EXERCISE WEEK 12:
PRACTICAL COURSE
MODELING, SIMULATION, OPTIMIZATION

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We consider the vibrating string in Figure 1 with a length of $L = 0.8$ [m], a tension $T = 10$ [N], and a mass of 0.01 [kg]. The transverse displacement of the string is denoted by $w(t, x)$. The two ends of the string are fixed, i.e. $w(t, 0) = w(t, L) = 0$. We can apply forces $u_1(t)$ and $u_2(t)$ at $x = \frac{3}{10}L$ and $x = \frac{1}{2}L$, respectively.

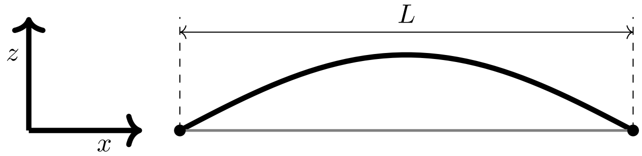

FIGURE 1. The considered vibrating string

We consider a FE model for the string based on 100 linear elements. The vector with free DOFs (so without the transversal displacement of the first and last node) is denoted by $\mathbf{w}_\mathrm{f}(t)$. The FE model then takes the form

$$\mathbf{M}_\mathrm{ff}\ddot{\mathbf{w}}_\mathrm{f}(t) + \mathbf{K}_\mathrm{ff}\mathbf{w}_\mathrm{f}(t) = \mathbf{F}_\mathrm{f}u(t), \qquad \mathbf{w}_\mathrm{f}(0) = \mathbf{w}_{\mathrm{init,f}}, \qquad \dot{\mathbf{w}}_\mathrm{f}(0) = \mathbf{0}.$$

We want to determine the forces $u(t) = [u_1(t), u_2(t)]^\top$ that minimize

$$J = \frac{1}{2}\int_0^T \left((\mathbf{w}_\mathrm{f}(t))^\top \mathbf{W}_\mathrm{ff}\mathbf{w}_\mathrm{f}(t) + (u(t))^\top \mathbf{R}u(t) \right) \,\mathrm{d}t.$$

The matrices $\mathbf{M}_\mathrm{ff}$, $\mathbf{K}_\mathrm{ff}$, and $\mathbf{W}_\mathrm{ff}$, and the vectors $\mathbf{F}_\mathrm{f}$ and $\mathbf{w}_{\mathrm{init,f}}$ are given in file `string_model.mat`. This file also contains the vector `x` with the positions of the nodes in FE model (including the constrained nodes) and the number of free DOFs `Nf`. We also fix $T = 0.25$ [s] and $\mathbf{R} = \mathbf{I}_{2\times 2}$.

a. (2pt) Rewrite the given problem that is constrained by a second-order ODE as a problem constrained by a first order ODE of the form

$$J = \frac{1}{2}\int_0^T \left((\mathbf{x}(t) - \mathbf{x}_d(t))^\top \mathbf{Q}(\mathbf{x}(t) - \mathbf{x}_d(t)) + (u(t))^\top \mathbf{R}u(t) \right) \,\mathrm{d}t.$$

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \qquad \mathbf{x}(0) = \mathbf{x}_{\mathrm{init}}.$$

Give explicit expressions for $\mathbf{E}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{x}_{\mathrm{init}}$, $\mathbf{x}_d(t)$, $\mathbf{Q}$, and $\mathbf{R}$ in terms of the given matrices $\mathbf{M}_\mathrm{ff}$, $\mathbf{K}_\mathrm{ff}$, and $\mathbf{W}_\mathrm{ff}$, and the vectors $\mathbf{F}_\mathrm{f}$ and $\mathbf{w}_{\mathrm{init,f}}$. Make sure that the matrices $\mathbf{E}$ and $\mathbf{A}$ are sparse.
Hint: introduce the vector $\mathbf{x}(t) = [(\mathbf{w}_\mathrm{f}(t))^\top, (\dot{\mathbf{w}}_\mathrm{f}(t))^\top]^\top$ and include an additional (trivial) equation $\mathbf{M}\dot{\mathbf{w}}_\mathrm{f}(t) = \mathbf{M}\dot{\mathbf{w}}_\mathrm{f}(t)$.

b. (3pts) Discretize the obtained first-order system with the Crank-Nicolson scheme and discretize the cost functional J with the combination of the trapezoid rule and midpoint rule from the lecture. Compute the discretely consistent gradient based on the adjoint state. Implement the resulting formulas in the files `compute_X`, `cost_function`, and `compute_Phi`, and the first part of `Week12_Exercise14bc`.

c. (3pts) Compute the coefficients G and H in the quadratic approximation of the cost functional $\beta \mapsto J(u_0 - \beta \nabla J)$. Compare the obtained quadratic approximation to the true values of the cost functional. Do you expect to see any difference?
Hint: you will need to complete the file `hessian` and the second part of `Week12_Exercise14bc`.

d. (2pts) Use the results from parts a, b, and c to develop and implement a gradient-based algorithm to minimize the functional J.
Hint: you can use the file `Week12_Exercise14d` as a starting point.