EXERCISE WEEK 7:
PRACTICAL COURSE
MODELING, SIMULATION, OPTIMIZATION

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We consider the aluminum plate in Figure 1 of $L_x = 0.4$ [m] by $L_y = 0.5$ [m] by $H = 0.001$ [m]. Aluminum has a Young's modulus of $E = 69 \cdot 10^9$ Pa and Poisson's ratio is $\nu = 0.3$. Because the plate is thin, the plate is assumed to be in a state of *plane stress*. We thus consider only the in-plane components of the displacement field $\vec{u}(x, y) = [u_x(x, y), u_y(x, y)]^\top$ which depend only on the in-plane coordinates x and y. The considered boundary conditions are as follows:

- At the edge $x = 0$, a force per unit length (unit [N/m])
$$-H \begin{bmatrix} \sigma_{xx} \\ \sigma_{yx} \end{bmatrix} = \vec{f}_{\text{left}}(y) = 100 \exp(-(y - \tfrac{1}{2} L_y)^2/a^2) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
with $a = 0.1$ [m] is applied in the positive x-direction.
- At the edge $y = 0$, a force per unit length (unit [N/m])
$$-H \begin{bmatrix} \sigma_{xy} \\ \sigma_{yy} \end{bmatrix} = \vec{f}_{\text{bot}}(x) = 100 \exp(-(x - \tfrac{1}{2} L_x)^2/a^2) \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
again with $a = 0.1$ [m], is applied in the positive y-direction.
- The edge $x = L_x$ is fixed, i.e. $u_x = u_y = 0$ when $x = L_x$.
- The edge $y = L_y$ is free, i.e.
$$H \begin{bmatrix} \sigma_{xy} \\ \sigma_{yy} \end{bmatrix} = \vec{f}_{\text{top}}(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

a. (2pts) Write down the weak form of the considered boundary value problem. Write the problem explicitly in terms of the components of the displacement field and do not use the stress and strain components σ_{xx}, σ_{xy}, σ_{yy}, ε_{xx}, ε_{xy}, and ε_{yy}.
Hint: Use the first tip at the end of `Week7_lecture`.

b. (2pts) Make a Galerkin discretization of the problem you formulated at a. Approximate the displacement field components $u_x(x, y)$ and $u_y(x, y)$ by

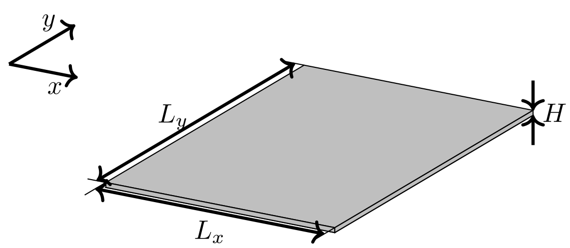

FIGURE 1. The considered aluminum plate

$\mathbf{N}(x,y)\mathbf{u}_x$ and $\mathbf{N}(x,y)\mathbf{u}_y$, respectively, where $\mathbf{N}(x,y)$ denotes a (row)vector of shape functions. Write the resulting equations (for the problem without Dirichlet boundary conditions) in the form

$$\underbrace{\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \mathbf{u}_x \\ \mathbf{u}_y \end{bmatrix}}_{\mathbf{u}} + \underbrace{\begin{bmatrix} \mathbf{f}_x \\ \mathbf{f}_y \end{bmatrix}}_{\mathbf{f}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

and give explicit expressions for the submatrices $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{21}$, and $\mathbf{A}_{22}$ and vectors $\mathbf{f}_x$ and $\mathbf{f}_y$.

c. (1pt) Explain how the matrix $\mathbf{A}$ and the vector $\mathbf{f}$ for problem without Dirichlet boundary conditions formulated at b. can be used to compute the solution with Dirichlet boundary conditions.

d. (4pts) Build a finite element model for the considered problem and compute (an approximation of) the resulting displacement field $\vec{u}$. Use $M_x = 32$ linear elements in the x-direction and $M_y = 42$ linear elements in the y-direction. You may again assume that the loading $f_{\text{left}}(y)$ and $f_{\text{bot}}(x)$ are constant in (the boundary of) each element.
Hint: You can use the file `Week7_exercised` as a starting point.

e. (1pt) What is the kernel of the matrix $\mathbf{A}$ for the problem without Dirichlet boundary conditions?
Hint: The kernel of the matrix $\mathbf{A}$ is spanned vectors that correspond to translation(s) and rotation(s) of the considered plate. For these rigid body modes, the corresponding strain components ε_{xx}, ε_{xy}, and ε_{yy} are zero.