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EXERCISE WEEK 9: PRACTICAL COURSE MODELING, SIMULATION, OPTIMIZATION

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We consider the aluminum rod in Figure 1 with a length of $L = 0.3$ [m], a cross sectional area of $A_{\text{cs}} = 0.01$ [m^2], and a thermal conductivity of $k = 237$ [W/m/K]. At five locations $x_{\text{meas},i} = (i-1)\frac{L}{4}$ $(i \in \{1, 2, 3, 4, 5\})$ along the rod, temperature sensors are installed. Our aim is to reconstruct the heat load applied to the rod based on the measured temperatures $\bar{T}_{\text{meas},i}$ at these five locations.

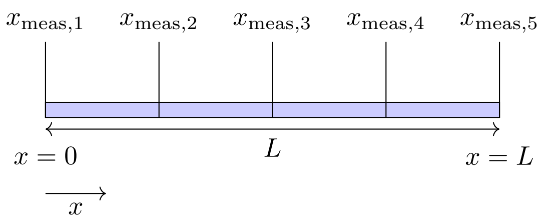

FIGURE 1. The considered aluminum rod

We reconstruct the applied heat load by minimizing the functional

$$\mathcal{J}(Q) = \frac{1}{2}\sum_{i=1}^{5}(T(x_{\text{meas},i}) - \bar{T}_{\text{meas},i})^2 + \frac{w}{2}\int_0^L (Q(x))^2 \, \mathrm{d}x,$$

where $w > 0$ is a weight and $T(x)$ is the steady-state temperature field resulting from the heat load $Q(x)$ which satisfies

$$kA_{\text{cs}}\frac{\mathrm{d}^2 T}{\mathrm{d}x^2}(x) + Q(x) = 0, \qquad kA_{\text{cs}}\frac{\mathrm{d}T}{\mathrm{d}x}(0) = hT(0), \qquad -kA_{\text{cs}}\frac{\mathrm{d}T}{\mathrm{d}x}(L) = hT(L).$$

A finite element discretization of this problem takes the form

$$J(\mathbf{u}) = \frac{1}{2}\mathbf{e}_{\text{meas}}^\top \mathbf{e}_{\text{meas}} + \frac{w}{2}\mathbf{u}^\top \mathbf{E}\mathbf{u},$$

$$\mathbf{e}_{\text{meas}} = \mathbf{E}_{\text{meas}}\mathbf{T} - \mathbf{T}_{\text{meas}}, \qquad \mathbf{A}\mathbf{T} + \mathbf{E}\mathbf{u} = \mathbf{0},$$

where $\mathbf{T}$ and $\mathbf{u}$ are vectors containing the nodal values of $T(x)$ and $Q(x)$, $\mathbf{T}_{\text{meas}} = [\bar{T}_{\text{meas},1}, \bar{T}_{\text{meas},2}, \bar{T}_{\text{meas},3}, \bar{T}_{\text{meas},4}, \bar{T}_{\text{meas},5}]^\top$ and the $5 \times N$-matrix $\mathbf{E}_{\text{meas}}$ (where N is the number of nodes) selects the nodal temperatures at the location of the temperature sensors. The matrices $\mathbf{E}$, $\mathbf{E}_{\text{meas}}$, and $\mathbf{A}$ for a FE model with $M = 100$ elements (and $N = 101$ nodes) are given in the file `rod_model.mat`. This file also contains the vector $\mathbf{T}_{\text{meas}}$ with the measured temperature values.

a. (2pts) Eliminate $\mathbf{e}_{\text{meas}}$ and $\mathbf{T}$ from the given expression of the discretized functional $J(\mathbf{u})$. Use the obtained expression to give explicit expressions for the Jacobian $\partial J / \partial \mathbf{u}$, the gradient ∇J w.r.t. the standard Euclidean inner product, and the gradient ∇J w.r.t. the weighted inner product based on the weighting matrix $\mathbf{W} = \mathbf{E}$.

b. (4pts) Implement the basic gradient descent algorithm given at the end of `lecture_week9`. Terminate the algorithm when the relative change in cost function and the relative change in $\mathbf{u}$ are below $\texttt{tol} = 10^{-3}$. Use $w = 10^{-6}$, an initial step size $\beta_0 = 100$, and the gradient w.r.t. to the weighted inner product with $\mathbf{W} = \mathbf{E}$.
Hint: You can use the file `Week9_exerciseb` as a starting point.

c. (2pts) Vary the weight w and explain what trend you observe in the obtained solutions. How does the choice of w influence the number of required iterations?

d. (2pts) Replace the gradient computed w.r.t. the weighted inner product with the gradient computed w.r.t. the standard innerproduct. What do you observe in the obtained optimal solution?