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EXERCISE WEEK 10: PRACTICAL COURSE MODELING, SIMULATION, OPTIMIZATION

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We consider the copper plate in Figure 1 with a length of $L_x = 0.5$ [m] in the x-direction, a length of $L_y = 0.75$ [m] in the y-direction, a thickness $H = 0.01$ [m], and a thermal conductivity $k = 400$ [W/K/m]. At six locations on the plate (indicated by the red crosses in the figure), temperature sensors are installed which measure the steady state temperatures $\bar{T}_{\mathrm{m},i}$ at these six locations.

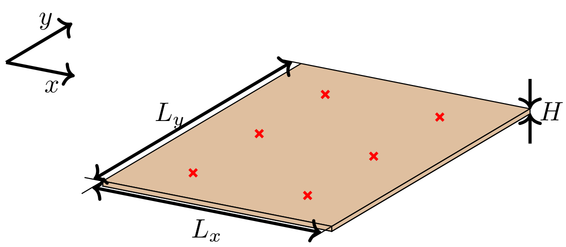

FIGURE 1. The considered copper plate

We reconstruct the heat load applied $Q(x, y)$ to the plate by minimizing the functional

$$\mathcal{J}(Q) = \frac{1}{2}\sum_{i=1}^{6}(T(x_{\mathrm{m},i}, y_{\mathrm{m},i}) - \bar{T}_{\mathrm{m},i})^2 + \frac{w}{2}\iint_{\Omega}(Q(x, y))^2 \,\mathrm{d}y\,\mathrm{d}x,$$

where $\Omega = [0, L_x] \times [0, L_y]$, $w > 0$ is a weight and $T(x, y)$ is the steady-state temperature field resulting from the heat load $Q(x, y)$. In particular, $T(x, y)$ is the solution of

$$kH\left(\frac{\partial^2 T}{\partial x^2}(x) + \frac{\partial^2 T}{\partial y^2}\right) + Q(x) = 0, \qquad -kH\nabla T \cdot \mathbf{n} = hT,$$

where the thermal conductance of the boundary $h = 5$ [W/K/m]. A finite element discretization of this problem takes the form

$$J(\mathbf{u}) = \frac{1}{2}\mathbf{e}_{\mathrm{m}}^{\top}\mathbf{e}_{\mathrm{m}} + \frac{w}{2}\mathbf{u}^{\top}\mathbf{E}\mathbf{u},$$

$$\mathbf{e}_{\mathrm{m}} = \mathbf{E}_{\mathrm{m}}\mathbf{T} - \mathbf{T}_{\mathrm{m}}, \qquad \mathbf{A}\mathbf{T} + \mathbf{E}\mathbf{u} = \mathbf{0},$$

where $\mathbf{T}$ and $\mathbf{u}$ are vectors containing the nodal values of $T(x, y)$ and $Q(x, y)$, $\mathbf{T}_{\mathrm{m}} = [\bar{T}_{\mathrm{m},1}, \bar{T}_{\mathrm{m},2}, \bar{T}_{\mathrm{m},3}, \bar{T}_{\mathrm{m},4}, \bar{T}_{\mathrm{m},5}, \bar{T}_{\mathrm{m},6}]^{\top}$, the $\mathbf{E}_{\mathrm{m}}$ is the $6 \times N$-matrix that selects the nodal temperatures at the location of the temperature sensors,

$$\mathbf{E} = \iint_{\Omega}(\mathbf{N})^{\top}\mathbf{N}\,\mathrm{d}y\,\mathrm{d}x,$$

$$\mathbf{A} = -kH \iint_{\Omega} \left(\left(\frac{\partial \mathbf{N}}{\partial x} \right)^{\top} \frac{\partial \mathbf{N}}{\partial x} + \left(\frac{\partial \mathbf{N}}{\partial y} \right)^{\top} \frac{\partial \mathbf{N}}{\partial y} \right) \, \mathrm{d}y \, \mathrm{d}x - h \int_{\partial\Omega} \mathbf{N}^{\top}\mathbf{N} \, \mathrm{d}y \, \mathrm{d}x.$$

with $\mathbf{N} = \mathbf{N}(x, y)$ the (row)vector of FE shape functions of length N.

The matrices $\mathbf{E}$, $\mathbf{E}_{\mathrm{m}}$, and $\mathbf{A}$ for a FE model on a nonuniform mesh with $M_x = 32$ elements in the x-direction and $M_y = 42$ elements in the y-direction are given in the file `Week10_plate_model.mat`. This file also contains the vector $\mathbf{T}_{\mathrm{m}}$ with the measured temperature values and the vectors $\mathbf{x}_{\mathrm{m}}$ and $\mathbf{y}_{\mathrm{m}}$ with the x- and y-coordinates of the sensor locations.

a. (2pts) Compute the gradient of the function J in the discretized problem w.r.t. the standard Euclidean inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\top}\mathbf{y}$ and the w.r.t. the weighted inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\top}\mathbf{E}\mathbf{y}$. Plot the obtained gradients in the point $\mathbf{u}_0 = \mathbf{0}$ against the spatial coordinates x and y.
Hint: there are two approaches to compute the Jacobian of J. One is to eliminate $\mathbf{e}_{\mathrm{meas}}$ and $\mathbf{T}$ from the expression for the discretized cost function given above and differentiate the resulting expression. The other is to use the chain rule.
Hint: You can use the file `Week10_exercisea` as a starting point.

b. (3pts) Determine the quadratic approximations of the functional $\beta \mapsto J(\mathbf{u}_0 - \beta \nabla J(\mathbf{u}_0))$ for the two gradients at $\mathbf{u}_0 = \mathbf{0}$ computed in part a. Compare the values of the quadratic approximation to the values of $\beta \mapsto J(\mathbf{u}_0 - \beta \nabla J(\mathbf{u}_0))$ for a range of step sizes β in a plot.
Hint: You can use the file `Week10_exerciseb` as a starting point.

c. (2pts) Develop and implement a gradient-based algorithm to minimize the functional $J(\mathbf{u})$. Use the gradient w.r.t. the weighted inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\top}\mathbf{E}\mathbf{y}$ computed in a. and use the quadratic approximation developed in b. to determine the step size β. Terminate the algorithm when the relative change in $\mathbf{u}$ and $J(\mathbf{u})$ are below 10^{-5}.
Hint: You can use the file `Week10_exercisec` as a starting point.
Hint: Do you need a line search to assure that the cost functional decreases in every iteration?

d. (3pts) Use the projected gradient method to assure that all entries of the vector $\mathbf{u}$ remain between 0 and 100.
Hint: You can use the file `Week10_exercised` as a starting point.
Hint: Do you now need a line search to assure that the cost functional decreases in every iteration?