On Decay Rates in Hypocoercive Kinetic Equations

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Overview

Two linear kinetic models

1. Fokker–Planck equation with linear drift
2. Goldstein–Taylor model

Similarities

1. "Defects" appearing
2. Hypocoercive dynamics

Tools

1. Spectral Theory
2. Entropy Method

Q: How fast do solutions converge to equilibrium?

Degenerate Fokker–Planck equation with linear drift

$$\partial_t f(x,t) = \operatorname{div}(\mathbf{D}\nabla f(x,t) + \mathbf{C}x f(x,t)) =: Lf, \quad x \in \mathbb{R}^d, t \geq 0$$

$$f(x,0) = f_0(x) \in L^1(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} f_0 dx = 1,\ f_0 \geq 0.$$

- $f(x,t)$ describes evolution of probability density of average particle of large particle system.

Two forces:

- **Collisions:** pos. semi-definite *diffusion matrix* $\mathbf{D} \in \mathbb{R}^{d\times d}$.
- **Transport:** *drift matrix* $\mathbf{C} \in \mathbb{R}^{d\times d}$.

Degenerate Fokker–Planck equation with linear drift

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Hypocoercivity conditions:

- $\mathbf{C}$ is positive stable, i.e. **spectral gap**

$$\mu_{\mathbf{C}} := \min\{\mathrm{Re}(\lambda) : \lambda \in \sigma(\mathbf{C})\} > 0.$$

- No eigenvector of $\mathbf{C}^T$ is in $\ker(\mathbf{D})$.
  (connection to Kalman rank condition)

e.g. [Villani '09] $\Longrightarrow$ there exists $\mathcal{C} \geq 1$ and $\alpha > 0$ s.t.

$$\|e^{-Lt} f(0)\|_{\mathcal{H}}^2 \leq \mathcal{C} e^{-2\alpha t} \|f(0)\|_{\mathcal{H}}^2, \quad t \geq 0.$$

Defectiveness

Our Focus: Sharp decay for defective drift matrix $\mathbf{C}$.

Definition:

$\lambda \in \sigma(\mathbf{C})$ is defective of order $n \in \mathbb{N}$, if

$$\text{(algebraic multiplicity)} - \text{(geometric multiplicity)} = n.$$

E.g. for $d = 2$:

$$\mathbf{C} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1},$$

where $\mathbf{J} = \begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix}$ is the **Jordan normal form**.

defective ODE of order 1:

$$\dot{x} = -\mathbf{C}x \implies |x(t)|^2 = |\mathbf{V}e^{-\mathbf{J}t}\mathbf{V}^{-1}x(0)|^2 \leq c|x(0)|^2(1 + t^2)e^{-2\mu_c t}$$

Theorem 1 (Arnold, Einav, W.)

Let $f_0 \in L^2(f_\infty^{-1}dx)$ *and let* $n \geq 0$ *be the maximum defect of* $\mathbf{C}$ *associated to* $\mu_{\mathbf{C}} := \min\{\mathrm{Re}(\lambda) : \lambda \in \sigma(\mathbf{C})\}$. *Then*

$$\|f(t) - f_\infty\|^2_{L^2(f_\infty^{-1}dx)} \leq c\|f_0 - f_\infty\|^2_{L^2(f_\infty^{-1}dx)}(1 + t^{2n})e^{-2\mu_{\mathbf{C}}t}.$$

Proof:

Builds on [Arnold, Erb '14]:

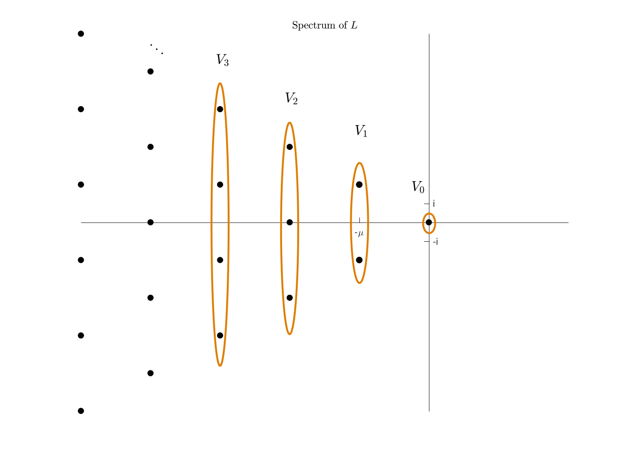
Drift matrix $\mathbf{C}$ defective in $\mu_{\mathbf{C}}$:

$$\|f(t) - f_\infty\|^2_{L^2(f_\infty^{-1}dx)} \leq c\|f_0 - f_\infty\|^2_{L^2(f_\infty^{-1}dx)}e^{-2(\mu_{\mathbf{C}}-\varepsilon)t}.$$

for arbitrary $\varepsilon > 0$.

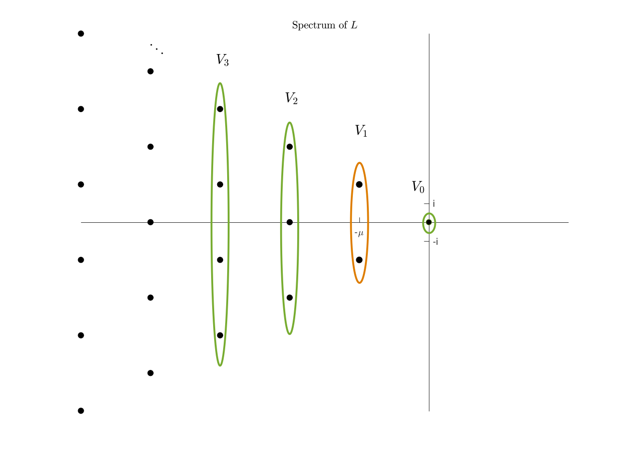
Proof: Spectral properties of L

$$\bigoplus_{m\geq 0} V_m = L^2(f_\infty^{-1}dx), \quad V_m \text{ are } L\text{-invariant}, \quad \dim(V_m) < \infty.$$



Proof: Splitting the solution

Idea: *Split the solution* $f = f_1 + f_2$ with $f_1 \in V_1$ and $f_2 \perp V_1$ has faster decay.



Fokker–Planck Equation with Added Uncertainty

Linear Fokker–Planck equation for $x \in \mathbb{R}$

$$\partial_t f(x,z,t) = \partial_x[\partial_x f(x,z,t) + C(z) x f(x,z,t)] := L(z) f, \quad t \geq 0.$$

- **Uncertain parameter** $z \in \mathbb{R}$ representing:
  - Model coefficient uncertainty,
  - Measurement errors etc.
- **Drift coefficient** $C \in C^1(\mathbb{R})$, $C(z) \geq \mu > 0$, with $\sup_{z \in \mathbb{R}} |\partial_z C(z)| < \infty$.

Question: *How sensitive is the large-time behaviour of solutions to the uncertainty?*

- Sensitivity analysis of the model w.r.t. the uncertain parameter z (here: no stochastic properties for z).
- $g(x,z,t) := \partial_z f(x,z,t)$, with $x, z \in \mathbb{R}, t \geq 0$.

Linear order sensitivity equation

$$\partial_t g(x,z,t) = L(z) g(x,z,t) + \partial_z L(z) f(x,z,t)$$

- **Goal:** Find sharp rate of decay for the system

$$\partial_t \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} L(z) & 0 \\ \partial_z L(z) & L(z) \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

uniform in $z \in \mathbb{R}$.

Theorem 2 (Arnold, Jin, W.)

Solutions $\Phi(x, z, t) = (f, \partial_z f)^T \in \mathbb{R}^2$ of the system of sensitivity equations for FPE with coefficient $C \in C^1(\mathbb{R})$, $\partial_z C \in L^\infty(\mathbb{R})$, $C(z) \geq \mu > 0$ converges uniformly in z to equilibrium:

$$\sup_{z\in\mathbb{R}} \|\Phi(z,t) - \Phi^\infty(z)\|^2_{L^2(f_\infty^{-1}dx)} \leq c(1+t^2)e^{-2\mu t} \sup_{z\in\mathbb{R}} \|\Phi(z,0) - \Phi^\infty(z)\|^2_{L^2(f_\infty^{-1}dx)}.$$

Proof:

- Eigenfunction expansion leads to ODEs on modal level.
- Construct **Lyapunov functionals for defective ODEs:** Time-dependent $P(t)$-norm.

The Two Velocity Goldstein–Taylor Model

Basic Setting

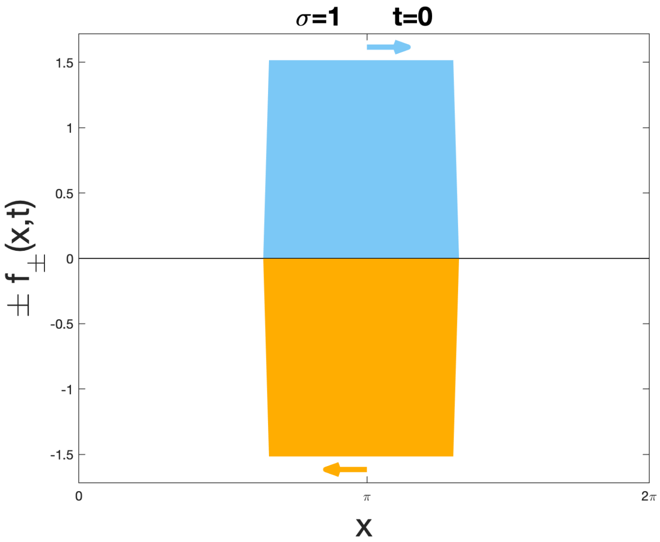
Two Velocity Goldstein–Taylor Model for $x \in \mathbb{T}$:

$$\begin{aligned} \partial_t f_+(x,t) + \partial_x f_+(x,t) &= \frac{\sigma(x)}{2}(f_-(x,t) - f_+(x,t)), \\ \partial_t f_-(x,t) - \partial_x f_-(x,t) &= -\frac{\sigma(x)}{2}(f_-(x,t) - f_+(x,t)). \end{aligned} \tag{GT}$$

- $f_\pm(\cdot, t)$ probability density of particles with $v = \pm 1$.
- Relaxation coefficient $\sigma(x) > 0$.

Goal: Find explicit decay rate for space-dependent relaxation $\sigma(x)$ via entropy method.

First: Understand $\sigma(x) \equiv \sigma > 0$.


σ=1 t=0
±f±(x,t)
1.5
1
0.5
0
-0.5
-1
-1.5
0
π
2π
x

Constant Relaxation $\sigma > 0$

Macroscopic variables

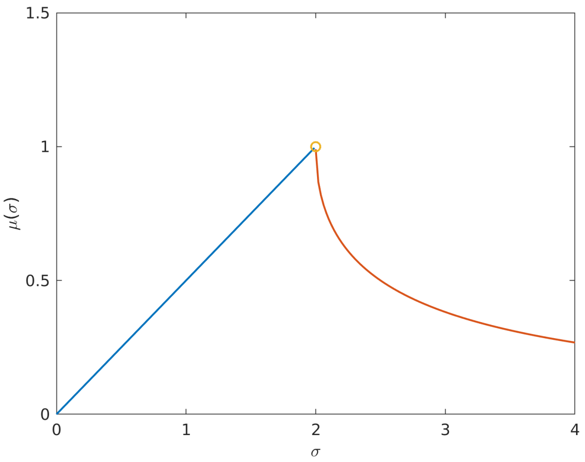
$$u := f_+ + f_-, \qquad w := f_+ - f_-.$$

Goldstein–Taylor Model in Fourier Space:

$$\partial_t \begin{pmatrix} \hat{u}_k(t) \\ \hat{w}_k(t) \end{pmatrix} = - \begin{pmatrix} 0 & ik \\ ik & \sigma \end{pmatrix} \begin{pmatrix} \hat{u}_k(t) \\ \hat{w}_k(t) \end{pmatrix}, \quad k \in \mathbb{Z}.$$

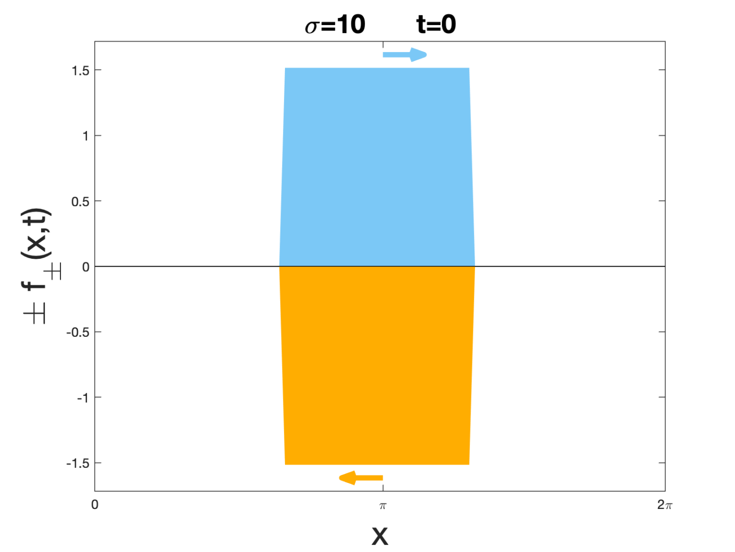
- uniform-in-k spectral gap

$$\mu(\sigma) := \min_{k \in \mathbb{Z}} \mu_k(\sigma) = \mu_1(\sigma) = \mathrm{Re}\left(\frac{\sigma}{2} - \sqrt{\frac{\sigma^2}{4} - 1}\right) \in (0, 1].$$



Sharp exponential decay rate of solutions

$$\mu(\sigma) = \mathrm{Re}\left(\frac{\sigma}{2} - \sqrt{\frac{\sigma^2}{4} - 1}\right).$$



σ=10 t=0
1.5
1
0.5
0
-0.5
-1
-1.5
±f±(x,t)
0
π
2π
x

Spatial Entropy functional

Entropy method:

$$\frac{d}{dt}E[u(t)-u^\infty, w(t)] \leq -\lambda E[u(t)-u^\infty, w(t)]$$
$$\implies \quad E[u(t)-u^\infty, w(t)] \leq e^{-\lambda t}E[u(0)-u^\infty, w(0)].$$

For parameter $\theta \in (0,2)$:

$$E_\theta[u,w] := \|u\|_{L^2}^2 + \|w\|_{L^2}^2 - \frac{\theta}{2\pi}\int_0^{2\pi} w\partial_x^{-1}u\,dx,$$

where

$$(\partial_x^{-1}u)(x) := \int_0^x u\,dx + c(u) \quad \text{with} \quad c(u) := -\frac{1}{2\pi}\int_0^{2\pi}\int_0^x u\,dy\,dx.$$

Spatial Entropy functional

Entropy method:

$\frac{d}{dt}E[u(t) - u^\infty, w(t)] \leq -\lambda E[u(t) - u^\infty, w(t)]$
$\implies \; E[u(t) - u^\infty, w(t)] \leq e^{-\lambda t} E[u(0) - u^\infty, w(0)].$

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"Norm equivalence":

$(1 - \frac{\theta}{2}) \|(u - u^\infty, w)\|_{L^2}^2 \leq E_\theta[u - u^\infty, w] \leq (1 + \frac{\theta}{2}) \|(u - u^\infty, w)\|_{L^2}^2.$

Decay estimates for constant $\sigma > 0$

Lemma 3 (Arnold, Einav, Signorello, W.)

Let $(u, w)^T$ *be a solution to (GT) with constant* $\sigma > 0$*.*

(i) *For* $\sigma \in (0, 2)$

$$E_\sigma[u(t) - u^\infty, w(t)] \leq E_\sigma[u(0) - u^\infty, w(0)]e^{-\sigma t}.$$

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(ii) $\sigma = 2$, *with* $\theta_{\varepsilon} := \frac{2(2-\varepsilon^2)}{2+\varepsilon^2}$,

$$E_{\theta_{\varepsilon}}[u(t) - u^{\infty}, w(t)] \leq E_{\theta_{\varepsilon}}[u(0) - u^{\infty}, w(0)]e^{-2(1-\varepsilon)t}.$$

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$$E_{\theta_{\varepsilon}}[u(t) - u^{\infty}, w(t)] \leq E_{\theta_{\varepsilon}}[u(0) - u^{\infty}, w(0)]e^{-2(1-\varepsilon)t}.$$

(iii) *For $\sigma > 2$ with $\mu = \frac{\sigma}{2} - \sqrt{\frac{\sigma^2}{4} - 1}$,*

$$E_{\frac{4}{\sigma}}[u(t) - u^{\infty}, w(t)] \leq E_{\frac{4}{\sigma}}[u(0) - u^{\infty}, w(0)]e^{-2\mu t}.$$

Space-dependent relaxation $\sigma(x)$

Theorem 4 (Arnold, Einav, Signorello, W.)

Let $(u, w)^T$ be a solution to (GT) with $u_0, w_0 \in L^2(\mathbb{T})$ and

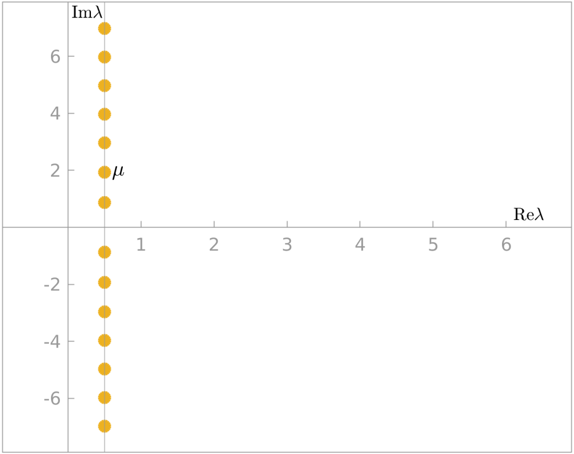
$$0 < \sigma_{\min} := \inf_{x\in\mathbb{T}} \sigma(x) \leq \sup_{x\in\mathbb{T}} \sigma(x) =: \sigma_{\max} < \infty.$$

Then, for $\theta^* = \min\{\sigma_{\min}, \frac{4}{\sigma_{\max}}\}$ exists an explicit decay rate $\alpha^*(\sigma_{\min}, \sigma_{\max}) > 0$ such that

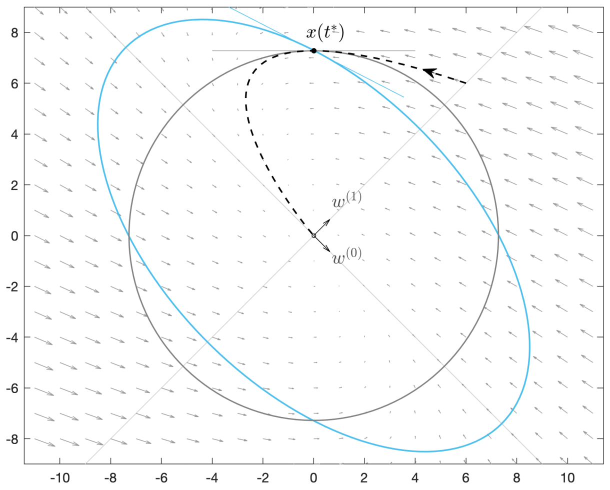
$$E_{\theta^*}[u(t) - u^\infty, w(t)] \leq e^{-\alpha^* t} E_{\theta^*}[u_0 - u^\infty, w_0].$$

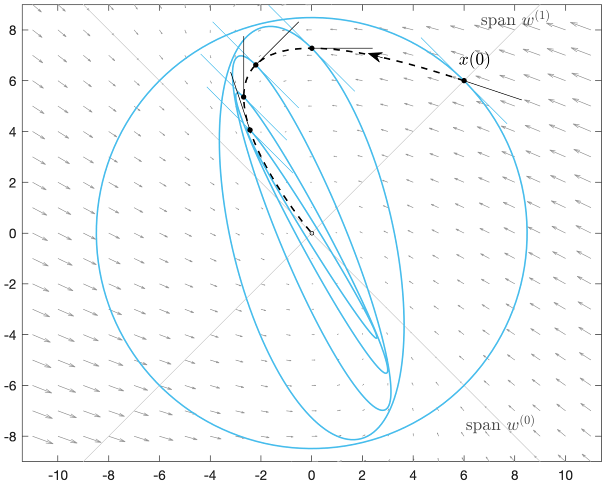
- Proof: perturbative approach
- Decay rate $\alpha^*(\sigma_{\min}, \sigma_{\max})$ is not sharp.
- Extends to multi-velocity models with $\sigma(x)$.

Thank you for your attention.



Eigenvalues of $A_k(\sigma = 1)$ for Fourier modes $k \in \mathbb{Z} \setminus \{0\}$.


$x(t^*)$
$w^{(1)}$
$w^{(0)}$
8
6
4
2
0
-2
-4
-6
-8
-10
-8
-6
-4
-2
0
2
4
6
8
10



span $w^{(1)}$
$x(0)$
span $w^{(0)}$
8
6
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2
0
-2
-4
-6
-8
-10
-8
-6
-4
-2
0
2
4
6
8
10