On the transport limit of singularly perturbed convection-diffusion problems on networks

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TRR 154 | Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks

Mini-Workshop at FAU
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Outline

1. Convection-diffusion problem on network of pipes
   coupling at pipe junctions, basic properties, well-posedness

2. Transport problem on network of pipes
   coupling at pipe junctions, basic properties, well-posedness

3. Asymptotic analysis for vanishing diffusion
   single pipe, boundary layers, asymptotic convergence, extension to networks

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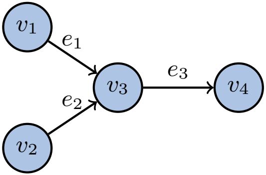
Convection-diffusion problem on network of pipes

Transport with diffusion on each pipe $e = (0, \ell^e)$**:**

$$a^e \partial_t u_\epsilon^e(x,t) + b^e \partial_x u_\epsilon^e(x,t) - \epsilon^e \partial_{xx} u_\epsilon^e(x,t) = 0, \quad x \in e,\ t > 0,$$
$$u_\epsilon^e(x,0) = u_0^e(x), \quad x \in e$$



Assumptions: $a^e,\ b^e,\ \epsilon^e > 0$ positive and constant on each pipe with $b^{e_1} + b^{e_2} = b^{e_3}$

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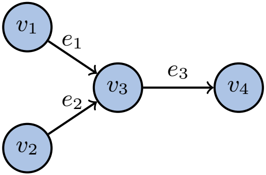

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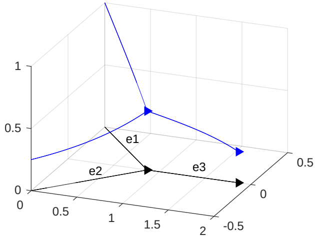
Coupling and boundary conditions:

- One boundary condition at each end of pipe



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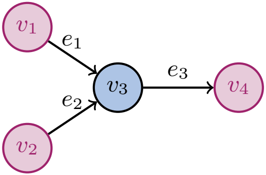

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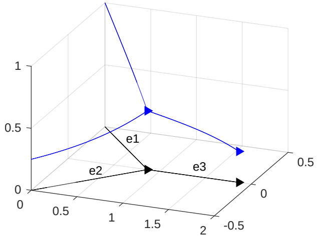

Convection-diffusion problem on network of pipes

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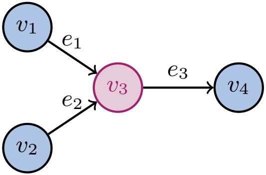

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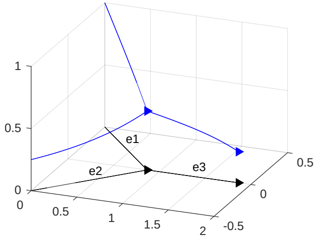

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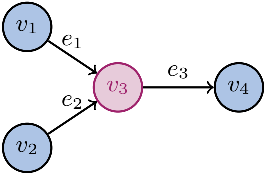

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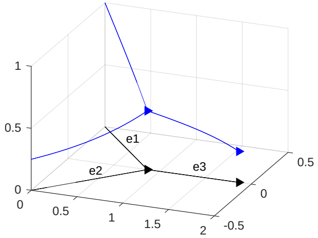
$$u^{e_i}(v_3, t) = \hat{u}^{v_3}(t), \qquad i = 1, 2, 3$$

- Conservation of diffusive flux at pipe junction v_3:

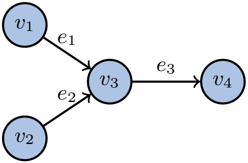
$$\sum_{i=1}^{3} (b^{e_i} u_\epsilon^{e_i}(v_3) - \epsilon^{e_i} \partial_x u_\epsilon^{e_i}(v_3)) n^{e_i}(v_3) = 0$$

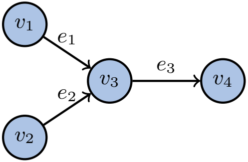


See [Mugnolo'14, Oppenheimer'00] for similar problems

Conservation of mass:

$$\frac{d}{dt}\int_{\mathcal{E}} au(x,t)\,dx = \sum_{v\in\mathcal{V}_\partial}\big(-b^e g^v + \epsilon^e \partial_x u_\epsilon^e(v)\big)n^e(v)$$





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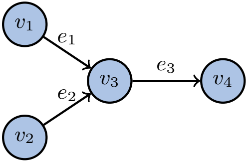
Dissipation of entropy: (due to diffusion)

$$\frac{d}{dt}\|a^{1/2}u\|^2_{L^2(\mathcal{E})} = -||\epsilon^{1/2}\partial_x u_\epsilon||^2_{L^2(\mathcal{E})} + \sum_{v\in\mathcal{V}_\partial}\big(-\tfrac{1}{2}b^e g^v + \epsilon^e\partial_x u^e_\epsilon(v)\big)\,g^v n^e(v)$$



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Theorem. (Well-posedness)

For any $g \in C^2(0,t_{\max};\mathcal{V}_\partial)$, $u_0 \in H^1(\mathcal{E})\cap H^2_{pw}(\mathcal{E})$ satisfying the coupling condition at junctions for $t=0$, the convection-diffusion problem on networks has a unique solution

$$u \in C^1([0,t_{\max}];L^2(\mathcal{E}))\cap C^0([0,t_{\max}];H^1(\mathcal{E})\cap H^2_{pw}(\mathcal{E})).$$

Proof: Convection-diffusion problem can be written as abstract evolution problem with dissipative operator $\Rightarrow$ Lumer-Phillips theorem yields well-posedness

Proof in [EggerPhilippi'20: On the transport limit of singularly perturbed convection-diffusion problems on networks. arXiv:2004.09490], see [EngelNagel'00, Pazy'83] for further details on semigroups

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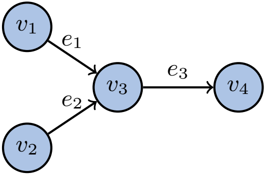
Assumptions: a^e and b^e positive and constant on each pipe with $b^{e_1} + b^{e_2} = b^{e_3}$



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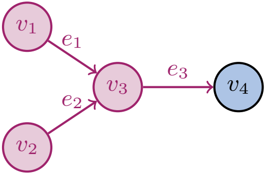

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Coupling and boundary conditions:

- Boundary condition at inflow boundary $x = 0$ of each pipe:

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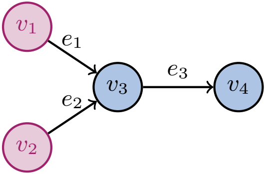

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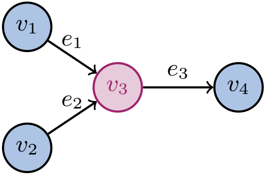

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- Coupling/mixing condition at pipe junction v_3:

$$\hat{u}^{v_3} = \frac{b^{e_1} u^{e_1}(v_3) + b^{e_2} u^{e_2}(v_3)}{b^{e_1} + b^{e_2}}$$

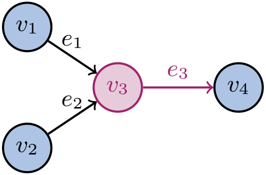
Coupling conditions

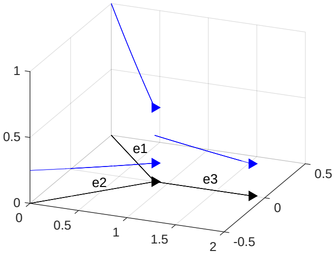
Pipe junctions: Mixing at v_3

$$\hat{u}^{v_3} = \frac{b^{e_1} u^{e_1}(v_3) + b^{e_2} u^{e_2}(v_3)}{b^{e_1} + b^{e_2}}$$

$\rightarrow$ Inflow boundary condition for pipe e_3

$\rightarrow$ Solution possibly discontinuous at v_3





Transport on networks was considered in [Dorn'08, DornKramarNagelRadl'10, EggerPhilippi'20, KramarSikolya'05, Mugnolo'14]

Coupling conditions

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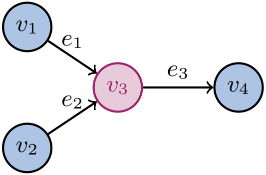

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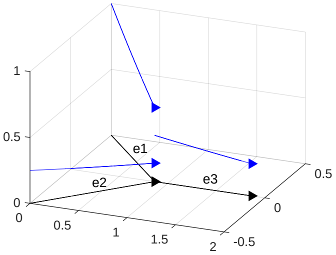

- Mass is conserved at pipe junction v_3

$$b^{e_1}u^{e_1}(v_3) + b^{e_1}u^{e_1}(v_3) = b^{e_3}u^{e_3}(v_3)$$

*since $b^{e_1} + b^{e_2} = b^{e_3}$

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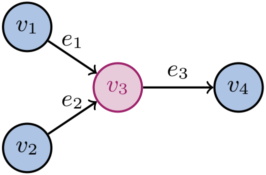

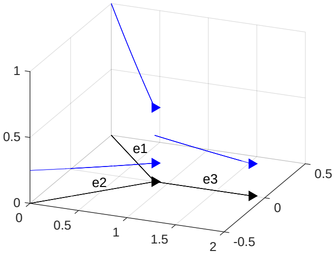

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- Entropy is dissipated at v_3 due to mixing

$$b^{e_1}|u^{e_1}(v_3)|^2 + b^{e_2}|u^{e_2}(v_3)|^2 \geq b^{e_3}|u^{e_3}(v_3)|^2$$

*by Jensen's inequality

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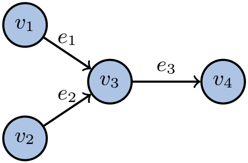

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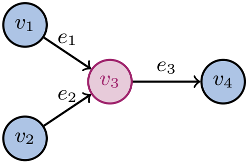
Basic properties and well-posedness



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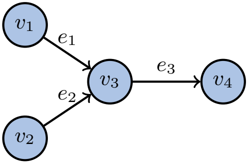

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Theorem. (Well-posedness)

For any $g \in C^2(0, t_{\max}; \mathcal{V}^{in}_{\partial})$ and $u_0 \in H^1_{pw}(\mathcal{E})$ satisfying the coupling condition at pipe junctions for $t = 0$, the transport problem on networks has a unique solution

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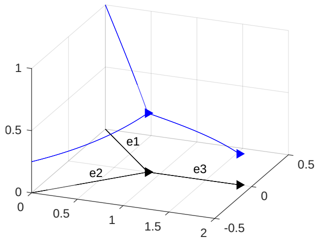
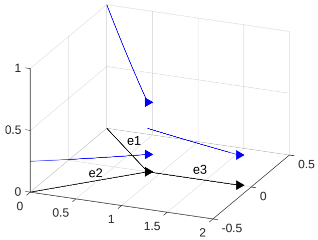
Convection-diffusion problem:



- Boundary condition at each end of pipe

Transport problem:



- Boundary condition at inflow boundary of pipe

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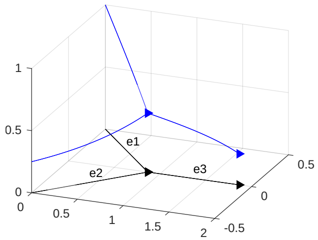

- Boundary condition at each end of pipe
- $|\mathcal{E}(v)| + 1$ coupling conditions at pipe junction

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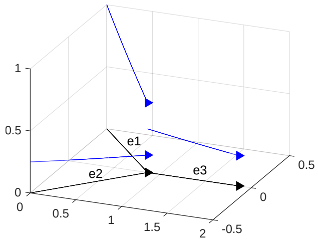

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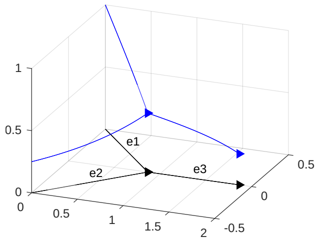

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- Continuity of solution across junctions

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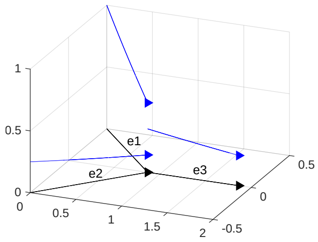
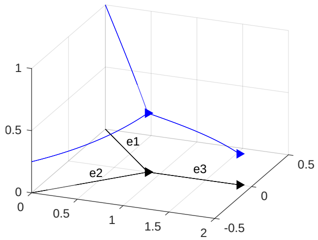


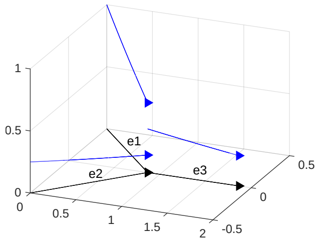
- Boundary condition at inflow boundary of pipe
- $|\mathcal{E}^{out}(v)| + 1$ coupling conditions at pipe junction
- Outflow continuity of solution at junctions

Convection-diffusion problem:



- Boundary condition at each end of pipe
- $|\mathcal{E}(v)| + 1$ coupling conditions at pipe junction
- Continuity of solution across junctions
- Conservation of mass

Transport problem:



- Boundary condition at inflow boundary of pipe
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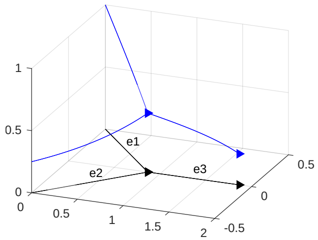

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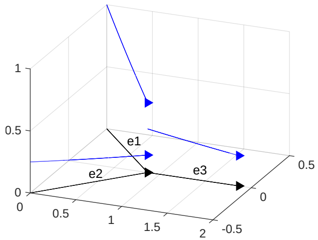

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Vanishing diffusion $\epsilon \to 0$ on single edge:

- Singular perturbed convection-diffusion problems intensively studied in literature
  see [GoeringFelgenhauerLubeRoosTobiska'83, Linß'09, MillerO'RiordanShishkin'12, RoosStynesTobiska'08]

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- Loss of regularity at outflow boundary, degenerate estimates in H^k-norm
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- Well-known asymptotic convergence result:

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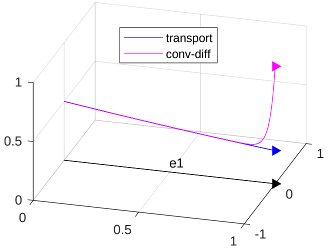

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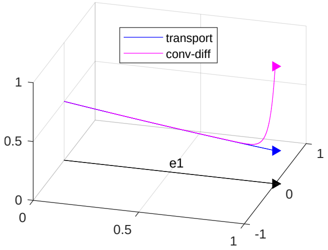


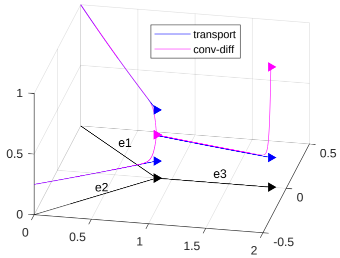
Question: Can we show analogous result on networks?

1. What are additional difficulties on networks?
2. How does the proof for single edges works?
3. Can we adopt techniques so that we can prove convergence estimate on networks?

Vanishing diffusion $\epsilon \to 0$ on networks:

- Boundary layer at outflow boundary of each edge due to
  - obsolete boundary conditions at network outflow boundary,
  - change in number and type of coupling conditions at network junctions

  $\Rightarrow$ Additional interior layers



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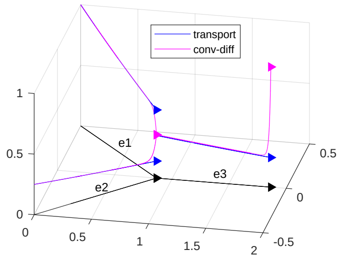

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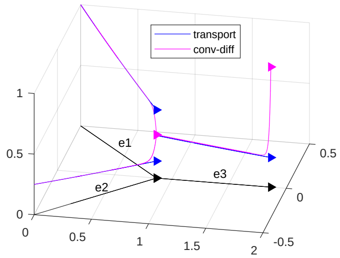

Proof on single edge

Asymptotic estimate

$$\|u_\epsilon - u\|_{L^\infty(0,T;L^2(0,\ell))} \leq C\sqrt{\epsilon} \quad \text{with } C \text{ indep. of } 0 < \epsilon \leq 1$$

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Sketch of proof:

- Split solution u_ϵ into regular (transport) part u and boundary layer w_ϵ
- Define suitable boundary layer function that approximates asymptotic behavior of solution:

$$w_\epsilon(x,t) := (g^\ell(t) - u(\ell,t))e^{-b(\ell-x)/\epsilon}$$

Important properties:

- $b\partial_x w_\epsilon - \epsilon\partial_{xx} w_\epsilon = 0$
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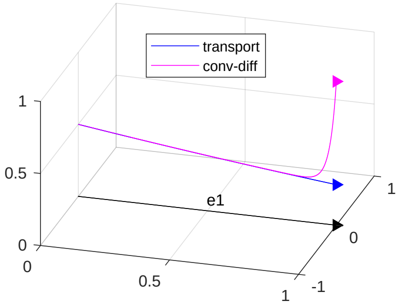

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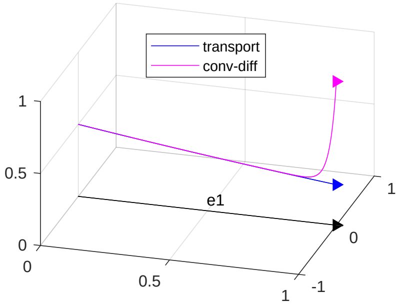

- Can estimate: $\|u_\epsilon - u\| \le \|u_\epsilon - u - w_\epsilon\| + \|w_\epsilon\| \le \|u_\epsilon - u - w_\epsilon\| + C\sqrt{\epsilon}$

  $\rightarrow$ Show that u_ϵ can be split into regular part u and boundary layer w_ϵ!

Complete proof in [Bobisud'67, RoosStynesTobiska'08]

Proof on single edge

Show: $\|u_\epsilon - u - w_\epsilon\|_{L^\infty(0,T;L^2(0,\ell))} \leq C\sqrt{\epsilon}$ with $w_\epsilon(x,t) := (g^\ell(t) - u(\ell,t))e^{-b(\ell-x)/\epsilon}$

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- Maximum principle yields boundedness of $u_\epsilon,\ \partial_t u_\epsilon$ and $\partial_x u_\epsilon(0)$ indep. of ϵ
- Investigate $u_\epsilon - u - w_\epsilon =: \eta_\epsilon$ at time $t = 0$ and at boundary
  - Initial time: $\eta_\epsilon(x,0) = 0$ for all $x \in (0,\ell)$
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- Estimate each term separately using
  - definition of boundary layer function,
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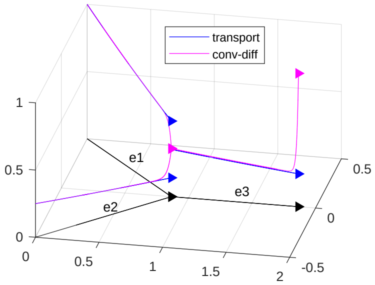
What is different compared to single edge?

- Additional interior layers due to change in number and type of coupling conditions at network junctions
- Values $u_\epsilon(v)$ at network junctions not prescribed a-priori but part of solution
- Transport solution discontinuous at network junctions for more than one inflowing edge



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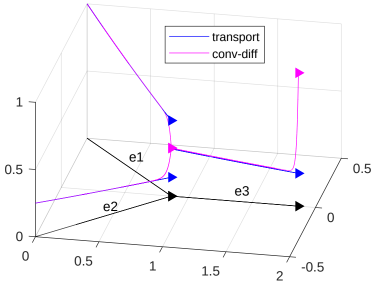
with $\hat{u}^{v_o^e}(t) := g^v(t)$ for $v \in \mathcal{V}_\partial^{\text{out}}$.

Compare with single edge:
$w_\epsilon(x,t) = (g^\ell(t) - u(\ell,t))e^{-b(\ell - x)/\epsilon}$



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  - $w_\epsilon^e(v_o^e,t) = (\hat{u}^{v_o^e}(t) - u^e(v_o^e,t))$
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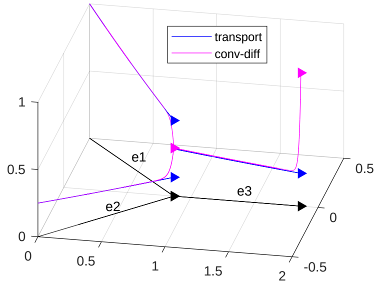

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Complete proof in [EggerPhilippi'20: On the transport limit of singularly perturbed convection-diffusion problems on networks.]

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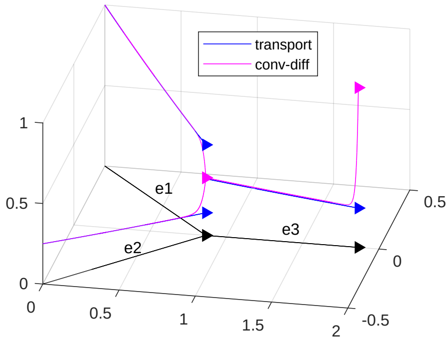
Numerical test

- Hybrid discontinuous Galerkin method for solving convection–diffusion and transport problem on network
- Maximum error at time $T = 4$ for vanishing diffusion
- Asymptotic estimate is sharp



ϵ	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
max error	0.1542	0.1109	0.0767	0.0534	0.0375	0.0264
rate	–	0.4764	0.5315	0.5214	0.5104	0.5061

Take-home messages

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- Extension to networks
  - Interior layer at junctions due to change in number and type of coupling conditions
  - Appropriate choice of boundary layer function at these interior layers allows similar analysis as on single edge