

**EXERCISE WEEK 5:
PRACTICAL COURSE
MODELING, SIMULATION, OPTIMIZATION**

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We again consider the steady-state temperature distribution in the aluminum rod in Figure 1 that we also considered in Exercise 1. The rod has a length of $L = 0.3$ [m], a cross sectional area of $A_{cs} = 0.01$ [m²], and a thermal conductivity of $k = 237$ [W/m/K]. Along the length of the rod, we apply a constant heat load

$$Q(x) = Q_0 \exp(-(x - \frac{1}{2}L)^2/a^2),$$

where $Q_0 = 100$ [W/m] and $a = 0.1$ [m].

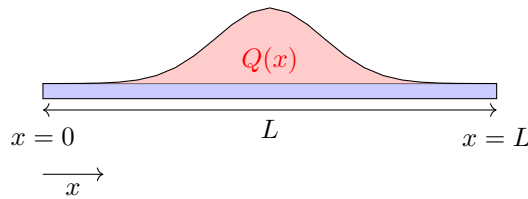


FIGURE 1. The considered aluminum rod

The temperature increase w.r.t. the reference temperature $T_0 = 293$ [K] is $T(x)$. At the left end of the rod, the temperature is fixed at the reference temperature T_0 , i.e. $T(0) = 0$. At the right end of the rod, the (outgoing) heat flow is proportional to the temperature increase, i.e. $A_{cs}q(L) = hT(L)$ [W], where $h = 3$ [W/K] is the cooling coefficient and the outgoing heat flux is $q(L) = -k\frac{dT}{dx}(L)$. The temperature increase $T(x)$ is thus the solution of the boundary value problem

$$kA_{cs}\frac{d^2T}{dx^2}(x) + Q(x) = 0, \quad T(0) = 0, \quad -kA_{cs}\frac{dT}{dx}(L) = hT(L).$$

- (2pts) Write down the weak form of the boundary value problem for the steady state temperature increase $T(x)$.
- (2pts) Assemble the stiffness matrix \mathbf{A} and the forcing vector \mathbf{f} for this problem from the element stiffness matrix \mathbf{A}^e and the element forcing vector \mathbf{f}^e . To simplify the construction of the forcing vector \mathbf{f} , you may approximate $Q(x)$ by a function that is constant in each element and equal to the value of $Q(x)$ in the center of each element. Use $M = 10$ linear \mathbb{P}_1 -elements of equal length. Compute and plot the steady-state temperature field $T(x)$.
Hint: You can use `Week5_exerciseb` as a starting point.
Hint: You can also try to construct the mass matrix \mathbf{E} because you will need this in part d.
- (3pts) Repeat question b., but now with $M = 10$ quadratic \mathbb{P}_2 elements of equal length. Construct the load vector \mathbf{f} based on the same assumption as in b.

Hint: Copy your code from part b. and make the necessary changes.

Hint: You can compute the element stiffness matrix \mathbf{A}^e and the element forcing vector \mathbf{f}^e by hand, or use the commands `syms` and `int` from the symbolic toolbox.

For this particular problem, we can solve the boundary value problem analytically and the solution takes the form

$$T(x) = T_{\text{part}}(x) + Ax + B,$$

where

$$T_{\text{part}}(x) = \frac{Q_0 a^2}{2A_{\text{cs}}k} \exp\left(\frac{-(x - \frac{1}{2}L)^2}{a^2}\right) + \frac{Q_0 a}{2A_{\text{cs}}k} \sqrt{\pi}(x - \frac{1}{2}L) \operatorname{erf}\left(\frac{x - \frac{1}{2}L}{a}\right),$$

and the constants A and B solve the linear system

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} T_{\text{part}}(0) \\ A_{\text{cs}}k \frac{dT_{\text{part}}}{dx}(L) + hT_{\text{part}}(L) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ A_{\text{cs}}k + hL & h \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}.$$

- d. (3pts) Inspect the convergence rates for the solutions computed based on \mathbb{P}_1 and \mathbb{P}_2 elements. Compute the solutions for $M = 10, 30, 100, 300$, and 1000 elements. Make two plots: 1) relative error vs. element size, 2) relative error vs. number of nodes. What are the convergence rates for both element types? Is this what you expect? Why is the second plot also interesting?

Hint: The most natural way to compute the L^2 -error is as follows: Evaluate the analytic solution in the nodes of the finite element model and determine the error. Interpolate this error using the FE shape functions and take the L^2 -norm. Do you see how the mass matrix \mathbf{E} appears?

Hint: You can use `Week5_exercised` as a starting point.

Because this exercise is already quite long, part e. is optional.

- e. (2pts) Improve the computation of the forcing vector \mathbf{f} by evaluating the integral in each element based on two Gauss points. Does this change the convergence rates for the \mathbb{P}_1 - and \mathbb{P}_2 -elements?