Practical Course: Modeling, Simulation, Optimization

Week 4

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Contents

4.A Motivation and preliminaries for von Neumann stability analysis

Motivation: we need information about $\sigma(\mathbf{A})$

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}(t) = \mathbf{A}\mathbf{u}(t) + \mathbf{f}(t), \qquad \mathbf{u}(0) = \mathbf{u}_0.$$

θ-scheme

$$\frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\Delta t} = (1-\theta)\left(\mathbf{A}\mathbf{u}^k + \mathbf{f}^k\right) + \theta\left(\mathbf{A}\mathbf{u}^{k+1} + \mathbf{f}^{k+1}\right), \qquad \mathbf{u}^0 = \mathbf{u}_0.$$

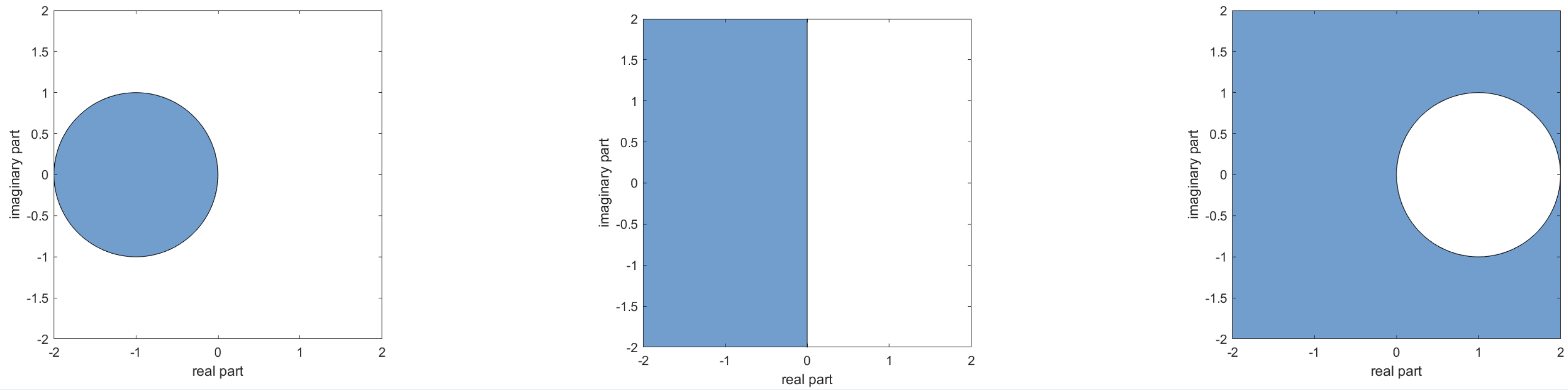
The scheme is stable iff

$$|1 + (1-\theta)\lambda\Delta t| \leq |1 - \theta\lambda\Delta t|, \qquad \text{for all } \lambda \in \sigma(\mathbf{A}).$$

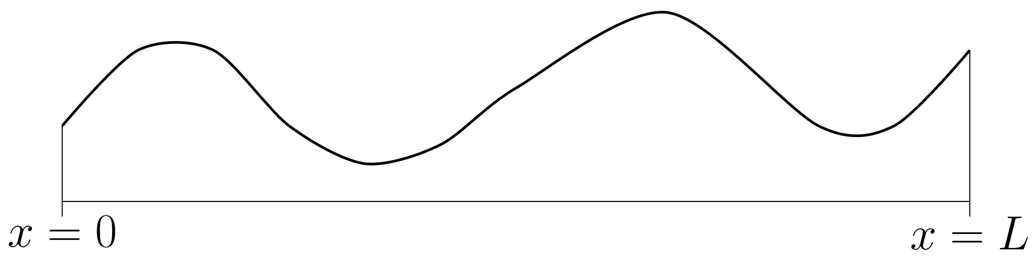
Forward Euler ($\theta = 0$) — $|1 + \lambda\Delta t| \leq 1$

Crank-Nicolson ($\theta = \frac{1}{2}$) — $|1 + \frac{1}{2}\lambda\Delta t| \leq |1 - \frac{1}{2}\lambda\Delta t|$

Backward Euler ($\theta = 1$) — $|1 - \lambda\Delta t| \geq 1$



Preliminary: Fourier Series



Any function $u(x)$ in $L^2([0,L])$ can be written as

$$u(x) = \sum_{n\in\mathbb{Z}} \hat{u}_n \exp(i2\pi \tfrac{x}{L} n), \qquad \hat{u}_n = \frac{1}{L}\int_0^L u(x)\exp(-i2\pi\tfrac{x}{L}n)\,\mathrm{d}x.$$

We effectively extend the function $u(x)$ defined on $[0,L]$ to a periodic function on $\mathbb{R}$. It therefore makes sense to compute the Fourier series of $v(x) = u(x+\Delta x)$.

$$\hat{v}_n = \frac{1}{L}\int_0^L u(x+\Delta x)\exp(-i2\pi\tfrac{x}{L}n)\,\mathrm{d}x = \frac{1}{L}\int_{\Delta x}^{L+\Delta x} u(\xi)\exp(-i2\pi\tfrac{\xi-\Delta x}{L}n)\,\mathrm{d}\xi$$
$$= \frac{\exp(i2\pi\frac{\Delta x}{L}n)}{L}\int_{\Delta x}^{L+\Delta x} u(\xi)\exp(-i2\pi\tfrac{\xi}{L}n)\,\mathrm{d}\xi = \exp(i2\pi\tfrac{\Delta x}{L}n)\hat{u}_n$$

A key observation

Shifting property

Extend $u \in L^2([0, L])$ to an L-periodic function on $\mathbb{R}$ and consider $v(x) = u(x + \Delta x)$.
Then

$$\hat{v}_n = \exp(i2\pi \frac{\Delta x}{L} n)\hat{u}_n.$$

A key observation

Shifting property

Extend $u \in L^2([0, L])$ to an L-periodic function on $\mathbb{R}$ and consider $v(x) = u(x + \Delta x)$. Then

$$\hat{v}_n = \exp(i2\pi \frac{\Delta x}{L} n)\hat{u}_n.$$

Now consider

$$w(x) = u(x + \Delta x) - 2u(x) + u(x - \Delta x)$$

A key observation

Shifting property

Extend $u \in L^2([0, L])$ to an L-periodic function on $\mathbb{R}$ and consider $v(x) = u(x + \Delta x)$. Then

$$\hat{v}_n = \exp(i2\pi\tfrac{\Delta x}{L}n)\hat{u}_n.$$

Now consider

$$w(x) = u(x + \Delta x) - 2u(x) + u(x - \Delta x)$$

Question: what is the Fourier series $(\hat{w}_n)_{n\in\mathbb{Z}}$?

A) $\hat{w}_n = \left(\exp(i2\pi\tfrac{\Delta x}{L}n) - 2 + \exp(i2\pi\tfrac{\Delta x}{L}n)\right)\hat{u}_n$

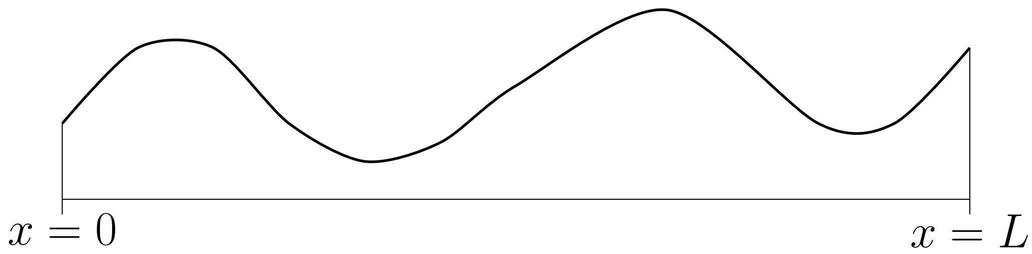
B) $\hat{w}_n = \left(\cos(2\pi\tfrac{\Delta x}{L}n) - 2\right)\hat{u}_n$

C) $\hat{w}_n = 2\left(\cos(2\pi\tfrac{\Delta x}{L}n) - 1\right)\hat{u}_n$

D) $\hat{w}_n = -4\sin^2(\pi\tfrac{\Delta x}{L}n)\hat{u}_n$

E) None of the above

Preliminary: Fourier Series



Any function $u(x)$ in $L^2([0, L])$ can be written as

$$u(x) = \sum_{n \in \mathbb{Z}} \hat{u}_n \exp(i2\pi \tfrac{x}{L} n), \qquad \hat{u}_n = \frac{1}{L}\int_0^L u(x) \exp(-i2\pi \tfrac{x}{L} n)\, \mathrm{d}x.$$

Parseval's theorem

For two functions $u, v \in L^2([0, L])$ with Fourier series $\{\hat{u}_n\}_{n\in\mathbb{Z}}$ and $\{\hat{v}_n\}_{n\in\mathbb{Z}}$

$$\frac{1}{L}\int_0^L \overline{u(x)} v(x)\, \mathrm{d}x = \sum_{n\in\mathbb{Z}} \bar{\hat{u}}_n \hat{v}_n$$

Preliminary: Fourier series

Parseval's theorem

For two functions $u, v \in L^2([0, L])$ with Fourier series $\{\hat{u}_n\}_{n\in\mathbb{Z}}$ and $\{\hat{v}_n\}_{n\in\mathbb{Z}}$

$$\frac{1}{L}\int_0^L \overline{u(x)}v(x)\,\mathrm{d}x = \sum_{n\in\mathbb{Z}} \bar{\hat{u}}_n \hat{v}_n$$

Again consider

$$w(x) = u(x+\Delta x) - 2u(x) + u(x-\Delta x)$$

and remember that $\hat{w}_n = -4\sin^2(\pi\frac{\Delta x}{L}n)\hat{u}_n$.

Question: which is the most precise statement about

$$\langle u, w\rangle = \frac{1}{L}\int_0^L \overline{u(x)}w(x)\,\mathrm{d}x.$$

A) $|\langle u, w\rangle| \leq 4\langle u, u\rangle$

B) $\langle u, w\rangle \in [0, 4\langle u, u\rangle]$

C) $\langle u, w\rangle \in [-4\langle u, u\rangle, 0]$

D) $\langle u, w\rangle \in [-4\sin^2(\pi\frac{\Delta x}{L}n)\langle u, u\rangle, 0]$

4.B Von Neumann stability analysis

The rough idea in von Neumann stability analysis

Consider a diffusion process

$$\frac{\partial u}{\partial t}(t,x) = \kappa \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x), \qquad (t,x) \in (0,T) \times (0,L),$$

As the result of a FD discretization, we obtain a system of ODEs for the free DOFs

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}(t) = \mathbf{A}\mathbf{u}(t) + \mathbf{f}(t), \qquad \mathbf{u}(0) = \mathbf{u}_0.$$

We need to bound $\sigma(\mathbf{A})$.

For 1-D finite differences of a Diffusion process, we have for most m that

$$(\mathbf{A}\mathbf{u})_m = \kappa \frac{u_{m+1} - 2u_m + u_{m-1}}{(\Delta x)^2}$$

Based on the result on the previous slide, we therefore expect that

$$\langle \mathbf{u}, \mathbf{A}\mathbf{u} \rangle \in [\tfrac{-4\kappa}{(\Delta x)^2} \langle \mathbf{u}, \mathbf{u} \rangle, 0]$$

Now suppose that $\mathbf{u}$ is an eigenvector, i.e. $\mathbf{A}\mathbf{u} = \lambda \mathbf{u}$, then

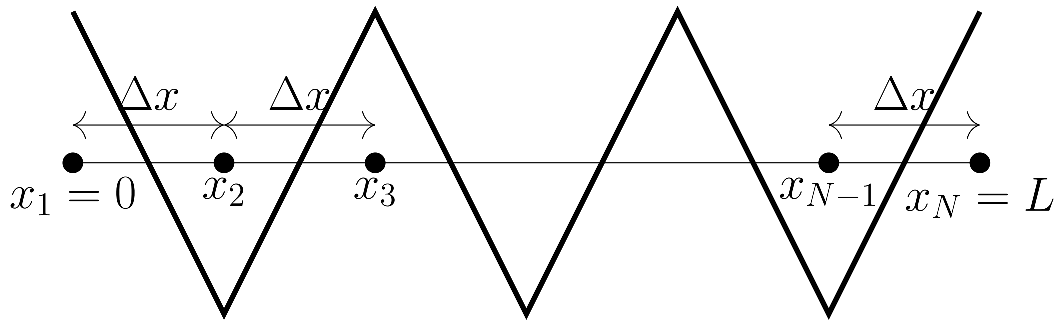
$$\lambda \langle \mathbf{u}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{A}\mathbf{u} \rangle \in [\tfrac{-4\kappa}{(\Delta x)^2} \langle \mathbf{u}, \mathbf{u} \rangle, 0], \qquad \Rightarrow \qquad \lambda \in [\tfrac{-4\kappa}{(\Delta x)^2}, 0].$$

Some remarks

- The relation between $\hat{w}_n$ and $\hat{u}_n$ is found by inserting the Fourier mode $\exp(i2\pi\frac{\Delta x}{L}n)$ into the numerical scheme.
- When we take the maximally oscillatory solution $u_m = (-1)^m$, we see that

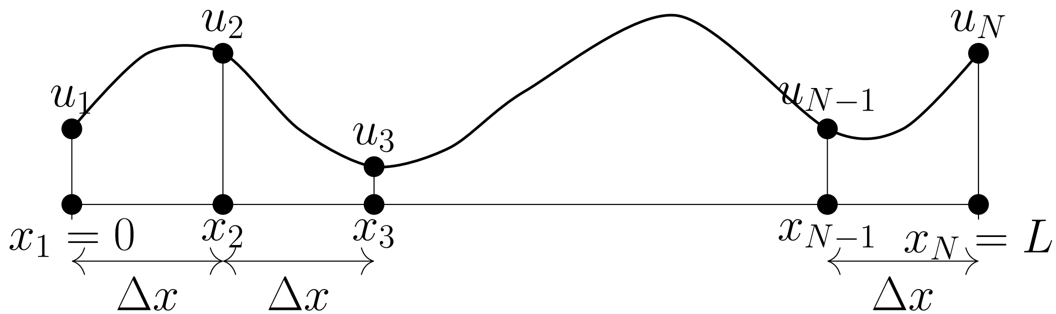
$$(\mathbf{Au})_m = \kappa\frac{u_{m+1} - 2u_m + u_{m-1}}{(\Delta x)^2} = \frac{-4\kappa}{(\Delta x)^2}u_m,$$

which shows that the derived bound is sharp.



- To make the arguments precise we also need
  - to consider the boundary conditions and
  - to replace the Fourier series by the Discrete Fourier Transform (DFT).

Discrete Fourier Transform



We now have a vector with samples $\mathbf{u} = [u_1, u_2, u_3, \ldots, u_{N-1}, u_N]^\top$.

$$u_m = \sum_{n=0}^{N-1} \hat{u}_n \exp(i2\pi \tfrac{m}{N} n), \qquad \hat{u}_n = \frac{1}{N} \sum_{m=1}^{N} u_m \exp(-i2\pi \tfrac{m}{N} n).$$

Shifting property

$$v_m = u_{m+1} \quad \Rightarrow \quad \hat{v}_n = \exp(i2\pi \tfrac{1}{N} n)\hat{u}_n.$$

Parseval's theorem

For two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^N$ with Discrete Fourier Transforms $\{\hat{u}_n\}_{n=0}^{N-1}$ and $\{\hat{v}_n\}_{n=0}^{N-1}$

$$\frac{1}{N}\mathbf{u}^H \mathbf{v} = \frac{1}{N} \sum_{m=1}^{N} \bar{u}_m v_m = \sum_{n=0}^{N-1} \bar{\hat{u}}_n \hat{v}_n.$$

Higher dimensions

We can use similar ideas based on higher dimensional versions of the Fourier transform.

$$u(x,y) = \sum_{r,s\in\mathbb{Z}} \hat{u}_{rs} \exp\left(i2\pi\left(\frac{\Delta x}{L_x}r + \frac{\Delta y}{L_y}s\right)\right)$$

For a 2-D diffusion process

$$\frac{\partial u}{\partial t}(t,x,y) = \kappa\left(\frac{\partial^2 u}{\partial x^2}(t,x,y) + \frac{\partial^2 u}{\partial y^2}(t,x,y)\right) + f(t,x,y).$$

We consider

$$w(x,y) = \frac{u(x+\Delta x,y) - 2u(x,y) + u(x-\Delta x,y)}{(\Delta x)^2} + \frac{u(x,y+\Delta y) - 2u(x,y) + u(x,y-\Delta y)}{(\Delta y)^2}.$$

$$\hat{w}_{rs} = \left(\tfrac{2}{(\Delta x)^2}(\cos(2\pi\tfrac{\Delta x}{L_x}r) - 1) + \tfrac{2}{(\Delta y)^2}(\cos(2\pi\tfrac{\Delta y}{L_y}s) - 1)\right)\hat{u}_{rs}$$

We therefore typically (also depending on the BCs) get that

$$\sigma(\mathbf{A}) \in \left[-4\kappa\left(\tfrac{1}{(\Delta x)^2} + \tfrac{1}{(\Delta y)^2}\right), 0\right].$$

4.C Spatio-temporal discretization of advection problems

Conservation mass

$$\frac{\partial \rho}{\partial t}(t,x) = -\frac{\partial \phi}{\partial x}(t,x).$$

To complete the model, we need a *constitutive relation*
that relates the mass flux $\phi(t,x)$ [kg/s] to the mass density $\rho(t,x)$ [kg/m].

Fick's law

$$\phi(t,x) = -\kappa \frac{\partial \rho}{\partial x}(t,x).$$

The coefficient κ [m^2/s] is called the diffusivity.

Advective transport

$$\phi(t,x) = v\rho(t,x).$$

The velocity v [m/s] is given.

We can also consider combinations of these two constitutive relations:

$$\phi(t,x) = -\kappa \frac{\partial \rho}{\partial x}(t,x) + v\rho(t,x), \qquad \frac{\partial \rho}{\partial t}(t,x) = -v\frac{\partial \rho}{\partial x}(t,x) + \kappa \frac{\partial^2 \rho}{\partial x^2}(t,x).$$

Advective transport

Suppose we want to discretize the transport equation with $v > 0$:

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= -v\frac{\partial u}{\partial x}(t,x), \qquad (t,x) \in (0,T)\times(0,L), \\ u(t,0) &= g(t), \qquad u(0,x) = u_0(x). \end{aligned}$$

Naively, we would do the spatial discretization as

$$\begin{aligned} \frac{\mathrm{d}u_m}{\mathrm{d}t}(t) &= -v\frac{u_{m+1}(t) - u_{m-1}(t)}{2\Delta x}, \qquad (t,x) \in (0,T)\times(0,L), \\ u_1(t) &= g(t), \qquad u_m(0) = u_0(x_m). \end{aligned}$$

Now want to apply the Von Neumann stability analysis to the resulting $\mathbf{A}$-matrix.
So consider

$$w(x) = u(x+\Delta x) - u(x - \Delta x).$$

Question: what is the relation between the Fourier coefficients $\{\hat{w}_n\}_{n\in\mathbb{Z}}$ and $\{\hat{u}_n\}_{n\in\mathbb{Z}}$?

A) $\hat{w}_n = 2\cos(2\pi\frac{\Delta x}{L}n)\hat{u}_n$, B) $\hat{w}_n = 2\sin(2\pi\frac{\Delta x}{L}n)\hat{u}_n$,
C) $\hat{w}_n = 2i\cos(2\pi\frac{\Delta x}{L}n)\hat{u}_n$ D) $\hat{w}_n = 2i\sin(2\pi\frac{\Delta x}{L}n)\hat{u}_n$ E) None of the above.

Advective transport

Suppose we want to discretize the transport equation with $v > 0$:

$$\frac{\partial u}{\partial t}(t,x) = -v\frac{\partial u}{\partial x}(t,x), \qquad (t,x) \in (0,T)\times(0,L),$$
$$u(t,0) = g(t), \qquad u(0,x) = u_0(x).$$

Naively, we would do the spatial discretization as

$$\frac{\mathrm{d}u_m}{\mathrm{d}t}(t) = -v\frac{u_{m+1}(t) - u_{m-1}(t)}{2\Delta x}, \qquad (t,x) \in (0,T)\times(0,L),$$
$$u_1(t) = g(t), \qquad u_m(0) = u_0(x_m).$$

Von Neumann stability analysis for the resulting $\mathbf{A}$-matrix. So consider

$$w(x) = u(x+\Delta x) - u(x-\Delta x).$$

$\hat{w}_n = 2i\sin(2\pi\frac{\Delta x}{L}n)\hat{u}_n$. So $\mathrm{Im}(\lambda(\mathbf{A})) \in [\frac{-v}{\Delta x}, \frac{v}{\Delta x}]$.

The proposed discretization is always unstable for Forward Euler!

A possible solution: the Lax-Friedrich scheme

$$\frac{2u_m^{k+1} - \left(u_{m+1}^k + u_{m-1}^k\right)}{2\Delta t} = -v\frac{u_{m+1}^k - u_{m-1}^k}{2\Delta x}.$$

Note that $u(t_k, x_{m+1}) + u(t_k, x_{m-1}) = 2u(t_k, x_m) + O((\Delta x)^2)$.

It can be shown that this scheme is stable for $\Delta t \leq \frac{\Delta x}{v}$
(because this is not a θ-scheme we would need to check the analysis in the previous lecture again to prove this)

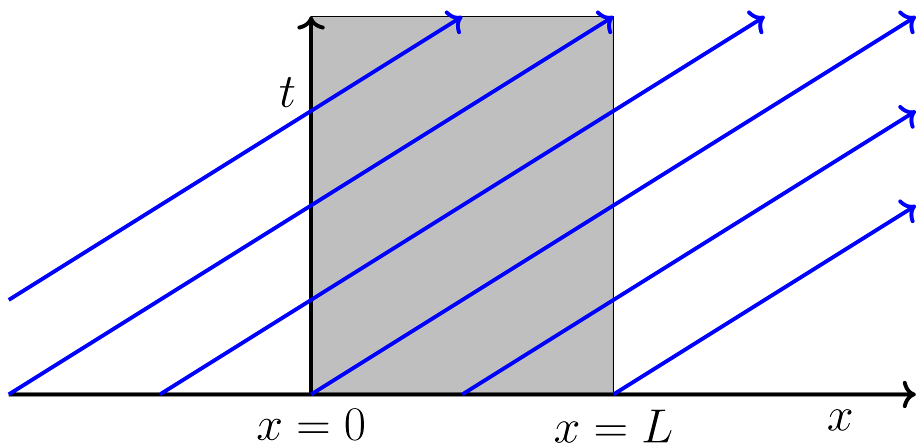
see also Allaire, Lemma 2.3.2, page 53

There are many other schemes for advection problems. For example Lax-Wendorf.

Another solution: upwinding

$$\frac{\partial u}{\partial t}(t,x) = -v\frac{\partial u}{\partial x}(t,x), \qquad (t,x) \in (0,T) \times (0,L),$$
$$u(t,0) = g(t), \qquad u(0,x) = u_0(x).$$

Idea: the solution $u(t,x) = h(x - vt)$ is constant along characteristics.



When we choose Δt and Δx such that $v\Delta t = \Delta x$, we have $u(t_{k+1}, x_{m+1}) = u(t_k, x_m)$. Consequently,

$$\frac{u(t_{k+1},x_m) - u(t_k,x_m)}{\Delta t} = v\frac{u(t_k,x_{m-1}) - u(t_k,x_m)}{\Delta x} = -v\frac{u(t_k,x_m) - u(t_k,x_{m-1})}{\Delta x}$$

Upwinding

Based on the previous slide, we would try to do the spatial discretization as

$$\frac{\mathrm{d}u_m}{\mathrm{d}t}(t) = -v\frac{u_m(t) - u_{m-1}(t)}{\Delta x}, \qquad (t,x) \in (0,T) \times (0,L),$$
$$u_1(t) = g(t), \qquad u_m(0) = u_0(x_m).$$

Now want to apply the Von Neumann stability analysis to the resulting $\mathbf{A}$-matrix.

$$w(x) = u(x) - u(x - \Delta x) \qquad \Rightarrow \qquad \hat{w}_n = \left(1 - \exp(-i2\pi\tfrac{\Delta x}{L}n)\right)\hat{u}_n.$$

Observe:

- The eigenvalues of $\mathbf{A}$ are in a disk with center $(-v/(\Delta x), 0)$ and radius $v/\Delta x$.
- When $v > 0$ ('upwinding'), all eigenvalues of $\mathbf{A}$ have negative real part
  $\Rightarrow$ Crank-Nicolson and Backward Euler are stable.
- When $v < 0$ ('downwinding'), all eigenvalues of $\mathbf{A}$ have positive real part
  $\Rightarrow$ all θ-schemes are unstable.
- Forward Euler is stable when
  $$\Delta t \leq \frac{v}{\Delta x}.$$
  This is also called the *Courant–Friedrichs–Lewy (CFL) condition*.

4.D Spatio-temporal discretization of advection-diffusion problems

Advection-Diffusion

Suppose we want to discretize the advection-diffusion equation

$$\frac{\partial u}{\partial t}(t,x) = -v\frac{\partial u}{\partial x}(t,x) + \kappa\frac{\partial^2 u}{\partial x^2}(t,x), \qquad (t,x) \in (0,T)\times(0,L),$$

We again consider centered finite differences

$$\frac{\mathrm{d}u_m}{\mathrm{d}t}(t) = -v\frac{u_{m+1}(t) - u_{m-1}(t)}{2\Delta x} + \kappa\frac{u_{m+1}(t) - 2u_m(t) + u_{m-1}(t)}{(\Delta x)^2},$$

For the Von Neumann stability analysis, we thus consider

$$w(x) = -\mathrm{Pe}\,(u(x+\Delta x) - u(x-\Delta x)) + (u(x+\Delta x) - 2u(x) + u(x-\Delta x)).$$

where

$$\mathrm{Pe} = \frac{v\Delta x}{2\kappa}.$$

is called the *mesh Péclet number*

Question: what is the relation between the Fourier coefficients $\{\hat{w}_n\}_{n\in\mathbb{Z}}$ and $\{\hat{u}_n\}_{n\in\mathbb{Z}}$?

A) $\hat{w}_n = \left(-\mathrm{Pe}\sin(2\pi\frac{\Delta x}{L}n) + (\cos(2\pi\frac{\Delta x}{L}n) - 2)\right)\hat{u}_n$,

B) $\hat{w}_n = \left(-\mathrm{Pe}\sin(2\pi\frac{\Delta x}{L}n) + 2(\cos(2\pi\frac{\Delta x}{L}n) - 1)\right)\hat{u}_n$,

C) $\hat{w}_n = \left(-2\mathrm{Pe}\sin(2\pi\frac{\Delta x}{L}n) + (\cos(2\pi\frac{\Delta x}{L}n) - 2)\right)\hat{u}_n$,

D) $\hat{w}_n = \left(-2\mathrm{Pe}\sin(2\pi\frac{\Delta x}{L}n) + 2(\cos(2\pi\frac{\Delta x}{L}n) - 1)\right)\hat{u}_n$,

E) None of the above.

Advection-Diffusion

Suppose we want to discretize the advection-diffusion

$$\frac{\partial u}{\partial t}(t,x) = -v\frac{\partial u}{\partial x}(t,x) + \kappa\frac{\partial^2 u}{\partial x^2}(t,x), \qquad (t,x) \in (0,T)\times(0,L),$$

We again consider centered finite differences

$$\frac{\mathrm{d}u_m}{\mathrm{d}t}(t) = -v\frac{u_{m+1}(t) - u_{m-1}(t)}{2\Delta x} + \kappa\frac{u_{m+1}(t) - 2u_m(t) + u_{m-1}(t)}{(\Delta x)^2},$$

For the Von Neumann stability analysis, we thus consider

$$w(x) = -\mathrm{Pe}\left(u(x+\Delta x) - u(x-\Delta x)\right) + \left(u(x+\Delta x) - 2u(x) + u(x-\Delta x)\right).$$

where

$$\mathrm{Pe} = \frac{v\Delta x}{2\kappa}.$$

is called the *mesh Péclet number*

$$\hat{w}_n = \left(-i2\mathrm{Pe}\sin(2\pi\tfrac{\Delta x}{L}n) + 2(\cos(2\pi\tfrac{\Delta x}{L}n) - 1)\right)\hat{u}_n.$$

Observe:

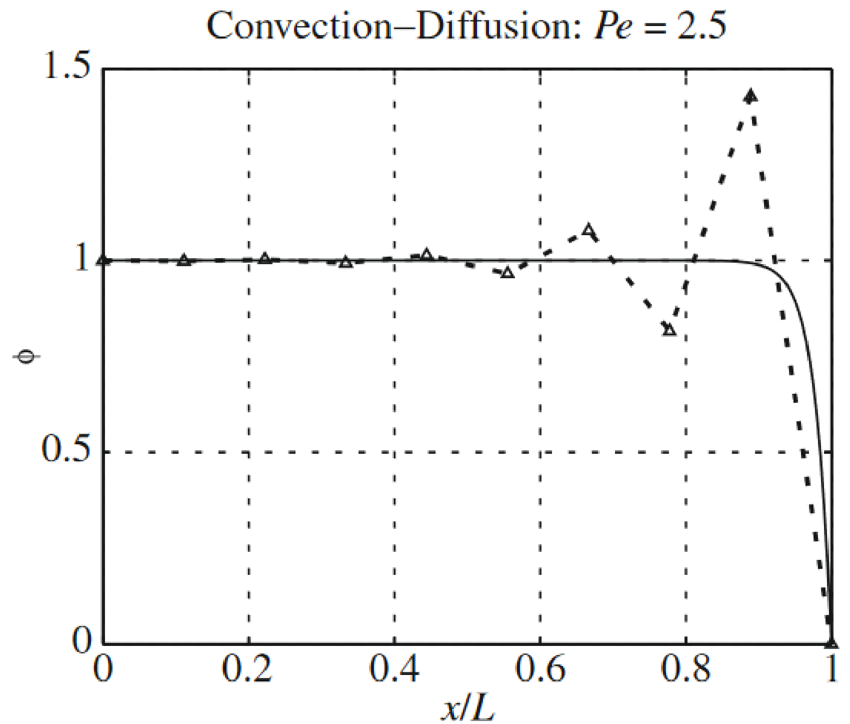
- The real part of the eigenvalues of $\mathbf{A}$ is always negative
  $\Rightarrow$ the Crank-Nicolson and Backward Euler schemes are unconditionally stable.
- It can also be verified that Forward Euler is stable when $|\mathrm{Pe}| \leq 1$.

The mesh Péclet number and steady state solutions

One should always keep $\mathrm{Pe} < 1$. When $\mathrm{Pe} > 1$, the steady-state solutions of our discretization contain spurious oscilations.

Example

$$-v\frac{\mathrm{d}u}{\mathrm{d}x} + \kappa\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = 0, \qquad u(0) = 1, \qquad u(L) = 0.$$



Analysis of the picture on the previous slide

Consider again the example:

$$-v\frac{\mathrm{d}u}{\mathrm{d}x} + \kappa\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = 0, \qquad u(0) = 1, \qquad u(L) = 0.$$

Discretization with centered finite differences:

$$-v\frac{u_{m+1} - u_{m-1}}{\Delta x} + \kappa\frac{u_{m+1} - 2u_m + u_{m-1}}{(\Delta x)^2} = 0,$$

$$-\mathrm{Pe}\,(u_{m+1} - u_{m-1}) + (u_{m+1} - 2u_m + u_{m-1}) = (1 - \mathrm{Pe})u_{m+1} - 2u_m + (1 + \mathrm{Pe})u_{m-1} = 0,$$

Observe that $u_m = \mu^m u_0$ satisfies the FD scheme iff

$$(1 - \mathrm{Pe})\mu^2 - 2\mu + (1 + \mathrm{Pe}) = 0, \qquad \Leftrightarrow \qquad \mu_\pm = \frac{1 \pm |\mathrm{Pe}|}{1 - \mathrm{Pe}}.$$

Therefore, any steady state solution of the FD-scheme is of the form

$$u_m = (\mu_-)^m u_0^- + (\mu_+)^m u_0^+,$$

for some constants u_0^- and u_0^+ that are determined by the BCs.

We conclude that u_m contains oscillations when either

$$\mu_- < 0 \text{ or } \mu_+ < 0, \qquad \Leftrightarrow \qquad |\mathrm{Pe}| > 1.$$

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Connection to upwinding

Recall the upwinding scheme for the advection equation we considered before:

$$\frac{\mathrm{d}u_m}{\mathrm{d}t}(t) = -v\frac{u_m(t) - u_{m-1}(t)}{2\Delta x},$$

Observe that

$$u_m - u_{m-1} = \tfrac{1}{2}(u_{m+1} - u_{m-1}) - \tfrac{1}{2}(u_{m+1} - 2u_m + u_{m-1})$$

Inserting this in the upwinding scheme, we find

$$\frac{\mathrm{d}u_m}{\mathrm{d}t}(t) = -v\frac{u_{m+1}(t) - u_{m-1}(t)}{4\Delta x} + v\frac{u_{m+1}(t) - 2u_m(t) + u_{m-1}(t)}{4\Delta x},$$

So the upwinding is equivalent to the centered differences scheme with

$$v^* = \tfrac{1}{2}v, \qquad \kappa^* = \tfrac{1}{2}v\Delta x, \qquad \mathrm{Pe}^* = \frac{v^*\Delta x}{2\kappa^*} = \frac{1}{2}.$$

Because $|\mathrm{Pe}^*| \leq 1$, we indeed expect the upwinding scheme is stable for all θ-schemes.

Important observation: upwinding leads to 'numerical' diffusion.

Final remark

Similar effects play a role on higher-dimensional spatial domain.

In higher dimensions, we can compute the Mesh Péclet number as

$$\mathrm{Pe} = \frac{|\mathbf{v}| \max\{\Delta x, \Delta y, \ldots\}}{2\kappa}.$$

It is then also advisable to keep $|\mathrm{Pe}| \leq 1$ to avoid spurious oscillations.