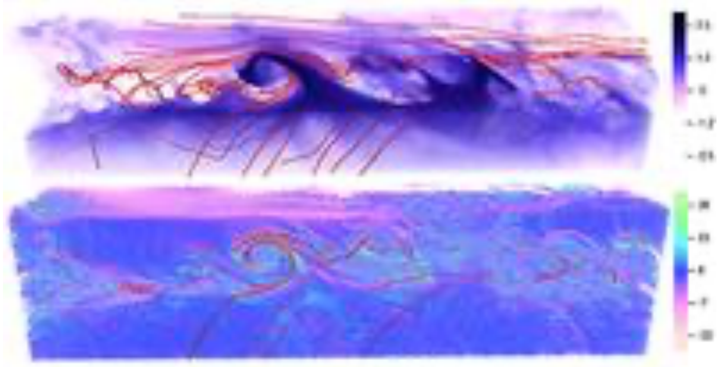
Some problems in the dynamics of stratified fluids



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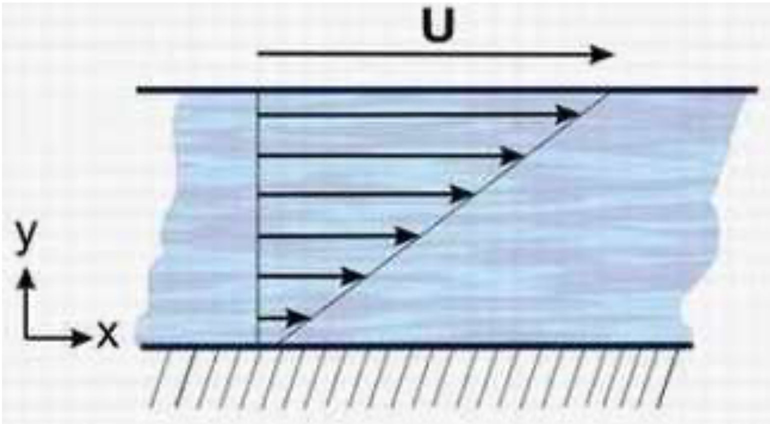
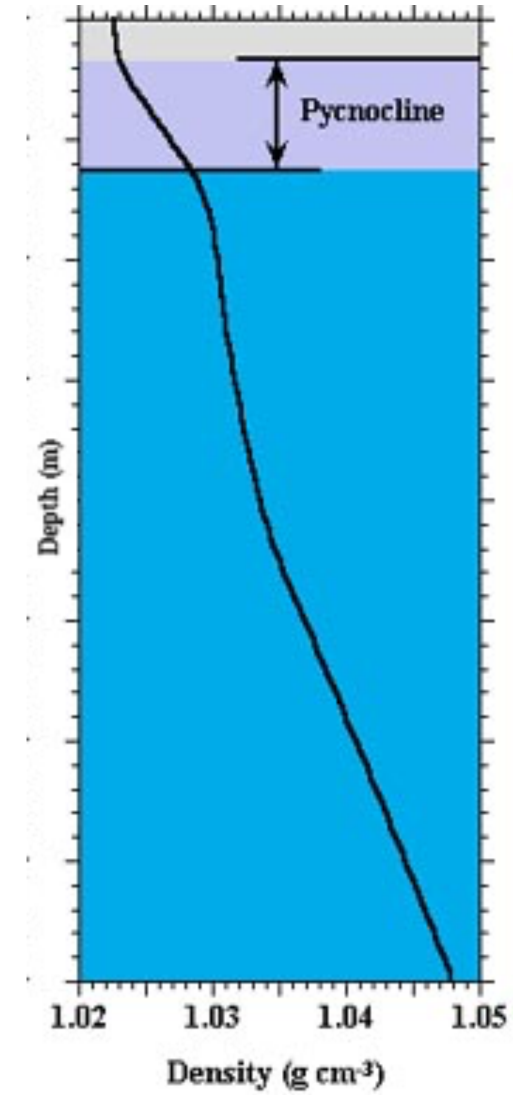


Stratified fluids

The incompressible Navier-Stokes + thermal diffusion 2D

$$\rho(\partial_t + \mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = -\rho\mathbf{g} + \nu\Delta\mathbf{u}$$
$$(\partial_t + \mathbf{u}\cdot\nabla)\rho = \kappa\Delta\rho$$
$$\nabla\cdot\mathbf{u} = 0$$

- $\mathbf{u} = (u^x, u^y)$ is the velocity field
- ρ is the density
- p is the pressure related to $\nabla\cdot\mathbf{u} = 0$
- $\mathbf{g} = (0, g)$ is gravity



Due to gravity, the lower density fluid is above and then density decreases with height (photo from http://ocp.ldeo.columbia.edu/climatekidscorner/whale_dir.shtml)

We study departures from the hydrostatic equilibrium:

$$(\bar\rho(y), \qquad \bar{\mathbf{u}}, \qquad \partial_y \bar p = -g\bar\rho)$$

where the stratification $\bar\rho(y)$ is stable —> $\bar\rho'(y) < 0$

Stratified fluids I

I) Linearization around the hydrostatic equilibrium with zero velocity

The perturbed variables are:

$$\rho = \overline{\rho}(y) + \tilde{\rho}(t,x,y) \qquad \mathbf{u} = (0 + \tilde{u}(t,x,y), 0 + \tilde{v}(t,x,y)) \qquad p = \overline{p} + \tilde{p}(t,x,y)$$

$$\partial_y \overline{p} = -\, g\overline{\rho}$$

RULE [Boussinesq]: density variations only counts for buoyancy effects, not inertia

(B)

$$\partial_t u + \delta\,(u\cdot\nabla)\,u + \nabla p + b\, e_3 = \nu\, \Delta u$$

$$\partial_t b + \delta\,(u\cdot\nabla)\, b + \frac{g}{\rho_0}\, u_3\, \overline{\rho}\,' = \underbrace{\kappa\, \Delta b}_{\text{due to thermal dissipation}}$$

(GRADIENTS of Temperature - Salinity create the STRATIFICATION)

$$\nabla\cdot u = 0\,,$$

where $u = (u_1, u_2, u_3)^T$, $e_3 = (0,0,1)^T$

ρ_0= reference density

b=buoyancy (normalized density perturbation)

—>Reference for formal derivation: book by M. Rieutord

—>Math papers on the (B) system: Lions, Temam, Wang, Beale, Bourgeois, Charve, Chemin, Abidi, Danchin, Hmidi, Paicu, Rousset…

Stable stratification

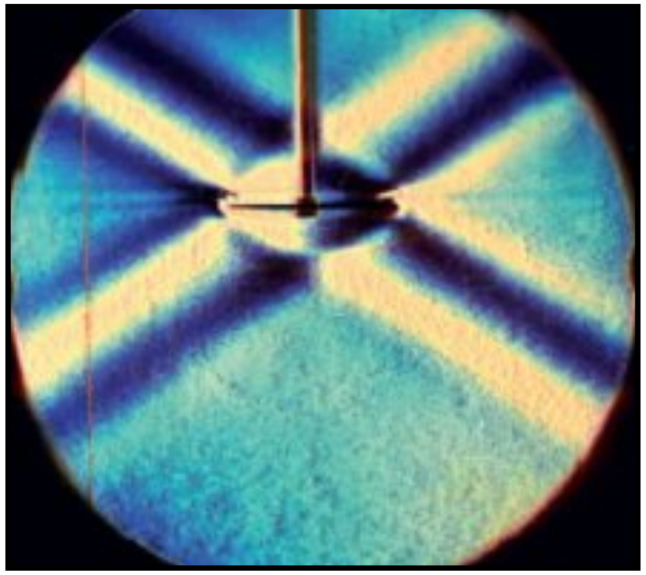
—> Stably stratified fluids —> the ocean, fluids in the core of the Earth

—> Why "stable" stratification? Consider the rest state

$$\bar{\rho}(y) \qquad \bar{\mathbf{U}} = (0,0) \qquad \partial_y \bar{p} = -\,\mathrm{g}\bar{\rho}$$

and linearize [linear inviscid approx] around it:

$$\partial_t b + \frac{g}{\rho_0} u_3 \bar{\rho}' = 0$$
$$\partial_t u + b\vec{e}_3 + \nabla P = 0$$
$$\nabla \cdot u = 0$$



The linearized system is spectrally stable if $\bar{\rho}'(y) < 0$: there is no eigenvalue ω s. t. $\Re(\omega) > 0$

* Dispersion relation: $\omega = \pm N \frac{|k_h|}{|\vec{K}|}$

*Brunt-Väisälä frequency: $N = \sqrt{\frac{-\bar{\rho}'}{\rho_0}}$

Dispersion relation of order 0 in k

The reflection law is unusual:
1) near-critical reflection from a slope

In the presence of dissipation,
2) anisotropy in the frequency variables

Stratified fluids II

II) Linearization around a shear flow in the inviscid regime

The perturbed variables are:

$$\rho = \bar{\rho}(y) + \tilde{\rho}(t,x,y) \qquad \mathbf{u} = (U(y) + \tilde{u}(t,x,y), 0 + \tilde{v}(t,x,y)) \qquad p = \bar{p} + \tilde{p}(t,x,y)$$

$$\partial_y \bar{p} = -g\bar{\rho}$$

(C)

$$\partial_t \tilde{\rho} + U(y)\,\partial_x \tilde{\rho} + \bar{\rho}'(y)\,\tilde{u}^y = 0$$

$$\bar{\rho}(y)\left(\partial_t \tilde{u}^x + U(y)\,\partial_x \tilde{u}^x + U'(y)\,\tilde{u}^y\right) = -\partial_x \tilde{p}$$

$$\bar{\rho}(y)\left(\partial_t \tilde{u}^y + U(y)\,\partial_x \tilde{u}^y\right) = -\partial_y \tilde{p} - g\tilde{\rho}$$

$$\partial_x \tilde{u}^x + \partial_y \tilde{u}^y = 0$$

The Miles-Howard criterion

* Seek for a solution $\tilde{\rho} = \rho(y)e^{\lambda t+ikx}$ $\tilde{u}^x = u^x(y)e^{\lambda t+ikx}$ $\tilde{u}^y = u^y(y)e^{\lambda t+ikx}$
* From the divergence-free condition, $u^x = -(i/k)\partial_y u^y$ and we can solve for p
* We end up with a 2nd order ODE in u^y
* Multiply it by the complex conjugate $(u^y)^*$ and integrate in y
* After a smart change of variable - [v is the new variable]

$$\Re(\lambda)\int_y \left\{ \overline{\rho}(y)(|\partial_y v|^2 + k^2|v|^2) + \frac{k^2\overline{\rho}(y)U'(y)^2}{|\lambda + ikU(y)|^2}\left(Ri(y) - \frac{1}{4}\right)|v|^2 \right\} dy = 0 \quad (*)$$

where

$$Ri(y) = \frac{-\overline{\rho}'(y)g}{\overline{\rho}(y)} \cdot \frac{1}{U'(y)^2} = \left(\frac{N(y)}{U'(y)}\right)^2$$

Richardson number

Brunt-Väisälä frequency

Since $\overline{\rho}' < 0$, then $\Re(\lambda) = 0$ when $Ri(z) > \frac{1}{4}$ and then there are no growing eigenvalues

Spectral stability Vs Asymptotic stability

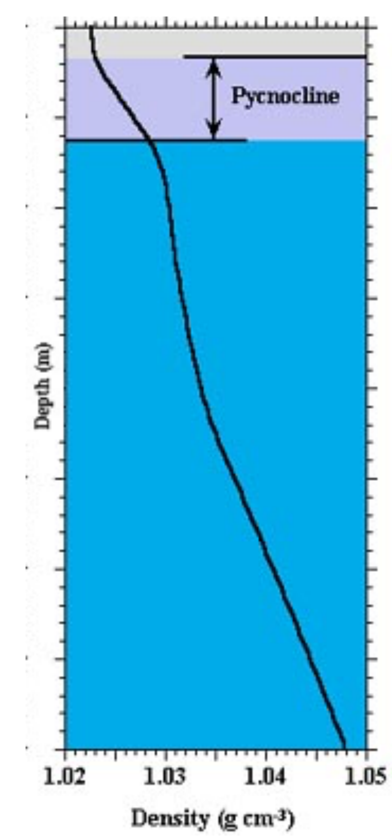
* Spectral stability (no growing eigenvalues) started 19th century [Rayleigh, Kelvin, Orr...]

BUT

it can be misleading in fluid dynamics,
where (linear) operators are often NON NORMAL (Orr mechanisms)

* We will investigate the (Lyapunov) asymptotic stability in $L^2(\mathbb{T}\times\mathbb{R})$
  for the linearized system
  * near the Couette flow $U(y) = y$ and where
  * the stratification is exponential $\bar{\rho}(y) = e^{-\beta y}$, $\beta > 0$

For small β, $\bar{\rho}(y) \approx 1 - \beta y$
and linear stratifications
are commonly considered



Stratified fluids I:
Near-critical reflection of internal waves from a sloping boundary

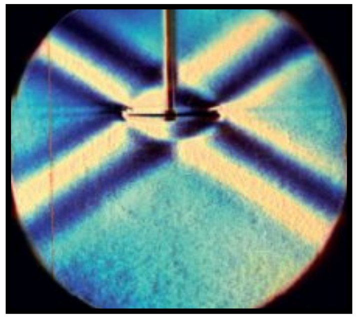
with Anne-Laure Dalibard and Laure Saint-Raymond

1) Reflection from a sloping boundary

- The dispersion relation fixes the angle of propagation, not the modulus of the wavelength

$$\omega = \pm N \frac{|k_h|}{|\vec{K}|} = \pm \sin\beta$$

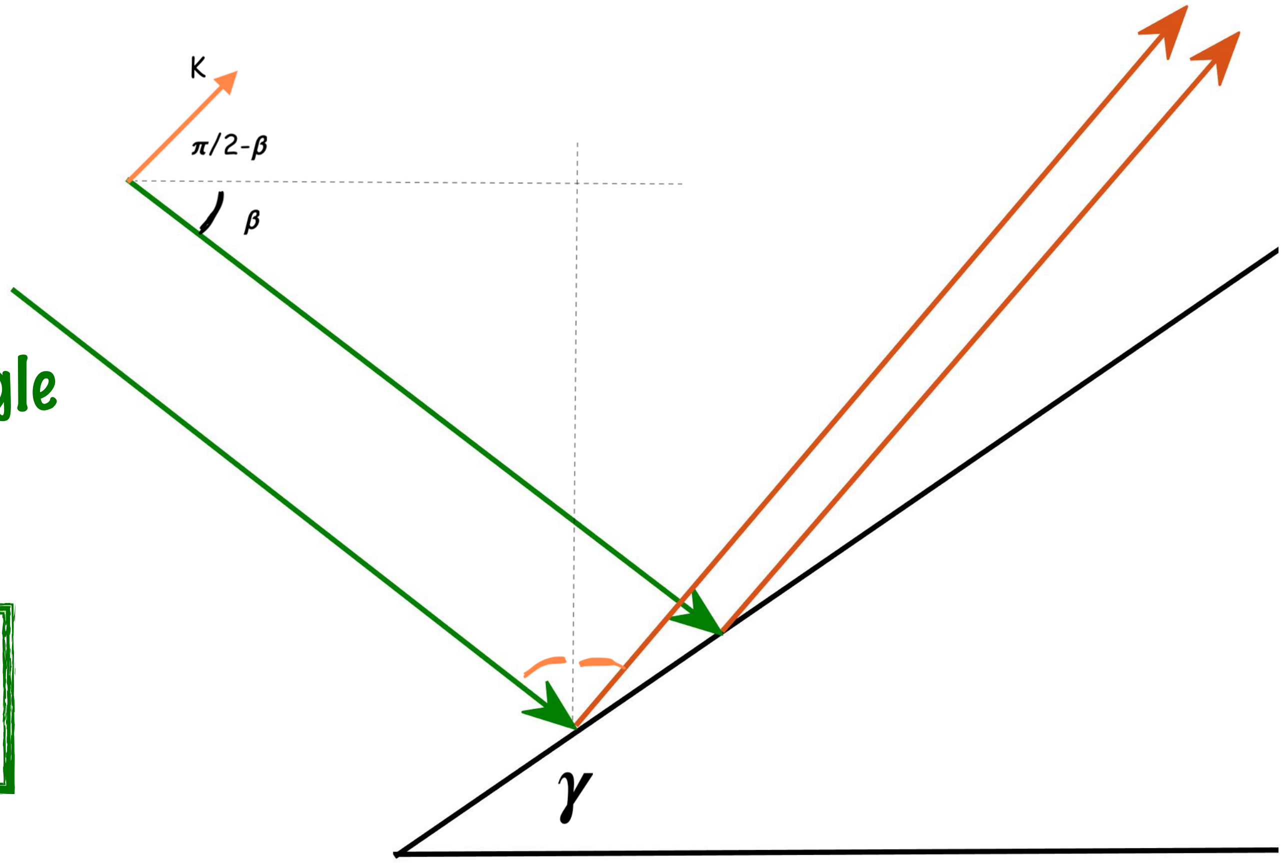
- The direction of propagation is orthogonal to K



- Reflection from a flat surface follows Descartes
- This is no longer true in the case of a SLOPE

The reflection law is: preserve the angle w.r.t. the vertical (gravity)

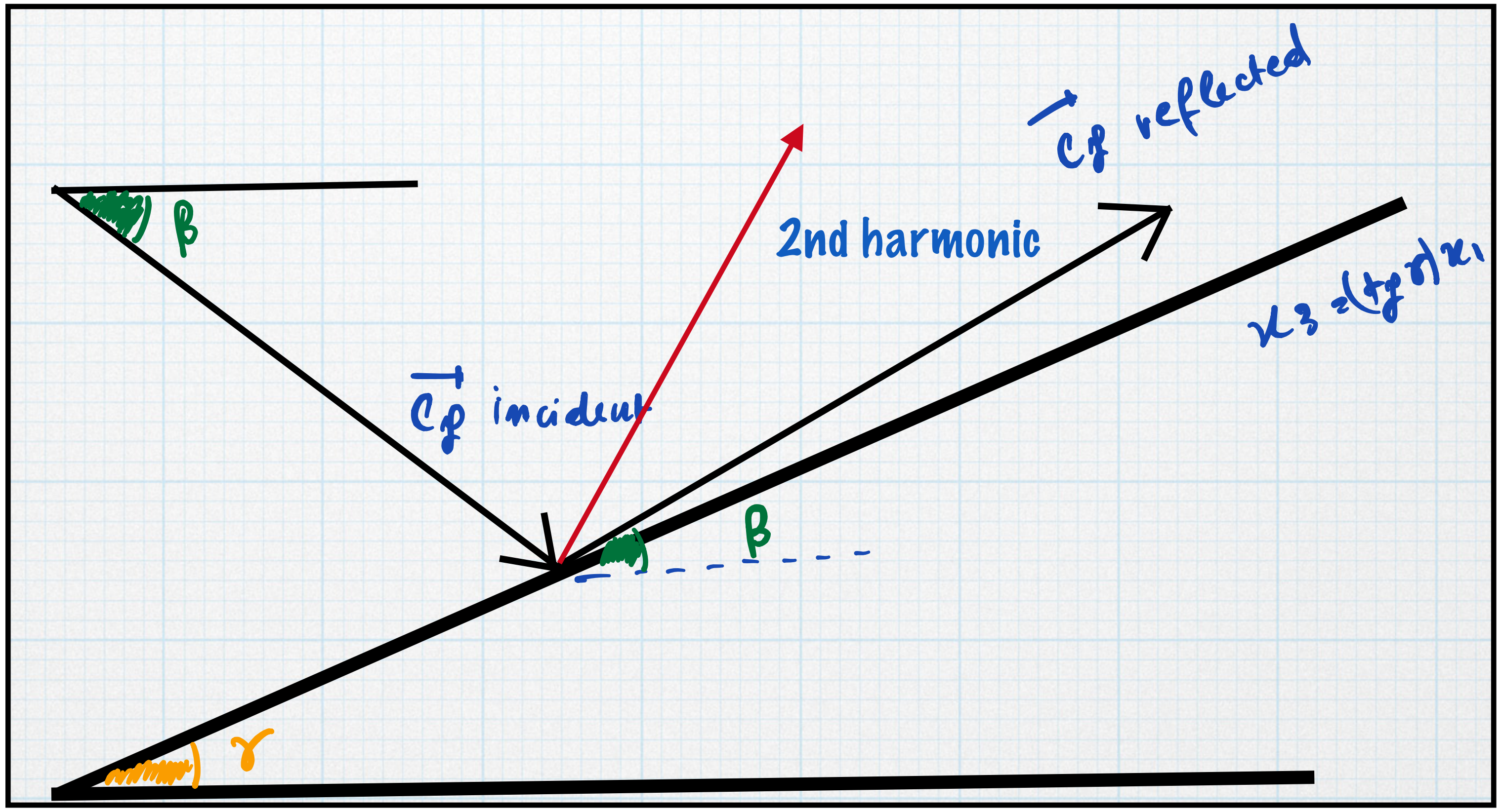
Focusing, accumulation of energy, growth of the amplitude of the waves



2 Main ingredients come into play: interaction with the boundary and role of the nonlinearity

Reflection from a slope: boundary + nonlinearity

Coming back to the simpler case of a sloping boundary… near-critical reflection



If $\gamma = \beta + O(\varepsilon^2)$ $(\varepsilon \ll 1)$

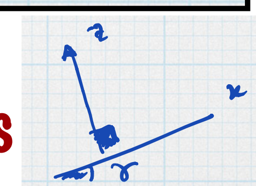
almost no space is left for the reflected wave, its vertical wavenumber is $O(\varepsilon^{-2})$

Then the wavelength is $O(\varepsilon^2)$

and its amplitude is $O(\varepsilon^{-2})$

FOCUSING… the "small" nonlinearity comes into play!

Weakly nonlinear system in the slope's coordinates



$$\begin{cases} \partial_t u - (\sin\gamma)\, b + \delta\, (\underline{u}\cdot\nabla)\, u + \partial_x p = \nu\, \Delta u \\ \partial_t w - (\cos\gamma)\, b + \delta\, (\underline{u}\cdot\nabla)\, w + \partial_z p = \nu\, \Delta w \\ \partial_t b + u(\sin\gamma) + w(\cos\gamma) + \delta(\underline{u}\cdot\nabla)\, b = \kappa\, \Delta b \end{cases}$$

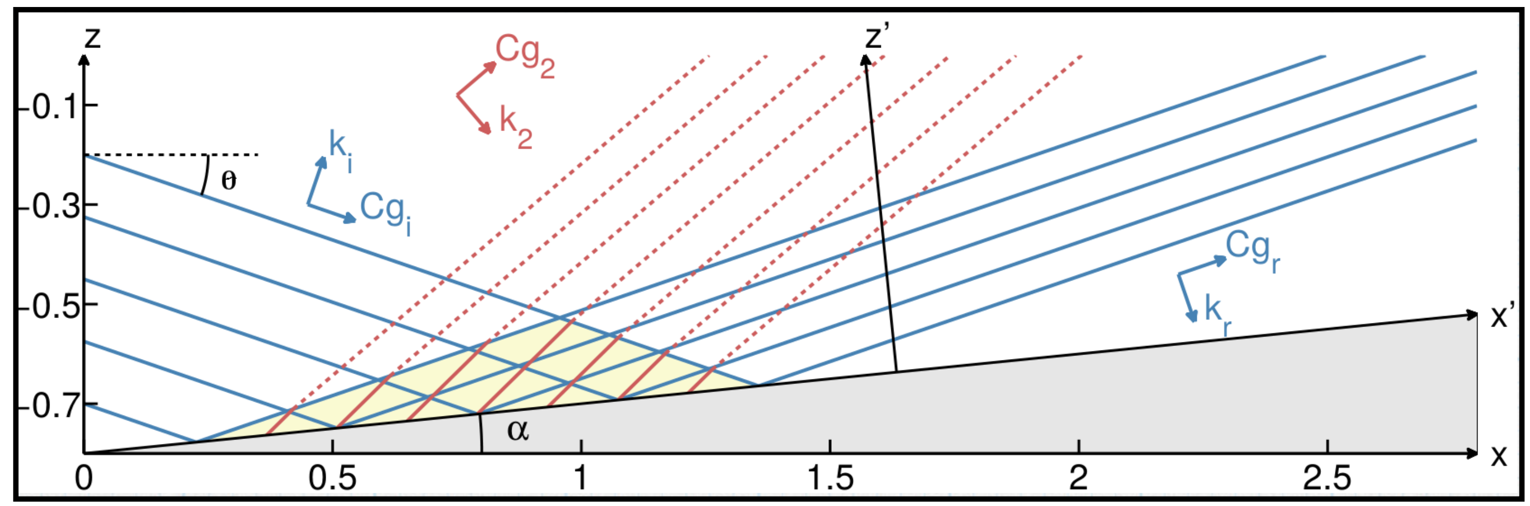
$$\boxed{\begin{array}{c} \nu \sim \kappa \ll 1 \\ \delta \ll 1 \end{array}}$$

+ Boundary conditions: $u|_{z=0} = w|_{z=0} = 0$ (no slip), $\partial_z b|_{z=0} = 0$ (no flux)

Near-critical reflection of internal waves from a sloping boundary

Since the amplitude of the "reflected wave" (boundary layers in the viscous case...) is "infinite", then the nonlinearity is no longer small and produces a second harmonic far from the boundary, in the outer region

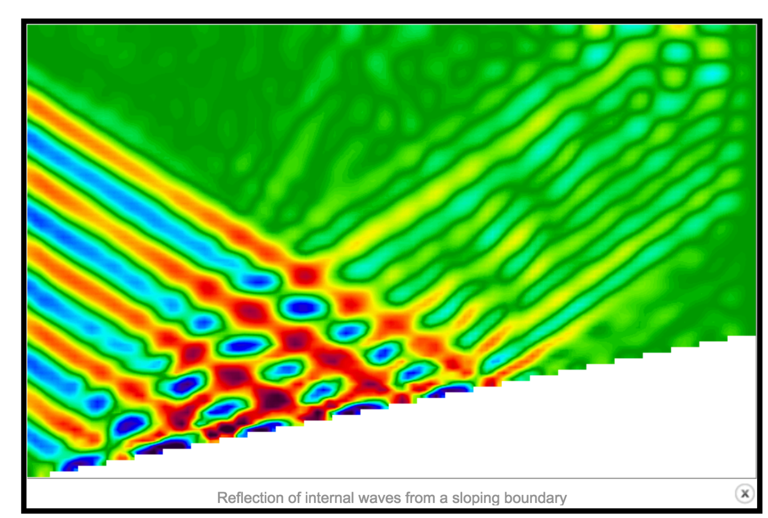
$$e^{-2i\omega_0 t + ik\cdot x}$$



* Formal study by Dauxois & Young (JFM 1999)

* GOAL: understand the interactions of IWs with the boundary (viscous case) and the role of the nonlinear term

-> construct an approximate solution which is CONSISTENT and STABLE, in some functional space with finite energy



[From LEGI, Grenoble]

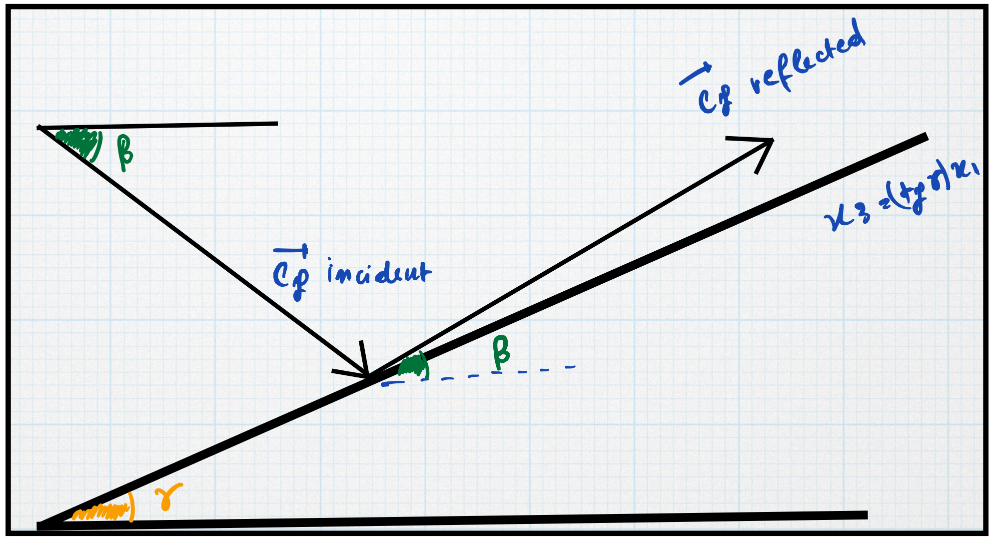
Main steps

1) Systematic boundary layers analysis of the linear viscous system
2) Fixed point iteration scheme (add correctors taking into account the nonlinearity)

A case study: linear analysis of the 2D INVISCID system

$$\partial_t u - (\sin\gamma) b + \partial_x p = 0$$
$$\partial_t w - (\cos\gamma) b + \partial_z p = 0$$
$$\partial_t b + u(\sin\gamma) + w(\cos\gamma) = 0$$
$$\partial_x u + \partial_z w = 0$$

BCs: $w|_{z=0} = 0$



Plane wave solution of the linear inviscid 2D system

$$\mathscr{L}(k,m)\begin{pmatrix} u_{refl} \\ w_{refl} \\ b_{refl} \end{pmatrix} = 0 \ \rightarrow \ \det(\mathscr{L}(k,m)) = 0 \ \rightarrow \omega^2 = \frac{(k\cos\gamma - m\sin\gamma)^2}{k^2+m^2} = \sin^2\beta = \sin^2\gamma + O(\varepsilon^2)$$

Dispersion Relation has 2 roots in m, and $m_{refl} = k\dfrac{\cos\gamma\sin\gamma + \cos\beta\sin\beta}{\sin^2\beta - \sin^2\gamma} = O(\varepsilon^{-2})$

Boundary Condition: $w_{inc}|_{z=0} + w_{refl}|_{z=0} = 0 \ \rightarrow \ e^{-i\omega_0 t + ik_0 x}(\alpha_{inc}\dfrac{k_0}{m_{inc}} + \alpha_{refl}\dfrac{k_0}{m_{refl}}) = 0$

Then α_{refl} has to be $O(\varepsilon^{-2})$. Thus, m>>1, amplitude >>1 —> critical case when $\beta \sim \gamma$

Linear analysis of the 2D VISCOUS system

(B2) $$\begin{cases} \partial_t u - (\sin\gamma)\, b + \partial_x p = \nu \Delta u \\ \partial_t w - (\cos\gamma)\, b + \partial_z p = \nu \Delta w \\ \partial_t b + u\,(\sin\gamma) + w\,(\cos\gamma) = \nu \Delta b \\ \partial_x u + \partial_z w = 0 \end{cases}$$

$$\boxed{\nu = \nu_0\, \varepsilon^6}$$

A plane wave $\begin{pmatrix} 1 \\ -\dfrac{k_0}{m_0} \\ \dfrac{i(k_0\cos\gamma - m_0 \sin\gamma)}{m_0 \sin\beta} \end{pmatrix} e^{-i\omega_0 t + ik_0 x + im_0 z}$

is an approximate solution **not satisfying the Boundary Conditions**

+BCs $$u|_{z=0} = w|_{z=0} = 0 \quad (\text{NO SLIP}), \quad \partial_z b|_{z=0} = 0 \quad (\text{NO FLUX})$$

Systematic approach: look for other linear solutions **(Viscous Boundary Layers)** to balance the trace, with $Re(\lambda) > 0$

$$\begin{pmatrix} u_\lambda \\ w_\lambda \\ b_\lambda \end{pmatrix} e^{-i\omega_0 t + ik_0 x - \lambda z}$$

System (B2) in Fourier
$$\mathscr{L}(k,\lambda)\big(u_\lambda\, w_\lambda\, b_\lambda\big) = 0$$

λ is "admissible" iff
$$det(\mathscr{L}(k,\lambda)) = 0, \quad Re(\lambda) > 0$$

- * $det(\mathscr{L}(k,\lambda)) = 0$ is a polynomial of degree 6 in λ
- * asymptotic of λ w.r.t. the small parameters: viscosity ν and **criticality parameter $\zeta = \omega^2 - \sin^2\gamma$**
- * **we need 3 admissible λ to lift the 3 BCs**

Linear boundary layers analysis of the viscous system

Proposition. Assume $\nu \ll 1$, $\zeta \ll 1$. One has the following roots in λ, with $Re(\lambda) > 0$

$\lvert\zeta\rvert \gtrsim 1$	$\nu^{\frac{1}{4}} \lesssim \lvert\zeta\rvert \ll 1$	$\nu^{\frac{1}{3}} \ll \lvert\zeta\rvert \ll \nu^{\frac{1}{4}}$	$\lvert\zeta\rvert \lesssim \nu^{\frac{1}{3}}$
• One reflected wave • One BL of size $\nu^{1/2}$ (lifting 2 conditions)	• One highly oscillating reflected wave (with slow decay) • One BL of size $(\nu/\zeta)^{1/2}$ • One BL of size $\nu^{1/2}$	• One BL of size $\lvert\zeta\rvert^4/\nu$ • One BL of size $(\nu/\zeta)^{1/2}$ • One BL of size $\nu^{1/2}$	• One BL of size $\nu^{1/3}$ • One BL of size $\nu^{1/2}$ (lifting 2 conditions)

-> Scaling studied by Dauxois and Young JFM 1999

Then find the amplitudes of the boundary layers by imposing zero trace on the boundary:

$$u_{inc}\big|_{z=0} + u_{BL}\big|_{z=0} = 0$$

$$w_{inc}\big|_{z=0} + w_{BL}\big|_{z=0} = 0$$

$$\partial_z b^{inc}\big|_{z=0} + \partial_z b_{BL}\big|_{z=0} = 0$$

This gives an (almost) exact sol to the linear viscous system

$$u_{lin} = a_{inc} X_{inc} e^{i(kx-\omega t) + m_{inc} z} + \sum_{j \in \{2,3,5\}} a_j X_j e^{i(kx-\omega t) - \lambda_j z} + c.c.$$

eigenvector of the incident wave

fixed

amplitudes

eigenvectors of the BLs

fixed

The weakly nonlinear system

OBS: The linear solution is an approximate solution to the weakly nonlinear system.

The boundary conditions are exactly satisfied, the only error comes from the weak nonlinearity (which is not so weak, because of the amplitudes of the BLs)

$\boxed{1^{st} \text{ step}}$ $\mathcal{U}_{app} = \mathcal{U}^0_{lin}$

WEAKLY NONLINEAR system in compact form

$$\boxed{\partial_t \mathcal{U}_{app} + \mathcal{L}^\varepsilon \mathcal{U}_{app} + \delta Q^\varepsilon(\mathcal{U}_{app}, \mathcal{U}_{app}) = \underbrace{\delta Q^\varepsilon(\mathcal{U}^0_{lin}, \mathcal{U}^0_{lin})}}$$

THIS IS THE ERROR TERM, BUT THERE IS NO ERROR ON THE BOUNDARY!

where

- $$\mathcal{L}^\varepsilon \begin{pmatrix} u \\ w \\ b \end{pmatrix} = \underset{\text{Leray Projector}}{\mathbb{P}} \begin{pmatrix} -b \sin\gamma \\ -b \cos\gamma \\ u \sin\gamma + w \cos\gamma \end{pmatrix} - \begin{pmatrix} \nu_0 \varepsilon^6 \Delta u \\ \nu_0 \varepsilon^6 \Delta w \\ \kappa_0 \varepsilon^6 \Delta b \end{pmatrix}$$

- $$Q(\mathcal{U}, \mathcal{U}') = \mathbb{P}\left((u\partial_x + w\partial_z)\, \mathcal{U}'\right)$$

Now the goal is to correct $\delta Q^\varepsilon(\mathcal{U}^0_{lin}, \mathcal{U}^0_{lin})$. Recall that $\mathcal{U}^0_{lin} = \mathcal{U}_{inc} + \mathcal{U}^0_{BL}$ then there are contributions of different orders...

The second harmonic and the mean flow

The nonlinear corrector has a non zero trace that has to be lifted.

$$\mathcal{U}_{inc} := \int_{\mathbb{R}^2} \widehat{A}(k,m) X_{k,m} \exp(i(kx - \omega_{k,m} t + my))\, dk\, dm, \qquad \widehat{A} \in \mathcal{S}(\mathbb{R}^2)$$

$$\begin{aligned}
\widehat{A}\widehat{A}' &= \widehat{A}(k,m)\widehat{A}(k',m') \\
&= \frac{1}{\varepsilon^4} \sum_{\eta,\eta' \in \{\pm 1\}} \chi\left(\frac{k+\eta k_0}{\varepsilon^2}, \frac{m+\eta m_0}{\varepsilon^2}\right) \chi\left(\frac{k'+\eta' k_0}{\varepsilon^2}, \frac{m'+\eta' m_0}{\varepsilon^2}\right) \\
&= \mathcal{A}_0(k,k',m,m') + \mathcal{A}_{II}(k,k',m,m'), \qquad \eta = \pm 1
\end{aligned}$$

where

$$\mathcal{A}_0(k,k',m,m') = \frac{1}{\varepsilon^4} \sum_{\eta \in \{\pm 1\}} \chi\left(\boxed{\frac{k+\eta k_0}{\varepsilon^2}}, \frac{m+\eta m_0}{\varepsilon^2}\right) \chi\left(\boxed{\frac{k'-\eta k_0}{\varepsilon^2}}, \frac{m'-\eta m_0}{\varepsilon^2}\right),$$

k+k'=0,
mean flow

$$\mathcal{A}_{II}(k,k',m,m') = \frac{1}{\varepsilon^4} \sum_{\eta \in \{\pm 1\}} \chi\left(\boxed{\frac{k+\eta k_0}{\varepsilon^2}}, \frac{m+\eta m_0}{\varepsilon^2}\right) \chi\left(\boxed{\frac{k'+\eta k_0}{\varepsilon^2}}, \frac{m'+\eta m_0}{\varepsilon^2}\right).$$

k+k'=+/- 2k0,
2nd harmonic

Then the first corrector with zero trace is given by:

$$\boxed{\mathcal{W}^1_{(a)} = \mathcal{W}^1_{BL,\varepsilon^2;(a)} + \mathcal{W}^1_{BL,\varepsilon^3;(a)} + \mathcal{W}^1_{II;(a)} + \mathcal{W}^1_{MF;(a)}}$$

The stability estimate

Theorem (R. Bianchini, A.-L. Dalibard, L. Saint-Raymond, to apper on Analysis and PDE).

Consider the Boussinesq equations in the scaling by Dauxois and Young (JFM 1999) in $\mathbb{R}^2_+ = \mathbb{R} \times \mathbb{R}_+$, with boundary conditions $u_{|_{z=0}} = w_{|_{z=0}} = \partial_z b_{|_{z=0}} = 0$.
Then there exists a **consistent approximate solution**

$$\mathscr{W}_{app} := (u_{app}, w_{app}, b_{app}) = \mathscr{W}_{inc} + \mathscr{W}_{BL} + \mathscr{W}^1_{II} + \mathscr{W}_{corr},$$

which solves the system with a remainder $O(\delta\varepsilon^2)$.

Moreover, denoting by $\mathscr{W}$ the unique weak solution to the Cauchy problem with initial data

$$\mathscr{W}|_{t=0} = \mathscr{W}_{app}|_{t=0},$$

we have the following **stability estimate:**

$$\|\mathscr{W} - \mathscr{W}_{app}\|_{L^2(\mathbb{R}^2_+)} \leq \delta\varepsilon^2 e^{(\delta\varepsilon^{-2}+1)t}. \quad (\longrightarrow \delta \leq \varepsilon^2)$$

Alternatively, $\|\mathscr{W} - \mathscr{W}_{app}\|_{L^2(\mathbb{R}^2_+)} \leq \delta^{\frac{1}{2}}\varepsilon^3 e^{\delta\varepsilon^{-2}t}.$

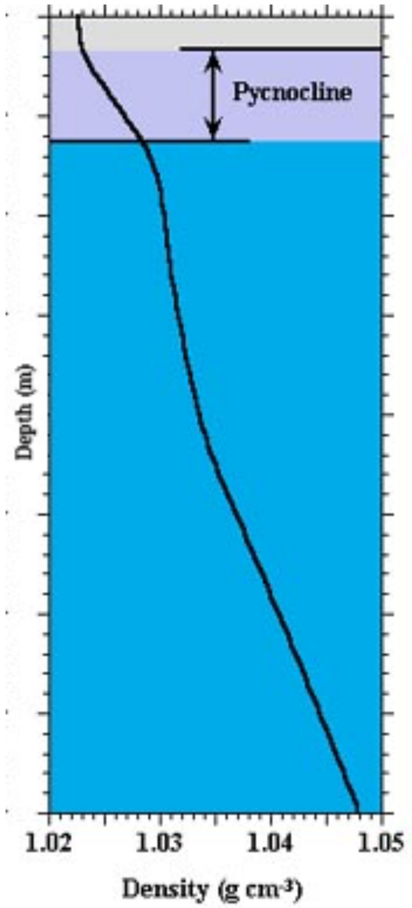
Stratified fluids II:
Linear inviscid damping of shear flows near Couette in the 2D stably stratified regime

with Michele Coti Zelati and Michele Dolce

2) Shear flows for exponentially stratified fluids

* Linearize around $\bar{\rho}(y) = e^{-\beta y}$ $\quad \bar{\mathbf{U}} = (U(y), 0)$ $\quad \partial_y \bar{p} = -\,\mathfrak{g}\bar{\rho}$
* Apply $\nabla^\perp = (\partial_y, -\partial_x)$ to the momentum equation so that

$$\boxed{\begin{aligned}(\partial_t + U(y)\partial_x)(\omega - \beta\partial_y\psi) - (U''(y) - \beta U'(y))\partial_x\psi &= -R\partial_x q,\\ (\partial_t + U(y)\partial_x)q &= \partial_x\psi,\\ \Delta\psi &= \omega,\end{aligned}}$$



where $\omega = \nabla^\perp \cdot (u^x, u^y)$, $\quad q = \dfrac{\beta\rho}{\bar{\rho}}$ and $\quad R = \beta\mathfrak{g}$

⟶ The Couette flow is $U(y) = y$. We consider shears near Couette:

$$\|U' - 1\|_{H^6} + \|U''\|_{H^5} \le \varepsilon$$

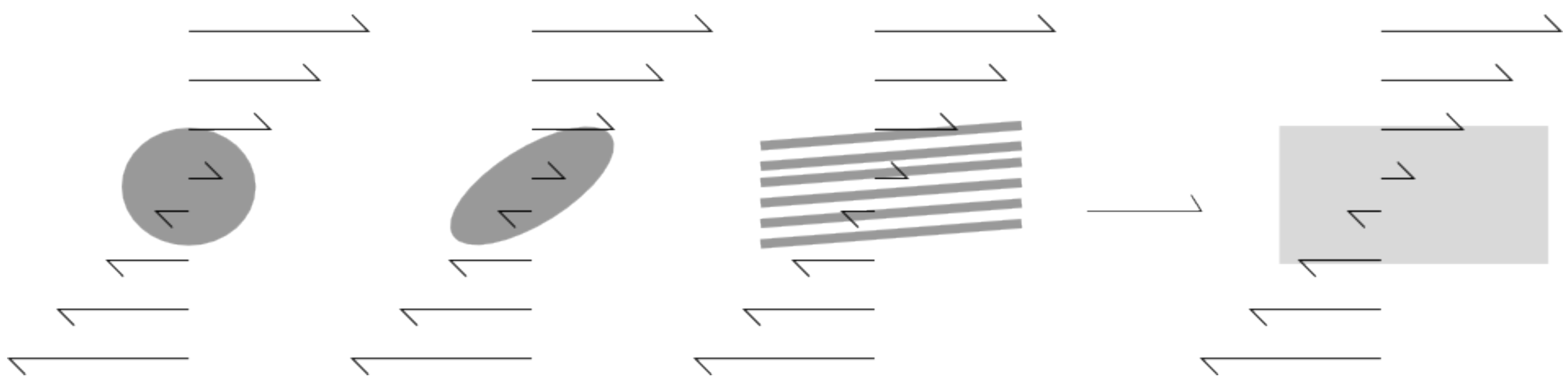
⟶ Recall that $Ri(y) = \dfrac{-\bar{\rho}'(y)\mathfrak{g}}{\bar{\rho}(y)} \cdot \dfrac{1}{U'(y)^2} = \beta\mathfrak{g} \cdot \dfrac{1}{U'(y)^2} \approx R = \beta\mathfrak{g}$

The Couette case U(y)=y with constant density

—> In the **constant density case**, the vorticity ω satisfies a **scalar transport eq**

$$\partial_t \omega + y \partial_x \omega = 0$$

$$\omega(t,x,y) = \omega_{in}(t, x - yt, y) \qquad \widehat{\omega}(t,k,\eta) = \widehat{\omega}_{in}(t,k,\eta + kt)$$



—> Mixing implies **time-decay** of the stream at **the cost of regularity**

$$|\widehat{\psi}(t,k,\eta)| = \left| \frac{\widehat{\omega}}{k^2+\eta^2} \right| \lesssim \frac{1}{\langle kt \rangle^2} |\langle \eta + kt \rangle^2 \widehat{\omega}_{in}(k, \eta + kt)|$$

—> Inviscid damping (predicted by Orr 1907)

[analogy with Landau damping for Vlasov-Poisson,
Mouhot-Villani 2011
Grenier, Nguyen, Rodnianski 2020]

$$\left\| u^x(t) - \int_{\mathbb{T}} u^x(t,\tau,y)\, d\tau \right\|_{L^2} + \langle t \rangle \|u^y(t)\|_{L^2} \lesssim \frac{1}{\langle t \rangle} \|\omega_{in}\|_{L^2}$$

—> In the NONLINEAR case, it was proved by Bedrossian & Masmoudi in 2013

What happens with stratified fluids and density variations?

* Hartman in 1975: decay rates
* Yang and Lin in 2018: hypergeometric functions

We will use a different strategy to improve the Couette case and handle shears close to Couette

1) Start with Couette, U(y)=y

$$(\partial_t + y\partial_x)(\omega - \beta\partial_y\psi) + \beta\partial_x\psi = -R\partial_x q,$$
$$(\partial_t + y\partial_x)q = \partial_x\psi,$$
$$\Delta\psi = \omega,$$

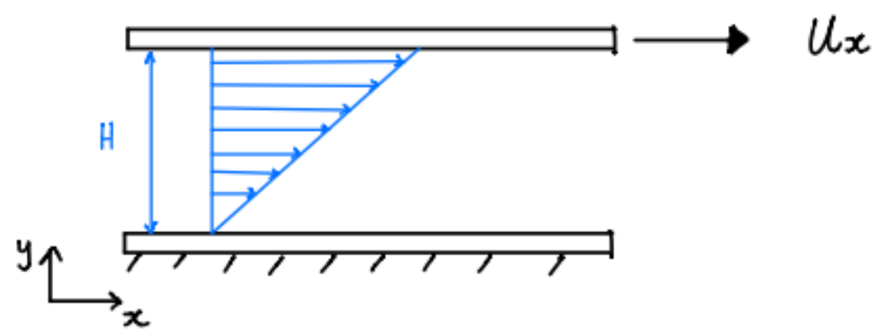
This is transported

2) Define $\theta = \omega - \beta\partial_y\psi = (I - \beta\partial_y\Delta^{-1})\omega$

$$(\partial_t + y\partial_x)\theta = -R\partial_x q - \beta\partial_x\Delta^{-1}(I - \beta\Delta^{-1})^{-1}\theta,$$
$$(\partial_t + y\partial_x)q = \partial_x\Delta^{-1}(I - \beta\Delta^{-1})^{-1}\theta$$

The Couette case

$$(\partial_t + y\partial_x)\theta = -R\partial_x q - \beta\partial_x\Delta^{-1}(I-\beta\Delta^{-1})^{-1}\theta,$$
$$(\partial_t + y\partial_x)q = \partial_x\Delta^{-1}(I-\beta\Delta^{-1})^{-1}\theta$$



—> **Follow the flow:**

- **new coordinates:** $X = x - yt, \quad Y = y$
- **new variables:** $\Omega(t,X,Y) = \omega(t, X+tY, Y), \quad \Theta(t,X,Y) = \theta(t,X+tY,Y)$

—> **Decoupling [Fourier] only in x**

$$\boxed{\begin{aligned} \partial_t\Theta_k &= -ikRQ_k - \beta\Delta_L^{-1}B_L\Theta_k \\ \partial_t Q_K &= ik\Delta_L^{-1}B_L\Theta_k \end{aligned}}$$

$$\Theta = B_L^{-1}\Omega$$

$$\boxed{\begin{aligned} \Delta_L &= \partial_{XX} + (\partial_Y - t\partial_X)^2 \\ B_L &= (I - \beta(\partial_Y - t\partial_X)\Delta_L^{-1})^{-1} \end{aligned}}$$

time-dependent
Fourier multipliers

The Couette case

* Look for a symmetrization of the non-autonomous dynamical system

$$Z_1 := p^{-\frac{1}{4}}\Theta_k, \qquad Z_2 := p^{\frac{1}{4}} i\sqrt{R}Q_k\,.$$

with $p = k^2 + (\eta - kt)^2$ the symbol of $-\Delta_L$. The system reads

matrix of remainders

$$\partial_t \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}\frac{p'}{p} & -k\sqrt{R}p^{-\frac{1}{2}} \\ k\sqrt{R}p^{-\frac{1}{2}} & \frac{1}{4}\frac{p'}{p} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \begin{pmatrix} \beta\frac{ik}{p}B_L & 0 \\ k\sqrt{R}p^{-\frac{1}{2}}(B_L - 1) & 0 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}.$$

In symbols, $B_L - 1 = \dfrac{i\beta(\eta - kt)}{p + i\beta(\eta - kt)} \approx t^{-1}$.

* The "matrix of remainders" is integrable in time.
* A Gronwall-type estimate [with a proper functional] will be enough.

The Couette case

* For any fixed frequency η, we define the point-wise energy functional

$$E(t) = \frac{1}{2}\left[|Z_1(t)|^2 + |Z_2(t)|^2 + \frac{1}{2k\sqrt{R}} \Re\left(p' p^{-\frac{1}{2}} Z_1(t) \overline{Z_2(t)} \right) \right]$$

* This is coercive for R>1/4, Miles-Howard

Theorem [R. Bianchini, M. Coti Zelati, M. Dolce 2020]
Let $\beta > 0$ and $k \neq 0$. Then

$$|p^{-\frac{1}{4}}\Theta_k(t)|^2 + |p^{\frac{1}{4}}Q_k(t)|^2 \approx |(k^2+\eta^2)^{-\frac{1}{4}}\Theta_k(0)|^2 + |(k^2+\eta^2)^{\frac{1}{4}}Q_k(0)|^2, \qquad \forall t \geq 0,$$

point-wise in η. In particular, back to Ω_k,

$$|p^{-\frac{1}{4}}\Omega_k(t)|^2 + |p^{\frac{1}{4}}Q_k(t)|^2 \approx |(k^2+\eta^2)^{-\frac{1}{4}}\Omega_k(0)|^2 + |(k^2+\eta^2)^{\frac{1}{4}}Q_k(0)|^2, \qquad \forall t \geq 0.$$

A Lyapunov instability of the vorticity

* From $$-\frac{1}{2\sqrt{R}-1}\frac{|k|^2}{p}E \lesssim \frac{d}{dt}E \lesssim \frac{1}{2\sqrt{R}-1}\frac{|k|^2}{p}E\,.$$

one has $|p^{-\frac{1}{4}}\Omega_k(t)|^2 \gtrsim |(k^2+\eta^2)^{-\frac{1}{4}}\Omega_k(0)|^2 + |(k^2+\eta^2)^{\frac{1}{4}}Q_k(0)|^2 =: \Xi_k(0)\,.$

* This implies that $|\Omega_k(t)| \gtrsim (k^2+(\eta-kt)^2)^{\frac{1}{4}}\Xi_k(0) \gtrsim \langle t\rangle^{1/2}\|\Xi(0)\|_{H^{\frac{1}{2}}},$

i.e. the vorticity grows after the critical time $t = \dfrac{\eta}{k}$.

Theorem [R. Bianchini, M. Coti Zelati, M. Dolce 2020]

Let $R > 1/4$ and $\beta > 0$.

$$\|q(t)-\langle q\rangle_x\|_{L^2} + \|v^x(t)-\langle v^x\rangle_x\|_{L^2} \lesssim \frac{1}{\langle t\rangle^{\frac{1}{2}}}\left(\|\omega_{in}-\langle\omega_{in}\rangle_x\|_{L^2} + \|q_{in}-\langle q_{in}\rangle_x\|_{H^1}\right),$$

$$\|v^y(t)\| \lesssim \frac{1}{\langle t\rangle^{\frac{3}{2}}}\left(\|\omega_{in}-\langle\omega_{in}\rangle_x\|_{H^{\frac{1}{4}}} + \|q_{in}-\langle q_{in}\rangle_x\|_{H^{\frac{5}{4}}}\right).$$

Shear flows near Couette

* Lin & Zeng 2010, shears near Couette: there are steady state in $H^{\frac{3}{2}}$, but not for H^s with $s > 3/2$.
* Bedrossian & Masmoudi in 2013 proved NONLINEAR inviscid damping for Couette in Gevrey 2^-

(2 is optimal, Deng & Masmoudi 2018)

$$\frac{d}{dt}E_s + \left(1 - \frac{1}{2\sqrt{R}}\right)\left[\|\sqrt{\frac{m'}{m}}Z_1\|_s^2 + \|\sqrt{\frac{m'}{m}}Z_2\|_s^2\right] \leq \frac{1}{4|k|\sqrt{R}}\left|\langle\left(p'p^{-\frac{1}{2}}\right)' Z_1, Z_2\rangle_s\right| + \sum_{i=1}^{8}\mathscr{R}_i,$$

R>1/4

<u>**Theorem [R. Bianchini, M. Coti Zelati, M. Dolce 2020]**</u>

Let $R > 1/4$ **and** $\beta > 0$. **There exists a small** $\varepsilon_0 = \varepsilon_0(\beta, R) \in (0,1)$ **s.t., if** $\varepsilon \in (0,\varepsilon_0]$ **and** $\|U' - 1\|_{H^6} + \|U''\|_{H^5} \leq \varepsilon$,

$$\|q(t) - \langle q\rangle_x|_{L^2} + \|v^x(t) - \langle v^x\rangle_x\|_{L^2} \lesssim \frac{1}{\langle t\rangle^{\frac{1}{2}-\delta_\varepsilon}}\left(\|\omega_{in} - \langle\omega_{in}\rangle_x\|_{L^2} + \|q_{in} - \langle q_{in}\rangle_x\|_{H^1}\right),$$

$$\|v^y(t)\| \lesssim \frac{1}{\langle t\rangle^{\frac{3}{2}-\delta_\varepsilon}}\left(\|\omega_{in} - \langle\omega_{in}\rangle_x\|_{H^{\frac{1}{4}}} + \|q_{in} - \langle q_{in}\rangle_x\|_{H^{\frac{5}{4}}}\right),$$

where $\delta_\varepsilon = 2\sqrt{\varepsilon}$.

loss of decay prescribed by the weight

Ongoing ...

* Instabilities of internal waves
* Nonlinear perturbations of the Couette flow in the 2D stably stratified regime

What about shears flows near Couette?

A review on the constant density case

—> Lin & Zeng in 2010, shears near Couette: there are steady state in $H^{\frac{3}{2}}$ near Couette, but not for H^s with $s > 3/2$.

—> Bedrossian & Masmoudi in 2013 proved NONLINEAR inviscid damping for Couette in Gevrey 2^- (2 is optimal, Deng & Masmoudi 2018)

Shears near Couette for stratified fluids

They use a nonlinear change of coordinates which follows the background shear.

In our LINEAR problem for shears near Couette we follow the shear as well.

—> We go back to the linearized system around

$$\bar{\rho}(y) = e^{-\beta y} \qquad \bar{\mathbf{u}} = (U(y), 0) \qquad \partial_y \bar{p} = -\mathfrak{g}\bar{\rho}$$

where $\|U' - 1\|_{H^6} + \|U''\|_{H^5} \leq \varepsilon$

Shear flows near Couette

$$(\partial_t + U(y)\partial_x)(\omega - \beta\partial_y\psi) - (U''(y) - \beta U'(y))\partial_x\psi = -R\partial_x q$$
$$(\partial_t + U(y)\partial_x)q = \partial_x\Delta^{-1}\omega$$

Define the new coordinates $X = x - U(y)t, \quad Y = y$ and the unknowns

$$\Theta(t,X,Y) = \theta(t,X+tY,Y), \quad Q(t,X,Y) = q(t,X+tY,Y), \quad \Omega(t,X,Y) = \omega(t,X+tY,Y)$$

$$\partial_t\Theta = -R\partial_X Q + \big(b(Y) - \beta g(Y)\big)\,\partial_X\Delta_t^{-1}B_t\Theta,$$
$$\partial_t Q = \partial_X\Delta_t^{-1}B_t\Theta\,.$$

Since $\partial_x \to \partial_X, \qquad \partial_y \to g(Y)(\partial_Y - t\partial_X),$ where $g(Y) = U'(U^{-1}(Y)), \quad b(Y) = U''(U^{-1}(Y)),$

—> then $\Delta_t = \partial_{XX} + g^2(Y)(\partial_Y - t\partial_X)^2 + b(Y)(\partial_Y - t\partial_X)$

—> and $B_t = (I - \beta g(Y)(\partial_Y - t\partial_X)\Delta_t^{-1})^{-1}$ where $\Theta = B_t^{-1}\Omega$

* Thanks to the hp $\|U' - 1\|_{H^6} + \|U''\|_{H^5} \le \varepsilon$, we have that $\Delta_t^{-1} = \Delta_L^{-1} + \mathscr{D}_\varepsilon$ and $B_t = B_L + \mathscr{B}_\varepsilon$

their H^s norm are $\mathcal{O}(\varepsilon)$

The energy functional

Couette: $Z_1 := p^{-\frac{1}{4}}\Theta_k, \qquad Z_2 := p^{\frac{1}{4}} i\sqrt{R}Q_k$. **The energy was point-wise in** (k,η).

Shears near Couette: $Z_1 := m^{-1}p^{-\frac{1}{4}}\Theta_k, \qquad Z_2 := m^{-1}p^{\frac{1}{4}} i\sqrt{R}Q_k$.

The energy is: $E_s(t) = \frac{1}{2}\left[\|Z_1(t)\|_s^2 + \|Z_2(t)\|_s^2 + \frac{1}{2k\sqrt{R}}\Re\langle p' p^{-\frac{1}{2}} Z_1(t), Z_2(t)\rangle_s\right]$.

Here $m = m_1 w^\delta$, where

—> m is a "ghost weight" (eating time-integrable remainders)

—> w is such that $\frac{w'}{w} = \frac{1}{4}\frac{|p'|}{p}$, so $(w^\delta)' = \frac{\delta_\varepsilon}{4}\frac{|p'|}{p}$,

where $\delta_\varepsilon = \mathcal{O}(\sqrt{\varepsilon})$.

It absorbs the non-integrable remainders
at the price of a small loss of time decay