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Belhassen Dehman · Enrique Zuazua

Boundary Sidewise Observability of the Wave Equation

Abstract. The wave equation on a bounded domain of $\mathbb{R}^n$ with non homogeneous boundary Dirichlet data or sources supported on a subset of the boundary is considered. We analyze the problem of observing the source out of boundary measurements done away from its support.

We first notice that due to the existence of solutions that are arbitrarily concentrated near the source, for any given integer N, these observability inequalities may not hold even if we allow a loss of N derivatives.

We then establish observability inequalities in Sobolev norms, under a suitable microlocal geometric condition on the support of the source and the measurement set, for sources fulfilling pseudo-differential conditions that exclude these concentration phenomena.

The proof relies on microlocal arguments and is essentially based on the use of microlocal defect measures.

1. Introduction

1.1. General setting

Let Ω be a bounded open domain of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^∞. We set

$$\mathcal{L} = \mathbb{R} \times \Omega \quad \text{and} \quad \partial\mathcal{L} = \mathbb{R} \times \partial\Omega.$$

We also introduce $A = (a_{ij}(x))$, a $n \times n$ matrix of C^∞ coefficients, symmetric, uniformly definite positive on a neighborhood of Ω.

Belhassen Dehman: Belhassen Dehman. Département de Mathématiques, Faculté des sciences de Tunis & Enit-Lamsin, Université de Tunis El Manar, 2092 El Manar, Tunisia.; belhassen.dehman@fst.utm.tn

Enrique Zuazua: ; [1] Chair for Dynamics, Control and Numerics - Alexander von Humboldt-Professorship, Department of Data Science, Friedrich-Alexander-Universität Erlangen-Nürnberg, 91058 Erlangen, Germany ,
[2] Chair of Computational Mathematics, Fundación Deusto, 48007 Bilbao, Basque Country, Spain,
[3] Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain.; enrique.zuazua@fau.de

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Finally, we take $g \in H^1(\partial\mathcal{L})$ and we assume that g is compactly supported in time in the interval $(0, +\infty)$.

We consider then the following wave system

$$\begin{cases} P_A u = -\partial_t^2 u + \sum_{i,j=1}^n \partial_{x_j}(a_{ij}(x)\partial_{x_i} u) = 0 \quad \text{in } \mathcal{L} \\ \\ u(t,.) = g(t,.) \quad \text{on } \partial\mathcal{L} \\ \\ u(0,.) = \partial_t u(0,.) = 0 \quad \text{in } \Omega. \end{cases} \tag{1}$$

This system is well posed in the classical energy space $C^0(\mathbb{R}, H^1(\Omega)) \cap C^1(\mathbb{R}, L^2(\Omega))$ equipped with the energy norm $\sup_{t\in\mathbb{R}} Eu(t)$, where

$$Eu(t) = \|u(t,.)\|_{H^1(\Omega)}^2 + \|\partial_t u(t,.)\|_{L^2(\Omega)}^2,$$

and

$$\|u(t,.)\|_{H^1(\Omega)}^2 = \sum_{i,j=1}^n \int_\Omega a_{ij}(x)\partial_{x_i} u \partial_{x_j} u dx,$$

see [15]. Actually, the solution u vanishes for $t \le 0$.

More precisely, the following energy estimate holds

$$\sup_{t\in\mathbb{R}} Eu(t) \le C||g||_{H^1(\partial\mathcal{L})}^2, \tag{2}$$

together with the added hidden regularity property of the trace of the normal derivative

$$\|\partial_n u_{|\partial\Omega}\|_{L^2((0,a)\times\partial\Omega)} \le C_a \|g\|_{H^1(\partial\mathcal{L})}, \tag{3}$$

valid for all $a > 0$.

Remark 1.1. *The constant appearing in estimate* (2) *and* (3) *depend on the metric attached to* $A = (a_{ij}(x))_{ij}$*, on the geometry of the domain* Ω *and, for* (3)*, also on the time-horizon* $a > 0$.

1.2. Geometry of the domain Ω

In this paper, we will deal with a particular class of domains Ω. This fact is made precise in the following condition.

Assumption A1

We assume that there exists a strictly concave (with respect to the metric attached to the matrix $A = (a_{ij}(x))_{ij}$*) open non empty subset* O *of the boundary* $\partial\Omega$*,* $\overline{O} \neq \partial\Omega$*.*

Geometrically, this guarantees that every geodesic of Ω for the metric $(a^{ij}) = (a_{ij})^{-1}$, that is tangent to O at some point m_0, has an order of tangency equal to 1; locally near this point and except for m_0, this geodesic lives in Ω.

For instance, if $A = Id$, we then deal with the euclidian metric, and Assumption A1 simply says that there exists a neighborhood V of O in $\mathbb{R}^n$, such that the set $V \setminus \Omega$ is strictly convex. See Fig.1.

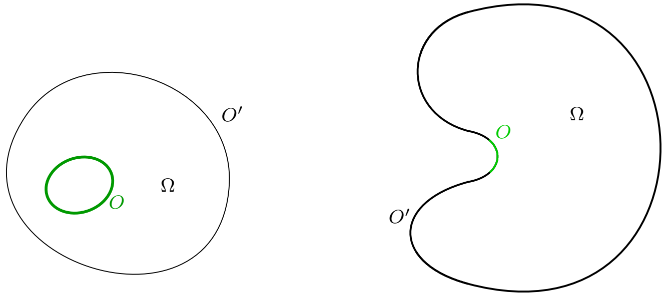

Fig. 1. Examples of strictly concave boundary subset O

Remark 1.2. (1) *Assumption A1, implicitly, substantially limits the class of domains* Ω *under consideration. For example, this condition excludes convex domains* Ω*. Indeed, for subsets* O *of the boundary of* Ω *to exist, so that they fulfil the assumption A1, the geometry of* Ω *needs to allow for some concavity zones of its boundary, as illustrated in Figure 1, and this excludes many domains* Ω*.*

(2) *In the literature, sets* O *fulfilling assumption A1 are sometimes said to be diffractive with respect to the metric attached to* $A = (a_{ij}(x))_{ij}$*.*

1.3. Motivation

From now, we will work under assumption A1. Let then O' be a non empty open subset of $\partial\Omega$ such that $\overline{O} \cap \overline{O'} = \emptyset$. We set

$$\Gamma = \mathbb{R} \times O, \qquad \Gamma' = \mathbb{R} \times O',$$

and for $a > 0$,

$$\mathcal{L}_a = (0, a) \times \Omega, \quad \Gamma_a = (0, a) \times O \quad \text{and} \quad \Gamma'_a = (0, a) \times O'.$$

In addition, we assume throughout the whole paper that the boundary data g is supported in $\overline{\Gamma}_M = [0, M] \times \overline{O}$ for some $M > 0$.

The aim of this paper is to analyze whether it is possible to observe the boundary data or source g in (1) from measurements done on the normal derivative $\partial_n u_{|\Gamma'}$ on the subset Γ' of the boundary. In other words, we are seeking for an estimate of the type

$$\|g\|_{H^1(\Gamma_M)} \leq C \|\partial_n u_{|\partial\Omega}\|_{L^2(\Gamma'_a)}, \tag{4}$$

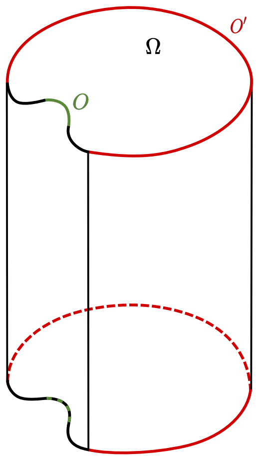

Fig. 2. Cylindrical domain where waves evolve. In green the support of the source g to be identified, and in red the subset of the boundary where measurements are done.

for some $a \geq M$.

Estimate (4) is the sidewise observability inequality object of analysis in this paper.

According to the classical hidden regularity property, see for instance [17], it is well known that the right hand side term of (4) is bounded above by

$$\|u\|_a^2 =: \sup_{t\in[0,a]} Eu(t) = \sup_{t\in[0,M]} Eu(t) = \|u\|_M^2.$$

More precisely, for every $a > 0$, there exists $C_a > 0$ such that every solution u of (1) satisfies

$$\|\partial_n u_{|\partial\Omega}\|_{L^2(\Gamma'_a)} \leq C_a \|u\|_M. \tag{5}$$

Therefore, a necessary condition for an estimate of the form (4) to hold is that the boundary data g under consideration needs to be observable out of the total interior energy $\|u\|_M$, namely, the existence of a constant $C > 0$ such that

$$\|g\|_{H^1(\Gamma_M)} \leq C\|u\|_M. \tag{6}$$

However, as we shall see, this inequality does not hold without additional structural conditions on the source term g under consideration. Indeed, in Theorem 2.5 and Corollary 2.6, we construct sequences of invisible sources (g_k) whose energy is essentially localized on the elliptic and/or glancing set of the boundary, such that

$$\|g_k\|_{H^1(\Gamma_M)} \to 1, \quad g_k \rightharpoonup 0 \quad \text{in } H^1, \quad \|u_k\|_M \longrightarrow 0, \tag{7}$$

which, of course, are an impediment for (6) to occur.

In fact, as we shall see, even the weaker version

$$\|g\|_{H^s(\Gamma_M)} \leq C\|\partial_n u_{|\partial\Omega}\|_{L^2(\Gamma'_a)} \tag{8}$$

may not for hold, for any fixed $s < 1$.

The lack of such sidewise observability inequalities is genuinely a multi-d phenomenon (see section 6). By the contrary, as shown in [22] and [25] by means of sidewise energy estimates, in 1-d , inequality (6) holds for BV coefficients and under natural conditions on the length of the time-interval. Counterexamples generated by waves concentrated on the support of the source may not arise in 1-d since light rays hitting the boundary are only of hyperbolic type.

Going back to the multi-d case under consideration, the lack of observability inequalities of the form (8) shows that, necessarily, an infinite number of derivatives may be lost on the measurement of the sources g, and thus, one has to impose some added restrictions on them to prevent concentration phenomena like (7) (see the pseudo-differential condition in assumption A3 below).

Within this class of sources g, the sidewise observability inequality (4) will be proved under a microlocal geometrical condition (see assumption A2 below), inspired (but different !) from the Geometric Control Condition introduced in [3]. Roughly, it guarantees that all rays emanating from the support of the source reach the observation region without earlier bouncing on the support of the source. This condition is sharp in terms of the geometry of the support of the sources O and the measurement subset O' and also in what concerns the sidewise observability time.

Finally, to close this section, let us note some contributions of the Inverse Problems community to this question of sidewise observation. Indeed, many papers investigated this problems, working essentially in a Lorentzian manifold with boundary. We quote among others, the paper by P. Stefanov and Y. Yang [23], where the authors prove that the local Dirichlet to Neumann operator is an elliptic pseudo-differential operator of order 1. See also Hintz-Uhlmann and Zhai [11]. However, these results are obtained on microlocal subsets of the hyperbolic region of the boundary and do not allow to observe the source data on the elliptic neither the glancing set.

1.4. Extensions and open problems.

The methods of this paper could be employed to handle other related problems such as:

- The simultaneous initial and boundary source sidewise observation. We refer to [25] for a complete analysis in 1-d.
- The problem treated in [4] where, on an annular domain $\Omega = A(R_1, R_2) = \{x \in \mathbb{R}^n, R_1 < |x| < R_2\}$ of $\mathbb{R}^n$, initial data are observed out of measurements on the exterior part of the boundary, under suitable conditions on the sources with support on the interior boundary.

Similar questions on the sidewise boundary observability and source identification are also of interest for other models such as, for instance Schrödinger, plate and heat equations, the elasticity system and thermoelasticiy, all of them rather well understood in the control of classical boundary control. However, to our best knowledge, similar sidewise observability estimates are not available, even in the scalar case such as Schrödinger or heat equations. Their analysis would certainly require of significant further developments.

1.5. Structure of the paper

The paper is organized as follows. In Section 2 we state the main results, and Section 3 is devoted to present some preliminary results. Most of the tools presented here are classical and we recall them in order to standardize the notations and make the paper self-contained. We start with the geometrical setting and we present in particular the generalized bicharacteristic curves and the partition of the cotangent space of the boundary $T^*\partial\mathcal{L}$. We also introduce the spaces of pseudo-differential symbols that will play the role of test functions on which we build the microlocal defect measures, of great importance in the proof. In Section 4, we present a geometric consequence of Assumption A2 and we perform a pseudo-differential multiplier calculus up to the boundary, in the spirit of [17], that will play a central role in the proof. Section 5 is mostly devoted to the proof of the main result namely Theorem 2.3. In Section 6, we present the proof of Theorem 2.5, essentially based on the microlocal behavior of the solutions to (1). We also present the proof of Corollary 2.6 where we construct a sequence of boundary data (g_k) concentrating on the glancing set.

2. Statement of the results

2.1. Sidewise observability

Let Ω be a domain of $\mathbb{R}^n$ admissible in the sense of assumption A1, and $\mathcal{O}$ a subset of the boundary $\partial\Omega$ strictly concave. And consider $\mathcal{O}'$ a subset of $\partial\Omega$ such that $\overline{\mathcal{O}} \cap \overline{\mathcal{O}'} = \emptyset$. We start with the geometric condition we will impose to the pair $\{\mathcal{O}, \mathcal{O}'\}$.

First, we recall that given the cylinder $\mathcal{L} = \mathbb{R} \times \Omega$ with Ω of class C^∞, we can define the Melrose-Sjöstrand compressed cotangent bundle of $\mathcal{L}$, $T_b^*\mathcal{L} = T^*\mathcal{L} \cup T^*\partial\mathcal{L}$. In addition, the matrix $A = (a_{ij}(x))$ being also of class C^∞, we have a flow on $T_b^*\mathcal{L}$, constituted of generalized bicharacteristic curves of the wave operator

$$P_A = -\partial_t^2 + \sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(x)\partial_{x_i}),$$

the celebrated Melrose-Sjöstrand flow (see [20]). We refer the reader to Section 3.2 for further details and precise definitions of these facts.

In particular, we recall the partition of the cotangent bundle of the boundary $T^*\partial\mathcal{L}$ into elliptic, hyperbolic and glancing sets :

$$T^*\partial\mathcal{L} = \mathcal{E} \cup \mathcal{H} \cup \mathcal{G}. \tag{9}$$

Now, consider an open subset $\mathcal{O}$ of $\partial\Omega$, strictly concave in the sense of assumption A1, such that $\overline{O} \subset \mathcal{O}$ and $\overline{O} \cap \overline{\mathcal{O}'} = \emptyset$. One can easily check that this is possible since A1 is an open condition.

Assumption A2: SGCC
We assume that there exists a time $T_0 > 0$ such that every generalized bicharacteristic curve issued from the boundary $\mathcal{O}$ at $t = 0$, intersects the boundary O' at a strictly gliding point , without intersecting $\overline{\Gamma}$, and before the time T_0.

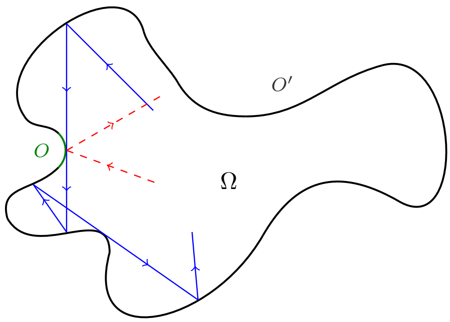

Fig. 3. Bicharacteristic rays passing throw O

Remark 2.1. (1) *The definition of strictly gliding point of the boundary will be given in Section 3.2.*

(2) *The notation (SGCC) stands for sidewise geometric control condition. In what follows, we provide some precisions.*

(3) *Set $\mathcal{U} = \mathbb{R} \times \mathcal{O}$. The generalized bicharacteristic curves issued from points of the boundary $\mathcal{U}$ are of two types and can be described through their projection on the basis, i.e the $(t, x)-$space. On one hand we have the curves that are transverse to $\partial\mathcal{L}$ and in this case we have two hyperbolic fibers issued from the same hyperbolic point $m_0 \in \partial\mathcal{L}$. At m_0, we have a hyperbolic reflection. On the other hand, the curve is tangent to $\partial\mathcal{L}$ at m_0 (one order tangency) and lies in $\mathcal{L} = \mathbb{R} \times \Omega$ otherwise. In the latter case, the generalized bicharacteristic curve can be interpreted as a "free bicharacteristic curve" since it's an integral curve of the hamiltonian field attached to the wave symbol (see Section 3.2).*
Condition (SGCC) requires that each one of these curves starting from $\mathcal{U}$ at $t = 0$, to intersect the boundary Γ' at a strictly gliding point , without intersecting $\overline{\Gamma}$, and before the time T_0. In this sense, this condition is stronger than the classical (GCC) of Bardos, Lebeau and Rauch [3] that needs the rays to hit $\partial\Omega$ at non diffractive points.

(4) *For instance if $\gamma = \gamma(s)$ is a ray issued from $\mathcal{U}$, we have $\gamma(0) = \rho \in T_b^*\mathcal{L}_{|\mathcal{U}}$, $\gamma(s_0) = \rho_1 \in T_b^*\mathcal{L}_{|\Gamma'}$ for some $s_0 \in]0, T_0[$, where ρ_1 is a strictly gliding point, and moreover $\gamma(s) \notin T_b^*\mathcal{L}_{|\overline{\Gamma}}$ for $0 < s < s_0$.*

In particular we can allow $\gamma(s)$ to live on the boundary, outside $T_b^*\mathcal{L}_{|\overline{\Gamma}}$ for some values of $s \in]0, s_0[$.

(5) *Notice that we don't make any assumption on the rays that don't intersect the open set $\mathcal{U}$ of the boundary. From this point of view, (SGCC) is weaker than the classical condition (GCC).*

(6) *Remark that if O is strictly convex, then obviously, (SGCC) cannot be satisfied (see Fig.4). Therefore, assumption A1 seems to be a well adapted framework to set up the microlocal condition A2.*

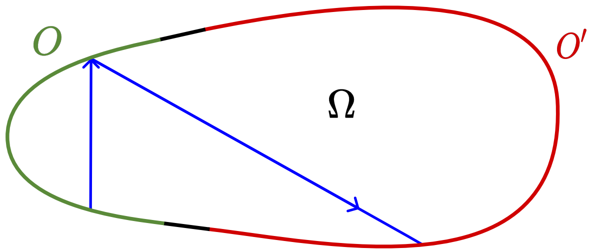

Fig. 4. Convex boundary. In blue, a geodesic ray.

Finally, we introduce the last assumption, namely a boundary condition on the data g. For this purpose, we recall that the lateral boundary $\partial\mathcal{L}$ of the cylinder $\mathcal{L} = \mathbb{R}\times\Omega$ is a submanifold of $\mathbb{R}^{n+1}$, of dimension n and class C^∞. We will denote by $(t, x') = (t, x'_1, ..., x'_{n-1})$ a system of local coordinates on $\partial\mathcal{L}$.

Assumption A3: Boundary condition fulfilled by observable sources

We assume one of the following conditions :

A3.a There exists a polyhomogeneous pseudo-differential operator $B_\alpha = b_\alpha(t, x'; D_t, D_{x'})$ on $\partial\mathcal{L}$, of order $\alpha > 0$, such that $Char B_\alpha \subset \mathcal{H}$ and

$$b_\alpha(t, x'; D_t, D_{x'})g = 0. \tag{10}$$

A3.b There exists a family of polyhomogeneous pseudo-differential operators $c_\alpha(t, x'; D_{x'})$ in the x'-variable on $\partial\mathcal{L}$, smooth with respect to respect to t, elliptic of order $\alpha > 0$ such that

$$c_\alpha(t, x'; D_{x'})g = 0. \tag{11}$$

A3.c There exists $\mathcal{U}_M$ an open neighborhood of $\overline{\Gamma}_M$ in $\partial\mathcal{L}$, there exists $\alpha > 0$ and a constant $C_\alpha > 0$ such that for every u solution of system (1), the boundary trace $(\partial_n u + \partial_t u)_{|\partial\mathcal{L}}$ satisfies

$$\|(\partial_n u + \partial_t u)_{|\partial\mathcal{L}}\|_{H^\alpha(\mathcal{U}_M)} \leq C_\alpha \|g\|_{H^1(\Gamma_M)}. \tag{12}$$

Remark 2.2. *For the definition of polyhomogeneous pseudo-differential operators on $\partial\mathcal{L}$, see Section 3.3. In particular, we recall that the characteristic set of $B_\alpha = b_\alpha(t,x';D_t,D_{x'})$ is given by*

$$Char B_\alpha = \{(t,x';\tau,\xi') \in T^*\partial\mathcal{L},\ \sigma(b_\alpha)(t,x';\tau,\xi') = 0\}$$

where $\sigma(b_\alpha)$ is the principal symbol of B_α.

We are now ready to state our main theorem.

Theorem 2.3. *Under assumptions A1, A2 and A3, for every $T > T_0$, there exists $C > 0$ such that every solution of (1), satisfies the observability estimate*

$$\|g\|_{H^1(\Gamma_M)} \le C\|\partial_n u_{|\Gamma'}\|_{L^2(\Gamma'_{M+T})}. \tag{13}$$

Remark 2.4. (1) *In case assumption A3.a is satisfied, we can relax assumptions A1 and A 2. Indeed, we may only assume the subset O of the boundary $\partial\Omega$ to be concave and not necessarily strictly concave. In particular, it can be locally a hyperplane. In addition, we may assume A2 only for transverse (hyperbolic) rays.*

(2) *Condition A3.b ensures some à priori spatial regularity on the data g, yielding microlocal regularity of g near the elliptic and the glancing sets of the boundary. For instance, it is fulfilled if g doesn't depend on the space variable x', i.e $g = g(t)$. In the same spirit, if we assume*

$$\|\nabla_{x'} u_{|\partial\mathcal{L}}\|_{H^\alpha(\mathcal{U}_M)} \le C_\alpha \|g\|_{H^1(\Gamma_M)},$$

for some $\alpha > 0$, we get the same positive conclusion, as a byproduct of the previous argument .

(3) *In Assumption A3.c , the open set $\mathcal{U}_M$ can be taken in the form $(-\varepsilon, M+\varepsilon)\times \mathcal{O}$, where $\mathcal{O}$ is an open neighborhood of $\overline{O}$ in $\partial\Omega$. This condition can be interpreted as a conditional stability assumption. See for instance V. Isakov* [14].

(4) *Obviously, the three conditions a), b) and c) of Assumption A3 are each of them sufficient and complementary. One could consider other assumptions guaranteeing the conclusion of Theorem* 2.3.

(5) *In the setting of assumption A3.a, one can for instance, consider the case where the boundary data g is subject to a wave equation. With $\chi = \chi(t,x) \in C_0^\infty(\Gamma_M)$, consider the system*

$$\begin{cases} P_A u = -\partial_t^2 u + \sum_{i,j=1}^n \partial_{x_j} a_{ij}(x)\partial_{x_i} u = 0 \quad \text{in } \mathcal{L} \\ u(t,.) = \chi(t,x) g(t,.) \quad \text{on } \partial\mathcal{L} \\ P'_A g = -\partial_t^2 g + \beta \sum_{i,j=1}^{n-1} \partial_{x'_j} a_{ij}(x',0)\partial_{x'_i} g = 0 \quad \text{on } \partial\mathcal{L} \\ u(0,.) = \partial_t u(0,.) = 0 \quad \text{on } \Omega \\ g(0,.) = g_0 \in H^1(\partial\mathcal{L}), \quad \text{and} \quad \partial_t g(0,.) = g_1 \in L^2(\partial\mathcal{L}) \end{cases} \tag{14}$$

where $\beta > 0$. One can easily check that assumption A3.a is fullfilled as soon as $\beta > 1$. However, if $\beta \leq 1$, the characteristic set of P'_A is contained in the union $\mathcal{E} \cup \mathcal{G}$ of the elliptic set and the glancing set. In this case, one can construct a sequence of sources (g_k) such that the corresponding sequence of solutions (u_k) to system (14) violates the observability estimate (13), with a loss of compactness located in $\mathcal{E}$ or $\mathcal{G}$, see Theorems 2.5 and 2.6.

(6) *To summarize: Even if, thanks to (SGCC), we can microlocally control the source g near the hyperbolic set of $\partial\mathcal{L}$, it still may develop singularities on the elliptic set, and/or travelling along some characteristic curves of the glancing set. In fact, as we will see in the proof of Theorem 2.3 the analysis on these sets requires a special attention. Assumption A3.a , A3.b or A3.c above are set to insure additional regularity on g that avoids the rising of such singularities.*

2.2. *On the lack of sidewise observability*

We present now the results concerning the lack of observability, even in the weaker version (8). These negative results ensure a loss of an infinite number of derivatives for all possible geometric configurations. Here we do not need any of the geometric conditions A1 or A2, that is, we work on a general bounded and smooth domain Ω and any partition of its boundary.

The proofs of these results will be given in Section 6.

Theorem 2.5. *For every $s < 0$, there exists a sequence of sources $(g_k)_{k\geq 1} \subset H^1(\partial\mathcal{L})$ supported in $\overline{\Gamma}_M$, such that the solutions (u_k) of system (1) satisfy*

$$\lim_{k\to\infty} \|g_k\|_{H^s(\Gamma_M)} = 1 \quad \textit{and} \quad \lim_{k\to\infty} \|\partial_n u_{k|\partial\Omega}\|_{L^2(\Gamma'_{M+T})} = 0, \tag{15}$$

for every $T > 0$. In particular, the lack of compactness of the sequence (g_k) in $H^s(\Gamma_M)$ is located in the elliptic set $\mathcal{E}$ of the boundary.

And we deduce from this theorem :

Corollary 2.6. *For every $s < 0$, there exists a sequence of sources $(g_k)_{k\geq 1} \subset H^1(\partial\mathcal{L})$ supported in $\overline{\Gamma}_M$, such that the solutions (u_k) of system (1) satisfy (15) for every $T > 0$.*

In particular, the lack of compactness of the sequence (g_k) in $H^s(\Gamma_M)$ is located in the glancing set $\mathcal{G}$ of the boundary.

Remark 2.7. *Actually, as we will see in the proof (cf. Section 6), we choose a sequence (g_k) supported in $\overline{\Gamma}_M = [0, M] \times \overline{O}$ such that for some fixed $\alpha > 1$, $\|g_k\|_{H^\alpha}$ is bounded outside the elliptic set $\mathcal{E}$ of the boundary. The propagation of the H^α-wave front will then provide the desired result.*

Remark 2.8. *In view of Theorem 2.5, we can not expect the sidewise observability estimate (13) to hold, unless an infinite number of derivatives is lost. Therefore, in order to get sidewise observability estimates in Sobolev norms, structural conditions on the sources need to be imposed, such as those of assumption A3.*

Remark 2.9. *To close this section and before going into the proofs, let us summarize the strategy one should follow to obtain sidewise observability for system* (1)*.*

First, we have to adress the problem only on well designed domains Ω*, i.e those satisfying assumption A1. Secondly, we choose the measurements domain, i.e a subset* O' *of the boundary* $\partial\Omega$*,* $\overline{O} \cap \overline{O'} = \emptyset$*, as sharp as possible, such that (SGCC) is fullfilled. For instance, in the case of the annular domain (Fig.1), if* O *is the interior boundary, then* O' *is the exterior boundary. And finally, we make sure that the boundary source* g *we aim to observe is admissible, i.e it satisfies some à priori condition in the spirit of condition A3, that prevents the presence of invisible solutions.*

3. Some Geometric and microlocal Facts, Operators and Measures

In this section we present a detailed description of the geometric framework and the various microlocal tools we use in this paper. In particular, we recall the geodesic local coordinates near the boundary and the cotangent bundle of Melrose-Sjöstrand $T_b^*\mathcal{L}$. Also, we detail the important notion of Melrose-Sjöstrand generalized bicharacteristic flow that governs all the propagation phenomena (of wave fronts sets and microlocal defect measures supports) . In addition, we define the spaces of pseudo-differential operators that we will use in the paper. All these notions are by now well known. We refer the reader to the book of L.Hörmander [13, Chapter 24] and the seminal article of Melrose-Sjöstrand [20] for further details on microlocal analysis and propagation of the wave front set. We also quote the papers of Gérard [9], Lebeau[16], Burq-Gérard [7] , Burq [5] and Burq-Lebeau [8] for an introduction of the microlocal defect measures and semi-classical measures and their propagation properties.

3.1. Geometry

Near a point m_0 of the boundary $\partial\Omega$, taking advantage of the regularity of Ω, we can define a system of geodesic local coordinates $x = (x_1, x_2, ..., x_n) \longrightarrow y = (y_1, y_2, ..., y_n)$ such that

$$\Omega = \{(y_1, y_2, ..., y_n),\ y_n > 0\}, \quad \partial\Omega = \{(y_1, y_2, ..., y_{n-1}, 0)\} = \{(y', 0)\}$$

where the wave operator is given by

$$P_A = -\partial_t^2 + \Big(\partial_{y_n}^2 + \sum_{1\le i,j\le n-1} \partial_{y_j} b_{ij}(y)\partial_{y_i}\Big) + M_0(y)\partial_{y_n} + M_1(y, \partial_{y'}).$$

Here, the matrix $(b_{ij}(y))_{ij}$ is of class C^∞, symmetric, uniformly definite positive on a neighborhood of m_0, $M_0(y)$ is a real valued function of class C^∞, and $M_1(y, \partial_{y'})$ is a tangential differential operator of order 1 with C^∞ coefficients.

In the sequel, we will come back to the notation $(t, x) = (t, x', x_n) = (t, y', y_n)$, and we shall write

$$P_A = \partial_n^2 + R(x_n, x', D_{x',t}) + M_0(x)\partial_n + M_1(x, \partial_{x'})$$

Notice that, in this coordinates system, the principal symbol of the wave operator P_A is given by

$$\sigma(P_A) = -\xi_n^2 + r(x,\tau,\xi') = -\xi_n^2 + \Big(\tau^2 - \sum_{1\le i,j\le n-1} a_{ij}(x)\xi_i\xi_j\Big).$$

We shall set $r_0(x',\tau,\xi') = r(x',0,\tau,\xi')$ and we denote $m_1 = m_1(x,\xi')$ the symbol of the vector field M_1.

3.2. Generalized bicharacteristic rays

Let us introduce the compressed cotangent bundle of Melrose-Sjöstrand $T_b^*\mathcal{L} = T^*\mathcal{L} \cup T^*\partial\mathcal{L}$. We recall that we have a natural projection

$$\pi \ : \ T^*\mathbb{R}^{n+1} \mid_{\overline{\Omega}} \longrightarrow T_b^*\mathcal{L} \tag{16}$$

and we equip $T_b^*\mathcal{L}$ with the induced topology.

Given the matrix $A(x) = (a_{ij}(x))$, we denote by $p_A(x;\tau,\xi) = \tau^2 -^t \xi A(x)\xi$, the principal symbol of the wave operator, and

$$\mathrm{Char}(P_A) = \{(t,x;\tau,\xi), p_A(x,\tau,\xi) = \tau^2 -^t \xi A(x)\xi = 0\},$$

the characteristic set, and $\Sigma_A = \pi(Char(P_A))$. In addition, we recall the hamiltonian field associated to p_A

$$H_{p_A} = 2\tau\partial_t - 2^t\xi A(x)\partial_x + \sum_{k=1}^{n} {}^t\xi\partial_{x_k}A(x)\xi\partial_{\xi_k}.$$

Also, we recall the following partition of $T^*\partial\mathcal{L}$ into elliptic, hyperbolic and glancing sets:

$$\#\Big\{\pi^{-1}(\rho) \cap Char(P_A)\Big\} = \begin{cases} 0 & if \quad \rho \in \mathcal{E} \\ 1 & if \quad \rho \in \mathcal{G} \\ 2 & if \quad \rho \in \mathcal{H} \end{cases} \tag{17}$$

For the sake of simplicity, we will develop the rest of this section in a system of local geodesic coordinates as introduced in section 3.1. We recall that we have locally

$$\mathcal{L} = \{(t,x) \in \mathbb{R}^{n+1}, x_n > 0\} \quad \text{and} \quad \partial\mathcal{L} = \{(t,x) \in \mathbb{R}^{n+1}, x_n = 0\}.$$

We also get :

$$\mathcal{E} = \{r_0 < 0\}, \qquad \mathcal{H} = \{r_0 > 0\}, \qquad \mathcal{G} = \{r_0 = 0\}.$$

Notice that using the projection π, one can identify the glancing set $\mathcal{G}$ with a subset of $T^*\mathbb{R}^{n+1}$.

Definition 3.1. (1) *A point* $\rho \in T^*\partial\mathcal{L}\backslash 0$ *is nondiffractive if* $\rho \in \mathcal{H}$ *or if* $\rho \in \mathcal{G}$ *and the free bicharacteristic* $(\exp sH_{p_A})\widetilde{\rho}$ *passes over the complement of* $\overline{\mathcal{L}}$ *for arbitrarily small values of* s*, where* $\widetilde{\rho}$ *is the unique point in* $\pi^{-1}(\rho) \cap Char(P_A)$.

(2) $\rho \in T^*\partial\mathcal{L}\backslash 0$ *is strictly gliding if* $\rho \in \mathcal{H}$ *or if* $\rho \in \mathcal{G}$ *and* $H^2_{p_A}(x_n)(\rho) < 0$.
In the latter case, the free bicharacteristic ray γ *issued from* ρ *leaves the boundary* $\partial\mathcal{L}$ *and enters in* $T^*(\mathbb{R}^{n+1} \setminus \overline{\mathcal{L}})$ *at* $\widetilde{\rho} = \pi^{-1}(\rho)$.

(3) $\rho \in T^*\partial\mathcal{L}\backslash 0$ *is strictly diffractive if* $\rho \in \mathcal{G}$ *and* $H^2_{p_A}(x_n)(\rho) > 0$.
This means that there exists $\varepsilon > 0$ *such that* $(\exp sH_{p_A})\widetilde{\rho} \in T^*\mathcal{L}$ *for* $0 < |s| < \varepsilon$.

Definition 3.2. *We shall denote by* $\mathcal{G}_d$ *the set of strictly diffractive points and by* $\mathcal{G}_{sg}$ *the set of strictly gliding points.*

Remark 3.3. (1) *Under assumption A1, we notice that over* Γ*, the glancing set* $\mathcal{G}$ *is reduced to* $\mathcal{G}_d$*, i.e*

$$\mathcal{G}_{|\Gamma} \subset \mathcal{G}_d.$$

Namely all generalized bicharacteristic curves issued from points of $\mathcal{G}_{|\Gamma}$ *have a first order tangency with the boundary .*

(2) *In local geodesic coordinates, the sets* $\mathcal{G}_d$ *and* $\mathcal{G}_{sg} \setminus \mathcal{H}$ *are given by*

$$\mathcal{G}_d = \{\xi_n = r_0 = 0,\, \partial_n r_{|x_n=0} > 0\}, \qquad \text{and} \qquad \mathcal{G}_{sg} \setminus \mathcal{H} = \{\xi_n = r_0 = 0,\, \partial_n r_{|x_n=0} < 0\}. \tag{18}$$

Definition 3.4. *A generalized bicharacteristic ray is a continuous map*

$$\mathbb{R} \supset I \setminus B \ni s \mapsto \gamma(s) \in T^*\mathcal{L} \cup \mathcal{G} \subset T^*\mathbb{R}^{n+1}$$

where I *is an interval of* $\mathbb{R}$*,* B *is a set of isolated points, for every* $s \in I \setminus B$*,* $\gamma(s) \in \Sigma_A$ *and* γ *is differentiable as a map with values in* $T^*\mathbb{R}^{n+1}$*, and*

(1) *If* $\gamma(s_0) \in T^*\mathcal{L} \cup \mathcal{G}_d$ *then* $\dot\gamma(s_0) = H_{p_A}(\gamma(s_0))$.

(2) *If* $\gamma(s_0) \in \mathcal{G} \setminus \mathcal{G}_d$ *then* $\dot\gamma(s_0) = H^G_{p_A}(\gamma(s_0))$*, where* $H^G_{p_A} = H_{p_A} + (H^2_{p_A}x_n/H^2_{x_n}p_A)H_{x_n}$.

(3) *For every* $s_0 \in B$*, the two limits* $\gamma(s_0 \pm 0)$ *exist and are the two different points of the same hyperbolic fiber of the projection* π.

Remark 3.5. (1) *We recall that if* Ω *has no contact of infinite order with its tangents, the Melrose-Sjöstrand flow is globally well defined.*

(2) *In the interior, i.e in* $T^*\mathcal{L}$*, a generalized bicharacteristic is simply a classical bicharacteristic ray of the wave operator whose projection on the basis is a geodesic of* Ω *equipped with the metric* $(a^{ij}) = (a_{ij})^{-1}$.

(3) *Finally,* γ *can be considered as a continuous map on the interval* I *with values in* $T^*_b\mathcal{L}$.

3.3. Pseudo-differential operators

In this section, we introduce the classes of pseudo-differential operators we shall use in this paper. We start with the operators on the cylinder $\mathcal{L}$.

Let $\mathcal{A}$ be the set of pseudo-differential operators of the form $Q = Q_i + Q_\partial$ where Q_i is a classical pseudo-differential operator , compactly supported in $\mathcal{L}$ and Q_∂ is a classical tangential pseudo-differential operator, compactly supported near $\partial\mathcal{L}$. More precisely, $Q_i = \varphi Q_i \varphi$ for some $\varphi \in C_0^\infty(\mathcal{L})$ and $Q_\partial = \psi Q_\partial \psi$ for some $\psi(t, x_n) \in C^\infty(\mathbb{R}\times] - \alpha, \alpha[)$. $\mathcal{A}^s$ will denote the elements of $\mathcal{A}$ of order s.

On the other hand, the boundary $\partial\mathcal{L} = \mathbb{R} \times \partial\Omega$ is a smooth manifold of dimension n without boundary. Following L.Hörmander [13] and using a system of local charts, we can define for $m \in \mathbb{R}$, the space of polyhomogeneous pseudo-differential operators $\Psi^m_{phg}(\partial\mathcal{L})$ on $\partial\mathcal{L}$, associated with symbols in $S^m_{phg}(T^*\partial\mathcal{L})$. These operators enjoy all classical properties of continuity and composition.

3.4. *Microlocal defect measures*

Here we use notations of section 3.2. Denote

$$\begin{cases} Z = \pi(CharP_A), & \hat{Z} = Z \cup \pi(T^*\overline{\mathcal{L}}_{|x_n=0}), \\ SZ = (Z \setminus \overline{\mathcal{L}})/\mathbb{R}^*_+, & S\hat{Z} = (\hat{Z} \setminus \overline{\mathcal{L}})/\mathbb{R}^*_+. \end{cases}$$

and for $Q \in \mathcal{A}^0$ with principal symbol $\sigma(Q) = q$, set

$$\kappa(q)(\rho) = q(\pi^{-1}(\rho)).$$

We define also for $u \in H^1(\mathcal{L})$

$$\phi(Q, u) = (Qu, u)_{H^1} = \int_{\mathcal{L}} \Big(\nabla_{t,x} Qu.\nabla_{t,x}\overline{u} + Qu.\overline{u}\Big) dxdt.$$

Finally, let (u_k) be a sequence of functions weakly converging to 0 in $H^1_{loc}(\mathcal{L})$. In [16] and [8], the authors prove the following result:

Theorem 3.6 (Burq-Lebeau [8]). *There exists a subsequence of (u_k) (still denoted by (u_k)) and a positive Radon measure μ on $S\hat{Z}$ such that*

$$\lim_{k\to\infty} \phi(Q, u_k) = \langle \mu, \kappa(q) \rangle, \qquad \forall Q \in \mathcal{A}^0.$$

We will refer to μ as a microlocal defect measure associated to the sequence (u_k).

On the other hand, on the boundary $\partial\mathcal{L}$, we can make use of the classical notion of microlocal defect measure introduced by P. Gérard in [9]. More precisely, for every sequence of functions (v_k) weakly converging to 0 in $H^1_{loc}(\partial\mathcal{L})$, there exists a positive Radon measure $\tilde{\mu}$ on $S^*(\partial\mathcal{L})$ such that we have, up to a subsequence

$$\lim_{k\to\infty} (Qv_k, v_k)_{L^2(\partial\mathcal{L})} = \langle \tilde{\mu}, |\eta|^{-2}\sigma(Q) \rangle, \qquad \forall Q \in \Psi^2_{phg}(\partial\mathcal{L}).$$

Here we have denoted by (y, η) the standard element of $T^*(\partial\mathcal{L}) \setminus 0$.

We will remind the properties of these measures in some steps of the proof later, see Section 5.3.

4. Preliminary results

4.1. A Geometric Lemma

Let O (resp. $\mathcal{O}$) be the open subset of $\partial\Omega$ introduced in the statement of Assumption A1 (resp. A2), and set $\mathcal{U} = \mathbb{R} \times \mathcal{O}$. Consider V a neighborhood of $\overline{O}$ in $\mathbb{R}^n$ such that $V \cap \partial\Omega \subset \mathcal{O}$. $\mathbb{R} \times V$ is an open neighborhood of $\overline{\Gamma} = \mathbb{R} \times \overline{O}$ in $\mathbb{R}^{n+1}$. In this setting $W = \mathbb{R} \times (V \cap \Omega) = (\mathbb{R} \times V) \cap \mathcal{L}$ is an interior neighborhood of the boundary $\overline{\Gamma}$ (see Figure 5). On the other hand, consider $\rho \in T^*W \cap Char(P_A)$ and denote $\gamma = \gamma(s)$ the generalized bicharacteristic issued from ρ, i.e $\gamma(0) = \rho$. In addition, we define by $\gamma^+ = \{\gamma(s), s > 0\}$, resp. $\gamma^- = \{\gamma(s), s < 0\}$ the outcoming half bicharacteristic and the incoming half bicharacteristic at ρ, see Figure 5.

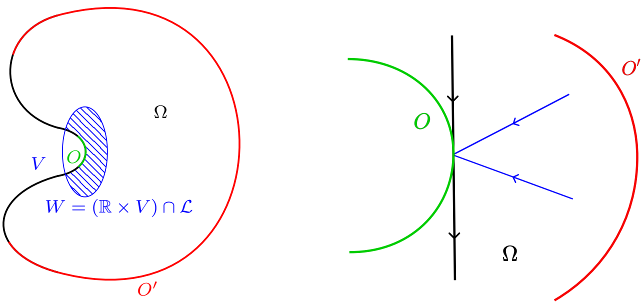

Fig. 5. On the left, interior neighborhood of Γ.
On the right, tangent (black) and hyperbolic (blue) half bicharacteristic rays.

Lemma 4.1. *With the notations above and under assumptions A1 and A2, for every $T > T_0$, there exists V neighborhood of $\overline{O}$ in $\mathbb{R}^n$, $V \cap \partial\Omega \subset \mathcal{O}$, such that for every $\rho \in T^*(W) \cap Char(P_A)$, one of the two half bicharacteristics issued from ρ, the outcoming one or the incoming one, travelling at speed one, intersects the boundary Γ' at a strictly gliding point, without intersecting the boundary $\overline{\Gamma}$, and before the time T.*

We will say that this half bicharacteristic satisfies (SGCC).

Proof. For $\rho \in T^*W \cap Char(P_A)$, denote by $\gamma_\rho = \{\gamma_\rho(s), s \in \mathbb{R}\}$ the generalized bicharacteristic issued from ρ. In particular, $\gamma_\rho(0) = \rho$. Assume that γ_ρ intersects $\mathcal{U}$ for some value $s_1 < 0$ at a hyperbolic or at a glancing point. According to assumption A2, we then get that for some $s \in \mathbb{R}$ such that $s - s_1 < T_0$, $\gamma_\rho(s)$ is a strictly gliding point of the boundary Γ' and, in addition $\{\gamma_\rho(s'), s_1 < s' < s\} \cap T_b^*\mathcal{L}_{|\overline{\Gamma}} = \emptyset$. In this case, we see that the statement of Lemma 4.1 is satisfied by the outcoming half bicharacteristic issued from ρ. Obviously, the case $s_1 > 0$ can be treated in a similar way. According to this, we may only focus on the points ρ close to $\overline{\Gamma}$ such that $\gamma_\rho = \{\gamma_\rho(s), s \in \mathbb{R}\}$ doesn't intersect $\overline{\Gamma}$ for $s \in]-T_0, T_0[$. In addition, due to the compactness of $\overline{O}$, it suffices to prove that every glancing point $\rho \in \mathcal{G}_{|\mathcal{U}} \subset T^*\partial\mathcal{L}_{|\mathcal{U}}$ admits a neighborhood V_ρ in $T^*(\mathbb{R}^{n+1})$ such that conclusion of Lemma 4.1 is valid for every $\rho' \in V_\rho \cap T^*\mathcal{L}$.

Before entering in the details of the proof, we warn the reader that if a generalized bicharacteristic γ_ρ hits the boundary transversally for some value s_0, that is at a hyperbolic point, we will denote this point by $\gamma_\rho(s_0)$, by abuse of notation.

Consider then $\rho \in \mathcal{G}_{|\mathcal{U}} \subset T^*\partial\mathcal{L}_{|\mathcal{U}}$ and let $s_0 \in]0, T_0[$ be a time such that the generalized bicharacteristic γ_ρ hits the boundary Γ' at a strictly gliding point. Here we have two possibilities : a) $\gamma_\rho(s_0)$ is a hyperbolic point or b) $\gamma_\rho(s_0)$ a glancing strictly gliding point. We will discuss each one of these cases, and in order to simplify the argument, we will work in local geodesic coordinates.

- Case a) : $\gamma_\rho(s_0)$ is a hyperbolic point. With the notations of Definition 3.4, $s_0 \in B_\rho$ where B_ρ is a set of isolated points in $\mathbb{R}$ such that the two limits $\gamma_\rho(s_0 \pm 0)$ exist and are the two different points of the same hyperbolic fiber of the projection π. Furthermore, we have

$$H_{p_A} x_n(\gamma_\rho(s_0 - 0)) = \frac{dx_n}{ds}(\gamma_\rho(s_0 - 0)) = -2\xi_n(\gamma_\rho(s_0 - 0)) < 0. \tag{19}$$

Consequently, for $\varepsilon > 0$ small enough, $\gamma_\rho(s_0 - \varepsilon)$ is an interior point, moreover, the x_n and ξ_n- coordinates satisfy

$$-2\xi_n(\gamma_\rho(s)) = \frac{dx_n}{ds}(\gamma_\rho(s)) \le -c, \quad \forall s \in [s_0 - \varepsilon, s_0[, \qquad \text{for some} \quad c > 0. \tag{20}$$

This yields

$$\xi_n(\gamma_\rho(s)) \ge c/2, \quad \forall s \in [s_0 - \varepsilon, s_0[. \tag{21}$$

In addition, we may assume that $0 < x_n(\gamma_\rho(s_0 - \varepsilon)) < \eta$ for some $\eta > 0$ to be chosen later. Now we fix $\varepsilon > 0$. Taking into account the continuity of the Melrose-Sjöstrand flow, it's clear that for $0 < \alpha < \frac{1}{4} x_n(\gamma_\rho(s_0 - \varepsilon))$, one can find V_ρ a small enough neighborhood of ρ in $T^*\mathbb{R}^{n+1}$, such that for all $\rho' \in V_\rho \cap T^*\mathcal{L} \cap Char(P_A)$,

$$|x_n(\gamma_\rho(s_0 - \varepsilon)) - x_n(\gamma_{\rho'}(s_0 - \varepsilon))| \le \alpha, \tag{22}$$

and

$$\xi_n(\gamma_{\rho'}(s)) \ge c', \quad \forall s \in [s_0 - \varepsilon, s_0[, \tag{23}$$

for some $c' > 0$. In particular, this means that $\gamma_{\rho'}(s_0 - \varepsilon)$ is an interior point since

$$x_n(\gamma_{\rho'}(s_0 - \varepsilon)) \ge \frac{3}{4} x_n(\gamma_\rho(s_0 - \varepsilon)) > 0. \tag{24}$$

In addition, notice that estimate (23) is valid as long as $x_n(\gamma_{\rho'}(s)) > 0$, so possibly for $s \in]s_0 - \varepsilon, s_0 + \beta[$, $\beta > 0$ small . Finally,

$$\begin{cases} x_n(\gamma_{\rho'}(s)) \le x_n(\gamma_{\rho'}(s_0 - \varepsilon)) - 2c'(s - s_0 + \varepsilon) \\ \le \frac{5}{4} x_n(\gamma_\rho(s_0 - \varepsilon)) - 2c'(s - s_0 + \varepsilon) \le \frac{5}{4}\eta - 2c'(s - s_0 + \varepsilon) \end{cases} \tag{25}$$

Consequently, we obtain that $x_n(\gamma_{\rho'}(s))$ vanishes for some $s \geq s_0 + \frac{5}{8c'}\eta - \varepsilon$, which means that the bicharacteristic ray $\gamma_{\rho'}$ leaves $\mathcal{L}$ at a hyperbolic point before the time $T > T_0$, as soon as $\frac{5}{8c'}\eta - \varepsilon < T - T_0$.

- Case b) : $\gamma_\rho(s_0)$ is a glancing strictly gliding point. According to Definition 3.1, we know in this case that

$$x_n(\gamma_\rho(s_0)) = r(\gamma_\rho(s_0)) = 0 \qquad \text{and} \qquad \frac{\partial r}{\partial x_n}(\gamma_\rho(s_0)) < 0. \tag{26}$$

Let then $B(\gamma_\rho(s_0), \varepsilon)$ be the open ball of $T^*\mathbb{R}^{n+1}$ with center $\gamma_\rho(s_0)$ and radius ε. It's clear that for ε and $c > 0$ suitable, one has

$$\frac{\partial r}{\partial x_n}(\zeta) \leq -c, \qquad \forall \zeta \in B(\gamma_\rho(s_0), \varepsilon). \tag{27}$$

Moreover, for $\eta \in]0, \varepsilon[$ small enough, using again the continuity of the Melrose-Sjöstrand flow, we may find V_ρ, a neighborhood of ρ in $T^*\mathbb{R}^{n+1}$ such that for all $\rho' \in V_\rho \cap T^*\mathcal{L} \cap Char(P_A)$,

$$\gamma_{\rho'}(s_0) \in B(\gamma_\rho(s_0), \eta). \tag{28}$$

In this setting, two cases may occur :
i) $\gamma_{\rho'}(s_0)$ is a boundary point and necessarily $r(\gamma_{\rho'}(s_0)) \geq 0$. If $r(\gamma_{\rho'}(s_0)) > 0$ then $\gamma_{\rho'}(s_0)$ is a hyperbolic point. Otherwise, $r(\gamma_{\rho'}(s_0)) = 0$ and then it's a glancing strictly gliding point thanks to (27).
ii) $\gamma_{\rho'}(s_0)$ is an interior point (see Figure 6 below).

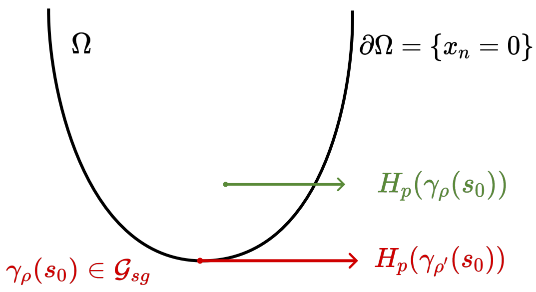

Fig. 6. Strictly gliding points

In this case, using the Hamiltonian field H_{p_A} , we get :

$$\frac{dx_n}{ds}(\gamma_{\rho'}(s_0)) = -2\xi_n(\gamma_{\rho'}(s_0)) \leq 2\eta. \tag{29}$$

Thus, if we denote in short $x_n(s) = x_n(\gamma_{\rho'}(s))$, we can perform a Taylor expansion and get in vue of (27) :

$$\begin{cases} x_n(s) = x_n(s_0) + \frac{dx_n}{ds}(s_0)(s - s_0) + \frac{1}{2}\frac{d^2x_n}{ds^2}(s_0)(s - s_0)^2 + o(s - s_0)^2 \\ \qquad\qquad \leq \eta + 2\eta(s - s_0) - c(s - s_0)^2 + o(s - s_0)^2. \end{cases} \tag{30}$$

Similarly, we obtain for the ξ_n - component of $\gamma_{\rho'}(s)$:

$$\begin{cases} \xi_n(s) = \xi_n(s_0) + \frac{d\xi_n}{ds}(s_0)(s-s_0) + o(s-s_0) \\ \qquad\qquad \geq -\eta + c(s-s_0) + o(s-s_0) \end{cases} \tag{31}$$

From (30) we deduce that $\gamma_{\rho'}(s)$ intersects the boundary before the time s_1 such that $s_1 - s_0 \approx \frac{1}{\sqrt{c}}\eta^{1/2}$. Furthermore, we conclude from (31) that $\xi_n(s) \geq \frac{\sqrt{c}}{2}\eta^{1/2}$ for s close to s_1, which means that $\gamma_{\rho'}(s_1)$ is a hyperbolic point of the boundary Γ'. Finally, we finish the argument by taking $\eta > 0$ such that $\frac{1}{\sqrt{c}}\eta^{1/2} < T - T_0$.

The proof of Lemma 4.1 is now complete. ■

4.2. *First computations*

We consider a family of pseudo-differential symbols in the class $\mathcal{A}^0$ introduced in section 3.3 above, tangential and classical. Since the result we seek is of local nature, we work in a system of geodesic coordinates near the boundary $\partial\mathcal{L}$ and choose these symbols in the form $q = q(x_n, x', t, \xi', \tau)$, and of class C^∞ with respect to x_n, real valued, compactly supported in (t, x', x_n), and independent of x_n in a strip $\{|x_n| < \beta\}$, $\beta > 0$ small enough. For instance, one may take q in the form $q(x_n, x', t, \xi', \tau) = \varphi(x_n)\tilde{q}(x', t, \xi', \tau)$, with $\varphi \in C_0^\infty(\mathbb{R})$, equal to 1 near $x_n = 0$. We shall denote by $Q = Q(x_n, x', t, D_{x',t})$ the corresponding tangential pseudo-differential operators .

In the proofs of theorem 2.3, we will make successive choices of symbols q.

We recall that in the system of local geodesic coordinates, the wave equation takes the form

$$\partial_n^2 u + R(x_n, x', D_{x',t})u + M_0(x)\partial_n u + M_1(x, \partial_{x'})u = 0. \tag{32}$$

We multiply the equation by $Q^2\partial_n\overline{u}$ and we integrate over $\mathcal{L}$.

$$\begin{cases} I_1 = \int_{\mathcal{L}} \partial_n^2 u\, Q^2\partial_n\overline{u} = -\int_{\partial\mathcal{L}} \partial_n u\, Q^2\partial_n\overline{u}\, d\sigma - \int_{\mathcal{L}} \partial_n u \partial_n\, Q^2\partial_n\overline{u} \\ = -\int_{\partial\mathcal{L}} \partial_n u\, Q^2\partial_n\overline{u}\, d\sigma - \int_{\mathcal{L}} \partial_n u[\partial_n,\, Q^2]\partial_n\overline{u} - \int_{\mathcal{L}} \partial_n u Q^2\partial_n^2\overline{u} \\ = -\int_{\partial\mathcal{L}} \partial_n u\, Q^2\partial_n\overline{u}\, d\sigma - \int_{\mathcal{L}} \partial_n u[\partial_n,\, Q^2]\partial_n\overline{u} - \int_{\mathcal{L}} Q^2\partial_n u\partial_n^2\overline{u} + \int_{\mathcal{L}} (Q^2 - Q^{*2})\partial_n u\partial_n^2\overline{u} \\ = -\int_{\partial\mathcal{L}} \partial_n u\, Q^2\partial_n\overline{u}\, d\sigma - \int_{\mathcal{L}} \partial_n u[\partial_n,\, Q^2]\partial_n\overline{u} - \int_{\mathcal{L}} Q^2\partial_n u\partial_n^2\overline{u} \\ \quad -\int_{\mathcal{L}} (Q^2 - Q^{*2})\partial_n u R\overline{u} - \int_{\mathcal{L}} M_0(Q^2 - Q^{*2})\partial_n u\partial_n\overline{u} - \int_{\mathcal{L}} (Q^2 - Q^{*2})\partial_n u M_1\overline{u} \end{cases} \tag{33}$$

$$\begin{cases} I_2 = \int_{\mathcal{L}} Ru\, Q^2 \partial_n \overline{u} = \int_{\mathcal{L}} Ru\, [Q^2, \partial_n]\overline{u} + \int_{\mathcal{L}} Ru\, \partial_n Q^2 \overline{u} \\ = -\int_{\partial\mathcal{L}} Ru\, Q^2 \overline{u} d\sigma - \int_{\mathcal{L}} (\partial_n R) u\, Q^2 \overline{u} - \int_{\mathcal{L}} \partial_n u\, R^* Q^2 \overline{u} - \int_{\mathcal{L}} Ru\, [\partial_n,\, Q^2]\overline{u} \\ = -\int_{\partial\mathcal{L}} Ru\, Q^2 \overline{u} d\sigma - \int_{\mathcal{L}} (\partial_n R) u\, Q^2 \overline{u} - \int_{\mathcal{L}} \partial_n u\, [R^*, Q^2]\overline{u} \\ \quad - \int_{\mathcal{L}} \partial_n u\, Q^2 R^* \overline{u} - \int_{\mathcal{L}} Ru\, [\partial_n,\, Q^2]\overline{u} \\ = -\int_{\partial\mathcal{L}} Ru\, Q^2 \overline{u} d\sigma - \int_{\mathcal{L}} (\partial_n R) u\, Q^2 \overline{u} - \int_{\mathcal{L}} \partial_n u\, [R^*, Q^2]\overline{u} - \int_{\mathcal{L}} Q^2 \partial_n u\, R\overline{u} \\ \quad - \int_{\mathcal{L}} (Q^{*2} - Q^2) \partial_n u\, R\overline{u} - \int_{\mathcal{L}} \partial_n u\, Q^2 (R^* - R)\overline{u} - \int_{\mathcal{L}} Ru\, [\partial_n,\, Q^2]\overline{u}. \end{cases} \tag{34}$$

Setting $f = M_0(x)\partial_n u + M_1(x, \partial_{x'})u$ and summarizing all the computations above, we obtain

$$\int_{\partial\mathcal{L}} \partial_n u Q^2 \partial_n \overline{u}\, d\sigma + \int_{\partial\mathcal{L}} Ru\, Q^2\, \overline{u}\, d\sigma + \int_{\mathcal{L}} (\partial_n R) u\, Q^2\ \overline{u} = 2\,\mathrm{Re} \int_{\mathcal{L}} f Q^2 \partial_n \overline{u} - \sum_{j=1}^{8} A_j. \tag{35}$$

We have $\int_{\mathcal{L}} f Q^2 \partial_n \overline{u} = \int_{\mathcal{L}} M_0 \partial_n u Q^2 \partial_n \overline{u} + \int_{\mathcal{L}} M_1 u Q^2 \partial_n \overline{u}$. The first term of the sum reads

$$\begin{aligned} \int_{\mathcal{L}} M_0 \partial_n u Q^2 \partial_n \overline{u} &= -\int_{\partial\mathcal{L}} M_0 u Q^2 \partial_n \overline{u}\, \partial\sigma - \int_{\mathcal{L}} (\partial_n M_0) u Q^2 \partial_n \overline{u} \\ &\quad - \int_{\mathcal{L}} M_0 u [\partial_n, Q^2] \partial_n \overline{u} - \int_{\mathcal{L}} M_0 u Q^2 \partial_n^2 \overline{u} \\ &= -\int_{\partial\mathcal{L}} M_0 u Q^2 \partial_n \overline{u}\, d\sigma - \int_{\mathcal{L}} (\partial_n M_0) u Q^2 \partial_n \overline{u} - \int_{\mathcal{L}} M_0 u [\partial_n, Q^2] \partial_n \overline{u} \\ &\quad + \int_{\mathcal{L}} M_0 u Q^2 R \overline{u} + \int_{\mathcal{L}} M_0 u Q^2 \overline{f}. \end{aligned} \tag{36}$$

Finally we obtain

$$\int_{\partial\mathcal{L}} \partial_n u Q^2 \partial_n \overline{u}\, d\sigma + \int_{\partial\mathcal{L}} Ru\, Q^2\, \overline{u}\, d\sigma + \int_{\mathcal{L}} u\, Q^2\, (\partial_n R)\, \overline{u} = \sum_{j=1}^{14} A_j \tag{37}$$

Remark 4.2. *In fact, we will see later that the remaining terms A_j for $j = 1, \ldots, 14$, as described below, do not play a role in our arguments, see Corollary 5.7 and Lemma 5.12.*

$$\begin{cases} A_1 = \int_{\mathcal{L}} \partial_n u[\partial_n, Q^2]\partial_n \overline{u},\ A_2 = -\int_{\mathcal{L}} \partial_n u(Q^{*2} - Q^2)R\overline{u},\ A_3 = \int_{\mathcal{L}}(Q^2 - Q^{*2})\partial_n u M_0 \partial_n u, \\ A_4 = \int_{\mathcal{L}}(Q^2 - Q^{*2})\partial_n u M_1 u,\ A_5 = \int_{\mathcal{L}} \partial_n u\,[R^*, Q^2]\overline{u},\ A_6 = \int_{\mathcal{L}}(Q^{*2} - Q^2)\partial_n u\, R\overline{u} \\ A_7 = \int_{\mathcal{L}} \partial_n u\, Q^2(R^* - R)\overline{u},\ A_8 = 2\,\mathrm{Re}\int_{\mathcal{L}}(\partial_n M_0)uQ^2\partial_n\overline{u},\ A_9 = 2\,\mathrm{Re}\int_{\partial\mathcal{L}} M_0 u Q^2 \partial_n \overline{u}\, d\sigma, \\ A_{10} = 2\,\mathrm{Re}\int_{\mathcal{L}} M_0 u[\partial_n, Q^2]\partial_n\overline{u},\ A_{11} = -2\,\mathrm{Re}\int_{\mathcal{L}} M_0 u Q^2 R\overline{u},\ A_{12} = -2\,\mathrm{Re}\int_{\mathcal{L}} M_0 u Q^2 \overline{f}, \\ A_{13} = -2\,\mathrm{Re}\int_{\mathcal{L}} M_1 u Q^2 \partial_n \overline{u}, \quad A_{14} = \int_{\mathcal{L}} Ru\,[\partial_n, Q^2]\overline{u} \end{cases} \tag{38}$$

5. Proof of Theorem 2.3

The proof relies on a classical strategy. We first establish a relaxed observability estimate, then we drop the compact term with the help of a unique continuation argument.

5.1. Relaxed observation and unique continuation

Proposition 5.1. *Under assumptions A1, A2 and A3, for every $T > T_0$, there exists $c > 0$ such that for every $g \in H^1(\partial\mathcal{L})$, $supp(g) \subset \overline{\Gamma}_M$, the solution u of (1), satisfies the observability estimate*

$$\|g\|_{H^1(\Gamma_M)} \leq c\|\partial_n u_{|\partial\Omega}\|_{L^2(\Gamma'_{M+T})} + c\|g\|_{L^2(\Gamma_M)}. \tag{39}$$

Also, we will need the following uniqueness result.

Lemma 5.2. *Assume that estimate (39) holds true for all $T > T_0$. Then for $g \in H^1(\partial\mathcal{L})$ with $supp(g) \subset \overline{\Gamma}_M$, if the solution u to system (1) satisfies $\partial_n u_{|\partial\Omega} \equiv 0$ on Γ'_{M+T}, then u vanishes identically. In particular, $g \equiv 0$.*

The proof of Lemma 5.2 is given at the end of this section and the proof of Proposition 5.1 will be the purpose of Section 5.2. Here, we first show how we can conclude the proof of Theorem 2.3 using these results.

For this , we use a contradiction argument. Assume that estimate (13) is false and consider a sequence of boundary data $(g_k) \in H^1(\partial\mathcal{L})$, $supp(g_k) \subset \overline{\Gamma}_M$, and (u_k) the sequence of associated solutions, with

$$\|\partial_n u_{k|\partial\Omega}\|_{L^2(\Gamma'_{M+T})} < \frac{1}{k}\|g_k\|_{H^1(\Gamma)}. \tag{40}$$

The sequence $v_k = \|g_k\|^{-1}_{H^1(\Gamma)} u_k$ then satisfies

$$P_A v_k = 0, \quad v_{k|\Gamma'} = 0, \quad \|v_{k|\partial\Omega}\|_{H^1(\Gamma)} = 1, \text{and} \quad \|\partial_n v_{k|\partial\Omega}\|_{L^2(\Gamma'_{M+T})} < \frac{1}{k}. \tag{41}$$

(v_k) is bounded in the energy space $C^0((0, M+T), H^1(\Omega)) \cap C^1((0, M+T), L^2(\Omega))$ accordingly to (2), thus we may assume that it converges weakly in the cylinder $\mathcal{L}_{M+T}$ to some function $v \in H^1(\mathcal{L}_{M+T})$.

In the same way, we assume that the sequence $\tilde{g}_k = v_{k|\partial\Omega}$ weakly converges to some $\tilde{g}$ in $H^1(\Gamma)$, with $supp(\tilde{g}) \subset \overline{\Gamma}_M$. Passing then to the limit $k \to \infty$ in (41), we obtain

$$P_A v = 0, \quad v_{|\partial\Omega} = \tilde{g}, \quad \text{and} \quad \partial_n v_{|\partial\Omega} = 0 \quad \text{on } \Gamma'_{M+T}. \tag{42}$$

The unique continuation result of lemma 5.2 then gives that the weak limits v and $\tilde{g}$ vanish identically. Coming back then to Proposition 5.1 and plugging v_k and $\tilde{g}_k$ in estimate (39), we get the contradiction

$$1 \le c\|\tilde{g}_k\|_{L^2(\Gamma_M)} \longrightarrow 0 \quad \text{as} \quad k \to \infty$$

thanks to the compact imbedding of $H^1(\Gamma_M)$ into $L^2(\Gamma_M)$.

Proof of the unique continuation. The proof is based on a classical argument of functional analysis. For $a \ge 0$ and $g \in H^1(\partial\mathcal{L})$ with $supp(g) \subset \overline{\Gamma}^a_M =: [-a, M] \times \overline{O}$, consider the system

$$\left\{ \begin{array}{c} P_A u = -\partial_t^2 u + \sum_{i,j=1}^n \partial_{x_j}(a_{ij}(x)\partial_{x_i} u) = 0 \quad \text{in } \mathcal{L} \\ u(t,.) = g(t,.) \quad \text{on } \partial\mathcal{L} \\ u(-a,.) = \partial_t u(-a,.) = 0 \quad \text{in } \Omega. \end{array} \right. \tag{43}$$

Clearly, the solutions of (43) satisfy a relaxed observability estimate similar to (5.1), namely

$$\|g\|_{H^1(\Gamma^a_M)} \le c\|\partial_n u_{|\partial\Omega}\|_{L^2(\Gamma'^a_{M+T})} + c\|g\|_{L^2(\Gamma^a_M)}. \tag{44}$$

for any $T > T_0$ and some $c > 0$. Here we have denoted $\Gamma^a_M = (-a, M) \times O$ and $\Gamma'^a_{M+T} = (-a, M+T) \times O'$.

Let us introduce the set

$$\mathcal{N}_a(T) = \left\{ g \in H^1(\partial\mathcal{L}),\ supp(g) \subset \overline{\Gamma}^a_M,\ u = u(g) \text{ solves (43) and } \partial_n u_{|\Gamma'^a_{M+T}} \equiv 0 \right\} \tag{45}$$

First we notice that thanks to (3), $\mathcal{N}_a(T)$ is a closed subset of $H^1(\Gamma^a_M)$. In addition, applying the relaxed observability (44) to an element of $\mathcal{N}_a(T)$ gives

$$\|g\|_{H^1(\Gamma^a_M)} \le c\|g\|_{L^2(\Gamma^a_M)}.$$

Using the compact imbedding $H^1(\Gamma^a_M) \hookrightarrow L^2(\Gamma^a_M)$, this implies that $\mathcal{N}_a(T)$ has a finite dimension, and thus is complete for any norm.

Now we come back to the initial problem. We pick $g \in \mathcal{N}_0(T)$, i.e $g \in H^1(\partial\mathcal{L})$ with support in $\overline{\Gamma}_M$, and we consider u, the associated solution of (1). Notice first that $g \in \mathcal{N}_a(T)$ for all $a > 0$. In what follows, we fix $a > 0$. In addition, for $\delta = \frac{1}{2}(T - T_0)$, we remark that estimate (44) is also satisfied by all functions $h \in \mathcal{N}_a(T - \delta)$. Moreover, for all $\varepsilon < \min(\delta, a)$, the function $g(t+\varepsilon, .)$ lies in $\mathcal{N}_a(T - \delta)$. We also have

$$h_\varepsilon = \frac{1}{\varepsilon}(g(t+\varepsilon,.) - g(t,.)) \underset{\varepsilon \to 0^+}{\to} \frac{\partial g}{\partial t} \quad \text{in} \quad L^2(\Gamma^a_M).$$

As a consequence, the sequence $(h_\varepsilon)_{\varepsilon>0}$ is a Cauchy sequence in $\mathcal{N}_a(T-\delta)$ endowed with the norm $\|.\|_{L^2(\Gamma_M^a)}$. As all norms are equivalent , the sequence $(h_\varepsilon)_{\varepsilon>0}$ is thus also a Cauchy sequence in $\mathcal{N}_a(T-\delta)$ endowed with the norm $\|.\|_{H^1(\Gamma_M^a)}$, which yields $\frac{\partial g}{\partial t} \in \mathcal{N}_a(T-\delta)$. In particular, $\frac{\partial g}{\partial t} \in H^1(\Gamma_M^a)$. This distribution is supported in $\overline{\Gamma}_M$, we get therefore $\frac{\partial g}{\partial t} \in \mathcal{N}_0(T-\delta)$. Finally if $u(\frac{\partial g}{\partial t})$ denotes the solution of system (43) with boundary data $\frac{\partial g}{\partial t}$, we write

$$\partial_n\big(u(\frac{\partial g}{\partial t})\big) = \partial_n\big(\frac{\partial u(g)}{\partial t}\big) = \partial_t\big(\frac{\partial u(g)}{\partial n}\big) = 0 \quad \text{on} \quad (0, M+T)\times O'.$$

Therefore we obtain that $\frac{\partial g}{\partial t} \in \mathcal{N}_0(T)$.

To summarize, we have proved that the time derivative $\frac{\partial}{\partial t}$ defines a linear operator on the finite dimensional space $\mathcal{N}_0(T)$. But we notice that this operator has no eigenvalue. Indeed, for $g \in \mathcal{N}_0(T)$, we have $supp(g) \subset \overline{\Gamma}_M$; therefore for all $\lambda \in \mathbb{C}$, the only solution of system

$$\frac{\partial g}{\partial t} = \lambda g, \qquad g(0,.) = 0$$

is the trivial one $g \equiv 0$. This concludes the proof of Lemma 5.2.

This also concludes the proof of Theorem 2.3 assuming the relaxed observation estimate (39). Accordingly, the next section is dedicated to the proof of Proposition 5.1. ■

5.2. *Proof of the relaxed observation*

In order to establish estimate 5.1, we use a contradiction argument. Assume that inequality (39) is false and consider a sequence of boundary data $(g_k) \in H^1(\partial\mathcal{L})$, $supp(g_k) \subset \overline{\Gamma}_M$, and (u_k) the sequence of associated solutions, with

$$\|\partial_n u_{k|\partial\Omega}\|_{L^2(\Gamma'_{M+T})} + \|g_k\|_{L^2(\Gamma_M)} < \frac{1}{k}\|g_k\|_{H^1(\Gamma_M)}. \tag{46}$$

The sequence $v_k = \|g_k\|^{-1}_{H^1(\Gamma)} u_k$ then satisfies

$$P_A v_k = 0, \quad v_{k|\partial\Omega} = \|g_k\|^{-1}_{H^1(\Gamma_M)} g_k, \quad \text{and} \quad \|\partial_n v_{k|\partial\Omega}\|_{L^2(\Gamma'_{M+T})} \longrightarrow 0. \tag{47}$$

(v_k) is bounded in $H^1(\mathcal{L}_T)$ and $(v_{k|\partial\Omega})$ is bounded in $H^1(\Gamma_M)$. Therefore we may assume that (v_k) weakly converges to some v in $H^1(\mathcal{L}_T)$ and $(v_{k|\partial\Omega})$ weakly converges to some $\tilde{g}$ in $H^1(\Gamma_M)$. Equations (47) then provides

$$P_A v = 0, \quad v_{|\partial\Omega} = \tilde{g}, \quad \text{and} \quad \partial_n v_{|\partial\Omega} = 0, \tag{48}$$

and Lemma 5.2 implies that v and $v_{|\partial\Omega} = \tilde{g}$ vanish identically. Thus, the weak limits are both equal to 0.

Our goal, will be to prove that in the contradiction setting assumed above, the sequence $(v_{k|\partial\Omega})$ strongly converges to 0 in $H^1(\Gamma)$, which is impossible since $\|v_{k|\partial\Omega}\|_{H^1(\Gamma_M)} = 1$ accordingly to (47).

For this purpose, we make use of a classical strategy. Following Burq-Lebeau [8], and coming back to the notation u_k instead of v_k, we attach to (u_k) a microlocal defect measure in $H^1(\mathcal{L}_{M+T})$ denoted by μ.

Also, we attach to (g_k) a microlocal defect measure on the boundary, in $H^1(\partial\mathcal{L})$, denoted by $\tilde{\mu}$. Finally, the sequence $\partial_n u_{k|\partial\Omega}$ weakly converges to 0 in $L^2_{loc}(\partial\mathcal{L})$. So we attach to it a microlocal defect measure in $L^2_{loc}(\partial\mathcal{L})$ denoted by ν.

Notice, that in the contradiction setting of (47), the measure ν vanishes identically over Γ'_{M+T}.

Finally, we will prove in several steps, that in the contradiction setting assumed above, the measure $\tilde{\mu}$ vanishes identically on Γ_M. Notice that in the different intermediate results we will prove below, we use this contradiction setting, without explicitly referring to it.

5.3. *Properties of the measures*

In the sequel we consider W an interior neighborhood of the boundary $\overline{\Gamma}$ as introduced in Section 4.1. We recall that $W = \mathbb{R} \times (V \cap \Omega) = (\mathbb{R} \times V) \cap \mathcal{L}$ where V is an open subset of $\mathbb{R}^n$, neighborhood of the spatial boundary $O \subset \partial\Omega$. We set

$$W^{\partial} = (\mathbb{R} \times V) \cap \partial\mathcal{L}. \tag{49}$$

In addition, for J an open interval of $\mathbb{R}$ such that $[0, M] \subset J$, we denote

$$W_J = \{(t,x) \in W,\ t \in J\} \quad \text{and} \quad W^{\partial}_J = \{(t,x) \in W^{\partial},\ t \in J\}. \tag{50}$$

The neighborhood W and the interval J will be fixed in the next Proposition.

Proposition 5.3. *Under assumptions A1 and A2, for every $T > T_0$, there exist W and J as above such that the measure μ vanishes identically near any interior point of W_J.*

Proof. Consider $T > T_0$. We take the interior neighborhood W of Γ satisfying the conclusion of Lemma 4.1 with $\frac{T+T_0}{2}$. In addition, we chose $J =]-\alpha, M+\alpha[$, where $0 < \alpha < \frac{T-T_0}{2}$. And we prove that $\rho \notin \mathrm{supp}(\mu)$ for all $\rho \in T^*W_J$. This fact is obvious if ρ is an elliptic point, thanks to the classical property of microlocal elliptic regularity. If $\rho \in Char(P_A)$, let $\gamma = \gamma(s)$ be the generalized half bicharacteristic starting at ρ and satisfying (SGCC). We know that for some s_0 (say $0 < s_0 < \frac{T+T_0}{2}$), $\gamma(s_0) = (t_0, x_0, \tau_0, \xi_0)$ is a strictly gliding point of the boundary Γ'_{M+T}. Consider U_0 a small neighborhood of (t_0, x_0) in $\mathbb{R}^{n+1}$ and denote by $\underline{u}_k$ the canonical extension of u_k to $\mathbb{R}^{n+1}$, i.e $\underline{u}_k = u_k$ in $\mathcal{L}$ and $\underline{u}_k = 0$ elsewhere. We have

$$\begin{cases} \underline{u}_k \rightharpoonup 0 \quad \text{in } H^1(U_0) \quad \text{weakly} \\ u_{k|\partial\Omega} = 0 \quad \text{on } U_0 \cap \partial\mathcal{L} \quad \text{and } \partial_n u_{k|\partial\Omega} \longrightarrow 0 \quad \text{on } U_0 \cap \partial\mathcal{L} \quad \text{strongly .} \end{cases} \tag{51}$$

Accordingly to the lifting lemma of Bardos, Lebeau and Rauch [3, Theorem 2.2] or Burq [5, Lemme 2.2], we know that $\underline{u}_k$ strongly converges to 0 in H^1 microlocally at $\gamma(s_0)$. Therefore we deduce that $\gamma(s_0) \notin supp(\mu)$ thanks to the work of Aloui [2, Lemme 3.1].

Now, accordingly to (SGCC), for $0 \leq s \leq s_0$, the bicharacteristic $\gamma(s)$ doesn't intersect the boundary Γ. It may only intersect $\partial\mathcal{L} \setminus \overline{\Gamma}$, on which we have homogeneous Dirichlet condition $u_{k|\partial\Omega} = 0$. Consequently, the measure propagation result of Lebeau [16] or Burq-Lebeau [8] is valid. Starting then backward from $\gamma(s_0)$, and using the propagation of the measure μ, we obtain that $\rho \notin \mathrm{supp}(\mu)$. Finally, the case $s_0 < 0$, $0 < |s_0| < \frac{T+T_0}{2}$, can be treated in a similar way. ∎

Remark 5.4. *In the rest of the proof, the neighborhood W and the interval J are fixed as in the proof of Proposition 5.3 above.*

Proposition 5.5. *Under assumptions A1 and A2, the measures μ, ν and $\tilde{\mu}$ vanish on the hyperbolic set of the boundary W_J^∂.*

Proof. The fact that $\mu \mathbf{1}_{\mathcal{H}} = 0$ is proved in Burq-Lebeau paper (see [8, Lemma 2.6]) and is independent of the boundary condition. It only needs the weak convergence of the sequence (u_k) to 0 in $H^1_{loc}(\mathcal{L})$. On the other hand, since $\mu = 0$ in the interior of W_J thanks to Proposition 5.3, the two hyperbolic fibers incoming to and outcoming from any hyperbolic point ρ_0 of the boundary W_J^∂ are not charged, i.e they don't intersect $supp(\mu)$. Therefore, the Taylor pseudo-differential factorization (see for instance Burq-Lebeau [8, Appendix]), shows that microlocally near ρ_0, $g_k = u_{k|\partial\Omega} \longrightarrow 0$ in H^1 and $\partial_n u_{k|\partial\Omega} \longrightarrow 0$ in L^2 strongly. So as a by-product, we get that ρ_0 is not in supp $\tilde{\mu}$ neither in supp ν. ∎

At this step, we can already conclude the proof of Theorem 2.3 under assumption A3.a.

Corollary 5.6. *Under assumptions A1, A2 and A3.a, the measure $\tilde{\mu}$ identically vanishes on the boundary W_J^∂.*

Proof. This result is a byproduct of Proposition 5.5 and we develop it for the convenience of the reader. First we recall a classical property of micolocal defect measures, namely the microlocal elliptic regularity. Let $\chi(t, x', \tau, \xi')$ and $\psi(t, x', \tau, \xi')$ two 0-order pseudo-differential symbols supported in $T^*(\partial\mathcal{L})_{|W_J^\partial} \setminus CharB_\alpha$, such that $\chi \equiv 1$ on $supp(\psi)$. Denoting $\chi = \chi(t, x', D_t, D_{x'})$ and $\psi = \psi(t, x', D_t, D_{x'})$, it's classical that one can find a pseudo-differential operator $B_{-\alpha}$, of order $(-\alpha)$ on $\partial\mathcal{L}$ such that

$$B_{-\alpha} B_\alpha \chi = \psi + R_{-\infty} \tag{52}$$

where $R_{-\infty}$ is infinitely smoothing. Consequently, we can write the elliptic estimate

$$\|\psi g_k\|_{H^1(\partial\mathcal{L})} \leq c_0 \|B_\alpha \chi g_k\|_{H^{1-\alpha}(\partial\mathcal{L})} + c_1 \|g_k\|_{L^2(\partial\mathcal{L})} \tag{53}$$

for some constants $c_0, c_1 > 0$. Therefore

$$\|\psi g_k\|_{H^1(\partial\mathcal{L})} \leq c_0 \|[B_\alpha, \chi] g_k\|_{H^{1-\alpha}(\partial\mathcal{L})} + c_1 \|g_k\|_{L^2(\partial\mathcal{L})} \leq c_2 \|g_k\|_{L^2(\partial\mathcal{L})} \tag{54}$$

for some $c_2 > 0$. We then deduce that $\psi g_k \longrightarrow 0$ strongly in $H^1(\partial\mathcal{L})$, which expresses that $supp(\tilde{\mu}) \subset CharB_\alpha$. Now , $CharB_\alpha \subset \mathcal{H}$ thanks to assumption A3.a, and $\tilde{\mu} \equiv 0$ on $\mathcal{H}$ accordingly to Proposition 5.5. Therefore, $\tilde{\mu}$ vanishes identically.

∎

The proof of Theorem 2.3 under assumption A3.a is complete.

Let us now continue the proof of Theorem 2.3 under assumption A3.b.

Denote by A_j^k the terms of (38) where we set u_k instead of u, and consider a pseudo-differential symbol $q = \sigma(Q) \in \mathcal{A}^0$ (see Section 3.3), chosen as in Section 4.2.

Corollary 5.7. *Under assumptions A1 and A2, if $q = \sigma(Q)$ is compactly supported in W_J, we have*

$$\lim_{k\to\infty} A_j^k = 0, \qquad \forall j \in \{1, 8, 9, 10, 12, 14\}. \tag{55}$$

Proof. We recall that the symbol $q = \sigma(Q)$ is independent of x_n in a strip $\{|x_n| < \beta\}$, $\beta > 0$ small. More precisely, we take q in the form $q(x_n, x', t, \xi', \tau) = \varphi(x_n)\tilde{q}(x', t, \xi', \tau)$, with $\varphi \in C_0^\infty(\mathbb{R})$, equal to 1 near $x_n = 0$. Therefore, if we choose β small enough, and assume that $\tilde{q}$ is supported in time in the interval J, the symbol of the bracket operator $[\partial_n, Q^2]$ is of order 0 and compactly supported in the interior of W_J. Thus, $\lim_{k\to\infty} A_j^k = 0$ for $j \in \{1, 10\}$ thanks to Proposition 5.3. The terms A_j^k, $j = 8, 9, 12$ are trivial. ∎

Remark 5.8. *In the rest of the proof, we will work henceforth, with this choice of symbol q, and we will choose successively, the localization of its support.*

Now, for the convenience of the reader, we recall the following result due to Burq-Lebeau [8].

In the system of geodesic coordinates introduced above, consider the function θ defined μ-almost everywhere on $S\hat{Z}$

$$\theta = \frac{\xi_n}{|(\tau, \xi')|} \quad \text{in} \quad x_n > 0, \qquad\qquad \theta = i\frac{\sqrt{-r_0}}{|(\tau, \xi')|} \quad \text{in} \quad \mathcal{E} \cup \mathcal{G}. \tag{56}$$

Lemma 5.9. [8, Lemma 2.7] *Let $Q_j \in \mathcal{A}^j$, $j = 1, 2$ be tangential pseudo-differential operators with principal symbols $\sigma(Q^j) = q_j$. Then we have with $\lambda^2 = |(\tau, \xi')|^2(1 + |\theta|^2)$*

$$\lim_{k\to\infty} \Big((Q_2 - iQ_1\partial_n)u_k \,|\, u_k\Big)_{L^2(\mathcal{L})} = \Big\langle \mu, \lambda^{-2}(q_2 + q_1\theta|(\tau, \xi')|)\Big\rangle \tag{57}$$

Proposition 5.10. *The measure μ vanishes on the elliptic set of the boundary W_J^∂.*

Proof. The elliptic microlocal regularity for measures or wave fronts is classical for elliptic interior points $\rho \in T^*W_J$. In what concerns the elliptic set of the boundary, we will invoke a result of Burq-Lebeau ([16, Lemma 2.6]), and we have to introduce some additional notations.

In the framework above, they define a boundary measure μ_∂^0 given by

$$\forall Q \in \mathcal{A}^0, \qquad \lim_k \int_{\partial\mathcal{L}} Qu_k\, \partial_n\overline{u}_k d\sigma = \Big\langle \mu_\partial^0, \sigma(Q)_{|x_n=0}\Big\rangle \tag{58}$$

Moreover, they provide the following link between the two measures μ and μ_∂^0 :

$$\mu_\partial^0 = -2\frac{|\theta|^2}{1+|\theta|^2}\,\mu\mathbf{1}_{|x_{n=0}}. \tag{59}$$

Therefore, we get

$$\mu_\partial^0 = \frac{2r_0(x';\tau,\xi')}{|(\tau,\xi')|^2 - r_0(x';\tau,\xi')}\mu\,\mathbf{1}_{|x_{n=0}} \quad \text{on} \quad \mathcal{E}\cup\mathcal{G}$$

But, since $u_{k|\partial\mathcal{L}} = g_k \longrightarrow 0$ in $L^2_{loc}(\partial\mathcal{L})$ strongly and $\partial_n u_{k|\partial\mathcal{L}}$ is bounded in $L^2_{loc}(\partial\mathcal{L})$, we easily get that $\mu_\partial^0 \equiv 0$. Consequently, we obtain $\mu \equiv 0$ on $\mathcal{E}$, since $r_0 < 0$ on this set. ∎

.

Remark 5.11. (1) *Notice that for this proposition, we have used none of the assumptions A_j, $j = 1, 2, 3$. We have only used the weak convergence $g_k \rightharpoonup 0$ in $H^1(\partial\mathcal{L})$ and subsequently $u_k \rightharpoonup 0$ in $H^1(\mathcal{L})$.*

(2) *One should be carefull that this proposition does not give any information about the behavior of the boundary data g_k on $\mathcal{E}\cup\mathcal{G}$. In other words, we have not yet any information about $\tilde{\mu}\mathbf{1}_{|\mathcal{E}\cup G}$.*

(3) *Up to now, we have proved that the measure μ vanishes in $T^*(W_J)$, i.e on interior points, and on the subset $\mathcal{H}\cup\mathcal{E}$ of $T^*(W_J^\partial)$. Therefore, μ is supported in the glancing set , that is $\mu = \mu\mathbf{1}_\mathcal{G}$.*

Lemma 5.12. *Under assumptions A1 and A2, and with a suitable choice of the pseudo-differential symbol $q = \sigma(Q)$, we have*

$$\lim_{k\to\infty} A_j^k = 0, \qquad \forall j \in \{2,3,4,5,6,7,11,13\}. \tag{60}$$

Together with (5.7)*, this implies that the right hand side of* (37) *tends to* 0 *as* $k \longrightarrow \infty$.

Proof. The proof essentially relies on the calculus Lemma 5.9 . If we detail the limit (57), we can write accordingly to Propositions 5.3, 5.5 and 5.10

$$\begin{cases} \lim_{k\to\infty}\Big(Q_2u\,|\,u\Big)_{L^2(\mathcal{L})} = \Big\langle \mu\mathbf{1}_\mathcal{G}, \lambda^{-2}q_2\Big\rangle \\ \lim_{k\to\infty}\Big(-iQ_1\partial_n u\,|\,u\Big)_{L^2(\mathcal{L})} = \Big\langle \mu\mathbf{1}_\mathcal{G}, \lambda^{-2}q_1\theta|(\tau,\xi')|\Big\rangle = \Big\langle \mu\mathbf{1}_\mathcal{G}, i\lambda^{-2}q_1\sqrt{-r_0}\Big\rangle = 0 \end{cases} \tag{61}$$

since $r_0 \equiv 0$ on the glancing set $\mathcal{G}$.

First, we take the pseudo-differential symbol $q = \sigma(Q)$ as in the proof of Corollary 5.7. With this choice, the terms $A_2^k, A_4^k, A_5^k, A_6^k, A_7^k, A_{11}^k$ and A_{13}^k can be treated with the second limit of (57) since the pseudo-differential operator $(Q^2 - Q^{*2})$, resp. $(R - R^*)$ is of order $\le (-1)$, *resp.* 1.

On the other hand, the term A_{11}^k tends to 0 thanks to the first limit of (57). Finally, for the term A_3^k, we have just to notice that $\partial_n u_k$ is bounded in $L^2_{x_n}(L^2_{t,x'})$ and converges weakly to 0 in this space, and use again the fact that $(Q^2 - Q^{*2})$ is of order $\le (-1)$. ∎

As a by-product, we have obtained the following lemma. We denote by $q = \sigma(Q)$ the symbol of the pseudo-differential operator $Q \in \mathcal{A}^0$.

Corollary 5.13. *Under assumptions A1 and A2, the measures $\mu, \tilde{\mu}$ and ν satisfy the following identity*

$$\left\langle \nu, q^2 \right\rangle + \left\langle \tilde{\mu}, |(\tau,\xi')|^{-2} q^2 r_0 \right\rangle = -\left\langle \mu \mathbf{1}_{\mathcal{G}}, |(\tau,\xi')|^{-2}\, q^2 (\partial_n r) \right\rangle, \tag{62}$$

for all 0*-order symbol q, supported in W_J.*

Now, we can conclude the study for the measure μ.

Proposition 5.14. *The measure μ vanishes identically over $T^*(W_J^{\partial})$.*
In particular, $u_k \longrightarrow 0$ strongly in $H^1(W_J)$ up to the boundary.

Proof. The proof relies on a specific choice of the symbol q. First, we recall the notation $r_0(x',\tau,\xi') = \tau^2 - \sum_{1\le i,j\le n-1} a_{ij}(x',0)\xi_i\xi_j$, see Section 3.1. In addition, it's clear that in formula (62), we adopt the notation $q = q_{|x_n=0}$. Let us then consider $\tilde{q}_0 \in C_0^\infty(]0,T[,\mathbb{R}_+)$ and a function $q_0 \in C_0^\infty(\mathbb{R},\mathbb{R})$, supported in $[-1,1]$, such that $q_0(s) = 1$ for $s \in [-1/2, 1/2]$. We set for $\varepsilon > 0$

$$q_\varepsilon(t,x',\tau,\xi) = \tilde{q}_0(t) q_0\Big(\frac{r_0(x',\tau,\xi)}{\varepsilon \sum_{1\le i,j\le n-1} a_{ij}(x',0)\xi_i\xi_j} \Big) \tag{63}$$

Plugging q_ε into (62) and letting $\varepsilon \longrightarrow 0^+$, we get by Lebesgue dominated convergence

$$\left\langle \nu, \mathbf{1}_{\mathcal{G}} \right\rangle = -\left\langle \mu \mathbf{1}_{\mathcal{G}}, |(\tau,\xi')|^{-2}\, (\partial_n r) \right\rangle \tag{64}$$

All points of the glancing set $\mathcal{G} = \mathcal{G}_d$ are strictly diffractive (see (18)) which gives $\partial_n r_{|\mathcal{G}} > 0$. Therefore the two members of this identity are of opposite sign and thus both are equal to zero. Consequently, the measure μ vanishes identically. ∎

Remark 5.15. (1) *Finally, summarizing previous results, we obtain that the measures equation (62) reads as follows :*

$$\left\langle \nu \mathbf{1}_{\mathcal{E}\cup\mathcal{G}}, q^2 \right\rangle + \left\langle \tilde{\mu} \mathbf{1}_{\mathcal{E}\cup\mathcal{G}}, |(\tau,\xi')|^{-2} q^2 r_0 \right\rangle = 0 \tag{65}$$

for all 0-order symbol q , supported in W_J.

(2) *Roughly speaking, this formula tells us that we have two ways to prove that $\tilde{\mu} \equiv 0$. Either, we set a condition on the data g itself, in other words, we make use of assumption A3.a or A3.b, or we we use a condition linking the two boundary data $\partial_n u_{|\partial\mathcal{L}}$ and $u_{|\partial\mathcal{L}} = g$, which is assumption A3.c.*

5.4. *End of the proof of Theorem 2.3*

Here we have reached the point where, for the first time, we make use of assumptions A3.b or A3.c .

Proposition 5.16. *Under assumptions A1, A2 and A3.b, the measures $\tilde{\mu}$ and ν vanish identically on the set $\mathcal{E} \cup \mathcal{G}$ and hence on the boundary $\partial\mathcal{L}$.*

Proof. In the setting of assumption A3.b, for every $t \in J$ we can write the classical elliptic estimate

$$\|g_k(t,.)\|_{H^1(\partial\Omega)} \leq c_0\|c(t,x',D_{x'})g_k(t,.)\|_{H^{1-\alpha}(\partial\Omega)} + c_1\|g_k(t,.)\|_{L^2(\partial\Omega)} = c_1\|g_k(t,.)\|_{L^2(\partial\Omega)} \tag{66}$$

for some constants $c_0, c_1 > 0$ independent of $t \in J$. We deduce that uniformly with respect to $t \in J$,

$$\|D_{x'_j} g_k(t,.)\|_{L^2(\partial\Omega)} \to 0 \quad \text{for} \quad k \to \infty$$

Therefore, integrating on t and taking the limit $k \to \infty$, we can write

$$\left\langle \tilde{\mu}, |(\tau,\xi')|^{-2}|\xi'|^2 \right\rangle = 0 \tag{67}$$

and this yields

$$\left\langle \tilde{\mu}, |(\tau,\xi')|^{-2}q^2\tau^2 \right\rangle = \left\langle \tilde{\mu}\mathbf{1}_{\mathcal{E}\cup\mathcal{G}}, |(\tau,\xi')|^{-2}q^2\tau^2 \right\rangle = 0 \tag{68}$$

since $\tau^2 \leq c|\xi'|^2$ in $\mathcal{E} \cup \mathcal{G}$. Together with the result of Proposition 5.5, this gives $\tilde{\mu} \equiv 0$ and $\nu \equiv 0$ accordingly to (65).

This completes the proof of Theorem 2.3 under assumption A3.b. ■

Proposition 5.17. *Under assumptions A1, A2 and A3.c, the measures $\tilde{\mu}$ and ν vanish identically on the set $\mathcal{E} \cup \mathcal{G}$ and hence on the boundary $\partial\mathcal{L}$.*

Proof. All identities we will handle in this proof take place on the boundary $\partial\mathcal{L}$. Therefore, we will simply write $\partial_n u_k$ (resp. u_k) instead of $\partial_n u_{k|\partial\mathcal{L}}$ (resp. $u_{k|\partial\mathcal{L}}$). In addition, without loss of generality, we may assume that $\mathcal{U}_M \subset W_J$. Denote $F_k = \partial_n u_k + \partial_t u_k$. Clearly, $F_k \rightharpoonup 0$ weakly in $L^2(\partial\mathcal{L})$. In addition, thanks to condition A3.c, F_k is bounded in $H^\alpha(\mathcal{U}_M)$, with $\alpha > 0$. Therefore we may assume that

$$\partial_n u_k + \partial_t u_k = F_k \to 0 \quad \text{strongly in} \quad L^2(\mathcal{U}_M). \tag{69}$$

Consider an elliptic point $\rho_0 \in T^*(\mathcal{U}_M)$. A classical analysis at elliptic points of the boundary , see for instance [8, Appendix], shows that microlocally near ρ_0, we have

$$\partial_n u_k - Op(\sqrt{-r_0(x',t,\tau,\xi')})u_k = o(1) \quad \text{in} \quad H^{1/2}, \quad \text{for} \quad k \to \infty. \tag{70}$$

Together with (69), this yields

$$\partial_t u_k + Op(\sqrt{-r_0(x',t,\tau,\xi')})u_k = o(1) \quad \text{in} \quad L^2, \quad \text{for} \quad k \to \infty. \tag{71}$$

Therefore $u_{k|\partial\mathcal{L}} = g_k \to 0$ strongly in H^1 near ρ_0 since the symbol $i\tau + \sqrt{-r_0(x', t, \tau, \xi')}$ is elliptic near this point. Consequently $\rho_0 \notin supp(\tilde{\mu})$ and using (70), $\rho_0 \notin supp(\nu)$. Thus we obtained

$$\nu = \nu \mathbf{1}_{\mathcal{G}} \quad \text{and} \quad \tilde{\mu} = \tilde{\mu}\mathbf{1}_{\mathcal{G}}. \tag{72}$$

Coming back to (65) and using a test symbol q elliptic near $\mathcal{G}$, we then get $\nu \equiv 0$ since $r_0 = 0$ on $\mathcal{G}$.

On the other hand, if Q is a 0-order polyhomogeneous pseudo-differential operator on $\partial\mathcal{L}$, with symbol q, real valued and supported in $\mathcal{U}_M$, we have

$$\Big(Q^2\partial_n u_k \,|\, \partial_n u_k\Big)_{L^2(\mathcal{U}_M)} = \Big(Q^2\partial_t u_k \,|\, \partial_t u_k\Big)_{L^2(\mathcal{U}_M)} + \Big(Q^2 F_k \,|\, F_k\Big)_{L^2(\mathcal{U}_M)} - 2Re\Big(Q^2 F_k \,|\, \partial_t u_k\Big)_{L^2(\mathcal{U}_M)} \tag{73}$$

Passing to the limit in k and taking into account (69) and (72), we obtain

$$\Big\langle \tilde{\mu}\mathbf{1}_{\mathcal{G}}, |(\tau, \xi')|^{-2} q^2 \tau^2 \Big\rangle = 0 \tag{74}$$

for all symbol q. And this gives $\tilde{\mu} \equiv 0$ since $\tau \neq 0$ near $\mathcal{G}$.

This completes the proof of Theorem 2.3 under assumption A3.c. ∎

6. Proof of Theorem 2.5 and Corollary 2.6

The proof is based on the wave front propagation theorem of Melrose-Sjöstrand , see [20]. We start with a general remark about solutions of system (1). Consider $g \in H^1(\partial\mathcal{L})$, with support in $\Gamma_M = (0, M) \times O$ and assume in addition that $WF(g)$, the C^∞-wave front of g, is contained in the elliptic set $\mathcal{E}$. First, we recall that the corresponding solution u vanishes identically for $t \leq 0$. Therefore u is of class C^∞ up to the boundary $\partial\mathcal{L}$, outside $\overline{\Gamma}_M$. Indeed, consider $\rho \in T_b^*(\mathcal{L})$, $\rho \notin T^*(\Gamma_M)$, and denote γ_ρ the generalized bicharacteristic curve issued from ρ. Following this curve backward in time, one enters in the region $\{t < 0\}$, say at some point $\gamma_\rho(-t_0)$, $t_0 > 0$, where u is smooth. Accordingly to the description of a generalized bicharacteristic curve given in Section 3.2, we have for $s_0 \in [-t_0, 0]$

- $\gamma_\rho(s_0)$ is an interior point, i.e it lies in the characteristic set $Char(P_A) \cap T^*(\mathcal{L})$,
- γ_ρ hits the boundary at a hyperbolic point for $s = s_0$,
- $\gamma_\rho(s_0)$ is a glancing point, i.e $\gamma_\rho \in \mathcal{G}$.

In all cases, $\gamma_\rho(s)$ never intersects the closed set $WF(g) \subset \mathcal{E}$. Hence by regularity propagation (see [20]), $\rho \notin WF(u)$. Moreover, this propagation property yields that the H^α norm of u is microlocally bounded near ρ, for every $\alpha \geq 1$.

In the sequel we use this property to prove that estimate (13) fails in general.

Take $s < 0$, $\alpha > 1$ and consider F a closed conical subset of $T^*(\Gamma_M)$, $F \subset \mathcal{E}$, and V_F a conical neighborhood of F in $T^*(\Gamma_M) \cap \mathcal{E}$. Finally, consider a symbol $b(t, x', \tau, \xi') \in \Psi^0_{phg}(\partial\mathcal{L})$, supported in V_F and equal to 1 on F. Denoting $B = b(t, x', D_t, D_{x'})$ the corresponding pseudo-differential operator, it's classical that one can construct a sequence of smooth functions $(f_k) \subset H^1(\partial\mathcal{L})$, compactly supported in Γ_M, satisfying

$$\|f_k\|_{H^s} = 1 \quad \text{and} \quad f_k \rightharpoonup 0 \quad \text{weakly in} \quad H^s(\Gamma_M), \tag{75}$$

and

$$\|Bf_k\|_{H^s} \to 1 \quad \text{for} \quad k \to \infty. \tag{76}$$

This simply means that the lack of compactness of (f_k) is located in $supp(b) \subset V_F \subset \mathcal{E}$.

We claim that with a suitable choice of the symbol $b(t, x', \tau, \xi')$, the sequence $g_k = Bf_k$ will be the key of our counterexample.

First, consider a symbol $q \in \Psi^0_{phg}(\partial\mathcal{L})$ such that $supp(q) \cap V_F = \emptyset$. Since the composition operator $Op(q)B$ is infinitely smoothing, it's classical that $\|Op(q)g_k\|_{H^\alpha}$ is uniformly bounded. More precisely, we have for some constant $c > 0$

$$\|Op(q)g_k\|_{H^\alpha} = \|Op(q)Bf_k\|_{H^\alpha} \le c\|f_k\|_{H^s} = c. \tag{77}$$

Moreover, accordingly to (75), we obtain that $Op(q)g_k \to 0$ strongly in $H^{\alpha'}(\Gamma_M)$ for all $\alpha' < \alpha$.

Let us now analyze the sequence (u_k) of solutions to the wave system (1) with (g_k) as boundary data .

$$\begin{cases} P_A u_k = 0 \quad \text{in} \quad \mathcal{L}, \quad u_{k|\partial\mathcal{L}} = g_k = Bf_k \\ \\ \qquad u_k(0) = \partial_t u_k(0) = 0. \end{cases} \tag{78}$$

We will need the following Lemma.

Lemma 6.1. *Consider $s < 0$ and for $c > 0$ denote $\mathcal{E}_c = \{(\tau, \xi) \in \mathbb{R}^n, \ |\tau| \le c|\xi|\}$. Then on the space $\{h \in H^s(\mathbb{R}^n), supp(\hat{h}) \subset \mathcal{E}_c\}$, $\|.\|_{L^2(\mathbb{R};H^s(\mathbb{R}^{n-1}))}$ is a norm, equivalent to its natural norm $\|.\|_{H^s(\mathbb{R}^n)}$.*

The proof is straightforward and left to the reader.

Now we choose the 0-order pseudo-differential operator B introduced above in the form

$$B = b(t, x', D_t, D_{x'}) = b_1(t, x')b_2(D_t, D_{x'}),$$

with $b_1 \in C_0^\infty(\Gamma_M)$, $b \equiv 1$ on F, and $supp(b) \subset V_F \subset \mathcal{E}$. Necessarily, we have for some $c > 0$,

$$supp(b_2(\tau, \xi')) \subset \{(\tau, \xi') \in \mathbb{R}^n, \ |\tau| \le c|\xi'|\}.$$

We then deduce that the sequence (Bf_k) is bounded in $L^2(0, M+T; H^s(O))$. Therefore system (78) is well posed and (u_k) is bounded in $C^0(0, M+T; H^s(\Omega))$ (see [15, Th.2.7]), and thus in $H^s(\mathcal{L}_{M+T})$. Using the propagation argument developed in the beginning of this section, we see that (u_k) is bounded in $H^\alpha(\mathcal{L}_{M+T})$ up to the boundary, except on the subset $V_F \subset \mathcal{E}$. In particular, this sequence is bounded in $H^\alpha(\mathcal{U})$ for any $\mathcal{U}$ interior neighborhood of the boundary observation region $\Gamma'_{M+T} = (0, M+T) \times O'$, ie :

$$\|u_k\|_{H^\alpha(\mathcal{U})} \le c \quad \text{for some} \quad c > 0. \tag{79}$$

Finally, since $u_k \rightharpoonup 0$ weakly in $H^s(\mathcal{L})$ thanks to (75), we obtain that $u_k \to 0$ strongly in $H^{\alpha'}(\mathcal{U})$ for any $\alpha' \in [1, \alpha[$, and this gives

$$\|\partial_n u_{k|\partial\Omega}\|_{L^2(\Gamma'_{M+T})} \to 0.$$

This concludes the proof of Theorem 2.5.

We come now to the proof of Corollary 2.6.

To this end, consider a glancing point $\omega_0 \in T^*(\Gamma_M) \cap \mathcal{G}$ and a sequence of elliptic points $(\omega_k)_{k\geq 1} \subset T^*(\Gamma_M) \cap \mathcal{E}$, converging to ω_0 for $k \to \infty$. For every $k \geq 1$, we apply Theorem 2.5 above with $F = \{\omega_k\}$. We then have a sequence $(g_k^p)_{p\geq 1} \subset H^s(\partial\mathcal{L})$ supported in Γ_M , weakly converging to 0 in $H^s(\Gamma_M)$, such that

$$\lim_{p\to\infty} \|g_k^p\|_{H^s} = 1 \quad \text{and} \quad \lim_{p\to\infty} \|\partial_n u^p_{k|\partial\Omega}\|_{L^2(\Gamma'_{M+T})} = 0, \tag{80}$$

where $(u_k^p)_{p\geq 1}$ denotes the sequence of associated solutions .

For $k = 1$, there exists $p_1 \geq 1$ such that

$$|\|g_1^p\|_{H^s} - 1| \leq 1/2 \quad \text{and} \quad \|\partial_n u^p_{1|\partial\Omega}\|_{L^2(\Gamma'_{M+T})} \leq 1/2, \quad \forall p \geq p_1.$$

Also, for $k = 2$, there exists $p_2 \geq p_1 + 1$ such that

$$|\|g_2^p\|_{H^s} - 1| \leq 1/3 \quad \text{and} \quad \|\partial_n u^p_{2|\partial\Omega}\|_{L^2(\Gamma'_{M+T})} \leq 1/3, \quad \forall p \geq p_2.$$

And more generally, for every $k \geq 2$, there exists $p_k \geq p_{k-1} + 1$ such that

$$|\|g_k^p\|_{H^s} - 1| \leq \frac{1}{k+1} \quad \text{and} \quad \|\partial_n u^p_{k|\partial\Omega}\|_{L^2(\Gamma'_{M+T})} \leq \frac{1}{k+1}, \quad \forall p \geq p_k.$$

It's then easy to see that the sequence $(h_k)_{k\geq 1} = (g_k^{p_k})_{k\geq 1}$ provides the counter-example of Corollary 2.6. Indeed,

$$\|h_k\|_{H^s} \to 1 \quad \text{and} \quad \|\partial_n u^k_{k|\partial\Omega}\|_{L^2(\Gamma'_{M+T})} \to 0, \quad \text{for} \quad k \to \infty.$$

and the lack of compactness is contained in the limit set $\{\omega_0\} \subset \mathcal{G}$.

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