EXERCISE WEEK 11: PRACTICAL COURSE MODELING, SIMULATION, OPTIMIZATION

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We consider heat conduction in the copper bar in Figure 1 with a length of $L = 0.3$ [m], a cross sectional area $A_{\text{cs}} = 10^{-4}$ [m^2], a thermal conductivity $k = 400$ [W/K/m], a mass density $\rho = 8960$, and a heat capacity $c = 385$ [J/kg/K]. The temperature increase in the bar (w.r.t. a constant temperature distribution at $t = 0$) is denoted by $T(x,t)$ [K]. In the part $[0, L/4]$, we can apply a uniform heat load $u(t)$ in [W]. We want to use the heat load $u(t)$ to keep the temperature at the right end $T(L,t)$ close to $T_d = 1$ [K].

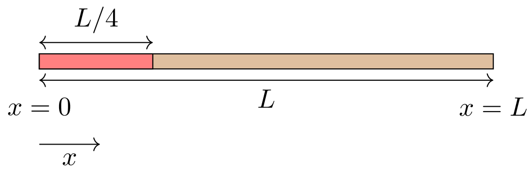

FIGURE 1. The considered copper bar

A FE element discretization of the bar with $M = 100$ elements and $N = 101$ nodes leads to a system of the form

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \qquad \mathbf{x}(0) = \mathbf{x}_{\text{init}}.$$

We want to determine the applied heat load $u(t)$ that minimizes

$$J = \frac{1}{2}\int_0^T \left((\mathbf{x}(t) - \mathbf{1})^\top \mathbf{Q}(\mathbf{x}(t) - \mathbf{1}) + (u(t))^2 \right) \, \mathrm{d}t.$$

The matrices $\mathbf{E}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{x}_{\text{init}}$, and $\mathbf{Q}$ are given in the file `bar_model.mat`. We also fix $T = 8$ [minutes] $= 480$ [s] and $\mathbf{1}$ denotes a vector of ones of length N.

a. (2pt) Compute the state $\mathbf{x}(t)$ resulting from the input $u(t) = \sin(\frac{\pi}{2}\frac{t}{T})$ using the Crank-Nicolson scheme. Use a time grid with $N_T = 201$ points. Use the obtained solution to evaluate the cost functional J for the input $u(t) = \sin(\frac{\pi}{2}\frac{t}{T})$ discretized with the combination of the trapezoid and the midpoint rule explained in the lecture.
Hint: You can use the file `Week11_exerciseab` as a starting point.

b. (2pts) Compute the adjoint state $\boldsymbol{\varphi}(t)$ (for the input $u(t) = \sin(\frac{\pi}{2}\frac{t}{T})$) that leads to discretely consistent gradients. Follow the procedure given in the lecture. Use the obtained (discretized) adjoint state to compute the gradient ∇J.
Hint: You can again use the file `Week11_exerciseab`.

c. (2pts) Compute the coefficients G and H in the quadratic approximation of the cost functional $\beta \mapsto J(u_0 - \beta\nabla J)$. Compare the obtained quadratic approximation to the true values of the cost functional. Do you expect to

see any difference?
Hint: you can again use the file `Week11_exercisec` as a starting point.

d. (3pts) Use the results from parts a,b, and c to develop and implement a gradient-based algorithm to minimize the functional J.
Hint: You can use the file `Week11_exercised` as a starting point.
Hint: Do you need a line search to assure that the cost functional decreases in every iteration?

e. (1pt) Run the algorithm you developed in d. with the matrix $\mathbf{Q}$ rescaled by a factor 0.01 and a factor 100. What do you observe in the obtained optimal controls? Try to explain your observations.