi}}(T^*) - \int_{T^*}^{t} F^{\mathcal{P}\bar{\phi}}(\tau) G^{\mathcal{P}\bar{\phi}}(\tau) dt \\ &\leq \mathbf{E}_{\mathcal{P}\bar{\phi}}(T^*) + \left| \int_{T^*}^{t} (|F^{\mathcal{P}\bar{\phi}}(\tau)|^2 + |G^{\mathcal{P}\bar{\phi}}(\tau)|^2) dt \right| \\ &\leq \mathbf{E}_{\mathcal{P}\bar{\phi}}(T^*) + \int_{T^*-L_{\mathcal{A}}}^{T^*+L_{\mathcal{A}}} (|\mathcal{P}F(\tau)|^2 + |\mathcal{P}G(\tau)|^2) dt \\ &\leq \mathbf{E}_{\mathcal{P}\bar{\phi}}(T^*) + \int_{T^*-L_{\mathcal{A}}}^{T^*+L_{\mathcal{A}}} (|\mathcal{P}F(\tau)|^2 + |\mathcal{Q}F(\tau)|^2) dt \\ &\leq C \int_{T^*-2L_{\mathcal{A}}}^{T^*+2L_{\mathcal{A}}} |F(\tau)|^2 dt \end{aligned}$$

(in the last step we have used Proposition IV.3(i) and the result of (i)). For the operator $\mathcal{Q}$ the proof is similar. □

REMARK IV.3. *When $\bar{\phi}$ is a solution of (7)-(11) (i.e., $G \equiv 0$), Lemma IV.1 gives $\mathbf{E}_{\mathcal{Q}\bar{\phi}}(t) = 0$. This implies that $\mathcal{Q}\bar{\phi}(t) = 0$. This relation may be viewed as a generalization of the time periodicity property of the solutions of the 1-d wave equation with homogeneous Dirichlet boundary conditions, which with our notations may be written as $\ell^- u(t) = 0$. As we have shown in Proposition IV.4, this generalized periodicity implies that all the essential L^2 information on $\bar{\phi}$ is contained in an interval of length $2L_{\mathcal{A}}$.*

3. The main theorem

In this section we prove the main result on the observability of the solutions of the homogeneous system (1)-(6).

For every non-empty multi-index $\bar{\alpha} = (\alpha_1, ..., \alpha_k) \in \mathcal{I}$ we define the operator $\mathcal{D}_{\bar{\alpha}}$ by

$$\mathcal{D}_{\bar{\alpha}} := \left(\prod_{i=1,\ i\neq\alpha_1}^{m} \mathcal{Q}_i \right) \left(\prod_{i=1,\ i\neq\alpha_2}^{m_{\alpha_1}} \mathcal{Q}_{\alpha_1,i} \right) \cdots \left(\prod_{i=1,\ i\neq\alpha_{k-1}}^{m_{\alpha_1,\ldots,\alpha_{k-1}}} \mathcal{Q}_{\alpha_1,\ldots,\alpha_{k-1},i} \right) \tag{37}$$

and for the empty index $\mathcal{D}$ is the identity operator. We recall that $\mathcal{Q}_{\bar{\beta}}$ is the operator constructed in the previous section for the sub-tree $\mathcal{A}_{\bar{\beta}}$ and that the products in (37) denote the composition of operators.

Note that for every $\bar{\alpha} \in \mathcal{I}$ the operator $\mathcal{D}_{\bar{\alpha}}$ is of type S with $s(\mathcal{D}_{\bar{\alpha}}) < L_{\mathcal{A}}$.

The observability result we will prove is

THEOREM IV.1. *There exists a constant $C > 0$ such that*

$$\mathbf{E}_{\mathcal{D}_{\bar{\alpha}}\bar{\phi}}(0) = \mathbf{E}_{\mathcal{D}_{\bar{\alpha}}\bar{\phi}}(t) \leq C \int_0^{2L_{\mathcal{A}}} |F(\tau)|^2 d\tau,$$

for every solution $\bar{\phi}$ of (1)-(5) and every $\bar{\alpha} \in \mathcal{I}_{\mathcal{S}}$.

The proof is based on

LEMMA IV.2. *There exists a positive constant C, such that for every $\bar{\alpha} \in \mathcal{I}_{\mathcal{S}}$ and every solution $\bar{\phi}$ of* (N)

$$\mathbf{E}_{\mathcal{D}_{\bar{\alpha}}\bar{\phi}}(t) \leq C \int_{t-2L_{\mathcal{A}}}^{t+2L_{\mathcal{A}}} \left(|F(\tau)|^2 + |G(\tau)|^2 \right) d\tau$$

for any $t \in \mathbb{R}$.

PROOF. We proceed by induction. For the case of a single string the assertion is an immediate consequence of the Proposition II.1.

Now fix $\bar{\alpha} = (\alpha_1, ..., \alpha_k) \in \mathcal{I}_{\mathcal{S}}$ and assume that the assertion of the theorem is true for the sub-tree $\mathcal{A}_{\alpha_1}$. That implies that

$$\mathbf{E}^{\alpha_1}_{\mathcal{D}^{\alpha_1}_{\bar{\alpha}}\bar{\phi}}(t) \leq C \int_{t-2L_{\alpha_1}}^{t+2L_{\alpha_1}} \left(|F_{\alpha_1}(\tau)|^2 + |G_{\alpha_1}(\tau)|^2 \right) d\tau \tag{38}$$

for any solution $\bar{\phi}$ of (N), where

$$\mathcal{D}^{\alpha_1}_{\bar{\alpha}} := \left(\prod_{i=1,\ i\neq\alpha_2}^{m_{\alpha_1}} \mathcal{Q}_{\alpha_1,i} \right) \cdots \left(\prod_{i=1,\ i\neq\alpha_{k-1}}^{m_{\alpha_1,\ldots,\alpha_{k-1}}} \mathcal{Q}_{\alpha_1,\ldots,\alpha_{k-1},i} \right) \tag{39}$$

is the operator $\mathcal{D}_{\bar{\alpha}}$ for the sub-tree $\mathcal{A}_{\alpha_1}$ with $\bar{\alpha} = (\alpha_2, ..., \alpha_k)$.

First, we estimate the energy $\mathbf{E}^{\alpha_1}_{\mathcal{D}_{\bar{\alpha}}\bar{\phi}}$ of $\mathcal{D}_{\bar{\alpha}}\bar{\phi}$ on the sub-tree $\mathcal{A}_{\alpha_1}$. To do this, we set

$$\bar{\omega} := (\prod_{j=1,\ j\neq\alpha_1}^{m} \mathcal{Q}_j)\bar{\phi}, \qquad \bar{\omega}_i := (\prod_{\substack{j=1,\ j\neq\alpha_1 \\ j\neq i}}^{m} \mathcal{Q}_j)\bar{\phi}, \quad i = 1, \ldots, m. \tag{40}$$

Note that these functions are also solutions of (N). They verify

$$\bar{\omega} = \mathcal{Q}_i\bar{\omega}_i, \qquad \mathcal{D}_{\bar{\alpha}}\bar{\phi} = \mathcal{D}^{\alpha_1}_{\bar{\alpha}}\bar{\omega}. \tag{41}$$

Besides, from (38)

$$\mathbf{E}^{\alpha_1}_{\mathcal{D}^{\alpha_1}_{\bar{\alpha}}\bar{\omega}}(t) \le C \int_{t-2L_{\alpha_1}}^{t+2L_{\alpha_1}} \left(|F^{\bar{\omega}}_{\alpha_1}(\tau)|^2 + |G^{\bar{\omega}}_{\alpha_1}(\tau)|^2\right) d\tau. \tag{42}$$

But, from the coupling formulas (16) we obtain that

$$\sum_{i=1}^{m} F^{\bar{\omega}}_i = \widehat{F}^{\bar{\omega}}, \qquad G^{\bar{\omega}}_i = \widehat{G}^{\bar{\omega}}, \tag{43}$$

so that it holds

$$F^{\bar{\omega}}_{\alpha_1} = \widehat{F}^{\bar{\omega}} - \sum_{i=1,\ i\neq\alpha_1}^{m} \mathcal{Q}_i F^{\bar{\omega}_i}_i = \widehat{F}^{\bar{\omega}} + \sum_{i=1,\ i\neq\alpha_1}^{m} \mathcal{P}_i \widehat{G}^{\bar{\omega}_i}, \qquad G^{\bar{\omega}}_{\alpha_1} = \hat{G}^{\bar{\omega}}. \tag{44}$$

and then, using the equalities (44), (42) gives

$$\mathbf{E}^{\alpha_1}_{\mathcal{D}^{\alpha_1}_{\bar{\alpha}}\bar{\omega}}(t) \le C \int_{t-2L_{\alpha_1}}^{t+2L_{\alpha_1}} \left(|\widehat{F}^{\bar{\omega}}(\tau) + \sum_{i=1,\ i\neq\alpha_1}^{m} \mathcal{P}_i \widehat{G}^{\bar{\omega}_i}(\tau)|^2 + |\hat{G}^{\bar{\omega}}(\tau)|^2 \right) d\tau$$

and this implies

$$\mathbf{E}^{\alpha_1}_{\mathcal{D}^{\alpha_1}_{\bar{\alpha}}\bar{\omega}}(t) \le C \int_{t-2L_{\alpha_1}}^{t+2L_{\alpha_1}} \left(|\widehat{F}^{\bar{\omega}}(\tau)|^2 + \sum_{i=1,\ i\neq\alpha_1}^{m} |\mathcal{P}_i \widehat{G}^{\bar{\omega}_i}(\tau)|^2 + |\hat{G}^{\bar{\omega}}(\tau)|^2 \right) d\tau. \tag{45}$$

Now, from the definition of $\bar{\omega}$ and the formulas (17), (18) we have

$$\widehat{F}^{\bar{\omega}} = (\prod_{j=1,\ j\neq\alpha_1}^{m} \mathcal{Q}_j)\widehat{F} = (\prod_{j=1,\ j\neq\alpha_1}^{m} \mathcal{Q}_j)\ell^+ F + (\prod_{j=1,\ j\neq\alpha_1}^{m} \mathcal{Q}_j)\ell^- G$$

and consequently

$$\int_{t-2L_{\alpha_1}}^{t+2L_{\alpha_1}} |\widehat{F}^{\bar{\omega}}|^2 d\tau \le 2 \int_{t-2L_{\alpha_1}}^{t+2L_{\alpha_1}} \left(|(\prod_{j=1,\ j\neq\alpha_1}^{m} \mathcal{Q}_j)\ell^+ F|^2 + |(\prod_{j=1,\ j\neq\alpha_1}^{m} \mathcal{Q}_j)\ell^- G|^2 \right) d\tau.$$

Observe that the operators $(\prod_{j=1,\ j\neq\alpha_1}^{m} \mathcal{Q}_j)\ell^+$ and $(\prod_{j=1,\ j\neq\alpha_1}^{m} \mathcal{Q}_j)\ell^-$ are of type S with $s < L_{\mathcal{A}} - L_{\alpha_1}$ so that, the latter inequality combined with Proposition IV.3 provides

$$\int_{t-2L_{\alpha_1}}^{t+2L_{\alpha_1}} |\widehat{F}^{\bar{\omega}}(\tau)|^2 d\tau \le C \int_{t-2L_{\mathbb{A}}}^{t+2L_{\mathcal{A}}} (|F(\tau)|^2 + |G(\tau)|^2) d\tau.$$

In a similar way it may be proved that

$$\int_{t-2L_{\alpha_1}}^{t+2L_{\alpha_1}} |\mathcal{P}_i \widehat{G}^{\bar{\omega}_i}(\tau)|^2 d\tau \le C \int_{t-2L_{\mathbb{A}}}^{t+2L_{\mathcal{A}}} (|F(\tau)|^2 + |G(\tau)|^2) d\tau$$

and

$$\int_{t-2L_{\alpha_1}}^{t+2L_{\alpha_1}} |\hat{G}^{\bar{\omega}}(\tau)|^2 d\tau \le C \int_{t-2L_{\mathbb{A}}}^{t+2L_{\mathcal{A}}} (|F(\tau)|^2 + |G(\tau)|^2) d\tau.$$

Therefore, these three inequalities together with (41) and (45) give

$$\mathbf{E}^{\alpha_1}_{\mathcal{D}_{\bar{\alpha}}\bar{\phi}}(t) \le C \int_{t-2L_{\mathbb{A}}}^{t+2L_{\mathcal{A}}} (|F(\tau)|^2 + |G(\tau)|^2) d\tau. \tag{46}$$

Now we proceed to estimate the energies ${}^{i}_{\mathcal{D}_{\bar\alpha}\bar\phi}$ of $\mathcal{D}_{\bar\alpha}\bar\phi$ on the remaining sub-trees $\mathcal{A}_i$ (i.e., for $i \neq \alpha_1$). According to Lemma IV.1, applied to $\bar\omega_i$ in the sub-tree $\mathcal{A}_i$, it holds that for every t' in $[t-L_i, t+L_i]$

$$\mathbf{E}^i_{\bar\omega}(t') = \mathbf{E}^i_{\mathcal{Q}_i\bar\omega_i}(t') \leq C\int_{t-2L_i}^{t+2L_i} |G_i^{\bar\omega_i}(\tau)|^2 d\tau, \tag{47}$$

for $i = 1, \ldots, m$. Taking into account that

$$G_i^{\bar\omega_i} = (\prod_{\substack{j=1,\ j\neq\alpha_1 \\ j\neq i}}^{m} \mathcal{Q}_j)\widehat{G} = (\prod_{\substack{j=1,\ j\neq\alpha_1 \\ j\neq i}}^{m} \mathcal{Q}_j)\ell^+ F + (\prod_{\substack{j=1,\ j\neq\alpha_1 \\ j\neq i}}^{m} \mathcal{Q}_j)\ell^- G, \tag{48}$$

we get from (47) and Proposition IV.3(i)

$$\mathbf{E}^i_{\bar\omega}(t') \leq C\int_{t-L_\mathcal{A}-L_i+L_{\alpha_1}}^{t+L_\mathcal{A}+L_i-L_{\alpha_1}} (|F(\tau)|^2+|G(\tau)|^2)d\tau \leq C\int_{t-2L_\mathcal{A}}^{t+2L_\mathcal{A}} (|F(\tau)|^2+|G(\tau)|^2)d\tau, \tag{49}$$

(here we have used the fact that the operators applied to F and G in the right hand term of (48) are of type S with $s = L_\mathcal{A} - L_{\alpha_1} - L_i$).

Now, if we apply Proposition IV.3(ii) with $\mathcal{B} = \mathcal{D}_{\bar\alpha}^{\alpha_1}$ to (49) (recall that $s(\mathcal{D}_{\bar\alpha}^{\alpha_1}) < L_{\alpha_1}$) we obtain, after choosing $t' = t$,

$$\mathbf{E}^i_{\mathcal{D}_{\bar\alpha}}\bar\phi(t) = \mathbf{E}^i_{\mathcal{D}_{\bar\alpha}^{\alpha_1}\bar\omega}(t') \leq\ C\int_{t-2L_\mathcal{A}}^{t+2L_\mathcal{A}} (|F(\tau)|^2+|G(\tau)|^2)d\tau. \tag{50}$$

Finally, from Proposition II.1 we obtain that the component ϕ of $\bar\phi$ verifies, for every $t' \in [t-L_\mathcal{A}, t+L_\mathcal{A}]$,

$$E_u(t') \leq\ C\int_{t-\ell-L_\mathcal{A}}^{t+\ell+L_\mathcal{A}} (|F(\tau)|^2+|G(\tau)|^2)d\tau.$$

Thus, using Proposition IV.3(ii), it holds that

$$E_{\mathcal{D}_{\bar\alpha}\bar\phi}(t') \leq\ C\int_{t-\ell-L_\mathcal{A}}^{t+\ell+L_\mathcal{A}} (|F(\tau)|^2+|G(\tau)|^2)d\tau,$$

for every $t' \in [t-L_\mathcal{A}+s(\ \mathcal{D}_{\bar\alpha}), t+L_\mathcal{A}-s(\ \mathcal{D}_{\bar\alpha})]$ and, since $s(\ \mathcal{D}_{\bar\alpha}) < L_\mathcal{A}$, this is true in particular for $t' = t$. Therefore,

$$E_{\ \mathcal{D}_{\bar\alpha}\bar\phi}(t) \leq\ C\int_{t-2L_\mathcal{A}}^{t+2L_\mathcal{A}} (|F(\tau)|^2+|G(\tau)|^2)d\tau. \tag{51}$$

Now, it suffices to combine (46), (50), (51) and the fact that

$$\mathbf{E}_{\ \mathcal{D}_{\bar\alpha}\bar\phi} = E_{\ \mathcal{D}_{\bar\alpha}\bar\phi} + \sum_{i=1}^{m} {}^{i}_{\mathcal{D}_{\bar\alpha}\bar\phi}$$

to conclude the proof. □

With the help of Lemma IV.2 the proof of Theorem IV.1 is simple.

Proof of Theorem IV.1. If $\bar\phi$ is a solution of (1)-(6), so is $\mathcal{D}_{\bar\alpha}\bar\phi$. In particular, the energy of $\mathcal{D}_{\bar\alpha}\bar\phi$ is conserved. Then, taking into account that $G \equiv 0$ for the solutions of (1)-(5), from Lemma IV.2 it holds

$$\mathbf{E}_{\ \mathcal{D}_{\bar\alpha}\bar\phi}(0) = \mathbf{E}_{\ \mathcal{D}_{\bar\alpha}\bar\phi}(2T_\mathcal{A}) \leq\ C\int_0^{4L_\mathcal{A}} |F(\tau)|^2 d\tau. \tag{52}$$

On the other hand, in this case $\mathcal{Q}F \equiv 0$ and then, using Proposition IV.4 (which may be applied to $\mathcal{Q}$ on the basis of Proposition IV.6) we have

$$\int_0^{4L_{\mathcal{A}}} |F(\tau)|^2 d\tau \leq C \int_0^{2L_{\mathcal{A}}} |F(\tau)|^2 d\tau.$$

With this, the assertion of the theorem follows from (52). □

4. Relation between $\mathcal{P}$ and $\mathcal{Q}$ and the eigenvalues

Our next objective is to express the inequality (52) in terms of the Fourier coefficients of the solution $\bar{u}$ of (1)-(6). This will lead to weighted observability inequalities with weights that depend on the eigenvalues μ_n of the operator $-\Delta_{\mathcal{A}}$. To study those weights we need some additional properties of the eigenvalues.

4.1. The eigenvalue problem. We consider the eigenvalue problem for the elliptic operator $-\Delta_{\mathcal{A}}$ associated to the hyperbolic problem (1)-(5):

$$-\theta^{\bar{\alpha}}_{xx}(x) = \mu\, \theta^{\bar{\alpha}}(x) \qquad x \in [0, \ell_{\bar{\alpha}}], \quad \bar{\alpha} \in \mathcal{I}, \tag{53}$$

$$\theta^{\bar{\alpha}\circ\beta}(0) = \theta^{\bar{\alpha}}(\ell_{\bar{\alpha}}) \qquad \bar{\alpha} \in \mathcal{I}_{\mathcal{M}}, \quad \beta = 1, \ \ldots, m_{\bar{\alpha}}, \tag{54}$$

$$\sum_{\beta=1}^{m_{\bar{\alpha}}} \theta^{\bar{\alpha}\circ\beta}_x(0) = \theta^{\bar{\alpha}}_x(\ell_{\bar{\alpha}}) \qquad \bar{\alpha} \in \mathcal{I}_{\mathcal{M}}, \tag{55}$$

$$\theta^{\bar{\alpha}}(\ell_{\bar{\alpha}}) = 0 \qquad \bar{\alpha} \in \mathcal{I}_{\mathcal{S}}, \tag{56}$$

$$\theta(0) = 0 \qquad \text{at the root } \mathcal{R}. \tag{57}$$

As it has been pointed out in Chapter I, the spectrum of $-\Delta_{\mathcal{A}}$ is formed by a positive, increasing sequence $\{\mu_k\}_{k\in\mathbb{Z}_+}$ of eigenvalues. We call it *spectrum of* $\mathcal{A}$ and denote it by $\sigma_{\mathcal{A}}$.

Clearly, we may consider the problem (53)-(57) for each sub-tree $\mathcal{A}_{\bar{\alpha}}$ of $\mathcal{A}$. The corresponding spectrum is called *spectrum of* $\mathcal{A}_{\bar{\alpha}}$ and is denoted by $\sigma_{\bar{\alpha}}$.

For technical reasons, as we did for the system (1)-(5), we will also consider smooth solutions of (53), which verify the boundary conditions (54)-(56) but not necessarily (57). For brevity, they are simply called *solutions of* (N_E) *corresponding to* μ.

PROPOSITION IV.9. *If μ is a common eigenvalue of two sub-trees $\mathcal{A}_{\bar{\alpha}\circ i}$, $\mathcal{A}_{\bar{\alpha}\circ j}$ $(i \neq j)$ with the same root $\mathcal{O}_{\bar{\alpha}}$ then μ is also an eigenvalue of $\mathcal{A}$. Moreover, there exists a non-zero eigenfunction $\bar{\theta}$ associated to μ such that*

$$\theta(0) = \theta_x(0) = 0.$$

PROOF. Let $\bar{\theta}^{\bar{\alpha}\circ i}$, $\bar{\theta}^{\bar{\alpha}\circ j}$ be non-zero eigenfunctions corresponding to the eigenvalue μ for the sub-trees $\mathcal{A}_{\bar{\alpha}\circ i}$ and $\mathcal{A}_{\bar{\alpha}\circ j}$, respectively. These functions are defined in the corresponding sub-trees but it will be sufficient to paste them conveniently to build up an eigenfunction of $\mathcal{A}$.

We may assume that the numbers $\theta^{\bar{\alpha}\circ i}_x(0)$, $\theta^{\bar{\alpha}\circ j}_x(0)$ are both different from zero. Indeed, if one of them, say $\theta^{\bar{\alpha}\circ i}_x(0)$, vanishes then the relations

$$\theta^{\bar{\alpha}\circ i}(0) = \theta^{\bar{\alpha}\circ i}_x(0) = 0$$

allow to ensure that the function $\bar{\theta}$, obtained by extending by zero the function $\bar{\theta}^{\bar{\alpha}\circ i}$ to the whole tree $\mathcal{A}$, satisfies (53)-(57) for that value of μ and then is an eigenfunction of $\mathcal{A}$.

Now define the function $\bar{\theta}$ by

$$\theta_{\bar{\alpha}'} = \begin{cases} \theta_x^{\bar{\alpha}\circ j}(0)\ \theta_{\bar{\beta}}^{\bar{\alpha}\circ i} & \text{if } \bar{\alpha}' = \bar{\alpha}\circ i\circ\bar{\beta}, \\ \text{-}\ \theta_x^{\bar{\alpha}\circ i}(0)\ \theta_{\bar{\beta}}^{\bar{\alpha}\circ j} & \text{if } \bar{\alpha}' = \bar{\alpha}\circ j\circ\bar{\beta}, \\ 0 & \text{otherwise,} \end{cases}$$

i.e., $\bar{\theta}$ coincides in the sub-tree $\mathcal{A}_{\bar{\alpha}\circ i}$ with $\theta_x^{\bar{\alpha}\circ j}(0)\bar{\theta}^{\bar{\alpha}\circ i}$, in $\mathcal{A}_{\bar{\alpha}\circ j}$ with $-\theta_x^{\bar{\alpha}\circ i}(0)\bar{\theta}^{\bar{\alpha}\circ j}$ and vanishes outside those sub-trees. It is easy to see that $\bar{\theta}$ satisfies the boundary conditions (54)-(55) at $\mathcal{O}_{\bar{\alpha}}$:

$$\sum_{k=1}^{m_{\bar{\alpha}}} \theta_x^{\bar{\alpha}\circ k}(0) = \theta_x^{\bar{\alpha}\circ j}(0)\theta_x^{\bar{\alpha}\circ i}(0) - \theta_x^{\bar{\alpha}\circ i}(0)\theta_x^{\bar{\alpha}\circ j}(0) = 0 = \theta_x^{\bar{\alpha}}(\ell_{\bar{\alpha}}).$$

As at the other nodes they are obviously satisfied, $\bar{\theta}$ is an eigenfunction of $\mathcal{A}$.

Finally observe that in both cases, the eigenfunction $\bar{\theta}$ constructed here is such that

$$\theta(0) = \theta_x(0) = 0,$$

and thus, $\theta \equiv 0$, i.e., $\bar{\theta}$ vanishes at the whole string containing the root of $\mathcal{A}$. □

REMARK IV.4. *Note that the eigenfunction constructed in the proof of Proposition IV.9 vanishes everywhere outside the sub-trees $\mathcal{A}_{\bar{\alpha}\circ i}$, $\mathcal{A}_{\bar{\alpha}\circ j}$. If we denote $\mathcal{A}_{\bar{\alpha}\circ i}\vee\mathcal{A}_{\bar{\alpha}\circ j}$ the tree formed by $\mathcal{A}_{\bar{\alpha}\circ i}$ and $\mathcal{A}_{\bar{\alpha}\circ j}$ in which the node $\mathcal{O}_{\bar{\alpha}}$ is considered as an interior point of a string of length $\ell_{\bar{\alpha}\circ i}+\ell_{\bar{\alpha}\circ j}$, we obtain that these sub-trees have a common eigenvalue if and only if there exists a an eigenfunction of $\mathcal{A}_{\bar{\alpha}\circ i}\vee\mathcal{A}_{\bar{\alpha}\circ j}$ that vanishes at the point $\mathcal{O}_{\bar{\alpha}}$.*

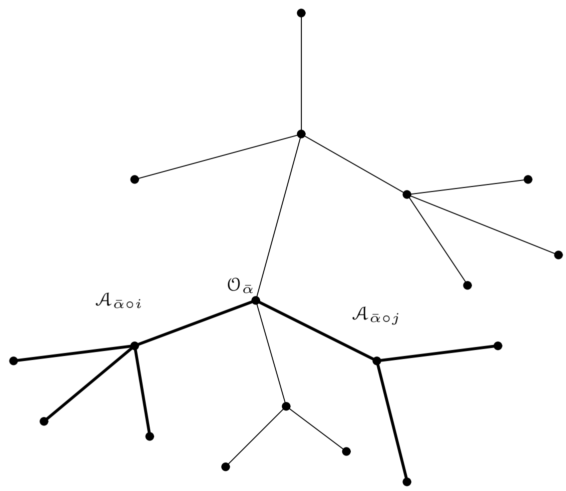

FIGURE 2. The sub-tree $\mathcal{A}_{\bar{\alpha}\circ i}\vee\mathcal{A}_{\bar{\alpha}\circ j}$

As it has been shown above, the operators $\mathcal{P}$ and $\mathcal{Q}$ are of type S with $s(\mathcal{P}) = s(\mathcal{Q}) = L_{\mathcal{A}}$. According to Remark IV.1, there exist functions p and q such that for all $t,\lambda \in \mathbb{R}$,

$$\mathcal{P}e^{i\lambda t} = p(\lambda)e^{i\lambda t}, \qquad \mathcal{Q}e^{i\lambda t} = q(\lambda)e^{i\lambda t}. \tag{58}$$

PROPOSITION IV.10. *Let $\lambda \in \mathbb{R} \setminus \{0\}$ and $f, g \in \mathbb{C}$ such that*

$$q(\lambda) f + i\lambda p(\lambda) g = 0. \tag{59}$$

If the tree $\mathcal{A}$ satisfies the property

$$|q_{\bar{\alpha}\circ i}(\lambda)| + |q_{\bar{\alpha}\circ j}(\lambda)| \neq 0 \text{ for any } \bar{\alpha} \in \mathcal{I}_{\mathcal{M}},\ i, j = 1, \ldots m_{\bar{\alpha}},\ i \neq j, \tag{60}$$

then there exists a unique solution $\bar{\theta}$ of (N_E) *corresponding to the value $\mu = \lambda^2$ such that*

$$\theta(0) = g \quad \text{and} \quad \theta_x(0) = f. \tag{61}$$

PROOF. First we construct the component θ of $\bar{\theta}$ (the one corresponding to the string $\mathbf{e}$). We set

$$\theta(x) = g \cos \lambda x + \frac{f}{\lambda} \sin \lambda x, \tag{62}$$

which clearly satisfies (61).

If the network consists of a single string of length ℓ, then

$$p(\lambda) = \cos \lambda \ell, \qquad q(\lambda) = i \sin \lambda \ell$$

and condition (59) becomes

$$if \sin \lambda \ell + ig\lambda \cos \lambda \ell = 0.$$

This implies that

$$\theta(\ell) = g \cos \lambda \ell + \frac{f}{\lambda} \sin \lambda \ell = 0,$$

what means that θ is a solution of (N_E) and so, the assertion is true in this case.

In the general case the remaining components of $\bar{\theta}$ are constructed by induction. Assume that the proposition is true for the sub-trees $\mathcal{A}_1, \ldots, \mathcal{A}_m$.

If we were able to choose numbers $f_1, \ldots, f_m$ verifying

$$\sum_{k=1}^{m} f_k = \theta_x(\ell) \text{ and } q_k(\lambda) f_k + i\lambda p_k(\lambda)\theta(\ell) = 0 \text{ for } k = 1, \ldots, m, \tag{63}$$

then, according to the induction assumption, we could find solutions $\bar{\theta}^1, \ldots, \bar{\theta}^m$, defined on the sub-trees $\mathcal{A}_1, \ldots, \mathcal{A}_m$, respectively, such that

$$\theta^k(0) = \theta(\ell), \ \ \theta^k_x(0) = f_k, \text{ for } k = 1, \ldots, m.$$

This would imply that

$$\sum_{k=1}^{m} \theta^k_x(0) = \theta(\ell) \text{ and } \quad \theta^k(0) = \theta(\ell), \quad \text{for } k = 1, \ldots, m.$$

Therefore, the function $\bar{\theta}$ defined on the tree by $\theta_{k\circ\bar{\alpha}} = \theta^k_{\bar{\alpha}}$ would be the solution of (N_E), whose existence is asserted in the proposition. Consequently, it remains to prove the possibility of the decomposition (63).

We remark that from the definition of p and q and formulas (28), (29) it follows that

$$p = \cos \lambda \ell \sum_{k=1}^{m} p_k \prod_{j \neq k} q_j + i \sin \lambda \ell \prod_{j=1}^{m} q_j, \tag{64}$$

$$q = i\sin\lambda\ell \sum_{k=1}^{m} p_k \prod_{j\neq k} q_j + \cos\lambda\ell \prod_{j=1}^{m} q_j, \tag{65}$$

Note that condition (60) implies that among the numbers $q_k(\lambda)$, $k = 1,...,m$, at most one may be equal to zero. Thus, we consider two cases: a) all the numbers $q_k(\lambda)$, $k = 1,...,m$, are different from zero and b) exactly one of those numbers, say, e.g., $q_1(\lambda)$ is equal to zero.

Case a). If we take

$$f_k = \frac{-i\lambda p_k(\lambda)\theta(\ell)}{q_k(\lambda)}$$

Then

$$\sum_{k=1}^{m} f_k = -i\lambda\theta(\ell)\sum_{k=1}^{m}\frac{p_k}{q_k} = -i\lambda(g\cos\lambda\ell + \frac{f}{\lambda}\sin\lambda\ell)\frac{\sum_{k=1}^{m} p_k \prod_{j\neq k} q_j}{\prod_{j=1}^{m} q_j}.$$

This equality, taking into account (64), (65), gives

$$\begin{aligned}\sum_{k=1}^{m} f_k &= -i\lambda g(\frac{p}{\prod_{j=1}^{m} q_j} - i\sin\lambda\ell) - f(\frac{q}{\prod_{j=1}^{m} q_j} - \cos\lambda\ell) \\ &= -\lambda g\sin\lambda\ell + f\cos\lambda\ell - \frac{i\lambda pg + qf}{\prod_{j=1}^{m} q_j} = -\lambda g\sin\lambda\ell + f\cos\lambda\ell = \theta_x(\ell).\end{aligned}$$

Thus, the numbers $f_1,...,f_m$ satisfy (63).

Case b). The relations (64), (65) together with $q_1(\lambda) = 0$ give

$$p(\lambda) = \cos\lambda\ell p_1(\lambda)\prod_{j\neq 1} q_j(\lambda), \qquad q(\lambda) = i\sin\lambda\ell p_1(\lambda)\prod_{j\neq 1} q_j(\lambda),$$

and from (59) we obtain

$$0 = q(\lambda)f + i\lambda p(\lambda)g = i\lambda(g\cos\lambda\ell + \frac{g}{\lambda}\sin\lambda\ell)p_1(\lambda)\prod_{j\neq 1} q_j(\lambda) = i\lambda\theta(\ell)p_1(\lambda)\prod_{j\neq 1} q_j(\lambda).$$

But $\prod_{j\neq 1} q_j(\lambda) \neq 0$ and then, necessarily, $\theta(\ell)p_1(\lambda) = 0$. It means that if we choose $f_1 = \theta_x(\ell)$ and $f_2,...,f_m$ verifying

$$\sum_{k=2}^{m} f_k = 0 \text{ and } q_k(\lambda)f_k + i\lambda p_k(\lambda)\theta(\ell) = 0 \text{ for } k = 2,...,m,$$

as in the previous case then the condition (63) is satisfied.

So far, we have proved the existence of a solution. It turns out that for the solutions satisfying (IV.10) we can give an explicit formula. Indeed, if we apply propositions IV.7 and IV.8 to the solution

$$\bar\theta(t,x) = e^{i\lambda t}\bar\theta(x)$$

of (N) we obtain

$$i\lambda p(\lambda)\theta^{\bar\alpha}(0) = \hat k(\lambda)\theta_x(0) = \hat k(\lambda)f, \qquad p(\lambda)\theta_x^{\bar\alpha}(0) = -l(\lambda)\theta_x(0) = -l(\lambda)f, \tag{66}$$

$$q(\lambda)\theta^{\bar\alpha}(0) = k(\lambda)\theta(0) = k(\lambda)g, \qquad q(\lambda)\theta_x^{\bar\alpha}(0) = i\lambda l(\lambda)\theta(0) = i\lambda l(\lambda)g, \tag{67}$$

where k, $\hat k$, l and r are the functions associated to the operators $\mathcal{K}$, $\hat{\mathcal{K}}$, $\mathcal{L}$ and $\mathcal{R}$, respectively, according to Remark IV.1.

On the other hand, the condition (60) implies that at least one of the numbers $p(\lambda)$ or $q(\lambda)$ is different from zero (see Proposition IV.12 below). Therefore, one of the equalities (66), (67) provides us with an explicit formula for the values of $\theta^{\bar{\alpha}}(0)$ and $\theta^{\bar{\alpha}}_x(0)$ for any $\bar{\alpha} \in \mathcal{I}_{\mathcal{M}}$ and thus, for the solution $\bar{\theta}$. In particular, if $f = g = 0$ the corresponding solution vanishes identically on $\mathcal{A}$, what clearly implies the uniqueness of the solution for arbitrary values of f and g. □

Remark IV.5. *The converse assertion is also true, even if the condition (60) is not fulfilled. Indeed, if* $\bar{\theta}$ *is a solution of* (N_E) *then*

$$\bar{\theta}(t,x) = e^{i\lambda t}\bar{\theta}(x)$$

is a solution of (N) *and*

$$\theta_t(t,0) = i\lambda e^{i\lambda t}\theta(0), \quad \theta_x(t,0) = e^{i\lambda t}\theta_x(0).$$

Then, from the relations (19) and (58) it follows

$$0 = \mathcal{P}\theta_t(t,0) + \mathcal{Q}\theta_x(t,0) = (ip\lambda\theta(0) + q\theta_x(0))e^{i\lambda t},$$

for every $t \in \mathbb{R}$. *Thus, (59) holds.*

Now we are ready to prove the following basic property.

Proposition IV.11. *Let* $0 \neq \lambda \in \mathbb{R}$. *Then* λ^2 *is an eigenvalue of* $\mathcal{A}$ *if and only if* $q(\lambda) = 0$.

Proof. First we prove that $q(\lambda) = 0$ implies that λ^2 is an eigenvalue, i.e., that there exists a non-zero solution of (53)-(57) for that value of λ. If the tree verifies (60) then this fact follows immediately from Proposition IV.10 choosing $g = 0$, $f \neq 0$. Note that the condition $0 \neq f = \theta_x(0)$ guarantees that $\bar{\theta}$ is not identically equal to zero. In particular, the assertion is true for a string, as it always verifies (60).

In the general case when the condition (60) may fail, we follow an induction argument: we suppose that the assertion has been proved for all the sub-trees $\mathcal{A}_{\bar{\alpha}}$ with non-empty $\bar{\alpha}$.

If $q_{\bar{\alpha}\circ i}(\lambda) = q_{\bar{\alpha}\circ j}(\lambda) = 0$ for some $\bar{\alpha} \in \mathcal{I}_{\mathcal{M}}$, $i \neq j$, then, according to the induction hypothesis, λ^2 is an eigenvalue of both $\mathcal{A}_{\bar{\alpha}\circ i}$ and $\mathcal{A}_{\bar{\alpha}\circ j}$. Then from Proposition IV.9 it follows that λ^2 is an eigenvalue of $\mathcal{A}$, too.

Let us see now the converse assertion. Let $\bar{\theta}$ be a non-zero eigenfunction corresponding to the eigenvalue λ^2. Then the function $\bar{u}(t,x) = e^{i\lambda t}\bar{\theta}(x)$ is a solution of (N). Choose $\bar{\alpha} \in \mathcal{I}$ such that one of the numbers $\theta^{\bar{\alpha}}(0)$ or $\theta^{\bar{\alpha}}_x(0)$ is different from zero (that is possible since, otherwise, it would be $\bar{\theta} \equiv 0$). For this solution of (N) we have for every $\bar{\alpha} \in \mathcal{I}$

$$F_{\bar{\alpha}}(t) = e^{i\lambda t}\theta^{\bar{\alpha}}_x(0), \qquad G_{\bar{\alpha}}(t) = i\lambda e^{i\lambda t}\theta^{\bar{\alpha}}(0),$$

and in particular, $G \equiv 0$. Then, from the Propositions IV.7 and IV.8 it follows that

$$0 = \mathcal{L}_{\bar{\alpha}}G = \mathcal{Q}F_{\bar{\alpha}} = \mathcal{Q}e^{i\lambda t}\theta^{\bar{\alpha}}_x(0) = q(\lambda)\theta^{\bar{\alpha}}_x(0),$$

$$0 = \mathcal{K}_{\bar{\alpha}}G = \mathcal{Q}G_{\bar{\alpha}} = \mathcal{Q}e^{i\lambda t}\theta^{\bar{\alpha}}(0) = i\lambda q(\lambda)\theta^{\bar{\alpha}}(0),$$

and therefore, necessarily, $q(\lambda) = 0$. □

4.2. Further properties of p and q.

PROPOSITION IV.12. *For every tree $\mathcal{A}$ the following properties hold:*
(i) one of the functions p, q is even and the other is odd;
(ii) there exists $\lambda_0 \in \mathbb{R}$ such that $p(\lambda_0) = q(\lambda_0) = 0$ if, and only if, there exist two sub-trees $\mathcal{A}_{\bar{\alpha}\circ i}$, $\mathcal{A}_{\bar{\alpha}\circ j}$, $i \neq j$, with common root $\mathcal{O}_{\bar{\alpha}}$ such that

$$q_{\bar{\alpha}\circ i}(\lambda_0) = q_{\bar{\alpha}\circ j}(\lambda_0) = 0.$$

PROOF. We proceed by induction. For a single string

$$p(\lambda) = \cos \lambda\ell, \qquad q(\lambda) = i \sin \lambda\ell.$$

In this case (i) is trivial. Assertion (ii) follows from the fact that $|p|^2 + |q|^2 = 1$.

Suppose now that (i), (ii) are true for the sub-trees $\mathcal{A}_1, \ldots, \mathcal{A}_m$.

Let h be a function, which is either even or odd. Denote

$$\rho(h) = \begin{cases} 1 & \text{if } h \text{ even,} \\ -1 & \text{if } h \text{ is odd.} \end{cases}$$

The function ρ is multiplicative:

$$\rho(h_1 h_2) = \rho(h_1)\rho(h_2).$$

According to the definitions of p and q and the formulas (28), (29) we have that

$$q(\lambda) = i \sin \lambda\ell \sum_{i=1}^{m} p_i(\lambda) \prod_{j\neq i} q_j(\lambda) + \cos \lambda\ell \prod_{i=1}^{m} q_i(\lambda), \tag{68}$$

$$p(\lambda) = \cos \lambda\ell \sum_{i=1}^{m} p_i(\lambda) \prod_{j\neq i} q_j(\lambda) + i \sin \lambda\ell \prod_{i=1}^{m} q_i(\lambda). \tag{69}$$

The hypotheses with respect to the sub-trees imply that $\rho(p_i) = -\rho(q_i)$, $i = 1, \ldots, m$. Then,

$$\rho(i \sin \lambda\ell \, p_i(\lambda) \prod_{j\neq i} q_j) = \prod_{i=1}^{m} \rho(q_i); \qquad \rho(\cos \lambda\ell \prod_{i=1}^{m} q_i) = \prod_{i=1}^{m} \rho(q_i).$$

From these relations and (69) we obtain

$$\rho(q) = \prod_{i=1}^{m} \rho(q_i).$$

In an analogous way it is proved that

$$\rho(p) = -\prod_{i=1}^{m} \rho(q_i).$$

From these two last equalities it holds $\rho(p) = -\rho(q)$. This proves the property (i).

We now prove (ii). If $p(\lambda_0) = q(\lambda_0) = 0$ then, from (68), (69) it follows that

$$0 = q(\lambda_0) = i \sin \lambda_0\ell \sum_{i=1}^{m} p_i(\lambda_0) \prod_{j\neq i} q_j(\lambda_0) + \cos \lambda_0\ell \prod_{i=1}^{m} q_i(\lambda_0),$$

$$0 = p(\lambda_0) = \cos \lambda_0\ell \sum_{i=1}^{m} p_i(\lambda_0) \prod_{j\neq i} q_j(\lambda_0) + i \sin \lambda_0\ell \prod_{i=1}^{m} q_i(\lambda_0).$$

This implies that

$$\sum_{i=1}^{m} p_i(\lambda_0) \prod_{j\neq i} q_j(\lambda_0) = 0, \tag{70}$$

$$\prod_{i=1}^{m} q_i(\lambda_0) = 0. \tag{71}$$

These equalities are verified if, and only if, for some i_0

$$q_{i_0}(\lambda_0) = 0, \qquad p_{i_0}(\lambda_0) \prod_{j\neq i_0} q_j(\lambda_0) = 0$$

and this is equivalent to the fact that one of the following assertions is true

(a) there exists $i_1 \neq i_0$ such that $q_{i_1}(\lambda_0) = 0$;

(b) $p_{i_0}(\lambda_0) = 0$.

In the first case assertion (ii) follows immediately. In (b), according to the induction assumption, there exist sub-trees of $\mathcal{A}_{i_0}$, and consequently also of $\mathcal{A}$, that verify condition (ii). $\square$

With the aid of the previous proposition it is possible to calculate how the operator $\mathcal{Q}$ acts on the functions $\sin\lambda t$ and $\cos\lambda t$.

COROLLARY IV.1. *The following equalities are verified*

$$\mathcal{Q}\sin\lambda t = \begin{cases} q(\lambda)\sin\lambda t & \text{if } q \text{ is even,} \\ -\,iq(\lambda)\cos\lambda t & \text{if } q \text{ is odd,} \end{cases}$$

$$\mathcal{Q}\cos\lambda t = \begin{cases} q(\lambda)\cos\lambda t & \text{if } q \text{ is even,} \\ iq(\lambda)\sin\lambda t & \text{if } q \text{ is odd.} \end{cases}$$

REMARK IV.6. *As a consequence of the previous formulas, when q is an even function then it is real valued, while, when it is odd then iq is real valued.*

5. Observability results

In this section we express the inequalities from Theorem IV.1 in terms of the initial data of the solution $\bar\phi$. This allows us to obtain weighted observability inequalities, with explicit weights on the Fourier coefficient of the initial data of the solution. Further, we study under what conditions those weights are different from zero.

5.1. Weighted observability inequalities. As stated above, a solution $\bar\phi$ of (1)-(5) is expressed in terms of the initial data $\bar\phi_0$, $\bar\phi_1$ by the formula

$$\bar\phi(t) = \sum_{k\in\mathbf{Z}_+} \left(\phi_{0,k}\cos\lambda_k t + \frac{\phi_{1,k}}{\lambda_k}\sin\lambda_k t\right)\bar\theta_k, \tag{72}$$

where $\{\phi_{0,k}\}$, $\{\phi_{1,k}\}$ are the sequences of Fourier coefficients of $\bar\phi_0$, $\bar\phi_1$ with respect to the orthonormal basis of eigenfunctions $\{\bar\theta_k\}_{k\in\mathbb{Z}_+}$ and $\lambda_k = \sqrt{\mu_k}$.

Besides, the energy of the solution $\bar\phi$ is given by

$$\mathbf{E}_{\bar\phi} = \frac{1}{2}\sum_{k\in\mathbf{Z}_+} \left(\lambda_k^2\phi_{0,k}^2 + \phi_{1,k}^2\right). \tag{73}$$

The operators $\mathcal{D}_{\bar\alpha}$ defined in Section 3 are of type S. Then, according to Remark IV.1, there exist functions $d_{\bar\alpha}$ such that

$$\mathcal{D}_{\bar\alpha} e^{i\lambda t} = d_{\bar\alpha}(\lambda) e^{i\lambda t}.$$

In particular, when $\bar\alpha$ is the empty index we have $d(\lambda) \equiv 1$.

These functions, taking into account (37) are expressed as

$$d_{\bar\alpha} := \left(\prod_{i=1,\ i\neq\alpha_1}^{m} q_i\right)\left(\prod_{i=1,\ i\neq\alpha_2}^{m_{\alpha_1}} q_{\alpha_1,i}\right)\cdots\left(\prod_{i=1,\ i\neq\alpha_{k-1}}^{m_{\alpha_1,\dots,\alpha_{k-1}}} q_{\alpha_1,\dots,\alpha_{k-1},i}\right), \tag{74}$$

and then Proposition IV.12 allows to ensure that, for every $\bar\alpha \in \mathcal{I}$, $d_{\bar\alpha}$ is an even or odd function. Moreover, from Corollary IV.1 we have the equalities

$$\begin{aligned}
\mathcal{D}_{\bar\alpha} \sin\lambda t &= \begin{cases} d_{\bar\alpha}(\lambda)\sin\lambda t & \text{for } d_{\bar\alpha} \text{ even},\\ -\,i d_{\bar\alpha}(\lambda)\cos\lambda t & \text{for } d_{\bar\alpha} \text{ odd},\end{cases}\\
\mathcal{D}_{\bar\alpha} \cos\lambda t &= \begin{cases} d_{\bar\alpha}(\lambda)\cos\lambda t & \text{for } d_{\bar\alpha} \text{ even},\\ i d_{\bar\alpha}(\lambda)\sin\lambda t & \text{for } d_{\bar\alpha} \text{ odd}.\end{cases}
\end{aligned} \tag{75}$$

Now fix $\bar\alpha \in \mathcal{I}_M$ and denote $\bar\omega = \mathcal{D}_{\bar\alpha}\bar\phi$. The function $\bar\omega$ is also a solution of (1)-(5) and, from (72),

$$\bar\omega(t) = \mathcal{D}_{\bar\alpha}\bar\phi(t) = \sum_{k\in\mathbf{Z}_+}\left(\phi_{0,k}\ \mathcal{D}_{\bar\alpha}\cos\lambda_k t + \frac{\phi_{1,k}}{\lambda_k}\ \mathcal{D}_{\bar\alpha}\sin\lambda_k t\right)\bar\theta_k.$$

Then, from (75) it follows that

$$\bar\omega(t) = \sum_{k\in\mathbf{Z}_+} d_{\bar\alpha}(\lambda_k)\left(\phi_{0,k}\cos\lambda_k t + \frac{\phi_{1,k}}{\lambda_k}\sin\lambda_k t\right)\bar\theta_k, \qquad \text{if } d_{\bar\alpha} \text{ is even},$$

$$\bar\omega(t) = \sum_{k\in\mathbf{Z}_+} i d_{\bar\alpha}(\lambda_k)\left(\phi_{0,k}\sin\lambda_k t - \frac{\phi_{1,k}}{\lambda_k}\cos\lambda_k t\right)\bar\theta_k, \qquad \text{if } d_{\bar\alpha} \text{ is odd}.$$

Thus, in both cases, the energy of $\bar\omega$ computed by the formula (73) is given by

$$\mathbf{E}_{\bar\omega} = \frac{1}{2}\sum_{k\in\mathbf{Z}_+} |d_{\bar\alpha}(\lambda_k)|^2\left(\lambda_k^2\phi_{0,k}^2 + \phi_{1,k}^2\right). \tag{76}$$

With this, the inequality of Theorem IV.1 may be written in terms of the initial data of the solution $\bar\phi$ as:

$$\sum_{k\in\mathbf{Z}_+} |d_{\bar\alpha}(\lambda_k)|^2\left(\lambda_k^2\phi_{0,k}^2 + \phi_{1,k}^2\right) \le C\int_0^{2T_{\mathcal{A}}} |F(t)|^2 dt = C\int_0^{2T_{\mathcal{A}}} |\phi_x(t,0)|^2 dt, \tag{77}$$

Consequently, if we define

$$c_k = \max_{\bar\alpha\in\mathcal{I}_{\mathcal{S}}} |d_{\bar\alpha}(\lambda_k)|, \tag{78}$$

we obtain:

Theorem IV.2. *There exists a positive constant C, such that*

$$\sum_{k\in\mathbf{Z}_+} c_k^2\left(\lambda_k^2\phi_{0,k}^2 + \phi_{1,k}^2\right) \le C\int_0^{2T_{\mathcal{A}}} |\phi_x(t,0)|^2 dt, \tag{79}$$

for every solution $\bar\phi$ with initial data $(\bar\phi_0, \bar\phi_1) \in V \times H$.

Remark IV.7. *It is easy to prove, using, e.g., formula (72) for the solutions, that if inequality (79) holds then for every* $\alpha, T \in \mathbb{R}$,

$$\sum_{k\in\mathbf{Z}_+} c_k^2 \left(\lambda_k^2\phi_{0,k}^2(T)+\phi_{1,k}^2(T)\right) \leq C\int_\alpha^{\alpha+2T_{\mathcal{A}}} |\phi_x(t,0)|^2 dt, \tag{80}$$

where $\phi_{0,k}(T)$ *and* $\phi_{1,k}(T)$ *are the Fourier coefficients of* $\bar{\phi}|_{t=T}$ *and* $\bar{\phi}_t|_{t=T}$, *respectively, in the basis* $\{\bar{\theta}_k\}_{k\in\mathbb{Z}_+}$.

5.2. Non-degenerate trees. In general, some of the coefficients c_k in the inequality (79) may vanish. That is why we consider a special class of trees for which all those numbers are different from zero.

Definition IV.1. *A tree* $\mathcal{A}$ *is said to be* ***non-degenerate*** *if the numbers* c_k, *defined for that tree by (78), are different from zero for every* $k \in \mathbb{Z}_+$. *Otherwise, the tree is said to be* ***degenerate***.

The following proposition provides us with a more transparent characterization of non-degenerate trees.

Proposition IV.13. *The tree* $\mathcal{A}$ *is non-degenerate if and only if the spectra* $\sigma_{\bar{\alpha}\circ i}$, $\sigma_{\bar{\alpha}\circ j}$ *of any two sub-trees* $\mathcal{A}_{\bar{\alpha}\circ i}$, $\mathcal{A}_{\bar{\alpha}\circ j}$ *of* $\mathcal{A}$ *with common* $\mathcal{O}_{\bar{\alpha}}$ *root are disjoint.*

Proof. Note that it takes place a more general fact: an inequality like (79) with different from zero coefficients c_k (not necessarily given by (78)) is impossible for a tree having two sub-trees with common root that share an eigenvalue μ. Indeed, in such case, with the help of Proposition IV.9 we can construct a non-zero solution $\bar{\phi}$ of (1)-(5) such that $\phi_x(t,0)\equiv 0$. With this, a (79)-like inequality would give

$$\sum_{k\in\mathbf{Z}_+} c_k^2\left(\lambda_k^2\phi_{0,k}^2+\phi_{1,k}^2\right)\leq 0,$$

what is false, since $\bar{\phi}$ is not identically equal to zero.

For the converse assertion we argue by contradiction. We will prove that if $c_k = 0$ for some $k\in\mathbb{Z}_+$ and any two sub-trees of $\mathcal{A}$ with common root have disjoint spectra then $d_{\bar{\alpha}}(\lambda_k) = 0$ for any $\bar{\alpha}\in\mathcal{I}$. In particular, $d(\lambda_k) = 0$, what would contradict the fact that $d(\lambda_k) = 1$.

Note firstly, that the property is immediate for exterior nodes, since

$$c_k \geq |d_{\bar{\alpha}}(\lambda_k)|$$

for $\bar{\alpha}\in\mathcal{I}_{\mathcal{S}}$.

For the interior nodes we follow a recursive argument: if $\bar{\alpha}\in\mathcal{I}_{\mathcal{M}}$ and $d_{\bar{\alpha}\circ\beta}(\lambda_k) = 0$ for all $\beta = 1,...,m_{\bar{\alpha}}$ then $d_{\bar{\alpha}}(\lambda_k) = 0$.

Indeed, we have that, for every $\beta = 1,...,m_{\bar{\alpha}}$,

$$d_{\bar{\alpha}\circ\beta} = d_{\bar{\alpha}}\prod_{i\neq\beta} q_{\bar{\alpha}\circ i}. \tag{81}$$

Assume that $d_{\bar{\alpha}}\neq 0$. Then (81) implies that

$$\prod_{i\neq 1} q_{\bar{\alpha}\circ i} = 0$$

and thus, there exists $i^*\neq 1$ such that

$$q_{\bar{\alpha}\circ i^*} = 0. \tag{82}$$

But then, from the equalities $d_{\bar{\alpha}\circ i^*} = 0$ and (81) it follows that there exists $j^* \neq i^*$ satisfying

$$q_{\bar{\alpha}\circ j^*} = 0. \tag{83}$$

However, the equalities (82) and (83) ensure, according to Proposition IV.11, that $\mu_k = \lambda_k^2$ is a common eigenvalue of the sub-trees $\mathcal{A}_{\bar{\alpha}\circ i^*}$ and $\mathcal{A}_{\bar{\alpha}\circ j^*}$. But that is impossible for the tree $\mathcal{A}$. Thus, $d_{\bar{\alpha}}(\mu_k) = 0$. This completes the proof of the proposition. □

Remark IV.8. *According to the previous proposition, if the spectra of some two sub-trees of $\mathcal{A}$ with common root have non-void intersection, inequality (79) degenerates; we can not recover information on the Fourier coefficients $\phi_{0,n}$, $\phi_{1,n}$ of the initial data of $\bar{\phi}$ from the observation of $\phi_x(t,0)$, for those values of n such that $c_n = 0$. However, as it has been indicated in the proof, this fact is not due to the technique used to obtain the inequality, since for degenerate trees no (79)-like inequality with all the coefficients c_k being different from zero, holds. Thus, Theorem IV.2 is sharp in the following sense:* it provides inequality (79) whenever one such inequality exists.

Corollary IV.2 (Unique continuation property). *If the tree $\mathcal{A}$ is non-degenerate and $\bar{\phi}$ is a solution of (7)-(11) such that $\phi_x(t,0) = 0$ for almost all $t \in [0, 2L_{\mathcal{A}}]$ then, $\bar{\phi} \equiv 0$.*

Remark IV.9. *Combining Propositions IV.12(ii) and IV.13, we obtain an alternative characterization of the non-degenerate trees: $\mathcal{A}$ is non-degenerate if, and only if,*

$$|p(\lambda)|^2 + |q(\lambda)|^2 \neq 0$$

for every $\lambda \in \mathbb{R}$.

Proposition IV.14. *If the tree $\mathcal{A}$ is non-degenerate then all its eigenvalues are simple.*

Proof. If λ_k^2 is an eigenvalue of a non-degenerate tree then, according to Proposition IV.11 and Remark IV.9,

$$q(\lambda_k) = 0, \qquad p(\lambda_k) \neq 0.$$

Consequently, if $\bar{\theta}_k$ is an eigenfunction of $\mathcal{A}$ corresponding to λ_k^2, formula (66) gives

$$\theta^{\bar{\alpha}}(0) = \frac{\hat{k}(\lambda_k)}{i\lambda_k p(\lambda_k)}\theta_x(0), \qquad \theta_x^{\bar{\alpha}}(0) = \frac{-l(\lambda_k)}{p(\lambda_k)}\theta_x(0),$$

Thus, $\bar{\theta}_k$ is determined, up to the constant factor $\theta_x(0)$, in a unique way. □

Remark IV.10. *Let $(\tilde{\mu}_k)_{n\in\mathbb{Z}_+}$ be the strictly increasing sequence of the eigenvalues μ_k of a tree without taking into account their multiplicity. In Chapter V we will prove that $\tilde{\mu}_k$ verifies $\mu_k^N \leq \tilde{\mu}_k \leq \mu_k^D$, for $k \in \mathbb{Z}_+$, where $\{\mu_k^D\}_{k\in\mathbb{Z}_+}$ and $\{\mu_k^N\}_{k\in\mathbb{Z}_+}$ are the ordered sequences formed by the distinct eigenvalues of the strings with Dirichlet or Neumann homogeneous boundary conditions, respectively. This fact will allow to prove that an inequality of type (79) is impossible for $T < 2L_{\mathcal{A}}$ (see Theorem V.1). Moreover, in this case the system (1)-(5) is not approximately controllable, and then, is not approximately controllable either.*

5.3. On the size of the set of non-degenerate trees. Now we give some information on the size of the set of degenerate trees. It turns out that almost all trees with the same topological structure are non-degenerate in the sense of a measure, defined in a natural way on the set of trees with that structure.

Let us be more precise. We shall say that two trees are *topologically equivalent* if their edges can be numbered with the same set of multi-indices. This means that they may differ only in the lengths of their edges. In particular, two equivalent trees have the same number of edges and vertices. The classes of topologically equivalent trees are called *topological configurations.*

Fix a topological configuration Σ with d edges. We assume that in the set of indices $\mathcal{I}$ for the elements of the trees belonging to Σ, a criterion of ordering have been defined and use the notation $<\mathcal{A}>$ for the corresponding ordered set of the lengths of the edges of $\mathcal{A}\in\Sigma$.

Then Σ may be identified with $(\mathbb{R}_+)^d$ by means of the canonical mapping $\pi : \Sigma \to \mathbb{R}^d$ defined by

$$\pi(\mathcal{A}) = <\mathcal{A}> \in \mathbb{R}^d.$$

Let μ_Σ be the measure induced in Σ by the Lebesgue measure of $\mathbb{R}^d$ through the mapping π. That is, if $B\subset\Sigma$ then

$$\mu_\Sigma(B) = m_d(\pi(B)),$$

where m_d is the usual Lebesgue measure in $\mathbb{R}^d$.

It takes place

Proposition IV.15. *Given a topological configuration Σ, almost every tree (in the sense of the measure μ_Σ) with that topological configuration is non-degenerate.*

Proof. Let $D^{i,j}_{\bar\alpha}\subset\Sigma$ denote the set of those trees $\mathcal{A}$, such that its sub-trees $\mathcal{A}_{\bar\alpha\circ i}$ and $\mathcal{A}_{\bar\alpha\circ j}$ are non-degenerate and have a common eigenvalue. Then the set $\Sigma_{deg}\subset\Sigma$ of degenerate trees may be decomposed as

$$\Sigma_{deg} = \bigcup_{\bar\alpha\in\mathcal{I}_\mathcal{M}} \bigcup_{i,j=1\ i\neq j}^{m_{\bar\alpha}} D^{i,j}_{\bar\alpha}. \tag{84}$$

We will prove that $\mu_\Sigma(D^{i,j}_{\bar\alpha}) = 0$, for every $\bar\alpha\in\mathcal{I}_\mathcal{M}$, $i,j = 1,...,m_{\bar\alpha}$, $i\neq j$. This fact, in view of (84), will imply $\mu_\Sigma(\Sigma_{deg}) = 0$. In what follows we consider that $\bar\alpha$, i and j are fixed.

The idea of the proof is simple. We fix a tree $\mathcal{B}$ having the structure[2] of $\mathcal{A}_{\bar\alpha\circ i}\vee\mathcal{A}_{\bar\alpha\circ j}$ (defined as in Remark IV.4) and extend it (i. e., we add edges) to a tree $\mathcal{A}\in D^{i,j}_{\bar\alpha}$. According to Remark IV.4, that is equivalent to choosing the node $\mathcal{O}_{\bar\alpha}$ of $\mathcal{A}\in\Sigma$ in a point of a string of $\mathcal{B}$ (precisely, of that string where it should be located to agree with the structure of Σ) where some eigenfunction of $\mathcal{B}$ vanishes. Once $\mathcal{O}_{\bar\alpha}$ has been chosen, the lengths of the remaining strings of $\mathcal{A}$ may be taken arbitrarily.

Observe that we may assume that no (non-identically zero) eigenfunction of $\mathcal{B}$ vanishes identically on the string that contains $\mathcal{O}_{\bar\alpha}$, since, otherwise, one of the sub-trees of $\mathcal{A}_{\bar\alpha\circ i}$ or $\mathcal{A}_{\bar\alpha\circ j}$ of the tree $\mathcal{A}$, obtained with this procedure, would be degenerate and thus, $\mathcal{A}\notin D^{i,j}_{\bar\alpha}$. This assumption implies that all the eigenfunctions

[2]That is, $\mathcal{B}$ is topologically equivalent to $\mathcal{A}_{\bar\alpha\circ i}\vee\mathcal{A}_{\bar\alpha\circ j}$.

of $\mathcal{B}$ are simple and then the node $\mathcal{O}_{\bar{\alpha}}$ should be chosen in a set of points, which is at most denumerable.

Thus, we have obtained, after some re-ordering if needed, that the set $\pi(D^{i,j}_{\bar{\alpha}})$ is contained in a set of the form

$$\left\{(h_1, h_2, ..., h_d) \in (\mathbb{R}_+)^d : \ h_1 + h_2 = h, \ h_1 \in \mathbf{N}(h, h_3, ..., h_d)\right\}, \tag{85}$$

where $\mathbf{N}(h, h_3, ..., h_d)$ is a denumerable set depending on h and $h_3, ..., h_d$.

It is easy to see, using, e.g., the Fubbini's theorem, that a set defined by (85) has d-dimensional Lebesgue measure equal to zero. Thus, the same is true of $\pi(D^{i,j}_{\bar{\alpha}})$ and then $\mu_\Sigma(D^{i,j}_{\bar{\alpha}}) = 0$. This completes the proof. □

Corollary IV.3. *The set $\Sigma \setminus \Sigma_{deg}$ of non-degenerate trees is dense in Σ provided with the metrics induced in Σ by the usual metrics of $\mathbb{R}^d$ through π.*

Remark IV.11. *The set Σ_{deg}, even though is small in the sense of μ_Σ, is dense in Σ. Indeed, it suffices to see that, if two edges of a tree with rationally dependent lengths have a common vertex and their other vertices are exterior then the tree is degenerate.*

6. Consequences concerning the controllability

Gathering the facts of the previous sections we obtain the following characterization of the controllability properties of trees.

Theorem IV.3. *Let $\mathcal{A}$ be a tree and $T > 0$. Then,*
a) If $T \geq 2L_{\mathcal{A}}$ the properties

1) the system (1)-(6) is spectrally controllable in time T;
2) the system (1)-(6) is approximately controllable in time T;
3) $\mathcal{A}$ is non-degenerate;
4) any two sub-trees of $\mathcal{A}$ with common root have disjoint spectra;

are equivalent and, when they are true, all the initial states of the space $\mathcal{W}$, defined by

$$\mathcal{W} = \left\{(\bar{u}_0, \bar{u}_1) \in V \times H' : \sum_{n \in \mathbb{N}} \frac{1}{c_n^2}\left(|u_n^0|^2 + \frac{1}{\mu_n}|u_n^1|^2\right) < \infty\right\},$$

where the weights c_n are given by (78), are controllable in time T. Besides, these properties are true for almost all tree, topologically equivalent to$\mathcal{A}$.

b) If $T < 2L_{\mathcal{A}}$ the spectral controllability property is false, independently on whether the tree is degenerate or not.

Proof. The assertion a) follows from the implications:

1) $\Rightarrow$ 2). This is a general fact, since $Z \times Z$ is dense in $H \times V'$ and then, the spectral controllability is a particular case of approximate controllability.

2) $\Rightarrow$ 3). Follows from the fact stated in Proposition IV.13 and Remark IV.8, that, for non-degenerate trees there exist non-zero eigenfunctions, which vanish identically on the string that contains the root. Those eigenfunctions do not satisfy the unique continuation property from the root.

3) $\Rightarrow$ 4). Has been proved in Proposition IV.13.

4) $\Rightarrow$ 1). It is also a consequence of Proposition IV.13. If any two sub-trees of $\mathcal{A}$ with common root have disjoint spectra, there exists a constant C, such that

$$\sum_{n\in\mathbb{N}} c_n^2 \left(\lambda_n^2 \phi_{0,n}^2 + \phi_{1,n}^2\right) \leq C \int_0^{2L_{\mathcal{A}}} |\phi_x(t,0)|^2 dt,$$

for every solution $\bar{\phi}$ of (7)-(12) with initial state $(\bar{\phi}_0,\, \bar{\phi}_1) \in V \times H$, where all the coefficients c_n are different from zero. This implies that the space of initial states

$$\mathcal{W} = \left\{ (\bar{u}_0, \bar{u}_1) \in V \times H' : \sum_{n\in\mathbb{N}} \frac{1}{c_n^2} \left(|u_n^0|^2 + \frac{1}{\mu_n} |u_n^1|^2 \right) < \infty \right\}$$

is controllable in any time $T \geq 2L_{\mathcal{A}}$. In particular, the space $Z \times Z$ will be controllable.

The assertion b) in the general case of arbitrary networks, whose structure is not necessarily a tree. This fact will be proved in Theorem V.1. $\Box$

7. Simultaneous observability and controllability of networks

The results of the previous sections allows to consider the one-node control problem for several (a finite number) of tree-shaped networks when the same control function is used to control all of them, i.e., when they are controlled simultaneously.

Let $\mathcal{A}^1$,...,$\mathcal{A}^R$ be the trees associated to the controlled networks. For the elements of the network whose graph is $\mathcal{A}^r$ we will use the same notations as in the preceding sections but adding the superscript r to them. Thus, the solution of (1)-(6) for the tree $\mathcal{A}^r$ (in what follows we shall briefly refer to this problem as $(1)_r$-$(6)_r$) is denoted by $\bar{u}^r$ and the spaces V and H constructed for that tree by V^r and H^r.

We define the space

$$\mathbf{W} = \prod_{r=1}^{R} V^r \times H^r,$$

endowed with the product Hilbert structure. The elements of $\mathbf{W}$ are called *simultaneous states.*

We shall say that the simultaneous state $\bar{w} \in \mathbf{W}$, is *controllable in time* T if it is possible to find a control function $v \in L^2(0,T)$ such that the solutions $\bar{u}^r$ of $(1)_r$-$(6)_r$ with initial states $(\bar{u}_0^r, \bar{u}_1^r)$ (the components of $\bar{w}$) and $v^r = v$ verify

$$\bar{u}^r(T,x) = \bar{u}_t^r(T,x) = \bar{0},$$

for every $i = 1, ..., R$.

Once again using HUM, the problem of characterizing the controllable simultaneous states is reduced to the study of observability inequalities for the corresponding homogeneous systems. Indeed, assume that there exist non-zero numbers c_n^k, $n \in \mathbb{Z}_+$, $k = 1, ..., R$, such that for every k the inequality

$$\int_0^T |\sum_{r=1}^{R} u_x^r(0,t)|^2 dt \geq \sum_{n\in\mathbf{Z}_+} (c_n^k)^2 \left(\mu_n^k |u_{0,n}^k|^2 + |u_{1,n}^k|^2\right), \tag{86}$$

holds for all the initial simultaneous states $\bar{w} \in \mathbf{W}$, where $\{u_{0,n}^r\}$ and $\{u_{1,n}^r\}$ are the sequences of Fourier coefficients of the components $\bar{u}_0^r$ and $\bar{u}_1^r$ of the initial state in the bases $\{\overline{\theta}_n^r\}$ of H^r, respectively, and $\bar{u}^r$ is the solution of $(1)_r$-$(6)_r$ with $v^r = 0$ and define the sets

$$\mathcal{W}^r = \{(\bar{u}_0^r, \bar{u}_1^r) \in V^r \times (H^r)' : \ \|(\bar{u}_0^r, \bar{u}_1^r)\|_r < \infty\}, \tag{87}$$

where

$$\|(\bar{u}_0^r, \bar{u}_1^r)\|_r^2 := \sum_{n \in \mathbb{Z}_+} \frac{1}{(c_n^r)^2} \left(|u_{0,n}^r|^2 + \frac{1}{\mu_n^r} |u_{1,n}^r|^2 \right)$$

Then, all the initial simultaneous states $\bar{w} \in \mathcal{W} = \prod_{i=1}^r \mathcal{W}^r$ are controllable in time T.

In particular, if the inequalities (86) hold then the initial simultaneous states $\bar{w} \in \prod_{1=1}^r Z^r \times Z^r$ are controllable (recall that Z^r is the set of all finite linear combinations of the eigenfunctions $\overline{\theta}_n^r$). In this case, the networks are said to be *simultaneously spectrally controllable.*

Moreover, the set of controllable simultaneous states in time T is dense in $\mathbf{W}$ (when that holds the networks are said to be *simultaneously approximately controllable in time* T) if and only if the following unique continuation property takes place:

$$\sum_{r=1}^R u_x^r(0,t) = 0 \ \ in \ \ L^2(0,T) \ \ implies \ \ (\bar{u}_0^r, \bar{u}_1^r) = \bar{0} \ \ for\ every \ \ r = 1, ..., R. \tag{88}$$

It is clear that, if a simultaneous state is controllable then each of its components is also controllable for the corresponding network. This implies that the if we expect at least the approximate controllability to hold, then we need to assume that all the trees supporting the networks are non-degenerate.

On the other hand, if two of the trees, say $\mathcal{A}^1$ and $\mathcal{A}^2$, have a common eigenvalue then, using the pasting procedure described in the proof of Proposition IV.9, we can construct non-zero solutions of $(1)_r$-$(6)_r$, $r = 1, 2$, such that

$$u_x^1(t,0) + u_x^2(t,0) = 0, \qquad t \in \mathbb{R}.$$

Therefore, choosing zero initial states for all the remaining trees $\mathcal{A}^r$, $r = 3, ..., R$, we obtain a simultaneous initial state in $\mathbf{W}$ for which inequalities (86) are impossible and moreover, for which the unique continuation property (88) fails.

Thus, the conditions that the trees $\mathcal{A}^r$, $r = 1, ..., R$, are non-degenerate and their spectra are pairwise disjoint are necessary for the simultaneous approximate controllability, and then for the spectral controllability. As we shall see, these conditions are also sufficient.

Put $T^* = \sum_{i=1}^r L^i$. For every $k = 1, ..., R$ we define the operator

$$\widehat{\mathcal{Q}}_k := \prod_{r=1,\ r \neq k}^R \mathcal{Q}^r,$$

where $\mathcal{Q}^r$ is the operator $\mathcal{Q}$ for the tree $\mathcal{A}^r$. Note that $\widehat{\mathcal{Q}}_k$ is an S-operator with $s(\widehat{\mathcal{Q}}_k) = T^* - L^k$.

Let $\widehat{q}_k$ be the function associated to $\widehat{\mathcal{Q}}_k$ according to Remark IV.1. Then

$$\widehat{q}_k = \prod_{r=1,\ r \neq k}^R q^r, \tag{89}$$

where q^r is the function corresponding to $\mathcal{Q}^r$.

PROPOSITION IV.16. *If for a given k there exist numbers c_n, $n \in \mathbb{Z}_+$, such that every solution of $(1)_k$-$(6)_k$ with $v_k = 0$ and initial state*

$$(\bar{u}_0^k, \bar{u}_1^k) = (\sum_{n\in\mathbb{Z}_+} u_{0,n}^k \bar{\theta}_n^k, \sum_{n\in\mathbb{Z}_+} u_{1,n}^k \bar{\theta}_n^k) \in V^k \times H^k$$

satisfies

$$\int_0^{2L_k} |u_x^k(0,t)|^2 dt \geq \sum_{n\in\mathbf{Z}_+} c_n^2 \left(\mu_n^k |u_{0,n}^k|^2 + |u_{1,n}^k|^2\right), \tag{90}$$

then

$$\int_0^{2T^*} |\sum_{r=1}^R u_x^r(0,t)|^2 dt \geq \sum_{n\in\mathbf{Z}_+} c_n^2 |\widehat{q}_k(\lambda_n^k)|^2 \left(\mu_n^k |u_{0,n}^k|^2 + |u_{1,n}^k|^2\right), \tag{91}$$

for every $(\bar{u}_0^r, \bar{u}_1^r) \in V^r \times H^r$, $r = 1, ..., R$.

PROOF. As $\widehat{\mathcal{Q}}_k$ is an S-operator with $s(\widehat{\mathcal{Q}}_k) = T^* - L^k$, using Proposition IV.3(i) we get

$$\int_0^{2T^*} |\sum_{r=1}^R u_x^i(0,t)|^2 dt \geq \int_{T^*-L^k}^{T^*+L^k} |\widehat{\mathcal{Q}}_k \sum_{r=1}^R u_x^i(0,t)|^2 dt. \tag{92}$$

But, as $\mathcal{Q}^r u_x^r(0,t) = 0$, then $\widehat{\mathcal{Q}}_k \bar{u}^k$ if $r \neq k$. Thus, inequality (92) becomes

$$\int_0^{2T^*} |\sum_{r=1}^R u_x^i(0,t)|^2 dt \geq \int_{T^*-L^k}^{T^*+L^k} |\widehat{\mathcal{Q}}_k u_x^k(0,t)|^2 dt. \tag{93}$$

Now we consider the function $\bar{\omega} = \widehat{\mathcal{Q}}_k \bar{u}$. As $\bar{\omega}$ is clearly a solution of $(1)_k$-$(5)_k$, then, according to (90) and Remark 80 it holds

$$\int_{T^*-L^k}^{T^*+L^k} |\omega_x(0,t)|^2 dt \geq \sum_{n\in\mathbb{Z}_+} c_n^2 \left(\mu_n^k \omega_{0,n}^2 + \omega_{1,n}^2\right). \tag{94}$$

On the other hand, it is simple to prove that the Fourier coefficients of the initial data of $\bar{u}$ and $\bar{\omega}$ are related by

$$\mu_n^k \omega_{0,n}^2 + \omega_{1,n}^2 = q_k^2(\lambda_n^k) \left(\mu_n^k u_{0,n}^2 + u_{1,n}^2\right). \tag{95}$$

Finally, combining (93)-(95) and the fact that $\omega_x(0,t) = \widehat{\mathcal{Q}}_k u_x^k(0,t)$, the inequality (91) is obtained. □

Now, if the trees $\mathcal{A}^1$,...,$\mathcal{A}^R$ are non-degenerate then we have for every $r = 1, ..., R$ inequalities (90) with non-zero coefficients c_n (depending on r), which are explicitly computed by formulas (78). Therefore, according to the Proposition IV.16, we shall also have inequalities (86) with explicitly computed coefficients

$$c_n^r = |\widehat{q}_r(\lambda_n^r)| c_n,$$

which are all different from zero whenever the spectra of any two of the trees $\mathcal{A}^r$ are disjoint, since $\widehat{q}_r(\lambda_n^r) \neq 0$ for all $r = 1, ..., R$ and $n \in \mathbb{Z}_+$. Indeed, if $\widehat{q}_r(\lambda_n^r) = 0$ for some r and n then equality (89) would imply that $q^i(\lambda_n^r) = 0$ for some $i \neq r$ and thus, from Proposition (IV.11), μ_n^r would be a common eigenvalue of the trees $\mathcal{A}^r$ and $\mathcal{A}^i$.

Consequently, we are able to construct, under those assumptions, a space

$$\mathcal{W} = \prod_{r=1}^{R} \mathcal{W}^r,$$

where $\mathcal{W}^r$ are defined by (87), of controllable simultaneous states in time $2T^*$. In particular,

Corollary IV.4. *The trees $\mathcal{A}^1$,...,$\mathcal{A}^R$ are simultaneously spectrally controllable in some time T (and then in time $2T^*$), if and only if they are spectrally controllable and their spectra are pairwise disjoint.*

8. Examples

8.1. Star-shaped network with n strings. In the framework of the study of the controllability of networks of strings from an exterior node, the star-shaped network with n strings constitutes the simplest example of a network with an arbitrary number of strings.

The star-shaped network with n strings is formed by n strings connected at one point. When $n = 3$, this network is the three string network studied in Chapter III.

Let us call $\mathcal{A}$ the star-shaped graph that supports the network. Following the numbering criterion introduced in Section 1 for trees, we will denote by$\mathcal{R}$ the controlled node and by $\mathcal{O}$ the interior node, that where the strings are coupled. The controlled string will denoted by $\mathbf{e}$ and its length by ℓ. The remaining $n-1$ exterior nodes are denoted by $\mathcal{O}_i$, $i = 1, ..., n-1$, the string that contains $\mathcal{O}_i$ by $\mathbf{e}_i$ and the length of this string by ℓ_i.

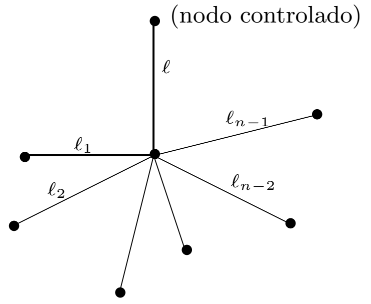

Figure 3. Star-shaped network with n strings

The only sub-trees of $\mathcal{A}$ are the strings $\mathbf{e}_i$, $i = 1, ..., n-1$:

$$\mathcal{A}_i = \{\mathbf{e}_i\}.$$

Therefore, the spectra σ_i of the sub-trees coincide with the eigenvalues of the homogeneous Dirichlet problem for a string and are given by

$$\sigma_i = \left\{ \left(\frac{k\pi}{\ell_i} \right)^2 : \quad k \in \mathbb{Z} \right\} \qquad i = 1, ..., n-1.$$

The non-degeneracy condition of $\mathcal{A}$ is $\sigma_i \cap \sigma_j = \emptyset$ for every pair i, j with $i \neq j$. This means

$$\frac{k\pi}{\ell_i} \neq \frac{m\pi}{\ell_j}$$

for all $k, m \in \mathbb{Z}$, which is equivalent to the fact that $\frac{\ell_i}{\ell_j}$ is irrational.

We can conclude, applying Theorem IV.3, that if $L_{\mathcal{A}}$ is the sum of the lengths of all the strings of the network then it takes place

COROLLARY IV.5. *The star-shaped network with n strings is approximately controllable in some time $T \geq 2L_{\mathcal{A}}$ (and then spectrally controllable in time $T = 2L_{\mathcal{A}}$) if, and only if, the ratio of any two of the lengths of the uncontrolled strings is an irrational number.*

Besides, when the non-degeneracy condition is fulfilled, all the initial states $(\bar{u}_0, \bar{u}_1) \in V' \times H$ satisfying

$$\sum_{k\in\mathbb{N}} \frac{1}{c_k^2} u_{0,k}^2 < \infty, \quad \sum_{k\in\mathbb{N}} \frac{1}{c_k^2 \mu_k} u_{1,k}^2 < \infty, \tag{96}$$

are controllable in time $T = 2L_{\mathcal{A}}$. Recall than in (96) $\mu_k = \lambda_k^2$ are the eigenvalues of the network and the coefficients c_k are defined by (78):

$$c_k = \max_{i=1,\dots,n-1} \prod_{j\neq i} |q_j(\lambda_k)|,$$

where q_j is the function associated to the operator $\mathcal{Q}_j$ for the sub-tree $\mathcal{A}_j$.

But, as $\mathcal{A}_j$ coincides with the string $\mathbf{e}_j$, the operator $\mathcal{Q}_j$ coincides with ℓ_j^- (see Subsection 2.1, where the operators $\mathcal{P}$ and $\mathcal{Q}$ are computed for a string) and then, from Remark IV.1,

$$q_j(\lambda_k) = i \sin \lambda_k \ell_j.$$

In conclusion,

$$c_k = \max_{i=1,\dots,n-1} \prod_{j\neq i} |\sin \lambda_k \ell_j|.$$

In Appendix A we pay special attention to the function

$$\mathbf{a}(\lambda, \ell_1, \dots, \ell_{n-1}) := \sum_{i=1}^{n-1} \prod_{j\neq i} |\sin \lambda \ell_j|.$$

There we provide conditions on the values of $\ell_1, \dots, \ell_{n-1}$ such that, for every $\lambda \in \mathbb{R}$, an inequality of the type

$$\mathbf{a}(\lambda, \ell_1, \dots, \ell_{n-1}) \geq C\lambda^{\alpha}$$

is satisfied. These conditions involve certain set $\mathbf{B}_{\varepsilon}$, $\varepsilon > 0$, which are defined in Appendix A, p. 160, where in addition, some conditions on the lengths, called *conditions* (S), are introduced.

As, obviously, $nc_k \geq \mathbf{a}(\lambda_k, \ell_1, \dots, \ell_{n-1})$ then, applying Corollary A.2 we obtain

COROLLARY IV.6. *If the numbers $\ell_1, \dots, \ell_{n-1}$ are such that for all values $i, j = 1, \dots, n-1$, $i \neq j$, the ratios $\frac{\ell_i}{\ell_j}$ belong to $\mathbf{B}_{\varepsilon}$ then, there exists a constant $C_{\varepsilon} > 0$ such that*

$$c_k \geq \frac{C_{\varepsilon}}{\lambda^{n-2+\varepsilon}}, \qquad k \in \mathbb{N}.$$

Therefore, all the initial states $(\bar{u}_0, \bar{u}_1) \in V^{n-2+\varepsilon} \times V^{n-3+\varepsilon}$ are controllable in time $T = 2L_{\mathcal{A}}$.

Under more restrictive assumptions on the lengths of uncontrolled strings, it is possible to guarantee the existence of a larger subspace of controllable initial states:

COROLLARY IV.7. *If the numbers $\ell_1, ..., \ell_{n-1}$ verify the conditions* (S) *then, for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$c_k \geq \frac{C_\varepsilon}{\lambda^{1+\varepsilon}}, \qquad k \in \mathbb{N}.$$

Therefore, all the initial states$(\bar{u}_0, \bar{u}_1) \in V^{1+\varepsilon} \times V^\varepsilon$ are controllable in time $T = 2L_{\mathcal{A}}$.

REMARK IV.12. *When $n = 3$ the results of Corollaries IV.6 and IV.7 coincide coincide with Corollary III.5 II relative to the three string network.*

8.2. Simultaneous control of n strings. This problem is quite similar to the previous one, though in fact it is simpler. It consists in controlling n strings $\mathbf{e}_1, ..., \mathbf{e}_n$ of lengths $\ell_1, ..., \ell_n$, which are not coupled; we just use the same function to control all the strings. This is the simplest example of simultaneous control of an arbitrary number of tree-shaped networks in the sense of Section 7. Let us note that the case $n = 2$ has been already studied in Section 2 of Chapter III. As it was pointed out there, this is the problem studied in [**6**], [**4**], [**10**], [**8**], [**9**], with the help of Theorem II.6. In [**27**] this problem was solved using the technique we describe here.

The controlled system is

$$\text{(97)} \qquad \begin{cases} u^i_{tt} - u^i_{xx} = 0 & (t,x) \in \mathbb{R} \times [0, \ell_i], \\ u^i(t, \ell_i) = 0, \quad u^i(t,0) = v(t) & t \in \mathbb{R}, \\ u^i(0,x) = u^i_0(x), \quad u^i_t(0,x) = u^i_1(x) & x \in [0, \ell_i], \end{cases}$$

for $i = 1, ..., n$.

Let us observe that this system may be viewed as a star-shaped network with n controlled from the interior node, that is, from the coupling point.

According to Corollary IV.4 of Section 7, the n strings are simultaneously spectrally controllable in some time T (and then also in time $T_0 = 2\sum_{i=1}^n \ell_i$) if, and only if, the spectra of any two strings are disjoint. This is equivalent to the fact that all the ratios $\frac{\ell_i}{\ell_j}$ with $i \neq j$ are irrational numbers.

It is possible to obtain additional information directly from Proposition IV.16. In this case, the controlled trees are strings: $\mathcal{A}_i = \{\mathbf{e}_i\}$. Then we have $\mathcal{Q}_i = \ell_i^-$ and therefore

$$\mathcal{Q}_i = \ell_i^-, \qquad \widehat{\mathcal{Q}}_i = \prod_{j \neq i} \mathcal{Q}_j = \prod_{j \neq i} \ell_j^-, \qquad |\widehat{q}_i(\lambda)| = \prod_{j \neq i} |\sin(\lambda \ell_j)|\,.$$

Besides, the eigenvalues (μ^i_k) and eigenfunctions (θ^i_k) of each $\mathcal{A}_i$ may be explicitly computed:

$$\mu^i_k = \left(\frac{k\pi}{\ell_i}\right)^2, \qquad \theta^i_k(x) = \sqrt{\frac{2}{\ell_i}} \sin(\frac{k\pi}{\ell_i} x).$$

On the other hand, if $(u^i_{0,k}), (u^i_{1,k})$ denote the sequences of Fourier coefficients of the initial state $(\bar{u}^i_0, \bar{u}^i_1)$ of the string $\mathbf{e}_i$ in the basis (θ^i_k), then

$$\text{(98)} \qquad \int_0^{2\ell_i} \left| u^i_x(t,0) \right|^2 \geq 4 \sum_{k \in \mathbb{N}} \left(\mu^i_k (u^i_{0,k})^2 + (u^i_{1,k})^2 \right),$$

for every solution

$$u^i_{tt} - u^i_{xx} = 0, \qquad u^i(t,0) = u^i(t,\ell_i) = 0,$$

with initial states $(\bar{u}^i_0, \bar{u}^i_1) \in Z_i \times Z_i$ (this is the observability inequality of a string from one of its extremes, see Proposition II.1).

Applying Proposition IV.16 to the inequalities (98) it hold that for every $i = 1, ..., n$, the inequalities

$$\int_0^{T^*} \left| \sum_{i=1}^n u^i_x(t,0) \right|^2 dt \geq 4 \sum_{k\in\mathbb{N}} \left|\widehat{q}_i(\lambda^i_k)\right|^2 \left(\mu^i_k (u^i_{0,k})^2 + (u^i_{1,k})^2\right),$$

are verified for every $i = 1, ..., n$ and every solution of the homogeneous version of (97) with simultaneous initial state $(\bar{u}^i_0, \bar{u}^i_1) \in Z_i \times Z_i$, $i = 1, ..., n$.

These are the observability inequalities associated to the problem (97): if for each $i = 1, ..., n$, the simultaneous initial state (u^i_0, u^i_1), $i = 1, ..., n$, satisfies

$$\sum_{k\in\mathbb{N}} \frac{(u^i_{0,k})^2}{\left|\widehat{q}_i(\lambda^i_k)\right|^2} < \infty, \qquad \sum_{k\in\mathbb{N}} \frac{(u^i_{1,k})^2}{\left|\widehat{q}_i(\lambda^i_k)\right|^2 \mu^i_k} < \infty,$$

then, that state is controllable in time T^*.

The numbers $\left|\widehat{q}_i(\lambda^i_k)\right|$ may be easily estimated:

$$\left|\widehat{q}_i(\lambda^i_k)\right| = \prod_{j\neq i} \left|\sin(\lambda^i_k \ell_j)\right| = \prod_{j\neq i} \left|\sin(k\pi \frac{\ell_j}{\ell_i})\right| \geq C \prod_{j\neq i} \left|\left|\left| k\frac{\ell_j}{\ell_i} \right|\right|\right|.$$

(Recall that $|||\eta|||$ denotes the distance from η to $\mathbb{Z}$.)

Then we obtain, in account of the results on Diophantine Approximation included in Appendix A, conditions that allow to identify subspaces of simultaneous controllable states in time $T^* = 2\sum_{i=1}^n \ell_i$:

Corollary IV.8. *If the numbers $\ell_1, ..., \ell_n$ are such that for all the values $i, j = 1, ..., n$, $i \neq j$, the ratios $\frac{\ell_i}{\ell_j}$ belong to $\mathbf{B}_\varepsilon$ then there exists a constant $C_\varepsilon > 0$ such that*

$$\left|\widehat{q}_i(\lambda^i_k)\right| \geq \frac{C_\varepsilon}{(\lambda^i_k)^{n-1+\varepsilon}}, \qquad k \in \mathbb{N}.$$

Therefore, the space of controllable simultaneous initial states in time $T^* = 2\sum_{i=1}^n \ell_i$ contains all those simultaneous states that verify

$$(u^i_0, u^i_1) \in V_i^{1+\varepsilon} \times V_i^\varepsilon,$$

for every $i = 1, ..., n$, where V_i^α is the space V^r defined in Chapter I by (I.27) for the string $\mathbf{e}_i$, that is,

$$V_i^\alpha = \left\{ \varphi = \sum_{k\in\mathbb{N}} \varphi_k \sin(\frac{k\pi}{\ell_i}x) : \quad \sum_{k\in\mathbb{N}} k^{2\alpha} \left|\varphi_k\right|^2 < \infty \right\}.$$

Corollary IV.9. *If the numbers $\ell_1, ..., \ell_n$ verify the conditions (S) then, for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*

$$\left|\widehat{q}_i(\lambda^i_k)\right| \geq \frac{C_\varepsilon}{(\lambda^i_k)^{1+\varepsilon}}, \qquad k \in \mathbb{N}.$$

Therefore, the space of controllable simultaneous initial states in time $T^* = 2\sum_{i=1}^n \ell_i$ *contains all those simultaneous states that verify*

$$(u_0^i, u_1^i) \in V_i^{1+\varepsilon} \times V_i^{\varepsilon},$$

for every $i = 1, ..., n$.

REMARK IV.13. *The problem of simultaneous control of* n *strings may be successfully studied with the aid of the method of moments. It suffices to note that the function*

$$F(z) = \prod_{i=1}^{n} \sin z\ell_i$$

is a generating function of the increasing sequence (σ_m) *formed by the numbers* λ_k^i, $i = 1, ..., n$, $k \in \mathbb{N}$ *(the positive square root of the eigenvalues of the strings).*

The function F *is bounded and of exponential type* $A = \sum_{i=1}^n \ell_i$. *Besides,*

$$|F'(\lambda_k^i)| = \prod_{j\neq i} \left| \sin(k\pi \frac{\ell_j}{\ell_i}) \right|.$$

This allows to obtain results similar to those of Corollaries IV.8 and IV.9.

8.3. A non star-shaped tree. Now let us consider a tree $\mathcal{A}$, which is not star-shaped, having a very simple structure as shown in Figure 4. We will assume in addition that $\ell_{1,2} = \ell_2$.

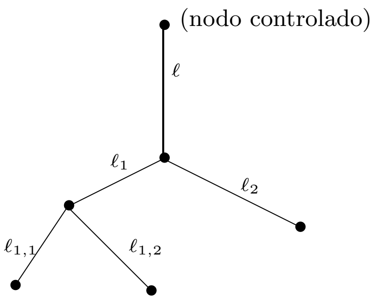

FIGURE 4. A tree which is not star-shaped

This tree contains four sub-trees. Two of them

$$\mathcal{A}_1 = \{\mathbf{e}_1, \mathbf{e}_{1,1}, \mathbf{e}_{1,2}\}, \qquad \mathcal{A}_2 = \{\mathbf{e}_2\},$$

have the common root $\mathcal{O}$. The other two

$$\mathcal{A}_{1,1} = \{\mathbf{e}_{1,1}\}, \qquad \mathcal{A}_{1,2} = \{\mathbf{e}_{1,2}\},$$

have the common root $\mathcal{O}_1$.

The operators $\mathcal{Q}$ corresponding to these sub-trees are

$$\mathcal{Q}_2 = \ell_2^-, \qquad \mathcal{Q}_{1,1} = \ell_{1,1}^-, \qquad \mathcal{Q}_{1,2} = \ell_{1,2}^-,$$
$$\mathcal{Q}_1 = \left(\ell_1^+\ell_{1,1}^-\ell_{1,2}^- + \ell_1^-\ell_{1,1}^+\ell_{1,2}^- + \ell_1^-\ell_{1,1}^-\ell_{1,2}^+\right).$$

The first three operators are obtained immediately, since the corresponding sub-trees are strings. The operator corresponding to $\mathcal{A}_1$, is the operator $\mathcal{Q}$ for a three string network and has been constructed in Chapter III.

The functions associated to these operators are

$$q_2(\lambda) = i\sin\lambda\ell_2, \qquad q_{1,1}(\lambda) = i\sin\lambda\ell_{1,1}, \qquad q_{1,2}(\lambda) = i\sin\lambda\ell_{1,2},$$

$$\begin{aligned} q_1(\lambda) &= -(\cos\lambda\ell_1\sin\lambda\ell_{1,1}\sin\lambda\ell_{1,2} + \sin\lambda\ell_1\cos\lambda\ell_{1,1}\sin\lambda\ell_{1,2} + \\ &\quad + \sin\lambda\ell_1\sin\lambda\ell_{1,1}\cos\lambda\ell_{1,2}). \end{aligned}$$

The functions d corresponding to the simple uncontrolled nodes are

$$d_{1,1}(\lambda) = q_2(\lambda)q_{1,2}(\lambda), \qquad d_{1,2}(\lambda) = q_2(\lambda)q_{1,1}(\lambda), \qquad d_2(\lambda) = q_1(\lambda).$$

Finally,

$$c_k = \max\left(|d_{1,1}(\lambda_k)|, |d_{1,2}(\lambda_k)|, |d_2(\lambda_k)|\right).$$

Now it is easy to see when $\mathcal{A}$ is degenerate. If $c_k = 0$ then,

$$|d_{1,1}(\lambda_k)| = |d_{1,2}(\lambda_k)| = |d_2(\lambda_k)| = 0.$$

Taking into account that $\ell_{1,2} = \ell_2$ from the latter equality it follows $\sin\lambda\ell_{1,2} = 0$ and then,

$$\sin\lambda\ell_1\sin\lambda\ell_{1,1} = 0.$$

If $\sin\lambda\ell_{1,1} = 0$ (resp., $\sin\lambda\ell_1 = 0$) then, necessarily, $\frac{\ell_{1,1}}{\ell_{1,2}}$ (resp. $\frac{\ell_1}{\ell_{1,2}}$) is a rational number. Consequently, we can ensure that $\mathcal{A}$ is non degenerate if the ratios $\frac{\ell_{1,1}}{\ell_{1,2}}$ and $\frac{\ell_1}{\ell_{1,2}}$ are irrational numbers.

Besides, we can give conditions on the lengths that guarantee that the coefficients c_k are not too smalls. If $\alpha \in \mathbb{R}$ is such that $\lambda_k^\alpha c_k \to 0$ then,

$$\lambda_k^\alpha |d_{1,1}(\lambda_k)| \to 0, \quad \lambda_k^\alpha |d_{1,2}(\lambda_k)| \to 0, \quad \lambda_k^\alpha |d_2(\lambda_k)| \to 0.$$

It is easy to see that this implies

$$\lambda_k^{\frac{\alpha}{2}} |\sin\lambda_k\ell_1\sin\lambda_k\ell_{1,1}| \to 0. \tag{99}$$

Then we can apply the results of Appendix A to conclude that (99) is impossible if

- the ratios $\frac{\ell_{1,1}}{\ell_{1,2}}, \frac{\ell_1}{\ell_{1,2}}$ and $\frac{\ell_1}{\ell_{1,1}}$ belong to some $\mathbf{B}_\varepsilon$ and $\alpha > 4 + \varepsilon$ or
- the numbers $\ell_1, \ell_{1,1}, \ell_{1,2}$ satisfy the conditions (S) and $\alpha > 2 + \varepsilon$.

Then, in the above cases we will have that there exists a positive constant C such that for every $k \in \mathbb{Z}$,

$$c_k \geq \frac{C}{\lambda_k^\alpha}.$$

Consequently, all the initial states $(\bar{u}_0, \bar{u}_1) \in V^\alpha \times V^{\alpha-1}$ are controllable in a time equal to twice the sum of the lengths of the strings.

CHAPTER V

Some observability and controllability results for general networks

In this chapter we have gathered some results of general character, which do not impose any restriction on the topological configuration of the networks.

The first of these results is described in Section 1, it concerns the spectral controllability from an exterior node of arbitrary networks, which may, in particular, contain cycles. A condition on the eigenfunctions of the network is given that guarantees the spectral controllability of the network in any time larger than twice its total length. For tree-shaped networks, that condition coincides with the spectral controllability criterion given in Chapter IV (Theorem IV.3), except by the fact that there it was possible to obtain information on what happens in the minimal control time.

However, for networks with more complex structures it is quite difficult to give an algebraic characterization of a condition guaranteeing the spectral controllability. This would require to take into account the specific structure of the graph that supports the network.

In Section 2 we present a result of general character, related to the control of an arbitrary network when we are allow controls to act on all its nodes. In spite of what may be expected at a first sight, such a large set of controlled points still does not guarantee the exact controllability of the network. That is why we will be concerned once again with the spectral controllability property of the system. It will be shown that in order to reach spectral controllability, it is sufficient to choose only four different control functions, simultaneously applied in several nodes of the network.

Finally, Section 3 is devoted to show that the Schmidt's theorem stated in Chapter I (Theorem I.1) is exact in the sense that, if a tree-shaped network has more than one uncontrolled nodes then it is not exactly controllable in any time $T > 0$.

1. Spectral controllability of general networks

1.1. Asymptotic behavior of the eigenfunctions. The eigenvalues of a networks cannot be explicitly computed. Let us recall that is was already impossible for the three string network: in that case, the eigenvalues are determined by the transcendental equation $q(\lambda_k) = 0$, where q is defined by the formula (III.52). However, it is not difficult to obtain certain information on the asymptotic behavior of the sequence of eigenvalues in the general case.

The idea is simple: the eigenvalues of the network may be compared with the eigenvalues of the strings with Dirichlet and Neumann boundary conditions.

To be more precisely, let us denote by $(\mu_n^{i,D}), (\mu_n^{i,N})$ the sequences of eigenvalues of the operator $-\Delta$ on the string $\mathbf{e}_i$ of length ℓ_i with homogeneous boundary conditions of Dirichlet and Neumann type, respectively. Let $(\mu_n^D), (\mu_n^N)$ be the strictly increasing sequences formed by the elements of the sets

$$\bigcup_{i=i}^{M}(\mu_n^{i,D}), \qquad \bigcup_{i=i}^{M}(\mu_n^{i,N}),$$

respectively.

Then, if $(\hat{\mu}_n)$ if the strictly increasing sequence of the eigenvalues of the network it holds

Proposition V.1. *For every $n \in \mathbb{N}$ the following inequalities are true*

$$\mu_n^N \leq \hat{\mu}_n \leq \mu_n^D.$$

This proposition is proved in [**62**], [**15**] for the general case of equations with variable coefficients. For vibrating strings, it seems to have been first stated by Camerer in 1980 (see reference [3] in [**15**]), though a detail study of this property has been also presented in [**66**]. A quite instructive application of these ideas for networks of beams is given in [**30**].

Let us observe that the eigenvalues $\mu_n^{i,D}$, $\mu_n^{i,N}$ may be computed explicitly:

$$\mu_n^{i,D} = \left(\frac{\pi n}{\ell_i}\right)^2, \qquad \mu_n^{i,N} = \left(\frac{\pi(n-1)}{\ell_i}\right)^2 \qquad n \in \mathbb{N}. \tag{1}$$

Thus, $\mu_n^{i,D} = \mu_{n+1}^{i,N}$ and the same is true for the sequences $(\mu_n^D), (\mu_n^N)$:

$$\mu_n^D = \mu_{n+1}^N.$$

With this, the inequality of Proposition V.1 becomes

$$\mu_n^N \leq \hat{\mu}_n \leq \mu_{n+1}^N. \tag{2}$$

If we denote $\hat{\lambda}_n := \sqrt{\hat{\mu}_n}$ we get as an immediate consequence of these inequalities the following property of generalized separation of the sequence of eigenvalues

$$\hat{\lambda}_{n+M+1} - \hat{\lambda}_n \geq \sqrt{\mu_{n+N}^N} - \sqrt{\mu_n^N} \geq \pi \min_{i=1,\dots,M}(\frac{1}{\ell_i}).$$

Indeed, from the inequalities (1) it follows that, for every $i = 1, ..., M$, and every $k \in \mathbb{N}$

$$\sqrt{\mu_{k+1}^{i,N}} - \sqrt{\mu_k^{i,N}} = \frac{\pi}{\ell_i}.$$

On the other hand, for each $n \in \mathbb{N}$, among the $M+1$ numbers

$$\sqrt{\mu_n^N}, \sqrt{\mu_{n+1}^N}, ..., \sqrt{\mu_{n+N}^N}$$

there are necessarily at least two corresponding to the same value i^* of i. Then

$$\sqrt{\mu_{n+N}^N} - \sqrt{\mu_n^N} \geq \sqrt{\mu_{k+1}^{i^*,N}} - \sqrt{\mu_k^{i^*,N}} = \frac{\pi}{\ell_{i^*}} \geq \pi \min_{i=1,\dots,M}(\frac{1}{\ell_i}).$$

In a similar way as it has been done in Chapter III for the three string network, it is possible to obtain asymptotic information on the sequence $(\hat{\mu}_n)$ from the inequalities (2). Indeed, if $n(r, (a_n))$ denotes the counting function of the sequence

(a_n), that is, $n(r,(a_n))$ is the number of elements of a_n contained on the interval $(0,r)$ then from the inequality (2) follows

$$n(r,(\sqrt{\mu_n^N}))-1 \le n(r,(\hat\lambda_n)) \le n(r,(\sqrt{\mu_n^N})). \tag{3}$$

On the other hand,

$$n(r,(\sqrt{\mu_n^N})) = \sum_{i=1}^{M} n(r,(\sqrt{\mu_n^{i,N}})). \tag{4}$$

But $\sqrt{\mu_n^{i,N}} = \frac{\pi(n-1)}{\ell_i}$, and thus

$$n(r,(\sqrt{\mu_n^{i,N}})) = [\frac{r\ell_i}{\pi}]+1$$

(here, $[\eta]$ denotes the integer part of the real number η). From this inequality we obtain

$$\frac{r\ell_i}{\pi} \le n(r,(\sqrt{\mu_n^{i,N}})) \le \frac{r\ell_i}{\pi}+1$$

and then, from (4),

$$\frac{r}{\pi}L \le n(r,(\sqrt{\mu_n^N})) \le \frac{r}{\pi}L+M.$$

Finally, replacing this estimate in (3) we obtain

$$\frac{r}{\pi}L-1 \le n(r,(\hat\lambda_n)) \le \frac{r}{\pi}L+M. \tag{5}$$

Let us observe that from the inequalities (5) it follows that the sequence $(\hat\lambda_n)$ has density:

$$D(\hat\lambda_n) := \lim_{r\to\infty}\frac{n(r,(\hat\lambda_n))}{r} = \frac{L}{\pi}. \tag{6}$$

It is possible to prove that (see, e.g., Problem 1, p. 142 in [**81**]) that for any sequence (a_n)

$$\lim_{r\to\infty}\frac{n(r,(a_n))}{r} = \lim_{n\to\infty}\frac{n}{a_n}.$$

Therefore, from (6) we obtain

$$\lim_{n\to\infty}\frac{\hat\lambda_n}{n} = \frac{\pi}{L}, \qquad \lim_{n\to\infty}\frac{\hat\mu_n}{n^2} = \left(\frac{\pi}{L}\right)^2;$$

that is, asymptotically, the eigenvalues of the network behave as those of one string of length L. This suggests, in view of the fact that for the pointwise control of a string of length ℓ the minimal control time is 2ℓ, that for the control of a network for one of its exterior nodes the minimal control time should be equal to $2L$. In Theorem V.1 we will prove that this fact is indeed true.

Summarizing the previous results we can formulate

Proposition V.2. *If $(\hat\lambda_n)$ is the strictly increasing sequence formed by the positive square roots of the eigenvalues of the network then ,*

1) The counting function of $(\hat\lambda_n)$ satisfies

$$\frac{r}{\pi}L-1 \le n(r,(\hat\lambda_n)) \le \frac{r}{\pi}L+M.$$

2) The sequence $(\hat{\lambda}_n)$ *has upper density*

$$D^+(\hat{\lambda}_n) = \frac{L}{\pi}.$$

3) The numbers $\hat{\lambda}_n$ *are separated in a generalized sense*

$$\hat{\lambda}_{n+M+1} - \hat{\lambda}_n \geq \pi \min_{i=1,\ldots,M} \left(\frac{1}{\ell_i}\right).$$

4) For every $T > 2\pi L$ *there exist positive numbers* γ_n*, such that*

$$\int_0^T \left| \sum_{n\in\mathbb{Z}} c_n e^{i\lambda_n t} \right|^2 dt \geq \sum_{n\in\mathbb{Z}} \gamma_n^2 \left|c_n\right|^2,$$

for every finite sequence (c_n)*.*

5) $\lim_{n\to\infty} \dfrac{\hat{\lambda}_n}{n} = \dfrac{\pi}{L}$

Let us note that the property 4 holds as an immediate consequence of Corollary II.5 of Theorem II.6.

Let us recall now a notion from the Theory of Non Harmonic Fourier Series. Let (λ_n) be a sequence of distinct real numbers and denote by Θ the set of all the finite linear combinations

$$f(t) = \sum c_n e^{i\lambda_n t}.$$

The number

$$R(\lambda_n) := \sup\{r : \quad \Theta \text{ is dense } C([-r,r])\}$$

is called *completeness radius* of (λ_n).

The information given by Proposition V.2 allows us to calculate the completeness radius of the sequence $(\pm\hat{\lambda}_n)$.

Proposition V.3. *The completeness radius of the sequence* $(\pm\hat{\lambda}_n)$ *is equal to* L*.*

This assertion is an direct consequence of the theorem 2.3.1 from [**34**] applied to the sequence $(\pm\hat{\lambda}_n)$. At the same time, that theorem from [**34**] is a consequence of the famous Beurling-Malliavin theorem allowing to express the completeness radius of a sequence in terms of its density (the details may be found in the original work of A. Beurling and P. Malliavin [**17**]).

In [**34**] the following proposition is also proved.

Proposition V.4 (Haraux and Jaffard, [**34**]). *Let* (λ_n) *be a sequence of real numbers. Then, the following properties are verified*

1) For every $T > 2R(\lambda_n)$ *and every* $n \in \mathbb{Z}$ *there exists a constant* $C_n > 0$ *such that*

$$\int_0^T \left| \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t} \right|^2 dt \geq C_n \left|a_n\right|^2, \tag{7}$$

for any finite sequence (a_n)*.*

2) If $T < 2R(\lambda_n)$ there is no finite set $I \subset \mathbb{Z}$ such that there exists a constant $C_I > 0$ with the property that, for some finite sequence $(\alpha_n)_{n\in I}$ the inequality

$$\int_0^T \left| \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t} \right|^2 dt \geq C_I \left| \sum_{n\in I} \alpha_n a_n \right|^2 , \tag{8}$$

is valid for every finite sequence (a_n).

If we apply this result to the sequence $(\pm\hat{\lambda}_n)$, we obtain, in view of Proposition V.3,

For every $T > 2\pi L$ there exist positive numbersC_n, $n \in \mathbb{Z}$, such that

$$\int_0^T \left| \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t} \right|^2 dt \geq C_n \left| a_n \right|^2 ,$$

for every finite sequence (a_n).

With respect to the similar result of the property 4) in Proposition, this latter approach has the disadvantage that it has been obtained in a non-constructive way, while the coefficients γ_n in Proposition V.2 4) may be, in principle, explicitly expressed in terms of the eigenvalues. The interest of this approach could be now considered mainly of historical character: it is based on result known for more than one decade, while the proof of Theorem II.6 have been recently published. However, the ready to use assertions and the simple proofs given in [**34**], will continue to be a constant reference in this kind of problems.

1.2. Application to the control of the network. The results on the asymptotic behavior of the sequence of eigenvalues allow to obtain the following information in connection to the control of arbitrary networks of strings from one exterior node.

THEOREM V.1. *a) For every $T > 2L$ the following properties of the system (I.11)-(I.16) are equivalent*

1) the system is approximately controllable in time T;
2) the system is spectrally controllable in time T;
3) the spectral unique continuation property: $\omega_x^1(\mathbf{v}_1) \neq 0$ is verified by any non-zero eigenfunction $\bar{\omega}$.

b) When $T < 2L$ the system (I.11)-(I.16) is not spectrally controllable; no element of $Z \times Z$ is controllable in time T.

PROOF. a) We will prove that 1) $\Rightarrow$ 3) $\Rightarrow$ 2). This, together with the immediate implication 2) $\Rightarrow$ 1) (the spectral controllability is a particular case of the approximate controllability), will give the assertion of the theorem.

1) $\Rightarrow$ 3). Let us observe that if $\varkappa_n = 0$ for some $n = n_0$ then for the solution of (I.17)-(I.21)

$$\bar{\phi}(t,x) = \cos\lambda_{n_0} t \; \bar{\theta}_{n_0}(x)$$

we will have $\phi_x^1(t,\mathbf{v}_1) = 0$ for every $t \in \mathbb{R}$. For this solution $\bar{\phi}$ the unique continuation property from the controlled node is not valid for any value of $T > 0$ and thus, the system (I.11)-(I.16) is not approximately controllable in any time $T > 0$. Therefore, 1) $\Rightarrow$ 3).

3) $\Rightarrow$ 2). From Chapter II we know that, if the observability inequality

$$\int_0^T \left|\phi_x^1(t, \mathbf{v}_1)\right|^2 dt \geq \sum_{n\in\mathbb{N}} c_n^2 \left(\mu_n \left|\phi_{0,n}\right|^2 + \left|\phi_{1,n}\right|^2\right), \tag{9}$$

is verified for every solution $\bar{u}$ of the homogeneous system (I.11)-(I.16) with initial data $(\bar{\phi}_0, \bar{\phi}_1) \in Z \times Z$ then, all the initial data $(\bar{u}_0, \bar{u}_1) \in H \times V'$ satisfying

$$\sum_{n\in\mathbb{N}} \frac{1}{c_n^2} \left|u_{0,n}\right|^2 < \infty, \qquad \sum_{n\in\mathbb{N}} \frac{1}{c_n^2 \mu_n} \left|u_{1,n}\right|^2 < \infty \tag{10}$$

are controllable in time T.

Using the formula (I.23) for the solutions of (I.17)-(I.21), the inequality (9) is written as

$$\int_0^T \left|\sum_{n\in\mathbb{N}} \varkappa_n (\phi_{0,n} \cos\lambda_n t + \frac{\phi_{1,n}}{\lambda_n} \sin\lambda_n t)\right|^2 dt \geq \sum_{n\in\mathbb{N}} c_n^2 \left(\mu_n \left|\phi_{0,n}\right|^2 + \left|\phi_{1,n}\right|^2\right), \tag{11}$$

where $(\phi_{0,n})$ and $(\phi_{1,n})$ are finite sequences and $\varkappa_n$ are the values of the normal derivatives of the eigenfunctions at the controlled node:

$$|\varkappa_n| = \theta_{n,x}^1(\mathbf{v}_1).$$

If we denote

$$a_n = \frac{1}{2}\left(\phi_{0,|n|} + \frac{\phi_{1,|n|}}{i\lambda_n}\right),$$

for $n \in \mathbb{Z}_*$, where $\lambda_n = -\lambda_{-n}$ if $n < 0$, we will have

$$\phi_{0,n} = a_n + a_{-n}, \quad \phi_{1,n} = (a_n - a_{-n}) i\lambda_n, \quad n \in \mathbb{N}.$$

With these notations, the inequality (11) becomes

$$\int_0^T \left|\sum_{n\in\mathbb{Z}_*} a_n \varkappa_{|n|} e^{i\lambda_n t}\right|^2 dt \geq 4 \sum_{n\in\mathbb{N}} c_n^2 \mu_n \left|a_n\right|^2, \tag{12}$$

for every finite sequence (a_n) of complex numbers with the property $a_{-n} = \overline{a_n}$.

Let us observe now that, as the network is such that no eigenfunction vanishes identically[1] on the controlled string, the eigenvalues μ_n are all simple. Indeed, if $\bar{\psi}$ and $\bar{\varphi}$ are two linearly independent eigenfunctions corresponding to the eigenvalue μ then the function

$$\bar{\omega} = \varphi_x^1(\mathbf{v}_1)\bar{\psi} - \psi_x^1(\mathbf{v}_1)\bar{\varphi}$$

is also and eigenfunction and is not identically equal to zero as $\bar{\psi}$ and $\bar{\varphi}$ are linearly independent. Besides

$$\omega_x^1(\mathbf{v}_1) = \varphi_x^1(\mathbf{v}_1)\psi_x^1(\mathbf{v}_1) - \psi_x^1(\mathbf{v}_1)\varphi_x^1(\mathbf{v}_1) = 0,$$

and this contradicts our hypothesis on the network.

Thus, the eigenvalue being simple, the sequences (λ_n) and $(\hat{\lambda}_n)$ coincide. Then, as a result of Proposition V.2 4) there exist positive numbers γ_n such that

$$\int_0^T \left|\sum_{n\in\mathbb{Z}_*} a_n \varkappa_n e^{i\lambda_n t}\right|^2 dt \geq 2 \sum_{n\in\mathbb{N}} \gamma_n^2 \varkappa_n^2 \left|a_n\right|^2.$$

[1] This condition is obviously equivalent to the fact that the normal derivative of the eigenfunction vanishes at the controlled node, since, by definition, the eigenfunctions are equal to zero at the controlled node and satisfy a second order ordinary differential equation.

Therefore, we can conclude that the inequality (12) is true with coefficients

$$c_n = \frac{\gamma_n |\varkappa_n|}{\sqrt{2}\lambda_n}.$$

Let us note that all these coefficients are different from zero, since the hypothesis of 3) guarantees that $\varkappa_n \neq 0$ for every n. Then, the initial states defined by (10) are controllable in time T and in particular, so are those of the space $Z \times Z$. This means that the system (I.11)-(I.16) is spectrally controllable in time T.

b) Let $I \subset \mathbb{N}$ be a finite set. If we apply Corollary II.2, it follows that the initial state

$$(\bar{u}_0, \bar{u}_1) = (\sum_{n\in I} \alpha_n \bar{\theta}_n, \sum_{n\in I} \beta_n \bar{\theta}_n) \in Z \times Z \tag{13}$$

is controllable in time T if, and only if, there exists a constant $C > 0$ such that

$$\int_0^T \left|\phi_x^1(t, \mathbf{v}_1)\right|^2 dt \geq C \left(\sum_{n\in I} \alpha_n \phi_{1,n} - \beta_n \phi_{0,n}\right)^2,$$

for every solution $\bar{\phi}$ of (I.17)-(I.21) with initial state $(\bar{\phi}_0, \bar{\phi}_1) \in Z \times Z$.

As a consequence of this, if the initial state $(\bar{u}_0, \bar{u}_1)$ defined by (13) is controllable in time T there exists a constant $C > 0$ such that

$$\begin{aligned}
\int_0^T \left|\sum_{n\in\mathbb{Z}_*} a_n \varkappa_n e^{i\lambda_n t}\right|^2 dt &\geq C \left(\sum_{n\in I} \alpha_n (a_n - a_{-n}) i\lambda_n - \beta_n (a_n + a_{-n})\right)^2 \\
&= C \left(\sum_{n\in I} (\alpha_n i\lambda_n - \beta_n) a_n + (-\alpha_n i\lambda_n - \beta_n) a_{-n}\right)^2 \\
&= C \left(\sum_{n\in I\cup -I} \rho_n a_n\right)^2,
\end{aligned} \tag{14}$$

for every finite sequence (a_n), where

$$\rho_n = \alpha_{|n|} i\lambda_n - \beta_{|n|}.$$

On the other hand, if $T < 2L$ then, since $R(\lambda_n) = L$ we have

$$T < 2R(\lambda_n).$$

Then, in account of Proposition V.4 2) we can ensure that there is no sequence satisfying (14). Therefore, the initial state $(\bar{u}_0, \bar{u}_1)$ defined by (13) is not controllable in time T if $T < 2L$. □

Remark V.1. *When $T > 2L$ and the spectral unique continuation property is verified, if we define the space $\mathcal{W}$ as the completion of $Z \times Z$ with the norm*

$$|||(\bar{\phi}_0, \bar{\phi}_1)||| := \left\{\int_0^T \left|\phi_x^1(t, \mathbf{v}_1)\right|^2 dt\right\}^{\frac{1}{2}},$$

then, all the initial states $(\bar{u}_0, \bar{u}_1) \in H \times V'$ such that $(\bar{u}_1, \bar{u}_0) \in \mathcal{W}\,'$ (the dual of $\mathcal{W}$) are controllable in time T.

In view of Proposition V.2, the space $\mathcal{W}$ *contains all those* $(\bar{u}_1, \bar{u}_0)$ *that satisfy*

$$\sum_{n\in\mathbb{N}} \frac{1}{\gamma_n^2 \varkappa_n^2} \left(|u_{0,n}|^2 + \frac{1}{\mu_n} |u_{1,n}|^2 \right) < \infty,$$

where the coefficients γ_n *are computed according to Corollary II.5 of Theorem II.6.*

REMARK V.2. *In general, when* $T < 2L$ *we do not know what happens with the approximate controllability of the system (I.11)-(I.16); possible, the available information on the sequences* (λ_n) *and* $(\varkappa_n)$ *is not sufficient to give an answer.*

For the three string network we were able to prove in Section 9 of Chapter III, that the approximate controllability does not hold whenever $T < 2L$*. Recall that in that case it was possible to construct explicitly a sequence for which the unique continuation property fails. The same construction may be done for the star-shaped network with* n *strings.*

In [**6**]*, the lack of simultaneous approximate controllability of* n *strings was obtained with the aid of Corollary II.4. This approach, however, is not appropriate for networks which are not star-shaped, since we do not have sufficient information on the sequence* $(\varkappa_n)$*.*

Finally, unlike the case of tree-shaped networks, we do not know whether the spectral controllability still holds in the minimal time $T = 2L$*.*

2. Colored networks

We consider now a network of N stings controlled at all of it nodes.

The motion of the network is described by the system

$$\begin{cases} u^i_{tt} - u^i_{xx} = 0 & \text{in } \mathbb{R}\times[0,\ell_k], \quad i = 1, ..., N, \\ u^i(t,0) = v^{k(\mathbf{v}_i^+)}(t) & t\in\mathbb{R}, \qquad\qquad i = 1, ..., N-1, \\ u^i(t,\ell_i) = v^{k(\mathbf{v}_i^-)}(t) & t\in\mathbb{R}, \\ u^i(0,x) = u^i_0(x), \;\; u^i_t(0,x) = u^i_1(x) & x\in[0,\ell_i], \qquad i = 1, ..., N. \end{cases} \tag{15}$$

Here we have denoted by $\mathbf{v}_i^+$, $\mathbf{v}_i^-$ the initial (corresponding to $x = 0$) and final $(x = \ell_i)$ nodes of the string $\mathbf{e}_i$, respectively, and $k(\mathbf{v})$ is the index of the node $\mathbf{v}$.

The problem (15) is well posed for initial states $(u^i_0, u^i_1) \in L^2(0,\ell_i)\times H^{-1}(0,\ell_i)$, $i = 1, ..., N$, and controls $v^k \in L^2(0,T)$.

This system, being controlled at a large number of points, is expected to have better controllability properties than the control systems studied up to now. As usually, we will say that the initial state $(u^i_0, u^i_1) \in L^2(0,\ell_i)\times H^{-1}(0,\ell_i)$, $i = 1, ..., N$, is controllable in time $T > 0$, if it is possible to choose controls $v^k \in L^2(0,T)$ such that the solution u^i, $i = 1, ..., N$, of (15) reaches the rest position in time T:

$$u^i(T,.) = u^i_t(T,.) = 0, \quad i = 1, ..., N.$$

For every $i = 1, ..., M$, we introduce the sets

$$X_i^+ = \left\{ j : \quad \mathbf{v}_i^+ \in \mathbf{e}_j \right\}, \qquad X_i^- = \left\{ j : \quad \mathbf{v}_i^- \in \mathbf{e}_j \right\},$$

which are, respectively, the sets of the indices of those strings that are incident at the initial and final nodes of the string $\mathbf{e}_i$.

A direct application of HUM guarantees that, if for every $i = 1, ..., M$ there exists a sequence of non-zero real numbers such that

$$\int_0^T (|\sum_{j\in X_i^+} \partial_n \phi^j(t, \mathbf{v}_i^+)|^2 + |\sum_{j\in X_i^-} \partial_n \phi^j(t, \mathbf{v}_i^-)|^2) dt \geq \sum_{n\in\mathbb{N}} (c_n^i)^2 \left(\mu_n^i (\phi_{0,n}^i)^2 + (\phi_{1,n}^i)^2\right),$$

for every solution $\bar{\phi} = (\phi^1, ..., \phi^N)$ of the homogeneous problem (15) then, the initial states (u_0^i, u_1^i), $i = 1, ..., N$, verifying

$$\sum_{n\in\mathbb{N}} \frac{(u_{0,n}^i)^2}{(c_n^i)^2} < \infty, \qquad \sum_{n\in\mathbb{N}} \frac{(u_{1,n}^i)^2}{\mu_n^i (c_n^i)^2} < \infty,$$

are controllable in time T.

Let us remark that the homogeneous problem is a set of N uncoupled wave equations with Dirichlet boundary conditions. The coupling in the original controlled system (15) is shown in the fact that the "observed quantity" in every node is the sum of the normal derivatives of the solutions corresponding to those strings that are coupled it that node. Let us observe that this is a local problem, in the sense that in the observability inequality for every string $\mathbf{e}_i$ only those solutions corresponding to strings that have common nodes with $\mathbf{e}_i$ are present.

Recall that for the simultaneous control of n strings we have proved that if T_i^+ and T_i^- are the sums of the lengths of all the strings that are incident to $\mathbf{v}_i^+$ and $\mathbf{v}_i^-$, respectively, then the following inequalities are verified

$$(16) \qquad \int_0^{2T_i^+} |\sum_{j\in X_i^+} \partial_n \phi^j(t, \mathbf{v}_i^+)|^2 dt \geq \sum_{n\in\mathbb{N}} (c_n^i)^2 \left(\mu_n^i (\phi_{0,n}^i)^2 + (\phi_{1,n}^i)^2\right),$$

$$(17) \qquad \int_0^{2T_i^-} |\sum_{j\in X_i^-} \partial_n \phi^j(t, \mathbf{v}_i^-)|^2 dt \geq \sum_{n\in\mathbb{N}} (c_n^i)^2 \left(\mu_n^i (\phi_{0,n}^i)^2 + (\phi_{1,n}^i)^2\right),$$

with coefficients[2] that may be explicitly computed whenever $\frac{\ell_p}{\ell_i}$ are irrational numbers for every $p, q \in X_i^+$ for (16) and $p, q \in X_i^-$ for (17).

Consequently, we can indicate conditions on the lengths of the strings, precisely those given for the simultaneous control of n strings, guaranteeing the controllability of the system (15) in explicitly characterized spaces. In particular, under the irrationality hypotheses mentioned above, the system is spectrally controllable in any time T^* that satisfies

$$T^* \geq 2\max\left\{T_i^+, T_i^-\right\} \qquad i = 1, ..., M.$$

Now we attempt to reduce the number of different functions used to control the system (15) by applying the same control function at several nodes. Let us assume that the in set $\{1, 2, ..., N\}$ of the indexes for nodes a partition is established:

$$\{1, 2, ..., N\} = K_1 \cup \cdots \cup K_r,$$

such that there is no string having its two nodes in the same set K_k. We will say that two nodes are equivalent if their indexes belong to the same class. A simple way of representing this partition of the set of nodes is to suppose that the nodes

[2]The coefficients c_n^i in the inequalities (16) and (17) son different. We have denoted them with the same symbols to avoid to make the notations even more difficult.

have been painted using r different colors such that no string has their nodes of the same color. With this, equivalent nodes are those of the same color.

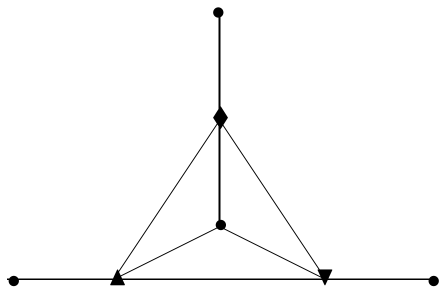

FIGURE 1. A network with colored nodes

In Figure 1 we have represented a network, whose nodes have been painted with four colors, shown with the symbols: $\blacklozenge, \blacktriangle, \blacktriangledown, \bullet$. For this network, four is the smallest number of colors that allows to paint the nodes without repeating the colors at the nodes of one string.

Now we add and additional restriction to system (15): $v_p = v_q$ if the nodes $\mathbf{v}_p$ and $\mathbf{v}_q$ are of the same color.

It is easy to see that this restriction leads to the same observability inequality as before, except by the fact that now the sets X_i^+ and X_i^- should be replaced by

$$X_i^+ = \bigcup_{\mathbf{v}\sim\mathbf{v}_i^+} \{j : \quad \mathbf{v} \in \mathbf{e}_j\}, \qquad X_i^- = \bigcup_{\mathbf{v}\sim\mathbf{v}_i^-} \{j : \quad \mathbf{v} \in \mathbf{e}_j\},$$

(the notation $\mathbf{v} \sim \mathbf{v}'$ indicates that the nodes $\mathbf{v}$ and $\mathbf{v}'$ are equivalent), which are the sets of the indices of the strings, which have some node of the same color as the initial node of $\mathbf{e}_i$ and of the final node $\mathbf{e}_i$, respectively.

We may conclude that, if the lengths of the strings satisfy $\frac{\ell_p}{\ell_q} \notin \mathbb{Q}$ for all the indices $p \neq q$ such that some of the nodes of the string $\mathbf{e}_p$ is of the same color as one of the nodes of $\mathbf{e}_q$ then the system (15) is spectrally controllable in any time

$$T \geq 2 \max_{k=1,\ldots,r} (T_k),$$

where T_k is the same of the lengths of all strings having some node of color k.

To avoid too detailed notations, we will replace the previous conditions by the following, clearly more restrictive ones:

1) the ratios $\frac{\ell_p}{\ell_q}$ are irrational numbers for all the indices $p \neq q$;
2) T is not smaller than twice the sum of the lengths of all the strings of the network.

Under these hypotheses, the minimal number of different control functions necessary to reach the spectral controllability of the network is equal to the minimal number of colors, which are necessary to paint the vertices of the graph such that no edge has its vertices of the same color. This is the classical problem on colored graphs (and this is equivalent to painting a map). The solution, the famous Four Colors Theorem, asserts that if the graph is planar, four colors are sufficient. This is an apparently trivial fact, but a "purely mathematical" rigorous proof is not known. Nowadays it has been proved with the aid of computers. The details may be found in [**1**].

Let us observe now that we have actually obtained two inequalities for every string: (16) and (17), while only one of them suffices to prove the corresponding observability inequality. That is why we may assume that the control associated with one of the colors is equal to zero (this is, indeed a particular choice of the control); to vary this control function is not necessary to control the system and the nodes where it is applied may remain fixed.

Summarizing the previous results we have obtained

PROPOSITION V.5. *If the network is supported on a planar graph and the lengths of its strings and T satisfy the conditions (1) and (2) then, four different functions are sufficient for the system (15) to be spectrally controllable in time T. Besides, one of those functions may be chosen identically equal to zero.*

REMARK V.3. *The condition requiring that the extremes of the strings are of distinct colors is natural if one expects at least the approximate controllability of the system, since it is impossible to control a string using the same control function in both of its extremes. Indeed, the observability inequality associated to that problem would be*

$$\int_0^T |\phi_x(t,0) - \phi_x(t,\ell)|^2 dt \geq \sum_{n\in\mathbb{N}} c_n^2 \left(\mu_n \phi_{0,n}^2 + \phi_{1,n}^2\right),$$

where ϕ is the solution of the wave equation $\phi_{tt} - \phi_{xx} = 0$ with boundary conditions $\phi(t,0) = \phi(t,\ell) = 0$. It suffices to take

$$\phi(t,x) = \cos\frac{2\pi}{\ell}t \ \sin\frac{2\pi}{\ell}x,$$

to see that this inequality cannot be true. Moreover, in this example we have the equality $\phi_x(t,0) - \phi_x(t,\ell) = 0$, and thus, the approximate controllability does not hold either.

3. Sharpness of the Schmidt's theorem

In this section we will prove that Theorem I.1 from Chapter I is sharp in the sense that, if in a tree-shaped network there is at least two uncontrolled nodes, then there exist initial states $(\bar{u}_0, \bar{u}_1) \in H \times V'$ of the network that are not controllable in any finite time T.

The proof is based on the fact that if there are two uncontrolled nodes, then a simple path may be found formed by consecutive strings and connecting those nodes. If there exists $T > 0$ such that every initial state $(\bar{u}_0, \bar{u}_1) \in H \times V'$ is controllable in time T then, we obtain the exact controllability of the system of serially connected strings with controls at the coupling points studied in Subsection 3.1. That is why we concentrate on studied in detail that latter system. We will prove that actually it is never exactly controllable, independently of the value of T or the lengths of the strings. From this will follow that a network with more than two uncontrolled nodes is never exactly controllable.

3.1. Simultaneous control of serially connected strings. Let us suppose that we have N strings of lengths $\ell_1, ..., \ell_N$, which are connected in series and that in every coupling point a control acts to determine the displacement of that point.

The motion of the strings of is described by the system of equations

$$(18)\qquad \begin{cases} u^k_{tt} - u^k_{xx} = 0 & \text{in } \mathbb{R} \times [0, \ell_k], \quad k = 1, ..., N, \\ u^k(t, \ell_k) = u^{k+1}(t, 0) = v^k(t) & t \in \mathbb{R}, \qquad\qquad k = 1, ..., N-1, \\ u^1(t, 0) = u^N(t, \ell_N) = 0 & t \in \mathbb{R}, \\ u^k(0, x) = u^k_0(x), \;\; u^k_t(0, x) = u^k_1(x) & x \in [0, \ell_k], \qquad k = 1, ..., N. \end{cases}$$

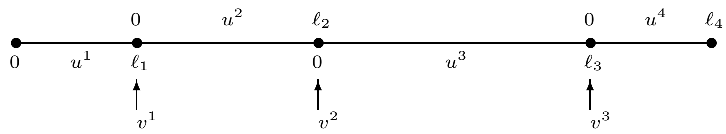

FIGURE 2. Four serially connected strings with controls v^1, v^2, v^3 at the connection points

For $T > 0$, this problem is well posed for initial states $(u^k_0, u^k_1) \in L^2(0, \ell_k) \times H^{-1}(0, \ell_k)$, $k = 1, ..., N$, and controls $v^k \in L^2(0, T)$. The corresponding homogeneous problem is also well posed for $(u^k_0, u^k_1) \in H^1_0(0, \ell_k) \times L^2(0, \ell_k)$.

Let us note that is problem is a particular case of the problem on colored networks studied in Section 2. Thus, we can indicate conditions on the lengths of the strings guaranteeing that the system is spectrally controllable. However, our in now to prove the existence of initial data $(u^k_0, u^k_1) \in L^2(0, \ell_k) \times H^{-1}(0, \ell_k)$, $k = 1, ..., N$, which are not controllable in any finite time $T > 0$, independently of the values of the lengths of the strings.

Once again applying HUM it follows that the system (18) is exactly controllable in time T if, and only if, there exists a constant $C > 0$ such that the solutions of the homogeneous system $\bar{\phi} = (\phi^1, ..., \phi^N)$, which in this case corresponds to N wave equations with homogeneous Dirichlet boundary conditions, verify

$$(19)\qquad \sum_{k=1}^{N-1} \int_0^T \left| \phi^k_x(t, \ell_k) - \phi^{k+1}_x(t, 0) \right|^2 dt \geq C \sum_{k=1}^{N} \mathbf{E}_k,$$

for $k = 1, ..., N$, where $\mathbf{E}_k$ is the energy of the solution ϕ^k, a conserved quantity.

Let us note that the inequality (19) cannot be true for arbitrary values of the lengths of the strings. Indeed, if, for example, all the lengths coincide and are equal to ℓ then the functions

$$\phi^k(t, x) = (-1)^k \sin \frac{t\pi}{\ell} \sin \frac{x\pi}{\ell}, \qquad k = 1, ..., N,$$

are solutions of the homogeneous system (18) and besides

$$\phi^k_x(t, \ell_k) - \phi^{k+1}_x(t, 0) = (-1)^k \frac{\pi}{\ell} \left(\sin \frac{t\pi}{\ell} \cos \frac{x\pi}{\ell}|_{x=\ell} + \sin \frac{t\pi}{\ell} \cos \frac{x\pi}{\ell}|_{x=0} \right) = 0,$$

for $k = 1, ..., N$. But the energy of these solutions does not vanish, since the solutions are non-trivial. Then, the inequality (19) is not true. Moreover, the unique continuation property being false, the system is not even approximately controllable.

Similar examples may be easily given whenever the lengths of the strings satisfy the conditions $\frac{\ell_{k+1}}{\ell_k} \in \mathbb{Q}$ for every k. On the other hand, if the ratio $\frac{\ell_{k+1}}{\ell_k}$ is an irrational number for some k, the the unique continuation property holds and then, so does the approximate controllability of (18).

However, as we have pointed out above, the inequality (19) is never valid, independently of the values of the lengths of the strings. Our aim is to proof this assertion.

For every $k = 1, ..., N$, the solution ϕ^k may be expressed as

$$\phi^k(t,x) = \sum_{n\in\mathbb{N}} \left(\phi^k_{0,n} \cos \lambda^k_n t + \frac{\phi^k_{1,n}}{\ell_k} \sin \lambda^k_n t \right) \sin \lambda^k_n x,$$

where $\lambda^k_n = \frac{n\pi}{\ell_k}$ are the eigenvalues of the k-th string and $(\phi^k_{0,n})$, $(\phi^k_{1,n})$ are the sequences of Fourier coefficients of the initial data ϕ^k_0, ϕ^k_1 in the basis $(\sin \lambda^k_n x)_{n\in\mathbb{N}}$ of $L^2(0,\ell_k)$.

Then

$$\phi^k(t,x) = \sum_{n\in\mathbb{Z}^*} a^k_n e^{i\sigma^k_n t} \sin \lambda^k_n x,$$

where

$$\sigma^k_n = sgn(n)\lambda^k_n, \qquad a^k_n = \frac{1}{2}(\phi^k_{0,|n|} + \frac{\phi^k_{1,|n|}}{i\lambda^k_n}).$$

The inequality (19) can now be written as

(20)

$$\sum_{k=1}^{N-1} \int_0^T \left| \sum_{n\in\mathbb{Z}^*} \left((-1)^n \lambda^k_n a^k_n e^{i\sigma^k_n t} - \lambda^{k+1}_n a^{k+1}_n e^{i\sigma^{k+1}_n t} \right) \right|^2 dt \geq C \sum_{k=1}^{N} \sum_{n\in\mathbb{Z}^*} \left| \lambda^k_n a^k_n \right|^2.$$

Our aim is to construct sequences (a^k_n), $k = 1, ..., N$ for which the inequality (20) is not verified.

In order to simplify the notations, we assume $N = 2$. Let (σ_n) be the increasing sequence formed by the elements of the sequences (σ^1_n) and (σ^2_n). Define (α_n) by

$$\begin{array}{lll} \alpha_n & = & (-1)^m \lambda^1_m a^1_m, \qquad \text{if } \sigma_n = \sigma^1_m, \\ \alpha_n & = & -\lambda^2_m a^2_m \qquad\qquad \text{if } \sigma_n = \sigma^2_m. \end{array}$$

With this, the inequality (20) becomes

$$\int_0^T \left| \sum_{n\in\mathbb{Z}^*} \alpha_n e^{i\sigma_n t} \right|^2 dt \geq C \sum_{n\in\mathbb{Z}^*} |\alpha_n|^2. \tag{21}$$

Note that the latter inequality could be obtained as a consequence of the classical Ingham inequality, if there would be some uniform separation between the numbers σ_n. We will see that this is not the case.

As $\sigma^1_{m+1} - \sigma^1_m = \frac{\pi}{\ell_1}$, $\sigma^2_{m+1} - \sigma^2_m = \frac{\pi}{\ell_2}$ we can ensure that $\sigma_{n+2} - \sigma_n \geq \pi \min\left\{\frac{1}{\ell_1}, \frac{1}{\ell_2}\right\}$; however, it could happen that $\liminf_{n\to\infty} (\sigma_{n+1} - \sigma_n) = 0$. Actually, this always happens. It suffices to note that the number $\frac{\ell_1}{\ell_2}$ may be approximated by rational numbers, that is, there exist sequences (p_k), (q_k) of entire numbers such that

$$\lim_{k\to\infty} \left(\frac{\ell_1}{\ell_2} - \frac{p_k}{q_k} \right) = 0.$$

This is equivalent to $\sigma^1_{p_k} - \sigma^2_{q_k} \to 0$. Thus, the elements of the sequence (σ_n) may get close.

This lack of uniform gap between the numbers σ_k not only makes impossible to apply the the Ingham inequality, but also that (21) is not true. It takes place

PROPOSITION V.6. *There is no positive constant C such that the inequality (19) is verified by all the solutions of the homogeneous system (18) with initial states $(\phi^k_0, \phi^k_1) \in H^1_0(0,\ell_k) \times L^2(0,\ell_k)$, $k = 1,...,N$.*

PROOF. The key element of the proof is the Dirichlet theorem of simultaneous approximation of real numbers by rationals (see more details in [**19**], Section I.5):

If $\xi^1, ..., \xi^M$ are real numbers then, for every $\varepsilon > 0$ and an infinite number of values of $p \in \mathbb{Z}$ there exist entire numbers $q_i(p)$, $i = 1, ..., M$ such that

$$|p\xi_i - q_i(p)| \leq \varepsilon \qquad i = 1, ..., M.$$

Let us fix $\varepsilon > 0$ and choose $\xi^i = \frac{\ell_{i+1}}{\ell_1}$, $i = 1,..,N-1$. Applying the Dirichlet theorem to the numbers ξ^i, $i = 1,..,N-1$ it holds that there exist infinite values of p for which the following inequality is verified

$$\left| \frac{p\pi}{\ell_1} - \frac{q_i(p)\pi}{\ell_{i+1}} \right| \leq \varepsilon_1 = \varepsilon \max_{i=1,..,N-1} \left(\frac{1}{\ell_{i+1}} \right),$$

and that is

$$\left| \lambda^1_p - \lambda^{i+1}_{q_i(p)} \right| \leq \varepsilon_1. \tag{22}$$

Now denote by (σ_n) the increasing sequence formed by the positive square roots of the eigenvalues λ^k_n of all the strings.

For each value p whose existence was ensured by Dirichlet theorem, let us define $m(p)$ by

$$\sigma_{m(p)} = \min \left\{ \lambda^1_p, \lambda^2_{q_1(p)}, ..., \lambda^i_{q_{N-1}(p)} \right\}.$$

Then, for infinite values of $p \in \mathbb{Z}$ the following inequality is true:

$$\left| \sigma_{m(p)+N-1} - \sigma_{m(p)} \right| \leq \varepsilon_1.$$

Since the elements $\sigma_{m(p)}, \sigma_{m(p)+1}, ..., \sigma_{m(p)+N-1}$ are close we can ensure that among them there is exactly one of the eigenvalues of every string. Let $n_k(p)$ be such that

$$\lambda^k_{n_k(p)} \in \left\{ \sigma_{m(p)}, \sigma_{m(p)+1}, ..., \sigma_{m(p)+N-1} \right\}$$

(this value is unique).

Then it will hold

$$\left| \lambda^k_{n_k(p)} - \lambda^{k'}_{n_{k'}(p)} \right| \leq \varepsilon_1, \tag{23}$$

for all $k, k' = 1, ..., N$.

Let us consider now for every $k = 1, ..., N-1$, the following solutions of the homogeneous version of (18)

$$\phi^k_p(t,x) = \frac{1}{2\lambda^k_{n_k(p)}} \cos 2\lambda^k_{n_k(p)} t \ \sin 2\lambda^k_{n_k(p)} x,$$

whose energy is

$$\mathbf{E}_k = \frac{\ell_k}{2}.$$

On the other hand,

$$\phi^k_{p,x}(t,\ell_k) - \phi^{k+1}_{p,x}(t,0) = \cos 2\lambda^k_{n_k(p)}t - \cos 2\lambda^{k+1}_{n_{k+1}(p)}t.$$

Then, from this inequality

$$\int_0^T \left|\phi^k_{p,x}(t,\ell_k) - \phi^{k+1}_{p,x}(t,0)\right|^2 dt \le \left|\lambda^k_{n_k(p)} - \lambda^{k+1}_{n_{k+1}(p)}\right|^2 \frac{T^3}{3}$$

(we have used the inequality

$$\int_0^T (\cos xt - \cos yt)^2 \le \frac{T^3}{3}|x-y|^2,$$

which is easily proved with the help of the mean value theorem).

In account of (23), we may conclude that

$$\int_0^T \left|\phi^k_{p,x}(t,\ell_k) - \phi^{k+1}_{p,x}(t,0)\right|^2 dt \le C\left|\lambda^k_{n_k(p)} - \lambda^{k+1}_{n_{k+1}(p)}\right|^2 \le \frac{T^3}{3}\varepsilon_1^2.$$

Finally, if the inequality (19) were true we would obtain

$$\frac{C}{2}\sum_{k=1}^N \ell_k = C\sum_{k=1}^N \mathbf{E}_k \le \frac{T^3}{3}\varepsilon_1^2 = \varepsilon^2 \max_{i=1,..,N-1}\left(\frac{1}{\ell_{i+1}}\right)^2 \frac{T^3}{3},$$

what is impossible, since ε may be chosen arbitrarily small. □

Remark V.4. *The problem of controlling N strings connected in a cycle with controls in all the nodes may be studied exactly in the same way. This problem is also described by the system (18) where the conditions $u^1(t,0) = u^N(t,\ell_N) = 0$ are replaced by $u^1(t,0) = u^N(t,\ell_N) = v_N(t)$. In Chapter VII in* [**2**] *the reader may find a proof of the lack of exact controllability in this case, based on the method of moments.*

CHAPTER VI

Simultaneous observation and control from an interior region

This chapter is devoted to the simultaneous control of strings with different densities from a common interior region of the strings.

This study is mainly motivated by the following fact. If we perform the changes of variables $x \to \ell_1 x$, $x \to \ell_2 x$ in the equations of system (III.10) for the simultaneous control from one of the exterior nodes of strings with density equal to one, we obtain

$$\text{(1)} \qquad \begin{cases} \ell_k^2 u_{tt}^1 - u_{xx}^k = 0 & \text{en } \mathbb{R} \times [0,1], \quad k = 1,2, \\ u^k(.,0) = v, \quad u^k(.,1) = 0 & \text{en } \mathbb{R}, \\ u^k(0,.) = u_0^k, \quad u_t^k(0,.) = u_1^k & \text{en } [0,1]. \end{cases}$$

Thus, the simultaneous control of two strings of lengths ℓ_1 and ℓ_2 from one of the exterior nodes may be also viewed as the simultaneous control from one end of two strings of lengths equal to one with densities ℓ_1 and ℓ_2.

As we have seen in Chapter III, the answer to this problem depends on the degree of irrationality of the number ℓ_1/ℓ_2. More precisely, ℓ_1/ℓ_2 needs to be irrational to guarantee that all the Fourier components of the solutions are observable, but, moreover, the space of data in which the controllability holds does also depends on ℓ_1/ℓ_2.

All this suggests to study the similar problem when the control acts over an interior region of the strings, a situation when one expects to be much more robust and not to depend on the ratio ℓ_1/ℓ_2. Section 1 is devoted to this problem. When the strings are of the same length and the control acts on the whole strings, then it is possible to control the system in arbitrarily small time. This fact is true even for membranes. We will study that problem in Section 2.

1. Simultaneous interior control of two strings

1.1. Statement of the problem. Let ℓ_1, ℓ_2 be positive numbers and $\boldsymbol{\omega}$ an interval contained in $(0,\ell_1) \cap (0,\ell_2)$.

Let us consider the system

$$\text{(2)} \qquad \begin{cases} \rho_k^2 u_{tt}^k - u_{xx}^k + f\chi_\omega = 0 & \text{in } \mathbb{R} \times [0,\ell_k], \\ u^k(.,0) = u^k(.,\ell_k) = 0 & \text{in } \mathbb{R}, \\ u^k(0,.) = u_0^k, \quad u_t^k(0,.) = u_1^k & \text{in } [0,\ell_k], \end{cases}$$

where $f \in L^2_{\text{loc}}(\mathbb{R}^2)$ and $\chi_{\boldsymbol{\omega}}$ is the characteristic function of the interval $\boldsymbol{\omega}$.

This system describes the motion of two strings $\mathbf{e}_1$ and $\mathbf{e}_2$ of lengths ℓ_1, ℓ_2 and densities ρ_1,ρ_2, respectively, which are simultaneously controlled by means of the same force localized on the interval $\boldsymbol{\omega}$.

The system (2) is well posed for initial states

$$(u_0^k, u_1^k) \in \mathcal{W}_k := H_0^1(0, \ell_k) \times L^2(0, \ell_k), \qquad k = 1, 2,$$

with a control force

$$f \in L^1(0, T; L^2(\boldsymbol{\omega})).$$

More precisely, under the previous assumptions on the initial data (u_0^k, u_1^k) and the control force f, the system (2) admits a unique solution in the energy space

$$(u^k, u_t^k) \in C([0, T]; \mathcal{W}_k), \qquad k = 1, 2.$$

We study the following control problem for (2): *given $T > 0$, to determine for which initial states (u_0^k, u_1^k),$k = 1, 2$, the function f may be chosen such that*

$$u^k(T, .) = u_t^k(T, .) = 0, \quad k = 1, 2.$$

We will say that the system (2) is *exactly controllable in time T* if all the initial states $(u_0^k, u_1^k) \in \mathcal{W}_k$, $k = 1, 2$, are controllable in time T.

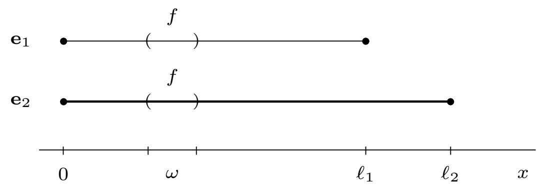

FIGURE 1. Two strings $\mathbf{e}_1$ and $\mathbf{e}_2$ of different densities controlled simultaneously from the common interval ω.

The application of the HUM guarantees that (2) is exactly controllable in time T if, and only if, there exists a constant $C > 0$ such that

$$C \int_{\boldsymbol{\omega}} \int_0^T |\phi^1(t, x) + \phi^2(t, x)|^2 dt dx \geq ||(\phi_0^1, \phi_1^1)||^2_{L^2 \times H^{-1}} + ||(\phi_0^2, \phi_1^2)||^2_{L^2 \times H^{-1}}, \tag{3}$$

for all the solutions ϕ^1, ϕ^2 of the homogeneous equations

$$\begin{cases} \rho_k^2 \phi_{tt}^k - \phi_{xx}^k = 0 & \text{in } \mathbb{R} \times [0, \ell_k], \quad k = 1, 2, \\ \phi^k(., 0) = \phi^k(., \ell_k) = 0 & \text{in } \mathbb{R}, \\ \phi^k(0, .) = \phi_0^k, \quad \phi_t^k(0, .) = \phi_1^k & \text{in } [0, \ell_k]. \end{cases} \tag{4}$$

The solutions of (4) are given by the formula

$$\phi^k(t, x) = \sum_{n \in \mathbb{N}} (\phi_{0,n}^k \cos \frac{n\pi}{\rho_k \ell_k} t + \frac{\rho_k \ell_k}{n\pi} \phi_{1,n}^k \sin \frac{n\pi}{\rho_k \ell_k} t) \sin \frac{n\pi}{\ell_k} x, \tag{5}$$

where $(\phi_{0,n}^k)$, $(\phi_{1,n}^k)$ are the sequences of Fourier coefficients of ϕ_0^k, ϕ_1^k, respectively, in the basis $(\sin \frac{n\pi}{\ell_k} x)_{n\in\mathbb{N}}$ of $L^2(0,\ell_k)$.

If we denote

$$a_n^k = \frac{1}{2}\left(\phi_{0,|n|}^k + \frac{i\rho_k \ell_k}{n\pi}\phi_{1,|n|}^k\right), \quad k=1,2, \quad n\in\mathbb{Z}_*,$$

the formula (5) may be rewritten as

$$\phi^k(t,x) = \sum_{n\in\mathbb{Z}_*} a_n^k e^{\frac{in\pi}{\rho_k\ell_k}t}\sin\frac{n\pi}{\ell_k}x.$$

Note that, by the definition of (a_k^n) we have $\overline{a_k^n} = a_{-k}^n$.

With these notations, the inequality (3) is equivalent to

$$C\int_{\omega}\int_0^T \left|\sum_{n\in\mathbb{Z}_*} a_n^1 e^{\frac{in\pi}{\rho_1\ell_1}t}\sin\frac{n\pi}{\ell_1}x + a_n^2 e^{\frac{in\pi}{\rho_2\ell_2}t}\sin\frac{n\pi}{\ell_2}x\right|^2 dtdx \geq \sum_{n\in\mathbb{N}}\left(|a_n^1|^2 + |a_n^2|^2\right), \tag{6}$$

for all finite complex sequences $(a_n^1)_{n\in\mathbb{Z}_*}$, $(a_n^2)_{n\in\mathbb{Z}_*}$ satisfying $a_{-n}^1 = \overline{a_n^1}$, $a_{-n}^2 = \overline{a_n^2}$.

Obviously, the inequality (6) is impossible if $\ell_1 = \ell_2$ and $\rho_1 = \rho_2$. Indeed, it suffices to take, e. g., $a_1^1 = -a_1^2 \neq 0$ and $a_n^1 = -a_n^2 = 0$ for $n \neq \pm 1$, to see that in this case, (6) is not satisfied. But note, that this extremely degenerate case corresponds to controlling simultaneously two identical strings with the same control and different initial configurations. This is obviously impossible in general, since the control depends in a very sensitive way on the initial data to be controlled.

Our aim is to prove that the inequality (6) is verified whenever $\rho_1 \neq \rho_2$ if T is sufficiently large. This is the object of the next subsection.

1.2. Control of strings with different densities. Now we consider the case when the densities of the strings are different, i.e., $\rho_1 \neq \rho_2$. The following holds

THEOREM VI.1. *If $\rho_1 \neq \rho_2$ and $T > T_0 := 2\max(\rho_1\ell_1, \rho_2\ell_2)$ then the inequality (6) is verified for all the finite complex sequences $(a_n^1)_{n\in\mathbb{Z}_*}$, $(a_n^2)_{n\in\mathbb{Z}_*}$ satisfying $a_{-n}^1 = \overline{a_n^1}$, $a_{-n}^2 = \overline{a_n^2}$.*

COROLLARY VI.1. *The strings $\mathbf{e}_1$ and $\mathbf{e}_2$ are simultaneously exactly controllable in time $T > T_0$ if $\rho_1 \neq \rho_2$.*

REMARK VI.1. *This result shows an important difference between the control from an extreme of the strings and the control from an arbitrarily small interior region. Recall that, according to Corollary III.1, all the initial states from $\left(H_0^1(0,1)\times L^2(0,1)\right)^2$ for the system (1) are controllable in time $T \geq 2(\ell_1+\ell_2)$ if, and only if, the ratio $\frac{\ell_1}{\ell_2}$ belongs to the set[1] $\mathcal{F}$, which is a set of null Lebesgue. Besides, the exact controllability of (1) in the space $\left(L^2(0,1)\times H^{-1}(0,1)\right)^2$ is never reached, independently of the values of ℓ_1 and ℓ_2. This shows that controlling from an interior subinterval provides a much more robust control mechanism than when the control is exerted at an of the extremes.*

[1] Recall that the elements of $\mathcal{F}$ are those real numbers having a development in continuous fraction $[a_0, a_1, ..., a_n, ...]$ with bounded (a_n).

In order to prove Theorem VI.1 we will use the scheme followed in the proof of Theorem III.2 relative to the simultaneous control of two strings from one extreme. The idea is quite simple: given and interval $\boldsymbol{\omega} \subset \mathbb{R}$, we construct another interval $\boldsymbol{\omega}' \subset \boldsymbol{\omega}$ and a continuous operator

$$\mathbf{B} : L^2((0,T) \times \boldsymbol{\omega}) \to L^2((0,T') \times \boldsymbol{\omega}')$$

such that, if ϕ^1, ϕ^2 are solutions of (4) then $\mathbf{B}\phi^1 = 0$ and, besides, there exists a constant $C > 0$ such that

$$C \int_{\boldsymbol{\omega}'} \int_0^{T'} |\mathbf{B}\phi^2|^2 dtdx \geq ||(\phi_0^2, \phi_1^2)||^2_{L^2 \times H^{-1}}.$$

Then we will have

$$C \int_{\boldsymbol{\omega}} \int_0^{T} |\phi^1 + \phi^2|^2 dtdx \geq C \int_{\boldsymbol{\omega}'} \int_0^{T'} |\mathbf{B}\phi^2|^2 dtdx \geq ||(\phi_0^2, \phi_1^2)||^2_{L^2 \times H^{-1}}.$$

The inequality

$$C \int_{\boldsymbol{\omega}} \int_0^{T} |\phi^1 + \phi^2|^2 dtdx \geq ||(\phi_0^1, \phi_1^1)||^2_{L^2 \times H^{-1}}$$

may be obtained in an analogous way.

Let us fix $\boldsymbol{\omega} = (\omega_1, \omega_2) \subset \mathbb{R}$ and define, for $a > 0$, the linear operator $\mathbf{B}_a$ that acts over a function $\phi(t,x)$ according to the formula

$$\begin{aligned} \mathbf{B}_a\phi(t,x) := \quad & \phi(t + 2a(x - \omega_1), x + a(x - \omega_1)) - \phi(t + a(x - \omega_1), x + 2a(x - \omega_1)) \\ & -\phi(t + a(x - \omega_1), x) + \phi(t, x + a(x - \omega_1)). \end{aligned}$$

Let us observe that, since $\omega_1 < \omega_2$ and $T > 0$, it is possible to chose for every $a > 0$ a number $\hat{\omega}_2 \in (\omega_1, \omega_2)$ such that

$$\hat{\omega}_2 < \frac{\omega_2 + 2a\omega_1}{1 + 2a} \quad \text{and} \quad \hat{T} := T - 2a(\hat{\omega}_2 - \omega_1) > 0. \tag{7}$$

PROPOSITION VI.1. *If $\hat{\omega}_2$ and $\hat{T}$ satisfy (7) then the operator $\mathbf{B}_a$ is continuous from $L^2((0,T) \times (\omega_1, \omega_2))$ to $L^2((0,\hat{T}) \times (\omega_1, \hat{\omega}_2))$, that is, there exists a constant $C > 0$ such that*

$$C \int_{\omega_1}^{\omega_2} \int_0^{T} |\phi(t,x)|^2 dtdx \geq \int_{\omega_1}^{\hat{\omega}_2} \int_0^{\hat{T}} |\mathbf{B}_a\phi(t,x)|^2 dtdx,$$

for every function ϕ for which both integrals are defined.

PROOF. Let us observe that

$$\mathbf{B}_a\phi(t,x) = \sum_{(p,q)\in S} (-1)^p \phi(t + pa(x - \omega_1), x + qa(x - \omega_1)),$$

where

$$S = \{(2,1), (1,2), (0,1), (1,0)\}.$$

then,

$$\int_{\omega_1}^{\hat{\omega}_2} \int_0^{\hat{T}} |\mathbf{B}_a\phi(t,x)|^2 dtdx \leq 4 \sum_{(p,q)\in S} \int_{\omega_1}^{\hat{\omega}_2} \int_0^{\hat{T}} |\phi(t+pa(x-\omega_1), x+qa(x-\omega_1))|^2 dtdx. \tag{8}$$

To estimate the integrals

$$\int_{\omega_1}^{\hat{\omega}_2}\int_0^{\hat{T}} |\phi(t+pa(x-\omega_1), x+qa(x-\omega_1))|^2 dtdx \tag{9}$$

we perform the change of variables

$$\xi = t+pa(x-\omega_1), \quad \eta = x+qa(x-\omega_1). \tag{10}$$

In these variables, (9) is written as

$$(1+qa)\int\int_{\Omega_{p,q}} |\phi(\xi,\eta)|^2 d\xi d\eta,$$

where $\Omega_{p,q}$ is the image of $(0,\hat{T})\times(\omega_1,\hat{\omega}_2)$ by the mapping defined by (10). Besides, in view of (7), for all $(p,q)\in S$,

$$\Omega_{p,q}\subset(0,T)\times(\omega_1,\omega_2).$$

Thus,

$$\int_{\omega_1}^{\omega_2}\int_0^T |\phi(\xi,\eta)|^2 d\xi d\eta \geq \int\int_\Omega |\phi(\xi,\eta)|^2 d\xi d\eta.$$

This fact, in account of the inequality (8), proves the proposition. □

The following proposition shows how the operators $\mathbf{B}_a$ act on the functions of the form $e^{\frac{in\pi t}{\rho\ell}}\sin\frac{in\pi x}{\ell}$. It is proved by simple calculations.

PROPOSITION VI.2. *For all $\rho,\ell\in\mathbb{R}$ and $n\in\mathbb{N}$ the following equality holds*

$$\mathbf{B}_a\left(e^{\frac{in\pi t}{\rho\ell}}\sin\frac{in\pi x}{\ell}\right) = 4e^{\frac{in\pi}{\rho\ell}(t+x-\omega_1)}\sin\frac{n\pi x}{\ell}\sin\left(\frac{n\pi(x-\omega_1)}{2\ell}\alpha\right)\sin\left(\frac{n\pi(x-\omega_1)}{2\ell}\beta\right).$$

where $\alpha=\left(\rho^{-1}+a\right)$ and $\beta=\left(\rho^{-1}-a\right)$.

REMARK VI.2. *If $\phi(t,x)$ is a solution of the wave equation*

$$\rho^2\phi_{tt}-\phi_{xx}=0, \qquad \phi(t,0)=\phi(t,\ell)=0,$$

whose initial data $\phi\,|_{t=0}$ and $\phi_t\,|_{t=0}$ are finite linear combinations of the eigenfunctions $\left(\sin\frac{in\pi x}{\ell}\right)$ then

$$\mathbf{B}_{\rho^{-1}}\phi(t,x)=0.$$

PROPOSITION VI.3. *Let $\ell,\alpha\neq\beta$ be positive numbers and I an interval in $\mathbb{R}$. Then, there exists a constant $C>0$ such that, for all $n\in\mathbb{R}$,*

$$\int_I\left|\sin\frac{n\pi x}{\ell}\sin\left(\frac{n\pi(x-\omega_1)}{2\ell}\alpha\right)\sin\left(\frac{n\pi(x-\omega_1)}{2\ell}\beta\right)\right|^2 dx\geq C.$$

This fact may be easily proved by computing the integral.

PROOF OF THE THEOREM VI.1. Let $(a_n^1)_{n\in\mathbb{Z}_*}$, $(a_n^2)_{n\in\mathbb{Z}_*}$ be complex finite sequences satisfying $a_{-n}^1=\overline{a_n^1}$, $a_{-n}^2=\overline{a_n^2}$ and

$$\phi^k(t,x)=\sum_{n\in\mathbb{Z}_*}a_n^k e^{\frac{in\pi t}{\rho_k\ell_k}}\sin\frac{n\pi x}{\ell_k}, \quad k=1,2.$$

Let us take $a=\rho_1^{-1}$. Since $T>T_0$, it is possible to choose $\hat{\omega}_2>\omega_1$ sufficiently close to ω_1 such that $\hat{\omega}_2$ and $\hat{T}$ satisfy (7) and, besides, $\hat{T}\geq T_0$.

Then, according to Proposition VI.1,

$$C\int_{\omega_1}^{\omega_2}\int_0^T |\phi^1+\phi^2|^2 dtdx \geq \int_{\omega_1}^{\hat{\omega}_2}\int_0^{\hat{T}} |\mathbf{B}_a\phi^1+\mathbf{B}_a\phi^2|^2 dtdx. \tag{11}$$

But from Remark VI.2,

$$\mathbf{B}_a\phi^1=0.$$

Thus, from the inequality (11) it follows

$$C\int_{\omega_1}^{\omega_2}\int_0^T |\phi^1+\phi^2|^2 dtdx \geq \int_{\omega_1}^{\hat{\omega}_2}\int_0^{\hat{T}} |\mathbf{B}_a\phi^2|^2 dtdx. \tag{12}$$

On the other hand, as

$$\phi^2(t,x)=\sum_{n\in\mathbb{Z}_*} a_n^2 e^{\frac{in\pi}{\rho_2\ell_2}t}\sin\frac{n\pi x}{\ell_2},$$

Proposition VI.2 guarantees that

$$\mathbf{B}_a\phi^2(t,x)=\sum_{n\in\mathbb{Z}_*} a_n^2 e^{\frac{in\pi}{\rho_2\ell_2}t}\Theta_n(x),$$

where

$$\Theta_n(x):=4e^{\frac{in\pi}{\rho_2\ell_2}(x-\omega_1)}\sin\frac{n\pi x}{\ell_2}\sin\left(\frac{n\pi(x-\omega_1)}{2\ell_2}\alpha\right)\sin\left(\frac{n\pi(x-\omega_1)}{2\ell_2}\beta\right)$$

with

$$\alpha=\frac{1}{\rho_2}+\frac{1}{\rho_1},\qquad \beta=\frac{1}{\rho_2}-\frac{1}{\rho_1}.$$

Moreover, in view of Proposition VI.3, there exists a constant $C>0$ such that for every $n\in\mathbb{N}$ the following inequality is verified.

$$\int_{\omega_1}^{\hat{\omega}_2}|\Theta_n(x)|^2dx\geq C. \tag{13}$$

Therefore, since $\hat{T}\geq T_0\geq 2\rho_2\ell_2$,

$$\begin{aligned}\int_{\omega_1}^{\hat{\omega}_2}\int_0^{\hat{T}}|\mathbf{B}_a\phi^2|^2dtdx &\geq \int_{\omega_1}^{\hat{\omega}_2}\int_0^{2\rho_2\ell_2}\left|\sum_{n\in\mathbb{Z}_*}a_n^2e^{\frac{in\pi}{\rho_2\ell_2}t}\Theta_n(x)\right|^2dtdx\\ &= 2\sum_{n\in\mathbb{N}}|a_n^2|^2\int_{\omega_1}^{\hat{\omega}_2}|\Theta_n(x)|^2dx,\end{aligned}$$

(we have used the fact that the functions $e^{\frac{in\pi}{\rho_2\ell_2}t}$ are orthogonal on $(0,2\rho_2\ell_2)$) and then, in view of (12) and (13),

$$C\int_{\omega_1}^{\omega_2}\int_0^T|\phi^1+\phi^2|^2dtdx\geq\sum_{n\in\mathbb{N}}|a_n^2|^2.$$

The inequality

$$C\int_{\omega_1}^{\omega_2}\int_0^T|\phi^1+\phi^2|^2dtdx\geq\sum_{n\in\mathbb{N}}|a_n^1|^2$$

is proved in a similar way, applying to $\phi^1+\phi^2$ the operator $\mathbf{B}_a$ with $a=\rho_2^{-1}$. This concludes the proof of the theorem. $\square$

1.3. Control of strings with equal densities. Theorem VI.1 does not provide any information on what happens when $\rho_1 = \rho_2$ but $\ell_1 \neq \ell_2$. This is due to the local character of the operators $\mathbf{B}_a$: they cannot distinguish between solutions of the wave equation that propagate at the same speed. This fact, however, is not purely technical. If $\rho_1 = \rho_2 = \rho$, the condition $\ell_1 \neq \ell_2$ is not sufficient for the inequality (6) to be true.

Indeed, let us assume that

$$\frac{\ell_1}{\ell_2} = \frac{p}{q}, \quad p, q \in \mathbb{N}.$$

Then, the solutions

$$\phi^1(t,x) = e^{\frac{ip\pi}{\rho\ell_1}t} \sin\frac{p\pi x}{\ell_1}, \quad \phi^2(t,x) = -e^{\frac{iq\pi}{\rho\ell_2}t} \sin\frac{q\pi x}{\ell_2}$$

satisfy

$$\phi^1(t,x) + \phi^2(t,x) \equiv 0.$$

Thus, an inequality of type (6) is impossible for any interval $\boldsymbol{\omega}$ and any time T, whenever the ratio $\frac{\ell_1}{\ell_2}$ is a rational number. It is even impossible to replace the right hand term in (6) by any other weaker norm of the initial data. In this sense, the problem turns out to be similar to that of the simultaneous control of two strings from one extreme, since the lengths of the strings do play a crucial role. When the ratio $\frac{\ell_1}{\ell_2}$ is an irrational number, it is possible to prove a weakened version of (6). We use the same technique as in Theorem III.2,

We denote by Z^k, $k = 1, 2$, the space of the finite linear combinations of the functions $\left(\sin\frac{n\pi x}{\ell_k}\right)_{n\in\mathbb{N}}$.

THEOREM VI.2. *Let $\rho_1 = \rho_2 = \rho$ and $T \geq 2\rho(\ell_1 + \ell_2)$. There exists a constant $C > 0$ such that*

$$C \int_{\boldsymbol{\omega}} \int_0^T |\phi^1(t,x) + \phi^2(t,x)|^2 dtdx \geq \sum \sin^2 \frac{\ell_2 n\pi}{\ell_1} \left((\phi_0^1)^2 + n^{-2}(\phi_1^1)^2\right), \tag{14}$$

$$C \int_{\boldsymbol{\omega}} \int_0^T |\phi^1(t,x) + \phi^2(t,x)|^2 dtdx \geq \sum \sin^2 \frac{\ell_1 n\pi}{\ell_2} \left((\phi_0^2)^2 + n^{-2}(\phi_1^2)^2\right), \tag{15}$$

for all the solutions ϕ^1, ϕ^2 of (4) with initial states in $Z^1 \times Z^1$ and $Z^2 \times Z^2$, respectively.

PROOF. The inequality (15) is equivalent to

$$C \int_{\boldsymbol{\omega}} \int_0^T |\Phi|^2 \, dtdx \geq \sum_{n\in\mathbb{N}} |a_n^2|^2 \sin^2 \frac{\ell_1 n\pi}{\ell_2},$$

where

$$\Phi := \sum_{n\in\mathbb{Z}_*} a_n^1 e^{\frac{in\pi t}{\rho\ell_1}} \sin\frac{n\pi x}{\ell_1} + a_n^2 e^{\frac{in\pi t}{\rho\ell_2}} \sin\frac{n\pi x}{\ell_2},$$

for all the finite complex sequences verifying $a_{-n}^1 = \overline{a_n^1}$, $a_{-n}^2 = \overline{a_n^2}$.

To prove this assertion, let us observe that, for every $x \in \omega$,

$$(\rho\ell_1)^{-}\left(\sum_{n\in\mathbb{Z}_*} a_n^1 e^{\frac{in\pi t}{\rho\ell_1}}\right) = 0, \tag{16}$$

where $(\rho\ell_1)^-$ is the operator defined by (II.7) for the number $\rho\ell_1$(the equality (16) corresponds to the $2\rho\ell_1$-periodicity in time of the solution ϕ^1). Then,

$$\begin{aligned}(\rho\ell_1)^-\Phi &= (\rho\ell_1)^-\sum_{n\in\mathbb{Z}_*} a_n^2 e^{\frac{in\pi t}{\rho\ell_2}}\sin\frac{n\pi x}{\ell_2}\\ &= \sum_{n\in\mathbb{Z}_*} a_n^2 e^{\frac{in\pi t}{\rho\ell_2}}\sin\frac{\ell_1 n\pi}{\ell_2}\sin\frac{n\pi x}{\ell_2}.\end{aligned}$$

Besides, from Proposition II.2 we obtain that for every $x\in\omega$,

(17)
$$\int_0^T |\Phi|^2\,dt \geq \int_{\rho\ell_1}^{T-\rho\ell_1} |(\rho\ell_1)^-\Phi|^2 dt = \int_{\rho\ell_1}^{T-\rho\ell_1} |\sum_{n\in\mathbb{Z}_*} a_n^2 e^{\frac{in\pi t}{\rho\ell_2}}\sin\frac{\ell_1 n\pi}{\ell_2}\sin\frac{n\pi x}{\ell_2}|^2 dt.$$

On the other hand, since $T\geq 2\rho(\ell_1+\ell_2)$ then,

$$\begin{aligned}&\int_{\rho\ell_1}^{T-\rho\ell_1}\left|\sum_{n\in\mathbb{Z}_*} a_n^2 e^{\frac{in\pi t}{\rho\ell_2}}\sin\frac{\ell_1}{\ell_2}n\pi\sin\frac{n\pi x}{\ell_2}\right|^2 dt\\ &\geq \int_{\rho\ell_1}^{\rho\ell_1+2\rho\ell_2}\left|\sum_{n\in\mathbb{Z}_*} a_n^2 e^{\frac{in\pi t}{\rho\ell_2}}\sin\frac{\ell_1 n\pi}{\ell_2}\sin\frac{n\pi x}{\ell_2}\right|^2 dt\\ &= 2\sum_{n\in\mathbb{N}}|a_n^2|^2\sin^2\frac{n\pi x}{\ell_2}\sin^2\frac{\ell_1 n\pi}{\ell_2}.\end{aligned}$$

(we have used here the fact that the functions $\left(e^{\frac{in\pi t}{\rho\ell_2}}\right)_{n\in\mathbb{Z}_*}$ are orthonormal on any interval of length $2\rho\ell_2$).

Further, in view of (17),

$$C\int_\omega\int_0^T |\Phi|^2\,dtdx \geq \sum_{n\in\mathbb{N}}|a_n^2|^2\sin^2\frac{\ell_1 n\pi}{\ell_2}\int_\omega \sin^2\frac{n\pi x}{\ell_2}dx. \tag{18}$$

Finally, let us observe that for any interval $\omega\subset\mathbb{R}$ there exists a constant $C=C(\omega)$ such that

$$\int_\omega \sin^2\frac{n\pi x}{\ell_2}dx \geq C.$$

Therefore, from (18) it holds

$$C\int_\omega\int_0^T |\Phi|^2\,dtdx \geq \sum_{n\in\mathbb{N}}|a_n^2|^2\sin^2\frac{\ell_1 n\pi}{\ell_2}.$$

The inequality (14) may be obtained in an analogous way. □

REMARK VI.3. *When the nuber $\frac{\ell_1}{\ell_2}$ is rational, some of the coefficients $\sin\frac{\ell_2 n\pi}{\ell_1}$ or $\sin\frac{\ell_1 n\pi}{\ell_2}$ entering in the right hand side of inequalities (14), (15) vanish. This agrees with the fact that in this case we cannot obtain an inequality of type (6).*

COROLLARY VI.2. *If the number $\frac{\ell_1}{\ell_2}$ is irrational and $T\geq 2\rho(\ell_1+\ell_2)$ then the system (2) is spectrally controllable in time T, that is, all the initial states $(u_0^1,u_1^1)\in Z^1\times Z^1$, $(u_0^2,u_1^2)\in Z^2\times Z^2$ are controllable in time T.*

If we have some additional information on the rational approximation properties of the ratio $\frac{\ell_1}{\ell_2}$, then it is possible to describe subspaces of controllable initial states in the same way as it was done in Subsection III.2.1.

Corollary VI.3. *a)If $\frac{\ell_1}{\ell_2} \in \mathbf{B}_\varepsilon$ then the subspace of initial states*

$$(u_0^i, u_1^i) \in \hat{H}^{2+\varepsilon}(0, \ell_i) \times \hat{H}^{1+\varepsilon}(0, \ell_i),$$

is controllable in any time $T \geq 2\rho(\ell_1 + \ell_2)$. In particular, if $\frac{\ell_1}{\ell_2}$ is an irrational algebraic number, this subspace is controllable for any $\varepsilon > 0$.

b) If $\frac{\ell_1}{\ell_2}$ admits a bounded development in continuous fractions, then the subspace of initial states

$$(u_0^i, u_1^i) \in \left[H^2(0, \ell_i) \cap H_0^1(0, \ell_i)\right] \times H_0^1(0, \ell_i),$$

is controllable in any time $T \geq 2\rho(\ell_1 + \ell_2)$.

2. Simultaneous control on the whole domain

Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and $f \in L^2_{\text{loc}}(\mathbb{R}^{n+1})$. Let us consider the system

$$\left\{\begin{array}{ll} \rho_k^2 u_{tt}^k - \Delta u^k + f = 0 & \text{in } \Omega \times \mathbb{R}, \\ u^k \,|_{\partial\Omega} = 0 & \text{in } \mathbb{R}, \\ u^k(0,.) = u_0^k, \quad u_t^k(0,.) = u_1^k & \text{in } \Omega. \end{array}\right. \tag{19}$$

The system (19) corresponds to the motion of N elastic membranes with densities $\rho_1, ..., \rho_N$ having at rest the same shape Ω and whose borders are fixed. Those membranes are controlled by means of a function f than acts on the whole domain Ω. When $n = 1$ the system (19) is a particular case of the system (2) with $\ell_1 = \ell_2$ and $\omega = (0, \ell_1)$.

The problem (19) is well posed for initial states $(u_0^k, u_1^k) \in H_0^1(\Omega) \times L^2(\Omega)$, $k = 1, ..., N$. When $f = 0$, (19) becomes the homogeneous system

$$\left\{\begin{array}{ll} \rho_k^2 \phi_{tt}^k - \Delta \phi^k + f = 0 & \text{in } \Omega \times \mathbb{R}, \\ \phi^k \,|_{\partial\Omega} = 0 & \text{in } \mathbb{R}, \\ \phi^k(0,.) = \phi_0^k, \quad \phi_t^k(0,.) = \phi_1^k & \text{in } \Omega, \end{array}\right. \tag{20}$$

which is also well posed for initial states $(\phi_0^k, \phi_1^k) \in L^2(\Omega) \times H^{-1}(\Omega)$, $k = 1, ..., N$.

If $(\mu_n)_{n\in\mathbb{N}}$ is the increasing sequence of the eigenvalues $-\Delta$ with Dirichlet homogeneous boundary conditions in Ω and $(\theta_n)_{n\in\mathbb{N}}$ is the orthonormal in $L^2(\Omega)$ sequences of the corresponding eigenfunctions, then the solutions of (20) are determined by the formulas

$$\phi^k(t,x) = \sum_{n\in\mathbb{Z}_*} a_n^k e^{\frac{\lambda_n t}{\rho_k}} \theta_{|n|}(x)$$

where

$$\lambda_n = \sqrt{\mu_{|n|}} \,\text{sgn}\, n, \quad n \in \mathbb{Z}_*,$$

$$a_n^k = \frac{1}{2}\left(\phi_{0,|n|}^k + i\lambda_n \phi_{1,|n|}^k\right), \quad k = 1, ..., N, \quad n \in \mathbb{Z}_*.$$

The control problem associated to the system (19) is: *given $T > 0$, to determine for which initial states $(u_0^k, u_1^k) \in H_0^1(\Omega) \times L^2(\Omega)$, $k = 1, ..., N$, there exists $f \in L^2((0,T) \times \Omega)$ such that the solution of (19) satisfies*

$$u^k \mid_{t=T} = u_t^k \mid_{t=T} = 0.$$

System (19) is said to be *exactly controllable in time T* when all the initial states from $H_0^1(\Omega) \times L^2(\Omega)$ are controllable in time T.

The application of HUM guarantees that the system (19) is exactly controllable in time T if, and only if, there exists a constant $C > 0$ such that the inequality

$$C \int_\Omega \int_0^T \left| \sum_{k=1}^N \phi^k(t,x) \right|^2 dtdx \geq \sum_{k=1}^N ||(\phi_0^k, \phi_1^k)||^2_{L^2(\Omega)\times H^{-1}(\Omega)} \tag{21}$$

is verified by all the solutions of (20) with initial states $(\phi_0^k, \phi_1^k) \in L^2(\Omega) \times H^{-1}(\Omega)$, $k = 1, ..., N$.

This fact is equivalent to the existence of a constant $C > 0$ such that

$$C \int_\Omega \int_0^T \left| \sum_{k=1}^N \sum_{n\in\mathbb{Z}_*} a_n^k e^{\frac{\lambda_n t}{\rho_k}} \theta_{|n|}(x) \right|^2 dtdx \geq 2 \sum_{k=1}^N \sum_{n\in\mathbb{N}} |a_n^k|^2, \tag{22}$$

for all the finite complex sequences $(a_n^k)_{n\in\mathbb{Z}_*}$, $k = 1, ..., N$, verifying $a_{-n}^k = \overline{a_n^k}$.

THEOREM VI.3. *The system (19) is exactly controllable in time $T > 0$ if, and only if, the numbers $\rho_1, ..., \rho_N$ are pairwise distinct.*

PROOF. If two of the numbers $\rho_1, ..., \rho_N$ coincide, say $\rho_1 = \rho_2$, and we choose

$$a_n^1 = -a_n^2, \quad a_n^k = 0, \quad k > N,$$

then the inequality (22) becomes

$$0 \geq 4 \sum_{n\in\mathbb{N}} |a_n^1|^2,$$

what is not true in general. Therefore, if two of the numbers $\rho_1, ..., \rho_N$ coincide (22) fails.

Let us observe that, due to the orthonormality in $L^2(\Omega)$ of the functions $(\theta_n)_{n\in\mathbb{N}}$, the inequality (22) may be written as

$$C \int_0^T \sum_{n\in\mathbb{N}} \left| \sum_{k=1}^N a_n^k e^{\frac{\lambda_n t}{\rho_k}} \right|^2 dt \geq \sum_{k=1}^N \sum_{n\in\mathbb{N}} |a_n^k|^2.$$

Then, for every $T > 0$ and distinct numbers $\rho_1, ..., \rho_N$ it suffices to apply Proposition VI.4 given below to obtain (22) and consequently, the proof of the theorem. $\square$

PROPOSITION VI.4. *Let $\rho_1, ..., \rho_N$ be distinct positive numbers and $(\lambda_n)_{n\in\mathbb{N}}$ a sequence of positive numbers that tends to infinite. Then, for every $T > 0$ there exists a constant $C = C(T, N, \rho_1, ..., \rho_N) > 0$ such that*

$$\int_0^T \left| \sum_{k=1}^N a^k e^{\frac{i\lambda_n t}{\rho_k}} \right|^2 dt \geq C \sum_{k=1}^N |a^k|^2,$$

for all $n \in \mathbb{N}$ and $(a^1, ..., a^N) \in \mathbb{R}^N$.

Proof. We proceed by induction with respect to the number N. For $N = 1$ the inequality is immediate. Let us suppose that the inequality is true for $N - 1$.

Let us denote

$$I_n = \sum_{k=2}^{N} a^k e^{\frac{i\lambda_n t}{\rho_k}}.$$

Then, according to our induction hypothesis, there exists a constant $C > 0$ such that, for every $n \in \mathbb{N}$,

$$\int_0^T |I_n|^2 \, dt \geq C \sum_{k=2}^{N} |a^k|^2.$$

On the other hand,

$$\int_0^T \left| \sum_{k=1}^{N} a^k e^{\frac{i\lambda_n t}{\rho_k}} \right|^2 dt = |a^1|^2 T + 2\Re\left(\int_0^T a^1 e^{\frac{i\lambda_n t}{\rho_1}} \overline{I_n} dt\right) + \int_0^T |I_n|^2 dt. \tag{23}$$

Let us observe that

$$\left| \Re \int_0^T a^1 e^{\frac{i\lambda_n t}{\rho_1}} \overline{I_n} dt \right| \leq |a^1| \left| \int_0^T e^{\frac{i\lambda_n t}{\rho_1}} \overline{I_n} dt \right| = |a^1| \left| \sum_{k=2}^{N} a^k \gamma_{n,k} dt \right|, \tag{24}$$

where

$$\gamma_{n,k} = \int_0^T e^{\left(\frac{1}{\rho_1} - \frac{1}{\rho_k}\right) i\lambda_n t} dt.$$

Besides,

$$|a^1| \left| \sum_{k=2}^{N} a^k \gamma_{n,k} \right| \leq |a^1| \sum_{k=2}^{N} |a^k| |\gamma_{n,k}| \leq \frac{1}{2} \sum_{k=2}^{N} \left(|a^1|^2 + |a^k|^2 \right) |\gamma_{n,k}|. \tag{25}$$

Combining (24) and (25) it holds

$$\left| \Re \int_0^T a^1 e^{\frac{i\lambda_n t}{\rho_1}} \overline{I_n} dt \right| \leq \frac{1}{2} \sum_{k=2}^{N} \left(|a^1|^2 + |a^k|^2 \right) |\gamma_{n,k}|,$$

which, in view of (23), implies

$$\int_0^T \left| \sum_{k=1}^{N} a^k e^{\frac{i\lambda_n t}{\rho_k}} \right|^2 dt \geq |a^1|^2 (T - \sum_{k=2}^{N} |\gamma_{n,k}|) + \sum_{k=2}^{N} |a^k|^2 (C - |\gamma_{n,k}|). \tag{26}$$

Let us observe now that[2], for every $k = 2, ..., N$,

$$|\gamma_{n,k}| \leq \frac{2|\rho_k - \rho_1|}{\rho_1 \rho_k \lambda_n} \underset{n\to\infty}{\to} 0.$$

Therefore, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\sum_{k=2}^{N} |\gamma_{n,k}| \leq \frac{T}{2}, \qquad |\gamma_{n,k}| \leq \frac{C}{2}, \quad k = 2, ..., N.$$

[2] It is precisely at this point of the proof where the condition that the numbers ρ_k, $k = 1, ..., N$, are all distinct is esential.

As a consequence of (26) it holds, for every $n \geq n_0$,

$$\int_0^T \left| \sum_{k=1}^N a^k e^{\frac{i\lambda_n t}{\rho_k}} \right|^2 dt \geq |a^1|^2 \frac{T}{2} + \frac{C}{2} \sum_{k=2}^N |a^k|^2 \geq C \sum_{k=1}^N |a^k|^2. \tag{27}$$

Finally, it suffices to note that the functions $e^{\frac{i\lambda_n t}{\rho_k}}$, $k = 1, ..., N$, are linearly independent over any interval, and thus

$$\int_0^T \left| \sum_{k=1}^N a^k e^{\frac{i\lambda_n t}{\rho_k}} \right|^2 dt > 0,$$

except when $a^1 = \cdots = a^N = 0$. This allows to apply a standard compactness argument to prove that, for every $n \in \mathbb{N}$, there exists a constant $C_n > 0$ such that

$$\int_0^T \left| \sum_{k=1}^N a^k e^{\frac{i\lambda_n t}{\rho_k}} \right|^2 dt \geq C_n \sum_{k=1}^N |a^k|^2.$$

Therefore, there exists $C > 0$ such that

$$\int_0^T \left| \sum_{k=1}^N a^k e^{\frac{i\lambda_n t}{\rho_k}} \right|^2 dt \geq C \sum_{k=1}^N |a^k|^2,$$

for every $n < n_0$. This fact, in view of (27), gives the assertion of the proposition.

$\square$

CHAPTER VII

Other equations on networks

In this chapter we study the observation and control problems for the heat, beam and Schrödinger equations on networks. We make emphasis on two main issues: the spectral observability/controllability of the corresponding systems and the possibility of identifying subspaces of controllable initial data for these equations with the aid of the information we have already obtained in previous chapters on the controllability of the system (I.11)-(I.16) of the network of strings.

The main spectral controllability result that we present asserts that whenever the system (I.11)-(I.16) is spectrally controllable in some time $T > 0$, the heat, beam and Schrödinger equations are also spectrally controllable in any time $\tau > 0$. Then, in view of Theorem V.1, the spectral controllability of those systems admits a spectral characterization: the systems are spectrally controllable in any time $\tau > 0$ if, and only if, no eigenfunction of the elliptic operator $-\Delta_G$ associated to the system (I.11)-(I.16) vanishes identically on the controlled string.

On the other hand, the possibility of describing subspaces of controllable initial data for the equations considered in this chapter from subspaces of controllable states for the system (I.11)-(I.16), and corollaries IV.5 and IV.6 for networks of strings allow, in particular, to identify subspaces of the form V^r (domains of powers of the operator $-\Delta_G$).

1. The heat equation

The following parabolic system will be called heat equation on a network:

$$u_t^i - u_{xx}^i = 0 \qquad \text{in } \mathbb{R} \times [0, \ell_i], \quad i = 1, ..., M, \tag{1}$$

$$u^1(t, \mathbf{v}_1) = h(t) \qquad t \in \mathbb{R}, \tag{2}$$

$$u^{i(j)}(t, \mathbf{v}_j) = 0 \qquad t \in \mathbb{R}, \quad j = 2, ..., N, \tag{3}$$

$$u^i(t, \mathbf{v}) = u^j(t, \mathbf{v}) \qquad t \in \mathbb{R}, \quad \mathbf{v} \in \mathcal{V}_{\mathcal{M}}, \ i, j \in I_{\mathbf{v}}, \tag{4}$$

$$\sum\nolimits_{i \in I_{\mathbf{v}}} \partial_n u^i(t, \mathbf{v}) = 0 \qquad t \in \mathbb{R}, \quad \mathbf{v} \in \mathcal{V}_{\mathcal{M}}, \tag{5}$$

$$u^i(0, x) = u_0^i(x) \qquad x \in [0, \ell_i], \quad i = 1, ..., M. \tag{6}$$

The problem (1)-(6) may be viewed as a model for the heat propagation in a network under the action of a controller on one of the exterior nodes of the network. For every $T > 0$, $h \in L^2(0, T)$ and $\bar{u}_0 = (u_0^1, ..., u_0^M) \in H$ the system (1)-(6) has a unique solution $\bar{u}$ that satisfies

$$\bar{u} \in C([0, T] : H) \cap L^2([0, T] : V).$$

When $h \equiv 0$, this solution is expressed by the formula

$$\bar{u}(t, x) = \sum_{n \in \mathbb{N}} u_{0,n} e^{-\mu_n t} \bar{\theta}_n(x),$$

if $\bar{u}_0 = \sum_{n\in\mathbb{N}} u_{0,n}\bar{\theta}_n$. Recall that $\mu_n = \lambda_n^2$ are the eigenvalues and $\bar{\theta}_n$ the eigenfunctions of the Dirichlet problem for the laplacian on the network, which is the same that corresponds to (I.11)-(I.15).

For the system (1)-(6) we consider the control problem: *determine for which initial data $\bar{u}_0 \in V'$, there exists a function $h \in L^2(0,T)$ such that the solution $\bar{u}$ of (1)-(6) satisfies*

$$\bar{u}(T,x) = \bar{0}.$$

When the initial datum $\bar{u}_0$ has this property it is said that $\bar{u}_0$ is *controllable to zero in time T*. If all the initial data $\bar{u}_0 \in Z$ are controllable to zero in time T (as before, Z is the set of all the finite lineal combinations of the eigenfunctions), we will say that the system (1)-(6) is spectrally controllable in time T.

Let us observe that, unlike it happens for the wave equation, or more general, for time-reversible equations, the fact that $\bar{u}_0$ and $\bar{u}_1$ are controllable to zero does not imply the existence of a function $h \in L^2(0,T)$ such that the solution of (1)-(6) with initial datum $\bar{u}_0$ coincides with $\bar{u}_1$ in time T. This is caused by the lack of time reversibility of the heat equation: due to the dissipative character of the heat operator, the solutions with initial data $\bar{u}_0 \in H$ satisfy $u^i(T) \in C^\infty((0,\ell_i))$ for $i = 2, ..., M$. Thus, only very smooth states of the system may be reached.

Proceeding as in the case of the wave equation, we obtain the following criterion of the controllability to zero of an initial datum:

Proposition VII.1. *The initial datum $\bar{u}_0 \in H$ is controllable to zero in time T with control $h \in L^2(0,T)$ if, and only if, for every $\bar{\phi}_0 \in Z$ the following inequality holds*

$$\langle \bar{u}_0, \bar{\phi}(T)\rangle_H = \int_0^T h(t)\partial_n\phi^1(T-t,\mathbf{v}_1)dt,$$

where $\bar{\phi}$ is the solution of the homogeneous system (1)-(6) with initial datum $\bar{\varphi}_0$.

Clearly, it is sufficient to check the equality of the previous proposition when $\bar{\phi}_0$ is one of the eigenfunctions. That is,

$$\int_0^T \varkappa_k e^{-\mu_k(T-t)}h(t)dt = u_{0,k}e^{-\mu_k T}, \qquad k \in \mathbb{N}, \tag{7}$$

where $\varkappa_k = \partial_n\theta_k^1(\mathbf{v}_1)$ is the normal derivative of the eigenfunction $\bar{\theta}_k$ at the controlled node $\mathbf{v}_1$.

After performing the change of variable $t \to \frac{T}{2} - t$, the control problem may be written equivalently as the following problem of moments:

Proposition VII.2. *The initial datum $\bar{u}_0 = \sum_{n\in\mathbb{N}} u_{0,n}\bar{\theta}_n \in H$ is controllable to zero in time T with control $h \in L^2(0,T)$ if, and only if, the following inequalities are satisfied*

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \varkappa_k e^{-\mu_k t}h(t)dt = u_{0,k}e^{-\mu_n\frac{T}{2}}, \qquad k \in \mathbb{N}. \tag{8}$$

This proposition allows to give the following characterization of the networks for which the system (1)-(6) is spectrally controllable.

Theorem VII.1. *The system (1)-(6) is spectrally controllable to zero in any time $T > 0$ if, and only if, $\varkappa_k \neq 0$ for every $k \in \mathbb{N}$.*

PROOF. The necessity of the condition $\varkappa_k \neq 0$ is immediate: if $\varkappa_k = 0$ for some value of k then, the equality (8) becomes

$$u_{0,k} = 0.$$

Consequently, it will not be possible to control an initial datum $\bar{u}_0 = \bar{\theta}_k \in Z$ with $u_{0,k} = 1$.

In order to prove that the condition $\varkappa_k \neq 0$ for every $k \in \mathbb{N}$ is sufficient for the spectral controllability to zero of the system (1)-(6), it is enough to prove that for every $T > 0$ there exists a sequence (w_n), which is biorthogonal to $(\varkappa_k e^{-\mu_k t})$ in $L^2(-\frac{T}{2}, \frac{T}{2})$.

According to Theorem V.1 the system (I.11)-(I.16) is spectrally controllable in time $T = 2L$ (recall that L is the total length of the graph). Then, using Proposition II.8, there exists a sequence $(v_n)_{n\in\mathbb{Z}_*}$ biorthogonal to $\left(\varkappa_k e^{i\lambda_k t}\right)$ in $L^2(-L, L)$.

From Theorem II.3 we conclude that for every $T > 0$ there exists a sequence (w_n) biorthogonal to $(\varkappa_k e^{-\mu_k t})$ in $L^2(-\frac{T}{2}, \frac{T}{2})$. $\square$

REMARK VII.1. *When the network graph is a tree, the condition $\varkappa_k \neq 0$ for all $k \in \mathbb{N}$ coincides with the fact that any two sub-trees with common root have disjoint spectra. Recall that this is the non-degeneracy condition introduced in Chapter IV.*

Following the procedure introduced by Russell in [**73**], it is possible to obtain additional information on the controllability of the system (1)-(6) as a consequence of the controllability of subspaces of initial states of the form $\mathcal{W}^r$ for the network of strings:

PROPOSITION VII.3. *If the subspace $\mathcal{W}^r$ is controllable for the system (I.11)-(I.16) in time $T > 0$ then all the initial data $\bar{u}_0 \in H$ are controllable to zero in any time $\tau > 0$ for the system (1)-(6).*

PROOF. According to Proposition II.9, if $\mathcal{W}^r$ is controllable for the system (I.11)-(I.16) in time $T > 0$ then there exists a sequence $(v_n)_{n\in\mathbb{Z}_*}$ biorthogonal to $\left(\varkappa_k e^{i\lambda_k t}\right)$ in $L^2(-\frac{T}{2}, \frac{T}{2})$. Besides, there exists a constant $C > 0$ such that for every $n \in \mathbb{Z}_*$, the sequence (v_n) satisfies

$$\|v_n\|_{L^2(-\frac{T}{2},\frac{T}{2})} \leq C\lambda_n^{r-1}. \tag{9}$$

From Russell's Theorem II.3 we obtain that for every $\tau > 0$ there exists a sequence (w_n) biorthogonal to $(\varkappa_k e^{-\mu_k t})$ in $L^2(-\frac{\tau}{2}, \frac{\tau}{2})$, for which there exist positive constants C_τ and γ such that

$$\|w_n\|_{L^2(-\frac{\tau}{2},\frac{\tau}{2})} \leq \|v_n\|_{L^2(-\frac{T}{2},\frac{T}{2})} e^{\gamma\lambda_n}, \tag{10}$$

for every $n \in \mathbb{N}$.

In view of (9), (10) we obtain

$$\|w_n\|_{L^2(-\frac{\tau}{2},\frac{\tau}{2})} \leq C\lambda_n^{r-1} e^{\gamma\lambda_n}. \tag{11}$$

Finally, applying Proposition II.6 to the problem of moments (8) it follows that all the initial data $\bar{u}_0 \in H$ satisfying

$$\sum_{n\in\mathbb{N}} \left| u_{0,n} e^{-\mu_n \frac{T}{2}} \right| \|w_n\|_{L^2(-\frac{\tau}{2},\frac{\tau}{2})} < \infty \tag{12}$$

are controllable.

In view of (11), after applying the Cauchy-Schwarz inequality we obtain that the convergence (12) is true if

$$\left(\sum_{n\in\mathbb{N}} |u_{0,n}|^2\right)\left(\sum_{n\in\mathbb{N}} \lambda_n^{2r-2} e^{2\gamma\lambda_n-\mu_n T}\right) < \infty.$$

Since $\mu_n = \lambda_n^2$ and $\lambda_n \to \infty$, the series

$$\sum_{n\in\mathbb{N}} \lambda_n^{2r-2} e^{2\gamma\lambda_n-\mu_n T}$$

is convergent for any $r \in \mathbb{R}$ and then, all the initial data, which verify

$$\sum_{n\in\mathbb{N}} |u_{0,n}|^2 < \infty;$$

are controllable. And that is, any $\bar{u}_0 \in H$ is controllable. □

Theorem VII.1 and Proposition VII.3 allows to immediately obtain information on the controllability of the heat equation on the star-shaped networks studied in Section 8 of Chapter IV, from the corollaries IV.5 and IV.6.

Corollary VII.1. *If the lengths $\ell_1, ..., \ell_{n-1}$ of the uncontrolled edges of the star-shaped network are such that*

1) *all the ratios $\frac{\ell_i}{\ell_j}$ with $i \neq j$ are irrational numbers, then the system (1)-(6) is spectrally controllable to zero in any time $T > 0$.*
2) *all the ratios $\frac{\ell_i}{\ell_j}$ with $i \neq j$ belong to some set $\mathbf{B}_\varepsilon$ then all the initial data $\bar{u}_0 \in H$ are controllable to zero in any time $T > 0$.*

2. Schrödinger equation

Let us consider the Schrödinger system on the network:

$$iu_t^k - u_{xx}^k = 0 \qquad \text{in } \mathbb{R}\times[0,\ell_k], \quad k = 1, ..., M, \tag{13}$$

$$u^1(t,\mathbf{v}_1) = h(t) \qquad t \in \mathbb{R}, \tag{14}$$

$$u^{k(j)}(t,\mathbf{v}_j) = 0 \qquad t \in \mathbb{R}, \quad j = 2, ..., N, \tag{15}$$

$$u^k(t,\mathbf{v}) = u^j(t,\mathbf{v}) \qquad t \in \mathbb{R}, \quad \mathbf{v} \in \mathcal{V}_{\mathcal{M}}, \ k, j \in k_{\mathbf{v}}, \tag{16}$$

$$\textstyle\sum_{k\in k_{\mathbf{v}}} \partial_n u^k(t,\mathbf{v}) = 0 \qquad t \in \mathbb{R}, \quad \mathbf{v} \in \mathcal{V}_{\mathcal{M}}, \tag{17}$$

$$u^k(0,x) = u_0^k(x) \qquad x \in [0,\ell_k], \quad k = 1, ..., M. \tag{18}$$

For every $T > 0$ and $\bar{\phi}_0 = (\phi_0^1, ..., \phi_0^M) \in V$ the homogeneous version of the system (13)-(18) with $h \equiv 0$ has a unique solution $\bar{\phi}$, which is expressed by the formula

$$\bar{\phi}(t,x) = \sum_{n\in\mathbb{N}} \phi_{0,n} e^{i\mu_n t} \bar{\theta}_n(x), \tag{19}$$

if $\bar{\phi}_0 = \sum_{n\in\mathbb{N}} \phi_{0,n}\bar{\theta}_n$. Once again, $\mu_n = \lambda_n^2$ are the eigenvalues and $\bar{\theta}_n$ are the eigenfunctions of the Dirichlet problem for the laplacian on the network. The homogeneous system (13)-(18) is well posed in any of the spaces V^r; the solution is also given by (19).

The non-homogeneous system (13)-(18) is well posed for any $T > 0$, $h \in L^2(0,T)$ and initial datum $\bar{u}_0 \in V'$: there exists a unique solution $\bar{u}$ of (13)-(18) satisfying

$$\bar{u} \in C([0,T] : V').$$

For the system (13)-(18) we consider the control problem: *determine for which initial data* $\bar{u}_0 \in V'$ *there exists a function* $h \in L^2(0,T)$ *such that the solution* $\bar{u}$ *of (13)-(18) satisfies*

$$\bar{u}(T,x) = \bar{0}.$$

When this is possible, it is said that *the initial datum* $\bar{u}_0 \in V'$ *is controllable in time* T.

The following proposition provides a characterization of the initial data that are controllable in time T.

Proposition VII.4. *The initial datum* $\bar{u}_0 \in V'$ *is controllable in time* T *with control* $h \in L^2(0,T)$ *if, and only if, for every* $\bar{\phi}_0 \in Z$ *the following inequality is satisfied*

$$i\langle \bar{u}_0, \overline{\bar{\phi}_0} \rangle_H = \int_0^T h(t)\overline{\partial_n \phi^1}(t, \mathbf{v}_1)dt,$$

where $\bar{\phi}$ *is the solution of the homogeneous system (13)-(18) with initial datum* $\bar{\phi}_0$.

This characterization may be written as a problem of moments:

Proposition VII.5. *The initial datum* $\bar{u}_0 = \sum_{n\in\mathbb{N}} u_{0,n}\bar{\theta}_n \in V'$ *is controllable in time* T *with control* $h \in L^2(0,T)$ *if, and only if, the following equalities are verified*

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \varkappa_n e^{-i\mu_n t} h(t)dt = u_{0,n} e^{-i\mu_n \frac{T}{2}}, \qquad n \in \mathbb{N}. \tag{20}$$

On the other hand, the technique of HUM allow us to give an alternative characterization:

Proposition VII.6. *There exist* $T > 0$ *and a sequence* $(c_n)_{n\in\mathbb{N}}$ *of positive numbers such that the inequality*

$$\int_0^T \left|\partial_n \phi^1(t, \mathbf{v}_1)\right|^2 dt \geq \sum_{k\in\mathbb{N}} c_k^2 \left|\phi_{0,k}\right|^2, \tag{21}$$

is verified by every solution $\bar{\phi}$ *of the homogeneous system (13)-(18) with initial datum* $\bar{\phi}_0 \in Z$*, or equivalently,*

$$\int_0^T \left|\sum_{n\in\mathbb{N}} \varkappa_n a_n e^{i\mu_n t}\right|^2 dt \geq \sum_{n\in\mathbb{N}} c_n^2 \left|a_n\right|^2, \tag{22}$$

for every finite sequence (a_n) *of complex numbers, if, and only if, the space*

$$\mathcal{W} = \left\{ \bar{u}_0 = \sum_{n\in\mathbb{N}} u_{0,n}\bar{\theta}_n \in V' : \sum_{n\in\mathbb{N}} \frac{1}{c_n^2} \left|u_{0,n}\right|^2 < \infty \right\}$$

is controllable in time T.

Theorem VII.2. *The system (13)-(18) is spectrally controllable in any time* $T > 0$ *if, and only if,* $\varkappa_n \neq 0$ *for every* $n \in \mathbb{N}$.

Proof. The necessity of the condition $\varkappa_n \neq 0$ is immediate: if $\varkappa_n = 0$ for some value of n then the equality of Proposition VII.5 becomes $u_{0,n} = 0$. Consequently, it is not possible to control an initial datum $\bar{u}_0 = \bar{\theta}_n \in Z$ with $u_{0,n} = 1$.

The proof of the sufficiency can be obtained from VII.5. The key element is provided by Proposition V.2, which ensures that

$$\lim_{n\to\infty} \frac{\mu_n}{n^2} = \frac{\pi^2}{L^2}.$$

This implies that the sequence (μ_n) satisfies

$$\sum_{n\in\mathbb{N}} \frac{1}{\mu_n} < \infty, \tag{23}$$

As it has been pointed out in the section 3 of Chapter II, the problem of moments (20) has a solution for any finite sequence

$$m_n = \frac{1}{\varkappa_n} u_{0,n} e^{-i\mu_n \frac{T}{2}}, \qquad n \in \mathbb{N}$$

if it is possible to find a biorthogonal sequence to $\left(e^{i\mu_n t}\right)$.

But the property (23) guarantees that for every $\tau > 0$ there exists a non-trivial entire function of exponential type at most τ, vanishing at every μ_n (see, e.g., Theorem 15, p. 139 in [**81**]). Then, for every $\tau > 0$ there exists a sequence biorthogonal to $\left(e^{i\mu_n t}\right)$ in $L^2(-\tau, \tau)$. □

Remark VII.2. *It is also possible to give a proof of the sufficiency of the condition $\varkappa_n \neq 0$ for every $n \in \mathbb{N}$ for the spectral controllability of the system (13)-(18) using Proposition VII.6. Indeed, as the sequence (λ_n) has finite upper density $D^+(\lambda_n)$, then $D^+(\mu_n) = 0$. If we apply Corollary II.5 of Theorem II.6 it follows that for every $T > 0$ there exist positive numbers γ_n, $n \in \mathbb{N}$, such that*

$$\int_0^T \left| \sum_{n\in\mathbb{N}} \varkappa_n a_n e^{i\mu_n t} \right|^2 dt \geq \sum_{n\in\mathbb{N}} \varkappa_n^2 \gamma_n^2 \left|a_n\right|^2,$$

for any finite sequence (a_n) of complex numbers. This is the inequality (21) with $c_n = \varkappa_n \gamma_n$; all these coefficients are positive if $\varkappa_n \neq 0$ for every $n \in \mathbb{N}$.

Corollary VII.2. *For every $T > 0$ the properties of the system (13)-(18):*

– unique continuation from the controlled node of the solutions of the homogeneous system:

$$\partial_n \phi^1(., \mathbf{v}_1) = 0 \quad \text{in } L^2(0;T) \quad \text{implies} \quad \bar{\phi}_0 = \bar{0};$$

– spectral unique continuation from the controlled node:

$$\varkappa_n \neq 0 \quad \text{for every } n \in \mathbb{N};$$

are equivalent.

Like in the case of the heat equation, for the system (13)-(18) it is possible to describe subspaces of controllable initial data based on similar information for the wave equation.

Proposition VII.7. *If the subspace $\mathcal{W}^r$ is controllable for the system (I.11)-(I.16) in some time $T > 0$ then all the initial data $\bar{u}_0 \in V^{2r-1}$ are controllable in any time $\tau > 0$ for the system (13)-(18).*

Proof. According to Remark II.6, if the subspace $\mathcal{W}^r$ is controllable for the system (I.11)-(I.16) in time $T>0$ then the inequality

$$\int_0^T \left|\sum_{n\in\mathbb{N}} \varkappa_n a_n e^{i\lambda_n t}\right|^2 dt \geq C \sum_{n\in\mathbb{N}} \lambda_n^{2(1-r)} |a_n|^2, \tag{24}$$

is valid for any finite complex sequence (a_n).

Let us observe that if $\mathcal{W}^r$ is controllable in time T for the system (I.11)-(I.16) then (I.11)-(I.16) is spectrally controllable and thus $\varkappa_n \neq 0$, $n\in\mathbb{N}$. Besides, proceeding as in Remark III.4, from (24) it follows that there exist constants $C_1, C_2 > 0$ such that for every $n \in \mathbb{N}$,

$$C_1\lambda_n^{1-r} \leq |\varkappa_n| \leq C_2\lambda_n. \tag{25}$$

Then the inequality (24) can be written in the equivalent form

$$\int_0^T \left|\sum_{n\in\mathbb{N}} a_n e^{i\lambda_n t}\right|^2 dt \geq C \sum_{n\in\mathbb{N}} \lambda_n^{2(1-r)} |a_n|^2 |\varkappa_n|^{-2}.$$

and from (25) we get

$$\int_0^T \left|\sum_{n\in\mathbb{N}} a_n e^{i\lambda_n t}\right|^2 dt \geq C \sum_{n\in\mathbb{N}} \lambda_n^{-2r} |a_n|^2. \tag{26}$$

If we apply Theorem II.7 to the inequality (26) we obtain

$$\int_0^T \left|\sum_{n\in\mathbb{N}} a_n e^{i\mu_n t}\right|^2 dt \geq C \sum_{n\in\mathbb{N}} \lambda_n^{-2r} |a_n|^2. \tag{27}$$

In view of (25), from (27) it follows

$$\int_0^T \left|\sum_{n\in\mathbb{N}} \varkappa_n a_n e^{i\mu_n t}\right|^2 dt \geq C \sum_{n\in\mathbb{N}} \lambda_n^{-2r} |a_n|^2 |\varkappa_n|^2 \geq C \sum_{n\in\mathbb{N}} \lambda_n^{2(1-2r)} |a_n|^2. \tag{28}$$

Now it suffices to note that, according to Proposition VII.6, the fact that the inequality (28) is valid for any finite complex sequence (a_n) is equivalent to the fact that all the initial data $\bar{u}_0 \in V^{2r-1}$ are controllable in time $\tau > 0$ for the system (13)-(18). $\square$

Theorem VII.2 and Proposition VII.7 allow to obtain immediate information for the Schrödinger equation on the star-shaped networks studied in the section 8 of Chapter IV, using the corollaries IV.5, IV.6 and IV.7.

Corollary VII.3. *If the lengths $\ell_1, ..., \ell_{n-1}$ of the uncontrolled edges of a star-shaped network are such that*

1) *all ratios $\frac{\ell_i}{\ell_j}$ with $i \neq j$ are irrational numbers, then the system (13)-(18) is spectrally controllable in any time $T > 0$.*
2) *all the ratios $\frac{\ell_i}{\ell_j}$ with $i \neq j$ belong to some set $\mathbf{B}_\varepsilon$, then the subspace $V^{2n-4+\varepsilon}$ of initial data for the system (13)-(18) is controllable in any time $T > 0$.*
3) *verify the conditions* (S), *then for every $\varepsilon > 0$ the subspace $V^{1+\varepsilon}$ of initial data for the system (13)-(18) is controllable in any time $T > 0$.*

REMARK VII.3. *All the results of this section are valid for the system obtained by replacing the equation (13) by* $iu_t^k + u_{xx}^k = 0$. *In this case, the corresponding observability inequality is*

$$\int_0^T \left| \sum_{n\in\mathbb{N}} \varkappa_n a_n e^{-i\mu_n t} \right|^2 dt \geq \sum_{n\in\mathbb{N}} c_n^2 \left|a_n\right|^2,$$

which, clearly, coincides with (22).

3. A model of network of beams

Now we will consider the following model of a network of flexible beams controlled from one exterior node.

(29)
$$u_{tt}^i + u_{xxxx}^i = 0 \qquad \text{in } \mathbb{R}\times[0,\ell_i], \quad i=1,...,M,$$
(30)
$$u^1(t,\mathbf{v}_1) = 0, \qquad \partial_n^2 u^1(t,\mathbf{v}_1) = h(t) \qquad t\in\mathbb{R},$$
(31)
$$u^{i(j)}(t,\mathbf{v}_j) = 0 \qquad t\in\mathbb{R}, \quad j=2,...,N,$$
(32)
$$u^i(t,\mathbf{v}) = u^j(t,\mathbf{v}), \qquad \partial_n^2 u^i(t,\mathbf{v}) = \partial_n^2 u^j(t,\mathbf{v}) \quad t\in\mathbb{R}, \quad \mathbf{v}\in\mathcal{V}_{\mathcal{M}},\ i,j\in I_{\mathbf{v}},$$
(33)
$$\sum\nolimits_{i\in I_{\mathbf{v}}} \partial_n u^i(t,\mathbf{v}) = \sum_{i\in I_{\mathbf{v}}} \partial_n^3 u^i(t,\mathbf{v}) = 0 \qquad t\in\mathbb{R}, \quad \mathbf{v}\in\mathcal{V}_{\mathcal{M}},$$
(34)
$$u^i(0,x) = u_0^i(x), \qquad u_t^i(0,x) = u_1^i(x) \qquad x\in[0,\ell_i], \quad i=1,...,M.$$

Let us observe that in this case the control acts through the normal derivative $\partial_n^2 u^1(.,\mathbf{v}_1)$ at the node $\mathbf{v}_1$.

The system (29)-(34) is well posed for $h\in L^2(0,T)$ and $\bar{u}_0\in V$, $\bar{u}_1\in V'$. The homogeneous version of (29)-(34) is also well posed for $\bar{u}_0\in V^2$, $\bar{u}_1\in H$.

We study the following control problem in time T for the system (29)-(34): *determine for which initial states* $(\bar{u}_0,\bar{u}_1)\in V\times V'$, *there exists* $h\in L^2(0,T)$ *such that the corresponding solution* $\bar{u}$ *of (29)-(34) satisfies*

$$\bar{u}(T,.) = \bar{u}_t(T,.) = \bar{0}.$$

Those initial states $(\bar{u}_0,\bar{u}_1)$ for which such a function h exists will be called *controllable in time* T. We will say that a subspace of $V\times V'$ is controllable in time T is so are all of its elements. In particular, if $Z\times Z$ is controllable in time T, we will say that the system (29)-(34) is *spectrally controllable in time* T.

Let us remark that Z denotes as previously the space of all the finite linear combinations of the eigenfunctions of the operator $\mathcal{D}_G$ associated to (29)-(34). This is the operator $\mathcal{D}_G : H\to H$ defined by

$$\mathcal{D}_G(u^1,...,u^M) = (u_{xxxx}^1,...,u_{xxxx}^M)$$

with the boundary conditions

$$u^{i(\mathbf{v})}(\mathbf{v}) = \partial_n^2 u^{i(\mathbf{v})}(\mathbf{v}) = 0,$$

at the exterior nodes and

$$u^i(\mathbf{v}) = u^j(\mathbf{v}), \qquad \partial_n^2 u^i(\mathbf{v}) = \partial_n^2 u^j(\mathbf{v}) \qquad i, j \in I_{\mathbf{v}},$$

$$\sum_{i\in I_{\mathbf{v}}} \partial_n u^i(\mathbf{v}) = \sum_{i\in I_{\mathbf{v}}} \partial_n^3 u^i(\mathbf{v}) = 0$$

at the interior ones..

The operator $\mathcal{D}_G$ coincides with the square of the elliptic operator $-\Delta_G$ associated to the problem (I.11)-(I.16). By this reason, the eigenfunctions of $\mathcal{D}_G$ coincide with the eigenfunctions of $(\bar\theta_n)$ de $-\Delta_G$ and the eigenvalues are $\left(\mu_n^2\right)$. In particular, the space Z for the equation (29)-(34) coincides with that of the equation (I.11)-(I.16). Besides, the solution of the homogeneous system (29)-(34) with initial data

$$\bar\phi_0 = \sum_{n\in\mathbb{N}} \phi_{0,n}\bar\theta_n, \qquad \bar\phi_1 = \sum_{n\in\mathbb{N}} \phi_{1,n}\bar\theta_n,$$

is expressed by the formula

$$\bar\phi(t,x) = \sum_{n\in\mathbb{N}} \left(\phi_{0,n}\cos\mu_n t + \frac{\phi_{1,n}}{\mu_n}\sin\mu_n t\right)\bar\theta_n(x).$$

Proposition VII.8. *The initial state $(\bar u_0, \bar u_1) \in V \times V'$ is controllable in time T with control $h \in L^2(0,T)$ if, and only if, for every $(\bar\phi_0, \bar\phi_1) \in Z \times Z$ the following equality is true*

$$\langle\bar\phi_1, \bar u_0\rangle_{V'\times V} - \langle\bar u_1, \bar\phi_0\rangle_{V'\times V} = \int_0^T h(t)\partial_n\phi^1(t,\mathbf{v}_1)dt, \tag{35}$$

where $\bar\phi$ is the solution of the homogeneous system (29)-(34) with initial state $(\bar\phi_0, \bar\phi_1)$.

Clearly, it is sufficient to check the equality (35) for the initial states of the form $(\bar 0, \bar\theta_n)$ and $(\bar\theta_n, \bar 0)$, $n \in \mathbb{N}$. Then, if we define $\mu_n = -\mu_{-n}$ for $n < 0$, Proposition VII.8 gives rise to a problem of moments:

Proposition VII.9. *The initial state $(\bar u_0, \bar u_1) \in V \times V'$ is controllable in time T with control $h \in L^2(0,T)$ if, and only the equalities*

$$\int_0^T \varkappa_{|n|} h(t) e^{i\mu_n t} dt = u_{1,|n|} - i\mu_n u_{0,|n|}, \tag{36}$$

are verified for every $n \in \mathbb{Z}_*$.

Let us observe that the problem of moments (36) coincides with the problem of moments for the Schrödinger equation, except by the fact that now the sequence $(\mu_n)_{n\in\mathbb{N}}$ should be replaced by $(\mu_n)_{n\in\mathbb{Z}_*} = (\pm\mu_n)_{n\in\mathbb{N}}$. That is why, proceeding as in the proof of Theorem VII.2 it is possible to prove

Theorem VII.3. *The system (29)-(34) is spectrally controllable in any time $T > 0$ if, and only if, $\varkappa_n \neq 0$ for every $n \in \mathbb{N}$.*

On the other hand, the technique of HUM allows us to obtain from Proposition VII.8

Proposition VII.10. *There exist $T > 0$ and a sequence $(c_n)_{n\in\mathbb{N}}$ of positive numbers such that the following inequality is verified*

$$\int_0^T \left|\partial_n \phi^1(t, \mathbf{v}_1)\right|^2 dt \geq \sum_{k\in\mathbb{N}} c_k^2 \left(\mu_n^2 \phi_{0,k}^2 + \phi_{1,k}^2\right), \tag{37}$$

by every solution $\bar{\phi}$ of the homogeneous system (29)-(34) with initial state $(\bar{\phi}_0, \bar{\phi}_1) \in Z \times Z$, or equivalently, the inequality

$$\int_0^T \left|\sum_{n\in\mathbb{Z}_*} \varkappa_{|n|} a_n e^{i\mu_n t}\right|^2 dt \geq \sum_{n\in\mathbb{N}} c_n^2 \left|a_n\right|^2, \tag{38}$$

is valid for every finite sequence of complex numbers (a_n) verifying $a_{-n} = \overline{a_n}$ if, and only if, the space

$$\mathcal{W} = \left\{(\bar{u}_0, \bar{u}_1) \in V \times V' : \sum_{n\in\mathbb{N}} \left(\frac{1}{c_n^2} u_{0,n}^2 + \frac{1}{c_n^2 \mu_n^2} u_{1,n}^2\right) < \infty\right\}$$

is controllable in time T.

Once again, it is possible to identify subspaces of controllable initial states for system (29)-(34) based on the similar information for the system (I.11)-(I.16).

Proposition VII.11. *If the subspace $\mathcal{W}^r$ is controllable for the system (I.11)-(I.16) in time $T > 0$ then all the initial states $(\bar{u}_0, \bar{u}_1) \in V^{2r+1} \times V^{2r-1}$ are controllable in any time $\tau > 0$ for the system (29)-(34).*

As a consequence of the previous result we obtain for star-shaped networks:

Corollary VII.4. *If the lengths $\ell_1, ..., \ell_{n-1}$ of the uncontrolled edges of a star-shaped network are such that*

1) *all the ratios $\frac{\ell_i}{\ell_j}$ with $i \neq j$ are irrational numbers, then the system (29)-(34) is spectrally controllable in any time $T > 0$.*
2) *all the ratios $\frac{\ell_i}{\ell_j}$ with $i \neq j$ belong to some set $\mathbf{B}_\varepsilon$, then the subspace $V^{2n-2+\varepsilon} \times V^{2n-4+\varepsilon}$ of initial states for the system (29)-(34) is controllable in any time $T > 0$.*
3) *verify the conditions* (S), *then for every $\varepsilon > 0$ the subspace $V^{3+\varepsilon} \times V^{1+\varepsilon}$ of initial states for the system (29)-(34) is controllable in any time $T > 0$.*

CHAPTER VIII

Final remarks and open problems

1. Brief description of the main results presented in this book

1.1. Networks of strings. The main result on the spectral controllability of arbitrary networks from an exterior node is given in Theorem V.1: *the network is spectrally controllable in some finite time if and only if the spectral unique continuation property from the controlled node is verified. Besides, when the spectral unique continuation property holds, the network is spectrally controllable in any time larger than twice the total length of the network; that is the minimal time allowing the spectral control.* From this point of view, the networks of strings behave, essentially, as a single string whose length coincides with the total length of the network. The main reason is that the sequence of eigenvalues of the network is asymptotically equivalent to the sequence of eigenvalues of a string with that length (Proposition V.2).

The main difference between those cases consists in that, for a string the spectral unique continuation property is always verified, while for any non-trivial topological configuration of the network there exists values of the lengths of the strings such that the spectral unique continuation fails. This leads to the fact that, in spite of the exact controllability of strings, the exact controllability of networks is never reached (Theorem I.2). In this sense, the exterior control of a network is analogous to the control of a string from an interior point.

The spectral controllability property allows to ensure the controllability of a subspace of initial states that may be explicitly described in terms of the eigenfunctions of the network and the values of the eigenfunctions at the controlled node (Remark V.1).

For networks with simple topological configurations it is possible to provide more precise information:

Tree-shaped networks. When the network graph is a tree it is possible to obtain a complete characterization of those trees for which the spectral unique continuation property is verified (Proposition IV.13) and thus, to characterize the trees which are spectrally controllable in a time equal to twice the total length of the network. The set of trees with a given topological configuration for which the spectral unique continuation fails has null Lebesgue measure (Proposition IV.15). Though these results could be obtained from Theorem V.1, the technique used in Chapter IV, essentially based on the representation of the solutions of the wave equation by means of the d'Alembert formula, allows to prove the spectral controllability in the minimal time (Theorem IV.2). Besides, it provides a weighted observability inequality with weights that are explicitly computed in terms of the eigenvalues.

Some of the results are of independent interest. That is the case, for instance, of the compatibility conditions $\mathcal{P}u_t(.,\mathbf{v}) + \mathcal{Q}u_x(.,\mathbf{v}) = 0$ at the controlled node

(Proposition IV.5). From those conditions it is possible to obtain an equation for the eigenvalues (Proposition IV.11) and the pseudo-periodicity property of the solutions of the homogeneous system (Remark IV.3), which implies that increasing the control time does not lead to improving the controllability results (Proposition IV.4).

Star-shaped networks. The star-shaped networks are a particular case of trees and then, the results of Theorem IV.3. In this case, the spectral unique continuation condition means that the ratios of the lengths of the uncontrolled strings are irrational numbers (Section IV.8.1). Besides, it is possible to identify subspaces of controllable initial states of the form $\mathcal{W}^r$ (which are, essentially, Sobolev spaces on the strings with appropriate boundary and coupling conditions at the multiple node). The existence of such subspaces depends on rational approximation properties of the ratios of the lengths.

For these networks it is possible to prove that when the observation time is smaller than twice the total lengths of the network, the unique continuation property from the controlled node fails for the solutions of the corresponding homogeneous system. This implies not only the lack of spectral controllability, but also of approximate. In Section III.9 we have given an example of a smooth solution of the homogeneous system for which the unique continuation property is not true when the observation time is small.

1.2. Simultaneous control of strings. Simultaneous control of trees from one exterior node. The results concerning the simultaneous control from an exterior node of a finite number of tree-shaped networks (Corollary IV.4) are similar to those corresponding to a single tree: *the networks are simultaneously spectrally controllable in some finite time, if and only if each of them is spectrally controllable and the spectra of the networks are pairwise disjoint. The minimal time to simultaneously control the system is the sum of the minimal control times of all the networks.*

When the networks are simultaneously spectrally controllable it is possible to indicate subspaces of controllable initial states, which are explicitly defined in terms of the eigenvalues of the networks (Proposition IV.16). In particular, if the simultaneously controlled networks are strings, then for certain values of the lengths of the strings it is possible to indicate Sobolev-type spaces of controllable initial states (Corollaries IV.8 and IV.9). Besides, it is proved (Corollary III.3) that, depending on the lengths of the strings, the space of controllable initial states may be arbitrarily small, that is, there exist initial states with Fourier coefficients increasing arbitrarily rapidly, which are not controllable in any finite time.

Control at all the nodes with only four functions. Using the results for the simultaneous control of strings, it is possible to solve the problem on how many different controls are necessary to reach the spectral controllability of the network. A simple application of the Four Colors Theorem allows to ensure that, under certain irrationality conditions of the lengths of the strings, four colors are sufficient (besides, one of them may be chosen identically equal to zero) to control the network in a time T^* (and then in any time larger than T^*), which is smaller than twice the total lengths of the network (Proposition V.5).

Simultaneous interior control of strings. The simultaneous control from an open set of two strings with different densities turns out to be a much more robust property: the networks are simultaneously exactly controllable in any time larger than the characteristic times of both strings (Corollary VI.1). However, when the string are of the same density, the results are completely analogous to those obtained for the simultaneous control of two string from one extreme (Corollaries VI.2 and VI.3).

1.3. Other equations on networks. For Shrödinger (Theorem VII.2), heat (Theorem VII.1) and beam (Theorem VII.3) equations the spectral unique continuation property from the controlled node is necessary and sufficient for the system to be spectrally controllable in any arbitrarily small time.

When we know spaces of controllable initial states for the wave equation of the type $\mathcal{W}^r$ (which is the case, for instance, of the star-shaped networks), then it is possible to identify subspaces of controllable initial data for the Schrödinger, heat and beam equations on that network (Propositions VII.3, VII.7, VII.11, respectively). In particular, the heat equation is exactly controllable.

2. Future lines of research and open problems

The context of multi-structures is extremely reach and provides a large number of problem of quite complex mathematical nature. In most cases, basic issues as the existence and uniqueness of the solutions remain open. The mathematical study of these problem will necessarily lead to new, more powerful mathematical tools.

In our opinion, the future development in connection with the problems addressed in this book should follow three main lines:

1.– Study of more complex models of string of beam networks, which provide a more realistic description of the motion of these objects and, in particular, take into account their three-dimensional character. The book [**51**] provides a valuable source of models of multi-body structures. We also refer to the article [**48**] for an account of the main developments in this field.

2.– Study of equations with variable coefficient on graphs. In view of the results in [**62**] and [**15**], we hope that the techniques used in Chapters V and VII may be adapted to this case.

3.– Study of systems of coupled multi-dimensional objects. This may be the case of networks as well as systems coupled through the boundary conditions.

All these problems required both a theoretical and numerical analysis.

To be more precise, we also mention some open problems which are directly related to the problems considered along this book.

1.– To study the observability properties of the wave equation with a potential

$$u_{tt} - u_{xx} - \alpha(x)u = 0$$

on graphs.

This problem has been recently considered in [**44**] in the case of simultaneous observation of strings for constant α. In that paper, using the generalized Ingham theorem II.4, it is proved that, generically in the set of all possible lengths of the strings, the unique continuation property and the spectral controllability hold. However, it would be interesting to describe the observed norm in terms of Sobolev spaces as it has been done for the case $\alpha = 0$. The case of variable α is a completely open question.

2.– Wave equation with variable coefficients

$$\rho(x)u_{tt} - (\sigma(x)u_x)_x = 0.$$

The results from the Beurling-Malliavin Theorem are expected to be still true, but the propagation arguments to apply them to the network are much more complex. We can try the usual trick of performing the change of variable to reduce the problem to an equation of the form

$$v_{tt} - v_{xx} + a(x)v = 0.$$

But then, we return to the difficulty mentioned in problem 1. In any case, we know that the coefficients should be at least of class BV. A smaller degree of smoothness is not sufficient even in the case of one string.

3.– When the equations of the network are of the type

$$u_{tt} - u_{xx} - \alpha(x,t)u = 0$$

the situation appears to be much more complex and the results we have presented here are not expected to be easily extended to that case.

The semilinear equation

$$u_{tt} - u_{xx} - f(u) = 0$$

seems to be a far from the the scope of the existing tools.

4.– To find a decomposition of a general graph allowing to give a description of the spectral unique continuation property in simpler terms as it was done for trees. The starting point should be the characterization of the spectral unique continuation property for a graph formed by a circle to with to strings are coupled. Even in that quite simple configuration, the property is not well understood.

5.– A quite rich and widely open field is the numerical implementation of the observability results for networks of strings in the spirit of works [**37**], [**65**], [**83**]. Some effective numerical techniques have been developed in [**49**] and [**50**] for observations from large sets, that is, when the original energy of the solution may be recovered from the observation. The situation seems to be more complicated in the case of weak observability considered along this book, since the irrationality conditions on the lengths is an obstacle to computational implementation.

APPENDIX A

Some consequences of diophantine approximation theorems

In this appendix we have gathered some results that have been used in the proof of several theorems in the main text. All of these results have a common feature: they are obtained as consequences of theorem related to the approximation of real numbers by rational numbers.

For $\eta \in \mathbb{R}$ we denote by $|||\eta|||$ the distance from η to the set $\mathbb{Z}$:

$$|||\eta||| := \min_{\eta - x \in \mathbb{Z}} |x|$$

and by $\mathbf{E}(\eta)$ the closest to η integer number[1]:

$$|\eta - \mathbf{E}(\eta)| = |||\eta|||.$$

Let us observe that $0 \leq |||\eta||| \leq \frac{1}{2}$ and besides, that η may be expressed as

$$\eta = \mathbf{E}(\eta) + \mathbf{F}(\eta), \tag{1}$$

where

$$|\mathbf{F}(\eta)| = |||\eta|||, \qquad -\frac{1}{2} \leq \mathbf{F}(\eta) \leq \frac{1}{2}.$$

For given real numbers $\ell_1, ..., \ell_N$, we define the function

$$\mathbf{a}(\lambda) = \mathbf{a}(\lambda, \ell_1, ..., \ell_N) := \sum_{i=1}^{N} \prod_{j \neq i} |\sin \lambda \ell_j| \, .$$

This function frequently appears in the problems we have studied. Our aim is to find conditions on the numbers $\ell_1, ..., \ell_N$ guaranteeing that for some $\alpha \in \mathbb{R}$ the function

$$\mathbf{a}(\lambda)\lambda^{\alpha}$$

remains bounded from below as $\lambda \to \infty$.

For $m \in \mathbb{N}$, $i = 1, ..., N$, we denote

$$\mathbf{z}^i(m) := \mathbf{z}^i(m, \ell_1, ..., \ell_N) := \prod_{j \neq i} |||\frac{\ell_j}{\ell_i} m|||,$$

$$\mathbf{m}^i(\lambda) := \mathbf{m}^i(\lambda, \ell_1, ..., \ell_N) := \mathbf{E}\left(\frac{\ell_i}{\pi}\lambda\right).$$

The following proposition allows to reduce the stated problem to one of approximation by rationals.

[1] If $|||\eta||| = \frac{1}{2}$ there are two integer numbers with that property: $\eta + \frac{1}{2}$ and $\eta - \frac{1}{2}$. In this case $\mathbf{E}(\eta)$ will denote one of these numbers.

PROPOSITION A.1. *There exists a positive constant C such that for every $\lambda \in \mathbb{R}$ the inequality*

$$\mathbf{a}(\lambda) \geq C \min_{i=1,\ldots,N} \mathbf{z}^i(\mathbf{m}^i(\lambda))$$

is satisfied.

PROOF. Let us remark first that every $x \in \mathbb{R}$ can be expressed in the form

$$x = \pi \mathbf{E}\left(\frac{x}{\pi}\right) + \pi \mathbf{F}\left(\frac{x}{\pi}\right).$$

Then it follows

$$\sin x = \sin \pi \mathbf{F}\left(\frac{x}{\pi}\right). \tag{2}$$

Taking into account that if $|x| \leq \frac{1}{2}$ then the inequalities

$$2\,|x| \leq |\sin \pi x| \leq \pi\,|x|\,,$$

are true, from (2) we obtain for every $x \in \mathbb{R}$

$$2|||\frac{x}{\pi}||| \leq |\sin x| \leq \pi|||\frac{x}{\pi}|||. \tag{3}$$

In view of this inequality, we have for every $\lambda \in \mathbb{R}$

$$2|||\mathbf{m}^i(\lambda)\frac{\ell_j}{\ell_i}||| \leq \sin\left(\mathbf{m}^i(\lambda)\frac{\ell_j}{\ell_i}\pi\right). \tag{4}$$

If we denote

$$\boldsymbol{\gamma}^i = \mathbf{F}\left(\frac{\ell_i}{\pi}\lambda\right),$$

then

$$\mathbf{m}^i(\lambda) = \mathbf{E}\left(\frac{\ell_i}{\pi}\lambda\right) = \frac{\ell_i}{\pi}\lambda - \mathbf{F}\left(\frac{\ell_i}{\pi}\lambda\right) = \frac{\ell_i}{\pi}\lambda - \boldsymbol{\gamma}^i.$$

Replacing this expression of $\mathbf{m}^i(\lambda)$ in the right hand term of (4) it follows

$$\begin{aligned} 2|||\mathbf{m}^i(\lambda)\frac{\ell_j}{\ell_i}||| &\leq \left|\sin\left(\ell_j\lambda - \boldsymbol{\gamma}^i\frac{\ell_j}{\ell_i}\pi\right)\right| = \left|\sin \ell_j\lambda \cos \boldsymbol{\gamma}^i\frac{\ell_j}{\ell_i}\pi - \cos \ell_j\lambda \sin \boldsymbol{\gamma}^i\frac{\ell_j}{\ell_i}\pi\right| \\ &\leq |\sin \ell_j\lambda| + \left|\boldsymbol{\gamma}^i\right|\frac{\ell_j}{\ell_i}\pi. \end{aligned}$$

On the other hand, from the inequality (3) we get

$$\left|\boldsymbol{\gamma}^i\right| = |||\lambda\frac{\ell_i}{\pi}||| \leq \frac{1}{2}\,|\sin \lambda\ell_i|\,.$$

Thus we may conclude that for every $i = 1, ..., N$ and every $j \neq i$

$$|||\mathbf{m}^i(\lambda)\frac{\ell_j}{\ell_i}||| \leq \frac{1}{2}\,|\sin \ell_j\lambda| + \frac{\pi\ell_j}{4\ell_i}\,|\sin \lambda\ell_i|\,.$$

Multiplying these inequalities we obtain that there exists a constant $C > 0$ such that for every $i = 1, .., N$

$$\mathbf{z}^i(\mathbf{m}^i(\lambda)) = \prod_{j\neq i}|||\frac{\ell_j}{\ell_i}\mathbf{m}^i(\lambda)||| \leq C\,|\sin \lambda\ell_i| + \frac{1}{2^{N-1}}\prod_{j\neq i}|\sin \ell_j\lambda|\,. \tag{5}$$

With this, it is now quite simple to prove the assertion of the proposition. We will prove that if (λ_n) is a sequence such that $\mathbf{a}(\lambda_n) \to 0$ then there exists a value i_0 of i such that $\mathbf{z}^{i_0}(\mathbf{m}^{i_0}(\lambda_n)) \to 0$.

Indeed, let us observe that if the sequence (λ_n) satisfies $\mathbf{a}(\lambda_n) \to 0$ then, for every i

$$\prod_{j \neq i} |\sin \ell_j \lambda_n| \to 0.$$

Then there would exist some i_0 such that $|\sin \lambda_n \ell_{i_0}| \to 0$. Thus, from the inequality (5) for $i = i_0$ it follows

$$\mathbf{z}^{i_0}(\mathbf{m}^{i_0}(\lambda_n)) \to 0.$$

This proves the proposition. □

COROLLARY A.1. *Let $\alpha \in \mathbb{R}$. If for every $i = 1, ..., N$, the ratio $\frac{\ell_j}{\ell_i}$, $j = 1, ..., N$, has the property that there exists a constant $C_i > 0$ such that, for every $m \in \mathbb{N}$,*

$$m^\alpha \prod_{j \neq i} |||\frac{\ell_j}{\ell_i} m||| \geq C_i, \tag{6}$$

then

$$\mathbf{a}(\lambda)\lambda^\alpha \geq C,$$

for every $\lambda > 0$.

In what follows we will see some rational approximation theorems. which provide sufficient conditions for an inequality like (6) to be true.

Let us recall that a real number ξ is said to be algebraic if ξ is the root of some polynomial with rational coefficients. The set $\mathbb{A}$ of all the algebraic numbers is a sub-fields of $\mathbb{R}$. The Lebesgue measure of $\mathbb{A}$ is equal to zero, but $\mathbb{A}$ is dense in $\mathbb{R}$. Besides, so is $\mathbb{A} \setminus \mathbb{Q}$. It is said that the algebraic number ξ is of order p if the polynomial with rational coefficients of minimal degree that vanishes at ξ is of degree p.

A classical problem in Number Theory is the following: given $\xi, \alpha \in \mathbb{R}$, determine whether the inequality

$$|||\xi m||| \leq \frac{1}{m^\alpha}$$

has solutions $m \in \mathbb{Z}$. This is equivalent to the existence of $m, n \in \mathbb{Z}$ such that

$$\left|\xi - \frac{n}{m}\right| \leq \frac{1}{m^{\alpha+1}}.$$

The relevance of this problem is related to the following theorem due to Liouville:

THEOREM A.1 (Liouville). *If ξ is an algebraic number of order $p \geq 2$, then the inequality*

$$\left|\xi - \frac{n}{m}\right| \leq \frac{1}{m^p}$$

has no solutions $n, m \in Z$.

This fact was used by Liouville to prove that not all the real numbers are algebraic; he constructed explicit examples of numbers $\xi \in \mathbb{R}$, known now as Liouville's numbers, one of which is

$$\xi = \sum_{n=1}^{\infty} 10^{-n^{n^2}},$$

such that the inequality (6) has solutions for every p. Clearly, in view of Theorem A.1, such numbers are not algebraic. Nowadays, this fact may seem to be rather simple, since the set of algebraic numbers has Lebesgue measure equal to zero and thus, the most of real numbers are not algebraic. However, at its time, this result had an outstanding scientific relevance.

Later on, Roth proved an stronger result:

THEOREM A.2 (Roth, [**72**]). *If ξ is an algebraic number of order $p \geq 2$, then the inequality*

$$\left|\xi - \frac{n}{m}\right| \leq \frac{1}{m^2} \tag{7}$$

has at most a finite number of solutions $n, m \in Z$.

The following results provide additional information

PROPOSITION A.2 ([**19**], p. 120). *For every $\varepsilon > 0$ there exists a set $\mathbf{B}_\varepsilon \subset \mathbb{R}$, such that the Lebesgue measure of $\mathbb{R} \setminus \mathbf{B}_\varepsilon$ is equal to zero, and a constant $C_\varepsilon > 0$ for which, if $\xi \in \mathbf{B}_\varepsilon$ then*

$$|||\xi m||| \geq \frac{C_\varepsilon}{m^{1+\varepsilon}}.$$

On the other hand, when $\varepsilon = 0$ a complete answer can be given. Let $\mathcal{F}$ be the set of all those irrational numbers $\eta \in \mathbb{R}$ such that if $[a_0, a_1, ..., a_n, ...]$ is the expansion of η in continuous fraction (see, e.g., [**19**] for a definition) then the sequence (a_n) is bounded. The set $\mathcal{F}$ is not denumerable and has Lebesgue measure equal to zero.

PROPOSITION A.3 ([**52**], Theorem 6, p. 24). *There exists a positive constant C such that*

$$|||\xi m||| \geq \frac{C}{m},$$

for every $m \in \mathbb{N}$, if, and only if, $\xi \in \mathcal{F}$.

In particular, $\mathcal{F}$ is contained in the sets $\mathbf{B}_\varepsilon$ for every $\varepsilon > 0$.

The following theorem due to W. Schmidt provides information on the simultaneous approximation of real numbers by rational numbers with the same denominator n.

THEOREM A.3 (W. Schmidt, [**77**]). *If the numbers $\xi_1, ..., \xi_N$ are algebraic and $1, \xi_1, ..., \xi_N$ are linearly independent over the field $\mathbb{Q}$, for every $\varepsilon > 0$, the inequality*

$$|||n\xi_1||| \cdot |||n\xi_2||| \cdots |||n\xi_N|||n^{1+\varepsilon} \leq 1$$

has at most a finite number of solutions $n \in \mathbb{N}$.

An immediate consequence of this theorem is that, if the numbers $\xi_1, ..., \xi_N$ are algebraic and $1, \xi_1, ..., \xi_N$ are $\mathbb{Q}$-linearly independent then, for each $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$|||n\xi_1||| \cdot |||n\xi_2||| \cdots |||n\xi_N|||n^{1+\varepsilon} \geq C_\varepsilon,$$

for every $n \in \mathbb{N}$.

As a counterpart, Schmidt proved the following more exact version of the Dirichlet theorem.

THEOREM A.4 (W. Schmidt, [**77**]). *If* $\xi^1, ..., \xi^M$ *are real numbers and* $\varepsilon < \frac{1}{M}$ *then, for an infinite number of values of* $p \in \mathbb{Z}$, *there exist integer numbers* $q_i(n)$, $i = 1, ..., M$ *such that*

$$|p\xi_i - q_i(p)| \leq \frac{1}{p^\eta} \qquad i = 1, ..., M. \tag{8}$$

DEFINITION A.1. *We will say that the real numbers* $\ell_1, ..., \ell_N$ verify the conditions (S) *if*

- $\ell_1, ..., \ell_N$ *are linearly independent over the field* $\mathbb{Q}$ *of rational numbers;*
- *the ratios* $\frac{\ell_i}{\ell_j}$ *are algebraic numbers for* $i, j = 1, ..., N$.

Let us observe that if $\ell_1, ..., \ell_N$ verify the conditions (S), then for every i the ratios $\frac{\ell_j}{\ell_i}$, $j = 1, ..., N$, satisfy the conditions of the Schmidt's theorem. Actually, if ℓ_i and ℓ_j are algebraic numbers, so is their ratio. Besides, if

$$\alpha_1 \frac{\ell_1}{\ell_i} + \cdots + \alpha_{i-1} \frac{\ell_{i-1}}{\ell_i} + \alpha_i \cdot 1 + \alpha_{i+1} \frac{\ell_{i+1}}{\ell_i} + \cdots + \alpha_N \frac{\ell_N}{\ell_i} = 0, \qquad \alpha_i \in \mathbb{Q},$$

then

$$\alpha_1 \ell_1 + \cdots + \alpha_N \ell_N = 0, \qquad \alpha_i \in \mathbb{Q},$$

and then, if $\ell_1, ..., \ell_N$ are linearly independent over $\mathbb{Q}$, it follows $\alpha_i = 0, i = 1, ..., N,$. Thus, $\frac{\ell_j}{\ell_i}$, $j = 1, ..., N$, are linearly independent over $\mathbb{Q}$, too.

Combining these results with Corollary A.1 we obtain

COROLLARY A.2. *Let* $\ell_1, ..., \ell_N$ *be positive numbers. Then*

1) *If for all values* $i, j = 1, ..., N$, $i \neq j$, *the ratios* $\frac{\ell_i}{\ell_j}$ *belong to* $\mathbf{B}_\varepsilon$ *then there exists a constant* $C_\varepsilon > 0$ *such that*
$$\mathbf{a}(\lambda) \geq \frac{C_\varepsilon}{\lambda^{N-1+\varepsilon}},$$
for every $\lambda > 0$. *In particular, if all the ratios belong to* $\mathfrak{F}$, *this inequality holds with* $\varepsilon = 0$.
2) *If the numbers* $\ell_1, ..., \ell_N$ *satisfy the conditions (S), then, for each* $\varepsilon > 0$ *there exists a constant* $C_\varepsilon > 0$ *such that*
$$\mathbf{a}(\lambda) \geq \frac{C_\varepsilon}{\lambda^{1+\varepsilon}},$$
for every $\lambda > 0$.

PROPOSITION A.4. *Let* (ω_n) *an unbounded sequence of positive solutions of the equation*

$$\sum_{i=0}^{N} \left(\cos \ell_i \omega \prod_{j \neq i} \sin \ell_j \omega \right) = 0 \tag{9}$$

and assume that the numbers $\ell_0, ..., \ell_N$ *satisfy the conditions (S). Then, for every* $\varepsilon > 0$ *there exists a constant* C_ε *such that for every* $n \in \mathbb{N}$ *and every* $i = 0, ..., N$, *the following inequality is true*

$$|\sin \ell_i \omega_n| \geq \frac{C_\varepsilon}{\omega_n^{1+\varepsilon}}.$$

PROOF. It is similar to the proof of Proposition A.1. We will show that there exists a constant $C > 0$ such that, for every $i = 0, ..., N$,

$$\prod_{j \neq i} |||\frac{\ell_j}{\ell_i}\mathbf{m}^i(\omega_n)||| \leq C \left|\sin \omega_n \ell_i\right|. \tag{10}$$

This, in account of the Schmidt's theorem, give the assertion of the proposition.

In order to prove (10), it is sufficient to see that, if $|\sin \omega_n \ell_i| \to 0$ then

$$\prod_{j \neq i} |||\frac{\ell_j}{\ell_i}\mathbf{m}^i(\omega_n)||| \to 0. \tag{11}$$

Indeed, if $|\sin \omega_n \ell_0| \to 0$ (we have taken $i = 0$ to simplify the notations) then from the equality (9) follows

$$\prod_{j=1}^{N} |\sin \ell_j \omega_n| \to 0. \tag{12}$$

On the other hand, the inequality (5) obtained in the proof of Proposition A.1, allows us to ensure that

$$\prod_{j=1}^{N} |||\frac{\ell_j}{\ell_0}\mathbf{m}^0(\omega_n)||| \leq C \left|\sin \omega_n \ell_0\right| + \frac{1}{2^N} \prod_{j=1}^{N} \left|\sin \ell_j \omega_n\right|.$$

From this, based on (12) and the fact $|\sin \omega_n \ell_0| \to 0$, we obtain the convergence (11). This proves the assertion. $\square$

Bibliography

1. K. Appel and W. Haken, *Every planar map is four colorable*, Illinois Journal of Mathematics **21** (1977), 429–567.
2. S. A. Avdonin and S. A. Ivanov, *Families of exponentials: The method of moments in controllability problems for distributed parameter systems*, Cambridge Univ. Press, Cambridge, 1995.
3. S. A. Avdonin and T. I. Seidman, *Pointwise and internal controllability for the wave equation*, Appl. Math. Optim. **46** (2002), 107–124.
4. S.A. Avdonin and W. Moran, *Simultaneous controllability of elastic systems*, in: I.I. Eremin, I. Lasiecka, V.I. Maksimov, (Eds.), Proceedings of the International Conference on Distributed Systems, Ekaterinburg, Russia, 30 May–2 June, 2000, pp. 28–31, IMM UrO RAN, 2000.
5. ———, *Ingham type inequalities and Riesz bases of divided differences .mathematical methods of optimization and control of large-scale systems (ekaterinburg, 2000)*, Int. J. Appl. Math. Comput. Sci. **11** (2001), no. 4, 803–820.
6. ———, *Simultaneous control problems for systems of elastic strings and beams*, Systems & Control Letters **44** (2001), no. 2, 147–155, por aparecer.
7. S.A. Avdonin and M. Tucsnak, *Simultaneous controllability in sharp time for two elastic strings*, ESAIM:COCV **6** (2001), 259–273.
8. C. Baiocchi, V. Komornik, and P. Loreti, *Ingham type theorems and applications to control theory*, Boll. Un. Mat. Ital. **B 8 (II-B)** (1999), 33–63.
9. ———, *Généralisation d'un théorème de Beurling et application à la théorie du contrôle*, C. R. Acad. Sci. Paris, Série I **330** (2000), 281–286.
10. C. Baiocchi, V. Komornik, and P. Loreti, *Ingham-Beurling type theorems with weakened gap conditions*, Acta Math. Hungar. **97** (2002), no. 1, 55–95.
11. C. Bardos, G. Lebeau, and Rauch J., *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary*, SIAM J. Control Optim. **30** (1992), 1024–1065.
12. N. K. Bari, *Biorthogonal systems and bases in Hilbert spaces*, Uĉen. Zap. Mosk. Gos. Univ. **4** (1951), no. 148, 68–107.
13. J. Von Below, *A characteristic equation associated to an eigenvalue problem on c^2-networks*, Linear Alg. Appl. **71** (1985), 309–325.
14. ———, *Classical solvability of linear parabolic equations on networks*, J. Diff. Eq. **72** (1988), 316–337.
15. ———, *Parabolic network equations*, Habilitation Thesis, Eberhard-Karls-Universität Tübingen, 1993.
16. A. Beurling, *The collected works of Arne Beurling*, L. Carleson, P. Malliavin, J. Neuberger, J. Wermer, (Eds.), vol. 2 Harmonic Analysis, Birkhäuser, Boston, 1989.
17. A. Beurling and P. Malliavin, *On the closure of characters and zeros of entire functions*, Acta Math. **118** (1967), 79–93.
18. H. Brésiz, *Análisis funcional: Teoría y aplicaciones*, Alianza Editorial, 1984.
19. J.M. Cassels, *An introduction to diophantine approximation*, Cambridge Univ. Press, Cambridge, 1966.
20. C. Castro, *Boundary controllability of the one-dimensional wave equation with rapidly oscillating density*, Asymptotic Analysis **20** (1999), 317–350.
21. C. Castro and E. Zuazua, *Une remarque sur les séries de Fourier non-harmoniques et son application à la contrôlabilité des cordes avec densité singulière*, C. R. Acad. Sci. Paris, Sèrie I **332** (1996), 365–370.
22. G. Chen, M. Delfour, A. Krall, and G. Payre, *Modelling, stabilization and control of serially connected beams*, SIAM J. Control Opt. **25** (1987), 526–546.

23. G. Chen and J. Zhou, *The wave propagation method for the analysis of boundary stabilization in vibrating structures*, SIAM J. Appl. Math. **50** (1990), 1254–1283.
24. R. Courant and D. Hilbert, *Methods of mathematical physics*, vol. I, Interscience, New York, 1962.
25. R. Dáger, *Observation and control of vibrations in tree-shaped networks of strings*, to appear.
26. R. Dáger and E. Zuazua, *Controllability of star-shaped networks of strings*, In: Bermúdez A. et al. (Eds), Fifth International Conference on Mathematical and Numerical Aspects of Wave Propagation, SIAM Proceedings (2000), 1006–1010.
27. ______, *Controllability of star-shaped networks of strings*, C. R. Acad. Sci. Paris, Sèrie I **332** (2001), 621–626.
28. ______, *Controllability of tree-shaped networks of vibrating strings*, C. R. Acad. Sci. Paris, Sèrie I **332** (2001), 1087–1092.
29. ______, *Spectral boundary controllability of networks of strings*, C. R. Acad. Sci. Paris, Sèrie I **334** (2002), no. 7, 545–550.
30. B. Dekoninck and S. Nicaise, *Control of networks of Euler-Bernoulli beams*, ESAIM:COCV **4** (1999), 57–81.
31. H. Fattorini, *On complete controllability of linear systems*, J. Diff. Eq. **3** (1967), 391–402.
32. H. O. Fattorini, *Estimates for sequences biorthogonal to certain complex exponentials and boundary control of the wave equation*, Lectures Notes in Control and Inform. Sci. **2** (1979), 111–124.
33. H.O. Fattorini and D.L.. Russell, *Uniform bounds on biorthogonal functions for real exponentials and application to the control theory of parabolic equations*, Quart. Appl. Math. **32** (1974), 45–69.
34. A. Haraux and S. Jaffard, *Pointwise and spectral control of plate vibrations*, Revista Matemática Iberoamericana **7:1** (1991), 1–24.
35. L. F. Ho, *Controllability and stabilizability of coupled strings with control applied to the coupled points*, SIAM J. Control Optim. **31** (1993), 1416–1433.
36. V. C. L. Hutson and J. S. Pym, *Applications of functional analysis and operator theory*, Mathematics in Science and Engineering, vol. 146, Acad. Press, London, 1980.
37. J. A. Infante and E. Zuazua, *Boundary observability for the space-discretizations of the one-dimensional wave equation*, Mathematical Modelling and Numerical Analysis **33** (1999), 407–438.
38. A. E. Ingham, *Some trigonometrical inequalities with applications in the theory of series*, Math. Z. **41** (1936), 367–379.
39. E. Isaacson and H.B .Keller, *Analysis of numerical methods*, John Wiley & Sons, New York, 1966.
40. S. Jaffard, M. Tucsnak, and E. Zuazua, *On a theorem of Ingham*, J. Fourier Anal. Appl. (1997), no. 3, 577–582.
41. ______, *Singular internal stabilization of the wave equation*, J. Differential Equations **145** (1998), 184–215.
42. A.G. Khapalov, *Interior point control and observation for the wave equation*, Abstract and Applied Analysis **1** (1996), no. 2, 219–236.
43. V. Komornik, *Exact controllability and stabilization. The multiplier method*, Masson, Paris and John Wiley & Sons, Chichester, 1994.
44. V. Komornik and P. Loreti, *Dirichlet series and simultaneous observability: two problems solved by the same approach*, Systems & Control Letters **48** (2003), no. 3-4, 221–227.
45. V. Komornik and M. Yamamoto, *Upper and lower estimates in determining point sources in a wave equation*, Inverse Problems **18** (2002), no. 2, 319–329.
46. W. Krabs, *On moment theory and controllability of one-dimensional vibrating systems and heating processes*, Lecture Notes in Control and Inform. Sci., vol. 173, Springer-Verlag, Berlin, 1992.
47. E. Lagnese, G. Leugering, and E.J. P.G. Schmidt, *Control of planar networks of Timoshenko beams*, SIAM J. Control Optim. **31** (1993), 780–811.
48. J. Lagnese, *Recent progress and open problems in control of multi-link elastic structures.*, Contemp. Math., vol. 209, pp. 161–175, Amer. Math. Soc., Providence, RI, 1997.
49. ______, *Domain decomposition in exact controllability of second order hyperbolic systems on 1-d networks. Recent advances in control of PDE.*, Control Cybernet **28** (1999), no. 3, 531–556.

50. J. Lagnese and G. Leugering, *Dynamic domain decomposition in approximate and exact boundary control in problems of transmission for the waves equations*, SIAM J. Control Optim. **38** (2000), no. 2, 503–537.
51. J. E. Lagnese, G. Leugering, and E.J.P.G. Schmidt, *Modelling, analysis and control of multi-link flexible structures*, Systems and Control: Foundations and Applications, Birhäuser-Basel, 1994.
52. S. Lang, *Introduction to diophantine approximations*, Addison-Wesley, Reading, MA, 1966.
53. G. Lebeau, *Contrôle de l'equation de Schrödinger*, J. Math. Pures Appl. **71** (1992), 267–291.
54. G. Leugering, *Reverberation analysis and control of networks of elastic strings*, Control of PDE and appl., Lect. Notes in Pure and Applied Math. (1996), 193–206.
55. G. Leugering and E.J.P.G. Schmidt, *On the control of networks of vibrating strings and beams*, Proc. Of the 28th IEEE Conference on Decision and Control, Vol. 3, IEEE, 1989, pp. 2287–2290.
56. ———, *On the modelling and stabilization of flows in networks of open canals*, SIAM J. Control Optim. **41** (2002), 164–180.
57. G. Leugering and E. Zuazua, *Exact controllability of generic trees*, "Control of Systems Governed by Partial Differential Equations", Nancy, France, March 1999, ESAIM Proceedings.
58. B. J. Levin, *Distribution of zeros of entire functions*, Translation of Math. Monographs, vol. 5, AMS, 1950.
59. J.-L. Lions, *Contrôlabilité exacte de systèmes distribués*, C. R. Acad. Sci. Paris, Sèrie I **302** (1986), 471–475.
60. ———, *Contrôlabilité exacte perturbations et stabilisation de systèmes distribués*, vol. I, Masson, Paris, 1988.
61. J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*, Dunod, Paris, 1968.
62. F. Ali Mehmeti, *A characterisation of generalized c^∞ notion on nets*, Int. Eq. and Operator Theory **9** (1986), 753–766.
63. ———, *Regular solutions of transmission and interaction problems for wave equations*, Math. Meth. Appl. Sci. **11** (1989), 665–685.
64. S. Micu and E. Zuazua, *Boundary controllability of a linear hybric system arising in the control of noise*, SIAM J. Control Optim. **35** (1997), no. 5, 1614–1638.
65. M. Negreanu and E. Zuazua, *Uniform boundary controllability of a discrete 1-D wave equation*, Systems & Control Letters **48** (2003), no. 3-4, 261–279.
66. S. Nicaise, *Spectre des réseaux topologiques finis*, Bull. Sc. math, 2^e série **111** (1987), 401–413.
67. R.E.A.C. Paley and N. Wiener, *Fourier transform in the complex domain*, Am. Math. Soc. Colloq. Publ., vol. 19, Am. Math. Soc., New York, 1934.
68. K.D. Phung, *Observability and control of Schödinger equations*, SIAM J. Control Optim. **40** (2001), no. 1, 211–230.
69. J. P. Puel and E. Zuazua, *Exact controllability for a model of multidimensional flexible structure*, Proc. Royal Soc. Edinburgh **123 A** (1993), 323–344.
70. R. M. Redheffer, *Elementary remarks on completeness*, Duke Math. J. **35** (1968), 103–116.
71. S. Rolewicz, *On controllability of systems of strings*, Studia Math. **36** (1970), no. 2, 105–110.
72. K. F. Roth, *Rational approximation to algebraic numbers*, Mathematika **2** (1955), 1–20.
73. D. L. Russell, *A unified boundary controllability theory for hyperbolic and parabolic partial differential equations*, Studies in Appl. Math. **52** (1973), 189–221.
74. ———, *Controllability and stabilizability theory for linear partial differential equations*, SIAM Review **20** (1978), 639–739.
75. ———, *The Dirichlet-Neumann boundary control problem associated with Maxwell's equations in a cylindrical region*, SIAM J. Control Optim. **24** (1986), 199–229.
76. E.J.P.G. Schimidt, *On the modelling and exact controllability of networks of vibrating strings*, SIAM J. Control. Opt. **30** (1992), 229–245.
77. W.M. Schmidt, *Simultaneous approximation to algebraic numbers by rationals*, Acta Math. **125** (1970), 189–202.
78. M. Tucsnak and G. Weiss, *Simultaneous exact controllability and some applications*, SIAM J. Control Optim. **38** (2000), 1408–1427.
79. U. Ullrich, *Divided differences and systems of nonharmonic Fourier series*, Proc. Amer. Math. Soc. **80** (1980), 47–57.
80. N. Wiener, *A class of gap theorems*, Ann. Scuola Norm. Sup. Pisa (1934), 367–372.

81. R.M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York, 1980.
82. E. Zuazua, *Some problems and results on the controllability of partial differential equations*, Progress in Mathematics, Birkhaüser Verlag, 1998, pp. 276–311.
83. E. Zuazua, *Boundary observability for the finite-difference space semi-discretizations of the 2-d wave equation in the square*, J. Math. Pures et Appliquées **78** (1999), 523–563.