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OPTIMAL SHAPE DESIGN AND PLACEMENT OF SENSORS VIA A GEOMETRIC APPROACH

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Motivation and Introduction

The optimal shape and placement of sensors frequently arises in industrial applications such as urban planning and temperature and pressure control in gas networks.

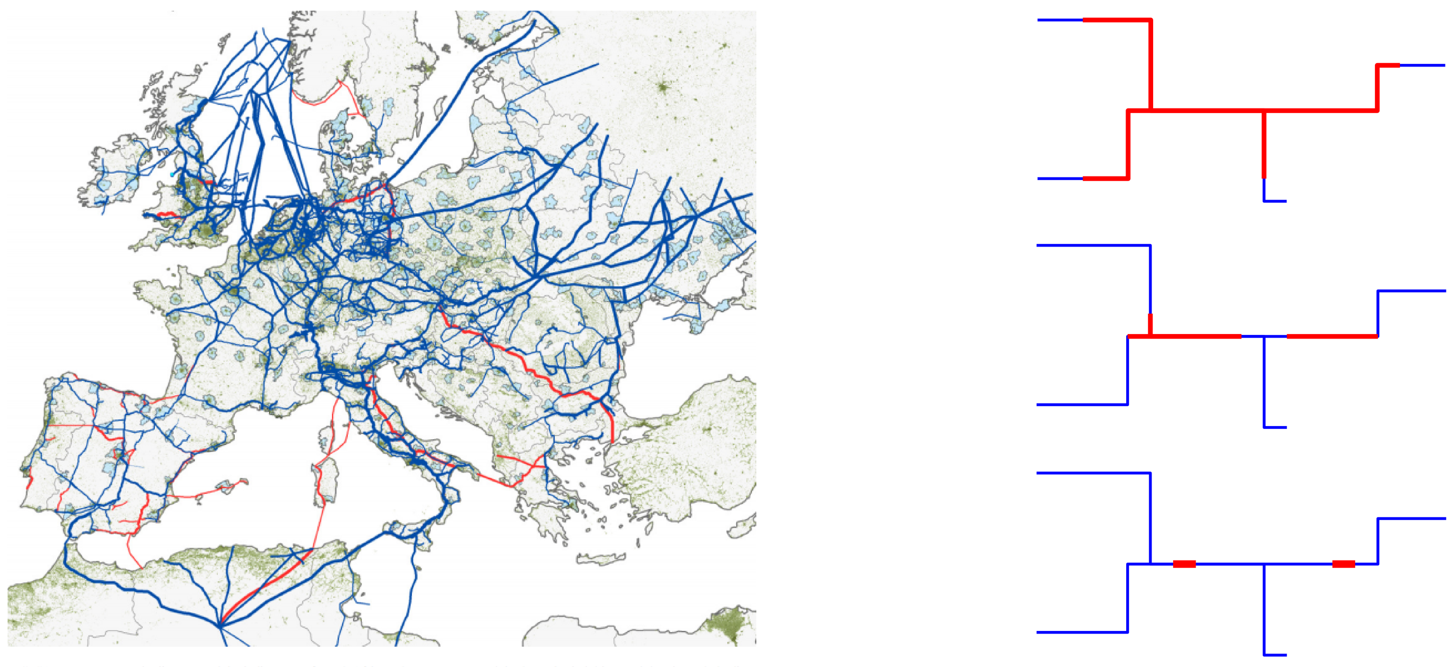

Fig 1. On the left, the European Gas Network and, on the right, optimal sensors (in red) over a graph corresponding to different penalisations of its length.

Roughly, a sensor is optimally designed and placed if it assures the maximum observation of the phenomenon under consideration. Naturally, it is often designed in a goal oriented manner, constrained by a suitable PDE, accounting for the physics of the process. Various optimization algorithms such as adjoint methods and random and heuristic search have been implemented. Here, we address the problem in a purely geometric setting, without involving the specific PDE model. We consider a simple and natural geometric criterion of performance, based on distance functions. But, as we shall see, tackling it will require to eventually consider a PDE inspired method.

Mathematical formulation of the problem

We adopt a shape optimization point of view. Given a region Ω in the euclidean space, we are interested in the optimal shape and placement of a sensor $\omega \subset \Omega$, of a given volume, in such a way to:

"minimize the maximal distance from the sensor ω to any point of the region Ω".

This problem can be formulated in the shape optimization framework as follows:

$$\inf\{d^H(\omega,\Omega)\ |\ |\omega| = c_0|\Omega| \text{ and } \omega\subset\Omega\}, \tag{1}$$

where $c_0 \in (0,1]$ and d^H is the Hausdorff distance defined as:

$$d^H(A,B) := \max(\sup_{x\in B} d_A(x), \sup_{x\in A} d_B(x)),$$

where $d_K : x \longmapsto \inf_{y\in K}\|x-y\|$ is the distance function to the set K.

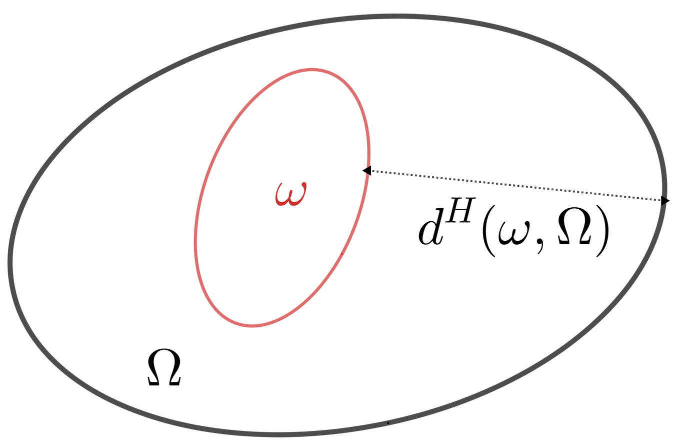

Fig 2. The Hausdorff distance between the sensor ω and the set Ω.

Additional constraints

By using a homogenization strategy, which consists in uniformly distributing the mass of the sensor over Ω (see **Figure 3**), we see that problem (1) does not admit a solution as the infimum vanishes and is asymptotically attained by a sequence of disconnected sets with an increasing number of connected components. Thus, it is necessary to impose additional constraints on ω in order to assure the existence of optimal solutions.
We investigate the problem for the following classes of sets:

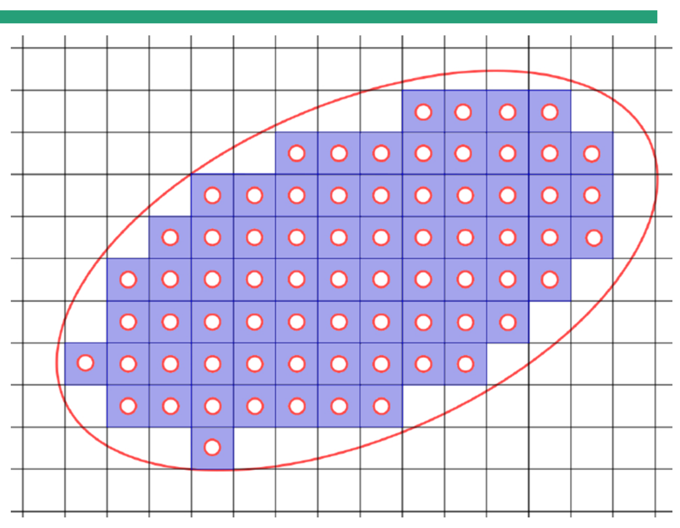

Fig 3. Homogenization strategy.

- $\mathcal{K} = \{\omega\subset\Omega |\ \omega \text{ is convex and } |\omega| = c_0|\Omega|\}$.
- $\mathcal{B}_N = \{\cup_{i=1}^N B_i \subset \Omega\ |\ (B_i)_i \text{ are disjoint balls s.t. } |B_i| = c_0|\Omega|/N \text{ for every } i\ \}$.

We are therefore interested in the problems:

$$\min_{\omega\in\mathcal{K}} d^H(\omega,\Omega) \quad\text{and}\quad \min_{\omega\in\mathcal{B}_N} d^H(\omega,\Omega). \tag{2}$$

Varadhan's approximation of the distance function

Addressing these problems requires to obtain a reliable numerical approximation of the Hausdorff distance. We do it using the following classical result:

Theorem 1. (Varadhan 67' [2])
Let U be an open subset of $\mathbb{R}^n$ and $\varepsilon > 0$, we consider the problem

$$\begin{cases} w_\varepsilon - \varepsilon\Delta w_\varepsilon = 0 & \text{in } U,\\ w_\varepsilon = 1 & \text{on } \partial U.\end{cases}$$

We have

$$\lim_{\varepsilon\to 0^+} -\sqrt{\varepsilon}\ln w_\varepsilon(x) = d(x,\partial U) := \inf_{y\in\partial U}\|x-y\|,$$

uniformly over any compact subset of U.

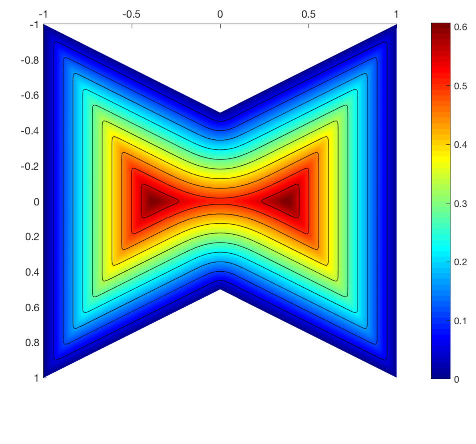

Fig 4. Approximation of the distance to the boundary on a non-convex sets, with $\varepsilon = 10^{-4}$.

In Figure 4 we observe that, when the distance increases, the level sets develop singularities.

A relaxed formulation

Given a domain $\Omega\subset\mathbb{R}^n$, $d(\Omega)$ being its diameter, we consider a large box B containing the set $\{x\in\mathbb{R}^n\ |\ d(x,\Omega)\le d(\Omega)\}$. For $\varepsilon>0$, we denote by w_ε the solution of the problem

$$\begin{cases} w_\varepsilon - \varepsilon\Delta w_\varepsilon = 0 & \text{in } B\backslash\omega,\\ w_\varepsilon = 1 & \text{on } \partial\omega\cup\partial B.\end{cases}$$

By **Theorem 1**, the function $v_\varepsilon \longmapsto -\sqrt{\varepsilon}\ln w_\varepsilon(x)$ uniformly converges to $d_\omega : x\longmapsto \inf_{y\in\omega}\|x-y\|$ on $\Omega\backslash\omega$. We then consider the following functionals

$$J_{p,\varepsilon}(\omega) := \|v_\varepsilon\|_p, \quad J_{p,0}(\omega) := \|d_\omega\|_p \quad\text{and}\quad J_{\infty,0} := d^H(\omega,\Omega) = \|d_\omega\|_\infty.$$

For both of the classes of sensors above $\mathcal{K}$ and $\mathcal{B}_N$, the following Γ-convergence result holds:

$$J_{p,\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} J_{p,0} \underset{p\to+\infty}{\longrightarrow} J_{\infty,0}.$$

This guarantees the convergence of the minimizers of the given functionals on those classes. It is then natural to address the following approximating shape optimization problems:

$$\min_{\omega\in\mathcal{K}} J_{p,\varepsilon}(\omega) \quad\text{and}\quad \min_{\omega\in\mathcal{B}_N} J_{p,\varepsilon}(\omega), \tag{3}$$

with p large and ε small. In contrast with (1), explicit formulae for the shape derivative of the relaxed functional $J_{p,\varepsilon}$ can be obtained, using classical tools on shape derivatives. This allows to develop efficient computational algorithms.

Numerical simulations

Using Matlab's routine `fmincon`, we solve (3) with $p = 30$ and $\varepsilon = 10^{-4}$.

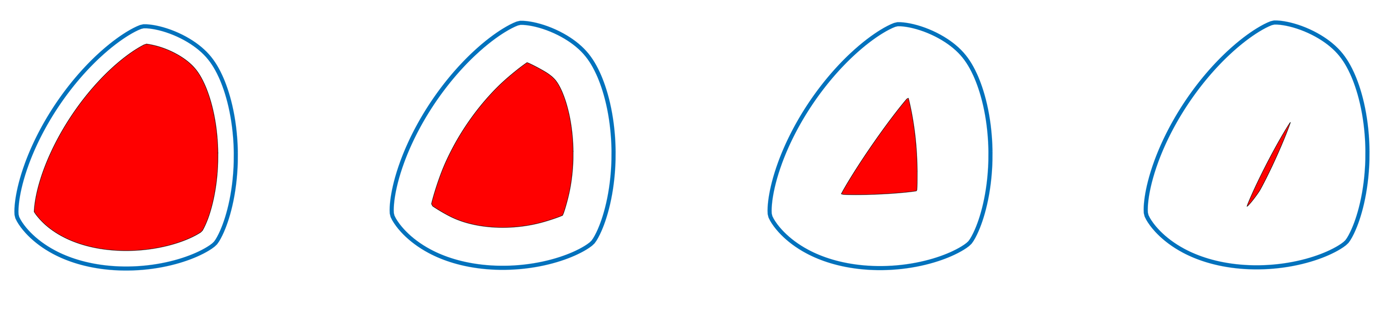

Fig 5. Optimal convex sensors obtained for the fractions $c_0 \in \{0.01, 0.1, 0.4, 0.7\}$.

When the volume fraction c_0 of the sensor ω is large, the optimal sensor corresponds to a level set of the distance function to $\partial\Omega$. But this is no longer true for small values of c_0, due to the singularities exhibited by the distance function.

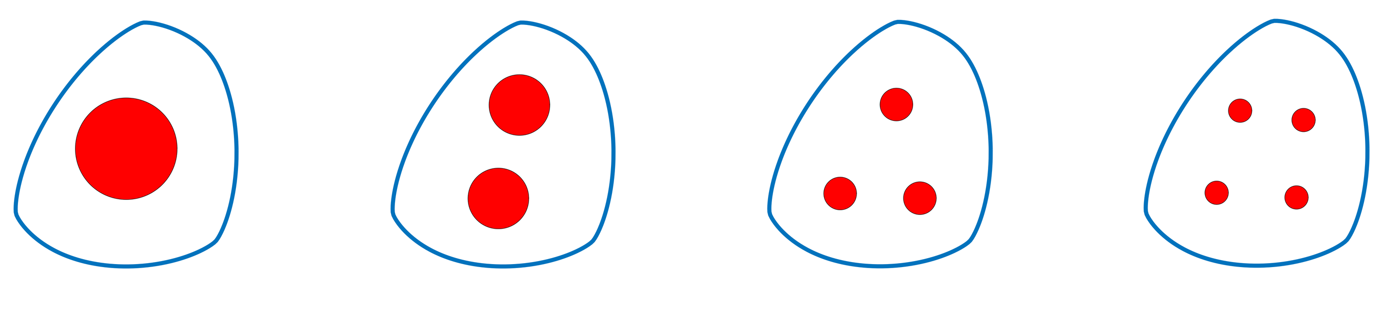

Fig 6. Optimal placement of spherical sensors for $N \in \{1,2,3,4\}$.

Conclusion

The problem of optimal shape and placement of sensors has been addressed in a purely geometric setting, independent of the physical process under consideration and in the absence of PDE restrictions. Problems are then recast in the context of the optimization of the Hausdorff distance, but the use of Varadhan's approximation theorem naturally leads to consider optimization problems constrained by the Laplacian. This allows to apply the classical analytical and computational tools in PDE shape design.

References

[1] I. Ftouhi, E. Zuazua (2022). **From the average to the maximal distance via Γ convergence.** In prepation.
[2] S. R. S. Varadhan (1967). **On the behavior of the fundamental solution of the heat equation with variable coefficients.** Communications on Pure and Applied Mathematics.

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