Analytic Properties of Heat Equation Solutions and Reachable Sets

Alden Waters

University of Groningen,
Bernoulli Institute
Groningen, Netherlands
in collaboration with
Professor Alexander Stromaier, University of Leeds

Outline of the talk

- What are we talking about? What exactly is reachability?
- Background on analytic functions
- Forward direction theorem and thermal layer potential theory
- Sharp example
- Analyticity and convergence of heat kernels in $\mathbb{C}^d$
- Converse direction theorem
- Where does this fit in?
- Future directions/open questions

What is the problem?

We consider the heat equation (Ω is a bounded Lipschitz domain)

$$\begin{aligned} \partial_t u(t,x) &= \Delta u(t,x) \quad \text{in} \quad [0,T]\times\Omega \\ u_0(0,x) &= u_0(x) \quad \text{in} \quad \Omega \\ u|_{[0,T]\times\partial\Omega}(t,x) &= h(t,x) \end{aligned}$$

with initial data $u_0 \in H_0^1(\Omega)$ on the time interval $[0, T]$, and with $u \in H^{1,\frac{1}{2}}([0,T]\times\Omega)$ and $h \in H^{\frac{1}{2},\frac{1}{4}}([0,T]\times\partial\Omega)$

The set

$$\mathcal{R}_\Omega(T, u_0) = \{v \in C^\infty(\Omega) \mid v(x) = u(T,x) \text{ where } u(t,x) \text{ solves "heat" } \text{ for some } h\}$$

is referred to as the *reachable set*. By null-controllabilty of the heat equation with boundary controls we have $\mathcal{R}_\Omega(T, u_0) = \mathcal{R}_\Omega(T, 0)$ and $\mathcal{R}_\Omega(T, 0)$ does not depend on $T > 0$.

Definition

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. The *reachable set* $\mathcal{R}_\Omega \subset C^\infty(\Omega)$ is defined as $\mathcal{R}_\Omega(T, 0)$ for some (and hence for all) $T > 0$.

Due to the linearity of the problem, the reachable set $\mathcal{R}_\Omega$ is a vector space.

Setup

Definition

For Ω an open and bounded Lipschitz domain in $\mathbb{R}^d$ we let $\mathcal{E}(\Omega)$ denote the set

$$\mathcal{E}(\Omega) = \left\{ z = x + iy \in \mathbb{C}^d \mid x \in \Omega,\ |y| < \operatorname{dist}(x, \partial\Omega) \right\}.$$

Setup (continued)

Notation:

- For an open subset $U \subset \mathbb{C}^d$ the set of holomorphic functions on U is denoted by $\mathcal{O}(U)$. We endow it with the topology of uniform convergence on compact subsets of U.
- For a subset $E \subset \mathbb{C}^d$ that is the closure of an open set we denote by $\mathcal{O}(E)$ the set

$$\mathcal{O}(E) = \bigcap_{U \supset E, U \text{ open}} \mathcal{O}(U).$$

- A function $f \in \mathcal{O}(\mathcal{E}(\Omega))$ is completely determined by its restriction to Ω and therefore we think of the set of analytic functions $\mathcal{O}(\mathcal{E}(\Omega))$ as a subset of $C^\infty(\Omega)$ without further mention.

Forward Direction

Theorem

If Ω is a bounded Lipschitz domain, then $\mathcal{R}_\Omega \subset \mathcal{O}(\mathcal{E}(\Omega))$. The domain $\mathcal{E}(\Omega)$ is optimal in the following sense. For any $w \in \mathbb{C}^d \setminus \overline{\mathcal{E}(\Omega)}$ there exists a function $u \in \mathcal{R}_\Omega$ which does not have an analytic extension to a connected open set containing w and Ω.

$\Rightarrow$ $\mathcal{E}(\Omega)$ is the optimal domain of extension!

Funny Mixed Sobolev Spaces

For a number $T > 0$ we define

$$Q = (0, T) \times \Omega \qquad \Sigma = (0, T) \times \partial\Omega.$$

Also let $\Omega_t = \{t\} \times \Omega$ so that

$$\partial Q = \overline{\Sigma} \cup \Omega_0 \cup \Omega_T.$$

For $r, s \geq 0$ we let

$$H^{r,s}(Q) = L^2((0, T); H^r(\Omega)) \cap H^s((0, T); L^2(\Omega)).$$

Free Space Green's Function

Let $G(t, x)$ be defined as

$$G(t,x) = \begin{cases} (4\pi t)^{-d/2} \exp(-\frac{|x|^2}{4t}) & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \leq 0. \end{cases}$$

Goal: write solutions to the heat equation on a bounded Lipschitz domain in terms of the free space Green's function.

Thermal Layer Theory

For sufficiently regular h the single layer potential for the heat equation is defined as follows:

$$S(h)(t,x) = \int_0^t \int_\Gamma G(t-s, x-y)h(s,y)\, dy\, ds$$

for $(t,x) \in Q$. The boundary layer potential operator is defined as

$$V(h)(t,x) = \int_0^t \int_\Gamma G(t-s, x-y)h(s,y)\, dy\, ds$$

with $(t,x) \in \Sigma$.

Proposition

The single layer potential operator S continuously extends to a map $S : H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to H^{1,\frac{1}{2}}(Q)$*. The boundary layer potential operator extends by continuity to an isomorphism*

$$V : H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma).$$

These properties come from a sequence of papers by M. Costabel for Lipschitz domains, c.f. *Boundary Integral Operators for the Heat Equation*, '90

Proposition

The trace map $\gamma : u \to u|_\Sigma$ is continuous and surjective from $\tilde{H}^{1,\frac{1}{2}}(Q)$ to $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$. For all $f \in \tilde{H}^{-1,-\frac{1}{2}}(Q)$ and $g \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ there exists a unique $u \in \tilde{H}^{1,\frac{1}{2}}(Q)$ with

$$(\partial_t - \Delta)u = f \quad in \quad Q$$
$$\gamma u = g \quad on \quad \Sigma.$$

In case $f = 0$ the solution u is given by $u = S(V^{-1}g) = S(\gamma_1 u) - D(\gamma u)$.

Heat kernel admits an analytic extension $\tilde{G}$:

$$\tilde{G}(t, z - y) = \left\{ \begin{array}{lll} (4\pi t)^{-d/2} \exp(-\frac{(z-y)^2}{4t}) & \text{if} & t > 0 \\ 0 & \text{if} & t \leq 0 \end{array} \right\}.$$

Note that $\operatorname{Re}(z - y)^2 = |x - w|^2 - |y|^2$. In the open set defined by $|y| < |x - w|$ the function $\tilde{G}(t, z - y)$ is smooth in t, and complex analytic in z.

For an integrable function f on Σ we define

$$\tilde{S}(f)(t,z) = \int_0^t \int_\Gamma \tilde{G}(t-s, z-y) f(s,y)\, dy\, ds.$$

Main idea: Whatever f is, then the kernel, $\tilde{G}$, is the only part which depends on z.

Sharp Example

Let $a \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$. Define

$$h_a(x,s) = \chi_{(0,\infty)}(s)\chi_{(0,\infty)}(1-s)\delta(x-x_0)e^{-\frac{a}{4-4s}}(1-s)^{\frac{d}{2}-1}.$$

By using $S(h_a)$ we get $u_t(x)$ that is a solution of the heat equation in $\mathbb{R} \times \Omega$ that extends to:

$$u(t,z) = \frac{1}{(4\pi)^{\frac{d}{2}}} \int_0^t (t-s)^{-\frac{d}{2}} e^{-\frac{(z-x_0)^2}{4(t-s)}} e^{-\frac{a}{4(1-s)}} (1-s)^{\frac{d}{2}-1} ds.$$

$\Rightarrow g(z) = u(1,x)$ is reachable.

Explicitly

$$g(z) = \frac{1}{(4\pi)^{\frac{d}{2}}} \int_0^1 s^{-1} e^{-\frac{(z-x_0)^2}{4s}} e^{-\frac{a}{4s}} ds =$$

$$\frac{1}{(4\pi)^{\frac{d}{2}}} E_1 \left(\frac{1}{4} \left((z - x_0)^2 + a \right) \right),$$

$$E_1(z) = -\gamma - \log(z) - \sum_{k=1}^{\infty} \frac{(-z)^k}{k(k!)},$$

This function is not analytic at $z = 0$, hence, $g(z)$ is not holomorphic at w when $(w - x_0)^2 + a = 0$, i.e. when $\mathrm{Re}(a) = (\mathrm{Im}(w))^2 - (\mathrm{Re}(w) - x_0)^2$ and $2\,\mathrm{Im}(w) \cdot x_0 = -\mathrm{Im}(a)$.

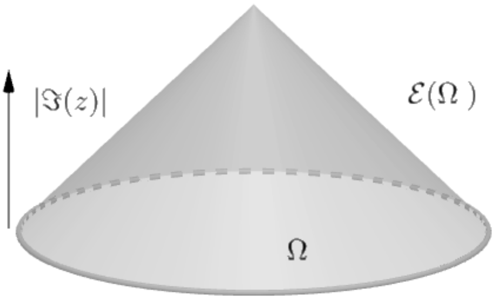

Figure: Analytic extension domain when $|\operatorname{Im}(z)|$ is projected onto the y axis.

Theorem

Assume that u_t is a solution of the heat equation (heat) with $u_0 \in \mathcal{O}(\mathcal{E}(\Omega))$. Then u_t converges to u_0 as $t \to 0_+$ uniformly on compact subsets of $\mathcal{E}(\Omega)$.

Heat equation kernel in the complex plane behaves like the kernel in the real plane, so can be considered the extension

Lemma

Assume that $u \in C^\infty(\Omega_1)$ has an analytic extension to $\mathcal{E}(\Omega_1)$. Then the analytic continuation of $K_t(u\chi)$ converges to the analytic continuation of u uniformly on compact subsets of $\mathcal{E}(\Omega)$ as $t \to 0_+$.

Same as for real heat kernel, the convolution solves the heat equation and then you need to check as $t \to 0_+$ this goes to the initial data.

***Tricker since the kernel is now in the complex plane!

The explicit formula for $K_t(u\chi)$ is

$$K_t(u\chi)(z) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{(z-w)^2}{4t}} g(w)dw,$$

where $g(w) = u(w)\chi(w)$. Set $v = \mathrm{i}\,\mathrm{Im}(z)$. Let Γ be the set $\mathbb{R}^d + \mathrm{i}v$, let $\mathcal{T}$ be the region $\mathbb{R}^d + \mathrm{i}[0,1]v = \{v + \mathrm{i}tv \mid t \in [0,1]\}$. For a smooth function φ in the complex plane we have

$$\int_{\partial Y} \varphi(x+\mathrm{i}y)d(x+\mathrm{i}y) = 2\mathrm{i} \int_Y \overline{\partial}_z \varphi(x+\mathrm{i}y)dxdy.$$

Thus, shifting the contour in the direction of v, we obtain

$$K_t(u\chi)(z) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{(z-w)^2}{4t}} g(w)dw =$$

$$\frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\Gamma} e^{-\frac{(z-w)^2}{4t}} g(w)dw + 2\mathrm{i}\frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathcal{T}} e^{-\frac{(z-w)^2}{4t}} u(w)\overline{\partial}_w \chi(w)d\sigma$$

$$= I_1(z) + I_2(z)$$

The first integral $I_1(z)$ equals

$$I_1(z) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{(\mathrm{Re}(z)-w)^2}{4t}} g(v+w)dw.$$

Since the heat kernel is a δ-family this integral converges to $g(\mathrm{Re}(z)+v) = g(z)$.

Backwards Direction

Theorem

Suppose that $\Omega = B_R(x_0)$ is a ball. Then

$$\mathcal{O}\left(\overline{\mathcal{E}(\Omega)}\right) \subset \mathcal{R}_\Omega \subset \mathcal{O}(\mathcal{E}(\Omega)).$$

In 1d, this theorem is due to Darde and Ervedoza. Also in 1d this result is sharp due to a result of Tucsnak, Hartmann and Orsoni.

Proof Sketch

Main Ideas:

- Analytic extension of the usual representation formula solves the heat equation in the y variable.
- Wick rotation $\Rightarrow$ solves backwards heat equation when restricted to the x axis.
- Starting from the complex axis, and then analytically extending to the real axis allows for this Wick rotation.
- Restriction to the boundary cylinder on the real axis gives the Dirichlet boundary controls.

Proof Sketch with more symbols

The function $u_t(x, y) = u_t(z)$ satisfies the Cauchy Riemann equations $(\nabla_x + i\nabla_y)u_t(x, y) = 0$ and is completely determined for $t > 0$ by $u_t(0, y) = \phi_t(y)$.
Indeed u_t converges to $u(x, y)$ uniformly on $\mathcal{E}(\Omega) = \mathrm{i}\mathcal{E}(\Omega)$ as $t \to 0_+$ by previous theorem we have uniform convergence on the compact set $\overline{\mathcal{E}(\Omega)}$.

By construction $(\Delta_x + \Delta_y)u_t(x, y) = 0$ and $u_t(x, y)$ solves the backward heat equation

$$
\begin{aligned}
&(\partial_t + \Delta_x)u_t(x, y) = 0 \\
&u_0(x, y) = u(x, y)
\end{aligned}
$$

on $\mathrm{i}\mathcal{E}(\Omega)$, the "Wick rotated" set. Since $\mathrm{i}\mathcal{E}(\Omega)$ contains Ω the function $u_t(x) := u_t(x, 0)$ solves

$$
\begin{aligned}
&(\partial_t + \Delta_x)u_t(x) = 0 \text{ in } Q \\
&u_0(x) = u(x) \\
&u_t(x)|_\Sigma = h(t, x)
\end{aligned}
$$

where h extends smoothly across $\partial\Omega$.

Where does this fit in?

Literature on controllability for heat equation:

- Zuazua, Lebeau-Robbiano, (null)controlability
- Fattorini and Russel, moment method, first limited results in 1d.
- Martin, Rosier, Rouchon theorems in 1d.
- Darde and Ervedoza, same converse theorem in 1d
- Tucsnak, Hartmann, Orsoni and students/coauthors, sharp converse theorem in 1d

$\Rightarrow$ No one has of yet computed the conditions for analytic extension for higher dimensional domains (!!)

Future Directions and Open Questions

- Manifolds- hyperbolic (Burq, precise asymptotics for such Kernels)
- Unbounded domains, upper half space (easier?)
- Manifolds, compact, (Mitrea layer potentials exist, Zelditch, behavior of kernels)
- Potentials, bounded/unbounded, propagator is different
- Specific geometries or different boundary conditions
- What is the sharp set for *generic* Lipschitz domain, not just a ball?