EXERCISE WEEK 3:
PRACTICAL COURSE
MODELING, SIMULATION, OPTIMIZATION

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We consider heat conduction in the steel plate in Figure 1. The plate is $L_x = 0.5$ [m] by $L_y = 0.4$ [m] and has a thickness of $H = 3$ [mm], a thermal conductivity $k = 57.7$ [W/m/K], a heat capacity *per unit mass* $c = 448$ [J/kg/K], and a mass density $\rho = 7840$ [kg/m^3]. Because the plate is thin, the temperature increase T only depends on the in-plane coordinates x and y and on the time t, i.e. $T(t, x, y)$.

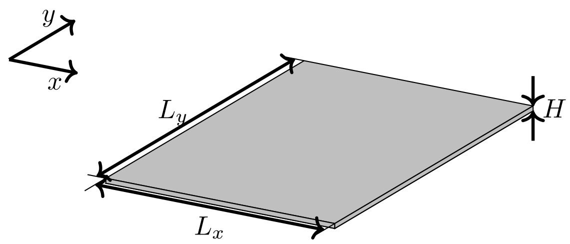

FIGURE 1. The considered steel plate

The boundaries of the plate are perfectly isolated, i.e. the heat flux $\mathbf{q} \cdot \mathbf{n} = 0$ on all edges of the plate. Here, the heat flux $\mathbf{q} = [q_x, q_y]^\top$ is given by Fourier's law of heat conduction and $\mathbf{n} = [n_x, n_y]^\top$ is the outward pointing unit normal. At time $t = 0$, the temperature increase in the plate is zero, i.e. $T(0, x, y) = 0$.

The plate is subjected to a heat load $Q(t, x, y)$ [W/m^2] of the form

$$Q(t, x, y) = P(t)\mathcal{B}(x, y),$$

where the applied power is $P(t) = P_0 \frac{1}{2} (\tanh(t_1 - t) + 1)$ [W] and the spatial shape of the heat load $\mathcal{B}$ [1/m^2] is given by

$$\mathcal{B}(x, y) = \frac{1}{\pi a^2} \exp\left(-\frac{(x - x_0)^2 + (y - y_0)^2}{a^2} \right),$$

where $P_0 = 50$ [W], $t_1 = 10$ [s], the center of the heat load is at $(x_0, y_0) = (0.7L_x, 0.8L_y)$, and the width parameter is $a = 0.05$ [m]. We are interested in the temperature increase $T(t, x, y)$ on the time interval $[0, T_{\text{sim}}]$ with $T_{\text{sim}} = 60$ [s].

a. (2pts) Write down the initial value problem for the temperature increase $T(t, x, y)$ in the plate. Do not use the unit normal $\mathbf{n}$ in the boundary conditions.

The focus of this exercise is on temporal discretization. The matrices resulting from a finite difference (with an explicit implementation of the boundary conditions) is therefore already provided in `Week3_spatial_discretization`. This file contains the matrix A, the load vector B, the grid vectors `x` and `y`, and the matrix `node_nmbrs` obtained for a finite difference discretization with $N_x = 81$ grid points in the x-direction and $N_y = 65$ grid points in the y-direction.

b. (3pts) Approximate the solution $T(t, x, y)$ on the time interval $[0, T_{\text{sim}}]$ using $N_T = 49$ grid points using three different time discretization schemes: Forward Euler, Crank-Nicolson, and Backward Euler. Are all three schemes stable?
Hint: you can use `Week3_Exercise4b` as a starting point.

c. (4pts) Plot the error in the solutions obtained with the three considered time discretization schemes for $N_T = 25, 49, 97, 193$, and 385, i.e. for $N_T = 12 \cdot 2^i + 1$ with $i \in \{1, 2, 3, 4, 5\}$. Use the solution obtained with the Crank-Nicolson scheme with $N_T = 769 = 12 \cdot 2^6 + 1$ as reference solution. Plot the relative error against the time increment Δt on a double log scale. What are the convergence rates you observe?
Hint: you can use `Week3_Exercise4c` as a starting point.
Hint: you can speed up simulations by precomputing an LU-factorization of matrices that you need to invert at every time step.

d. (1pt) We now replace the function $P(t)$ by the discontinuous function

$$\bar{P}(t) = \begin{cases} 1 & \text{when } t \leq t_1, \\ 0 & \text{otherwise.} \end{cases}$$

Will this change affect the convergence rates of the three considered time discretization schemes?
Hint: You can also answer this question if you were not able to complete b. and/or c.