Practical Course: Modeling, Simulation, Optimization

Week 6

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Contents

6.A The weak form in 2-D

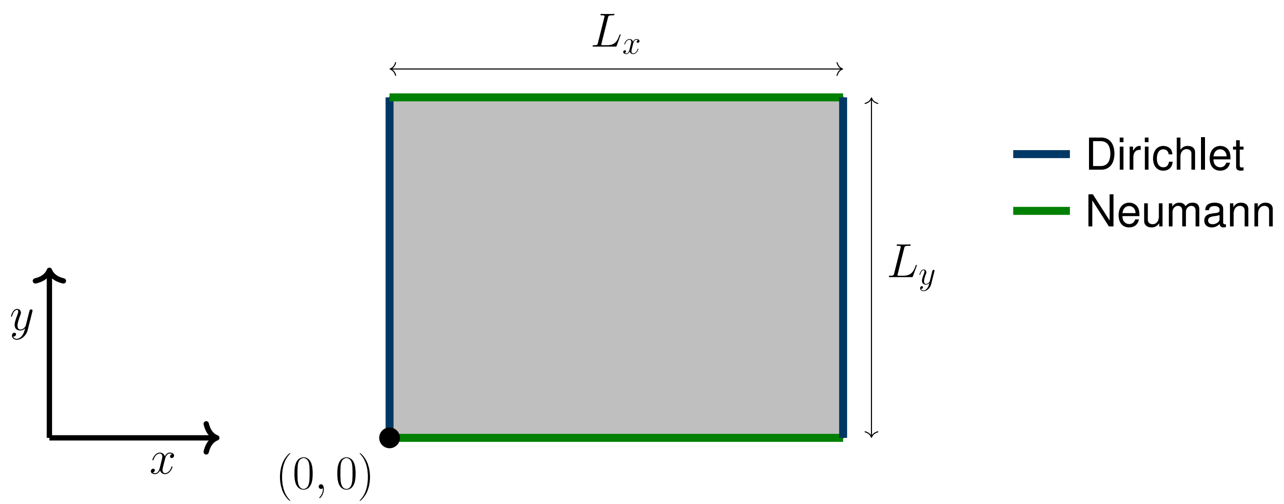
A sample problem

Consider the 2-D heat equation on $(x, y) \in [0, L_x] \times [0, L_y]$

$$\frac{\partial u}{\partial t}(t,x,y) = \kappa \left(\frac{\partial^2 u}{\partial x^2}(t,x,y) + \frac{\partial^2 u}{\partial y^2}(t,x,y) \right) + f(t,x,y),$$

$$u(t,0,y) = u(t,L_x,y) = 0, \qquad -\kappa \frac{\partial u}{\partial y}(t,x,0) = -\kappa \frac{\partial u}{\partial y}(t,x,L_y) = 0,$$

$$u(0,x,y) = u_0(x,y).$$



Integration by parts

Consider the 2-D heat equation on $(x,y) \in [0,L_x] \times [0,L_y]$

$$\frac{\partial u}{\partial t}(t,x,y) = \kappa \left(\frac{\partial^2 u}{\partial x^2}(t,x,y) + \frac{\partial^2 u}{\partial y^2}(t,x,y) \right) + f(t,x,y),$$
$$u(t,0,y) = u(t,L_x,y) = 0, \qquad -\kappa \frac{\partial u}{\partial y}(t,x,0) = -\kappa \frac{\partial u}{\partial y}(t,x,L_y) = 0,$$
$$u(0,x,y) = u_0(x,y).$$

Multiply by a test function $w(x,y)$ and integrate over $(x,y) \in [0,L_x] \times [0,L_y]$

- For the first term on the RHS, integration by parts over x shows that

$$\int_0^{L_y} \int_0^{L_x} w(x,y) \frac{\partial^2 u}{\partial x^2}(t,x,y)\, \mathrm{d}x\, \mathrm{d}y = \int_0^{L_y} w(x,y) \frac{\partial u}{\partial x}(t,x,y) \Big|_{x=0}^{L_x} \mathrm{d}y - \int_0^{L_y} \int_0^{L_x} \frac{\partial w}{\partial x}(x,y) \frac{\partial u}{\partial x}(t,x,y)\, \mathrm{d}x\, \mathrm{d}y$$

- For the first term on the RHS, integration by parts over y shows that

$$\int_0^{L_y} \int_0^{L_x} w(x,y) \frac{\partial^2 u}{\partial y^2}(t,x,y)\, \mathrm{d}x\, \mathrm{d}y = \int_0^{L_x} w(x,y) \frac{\partial u}{\partial y}(t,x,y)\, \mathrm{d}x \Big|_{y=0}^{L_y} - \int_0^{L_y} \int_0^{L_x} \frac{\partial w}{\partial y}(x,y) \frac{\partial u}{\partial y}(t,x,y)\, \mathrm{d}x\, \mathrm{d}y$$

The resulting weak form

Consider the 2-D heat equation on $(x, y) \in [0, L_x] \times [0, L_y]$

$$\frac{\partial u}{\partial t}(t,x,y) = \kappa \left(\frac{\partial^2 u}{\partial x^2}(t,x,y) + \frac{\partial^2 u}{\partial y^2}(t,x,y) \right) + f(t,x,y),$$
$$u(t,0,y) = u(t,L_x,y) = 0, \qquad -\kappa \frac{\partial u}{\partial y}(t,x,0) = -\kappa \frac{\partial u}{\partial y}(t,x,L_y) = 0,$$
$$u(0,x,y) = u_0(x,y).$$

A weak solution $u \in L^2([0,T], V)$ of the above problem satisfies

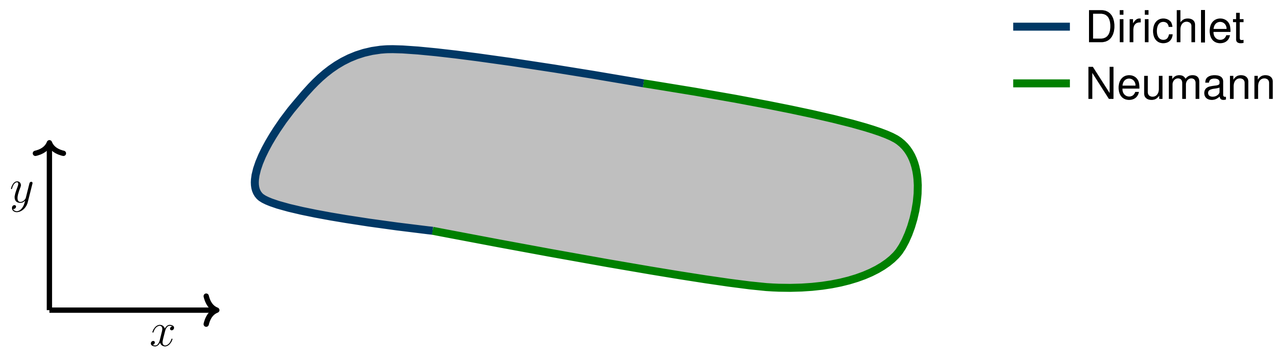
$$\int_0^{L^y} \int_0^{L^x} w(x,y) \frac{\partial u}{\partial t}(t,x,y)\, \mathrm{d}x\, \mathrm{d}y = -\kappa \int_0^{L_y} \int_0^{L_x} \frac{\partial w}{\partial x}(x,y) \frac{\partial u}{\partial x}(t,x,y)\, \mathrm{d}x\, \mathrm{d}y$$
$$- \kappa \int_0^{L_y} \int_0^{L_x} \frac{\partial w}{\partial y}(x,y) \frac{\partial u}{\partial y}(t,x,y)\, \mathrm{d}x\, \mathrm{d}y + \int_0^{L_y} \int_0^{L_x} w(x,y) f(t,x,y)\, \mathrm{d}x\, \mathrm{d}y$$
$$u(0,x,y) = u_0(x,y),$$

for all $w \in V = \{ w \in H^1([0,L_x] \times [0,L_y]) \mid w(0,\cdot) = w(L_x,\cdot) = 0 \}$ and a.a. $t \in [0,T]$.

A more general setting

Consider the 2-D heat equation on $(x, y) \in \Omega \subset \mathbb{R}^2$

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = \kappa \nabla^2 u + f, && (x, y) \in \Omega, t \in [0, T], \\
&u = 0, && (x, y) \in \partial\Omega_{\mathrm{D}}, t \in [0, T] \\
&-\kappa \nabla u \cdot \mathbf{n} = 0, && (x, y) \in \partial\Omega_{\mathrm{N}}, t \in [0, T], \\
&u(0) = u_0, && (x, y) \in \Omega.
\end{aligned}
$$



The resulting weak form

Consider the 2-D heat equation on $(x, y) \in \Omega \subset \mathbb{R}^2$

$$\begin{aligned} \frac{\partial u}{\partial t} &= \kappa \nabla^2 u + f, && (x,y) \in \Omega, t \in [0,T], \\ u &= 0, && (x,y) \in \partial\Omega_{\mathrm{D}}, t \in [0,T] \\ -\kappa \nabla u \cdot \mathbf{n} &= 0, && (x,y) \in \partial\Omega_{\mathrm{N}}, t \in [0,T], \\ u(0) &= u_0, && (x,y) \in \Omega. \end{aligned}$$

Multiply by a testfunction $w(x, y)$ and integrate over $(x, y) \in \Omega$.
For the first term on the RHS, we find using Green's first identity

$$\iint_\Omega w \nabla^2 u \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} w(\nabla u \cdot \mathbf{n}) \,\mathrm{d}S - \iint_\Omega \nabla w \cdot \nabla u \,\mathrm{d}x\,\mathrm{d}y.$$

A weak solution $u \in L^2([0, T], V)$ of the above problem satisfies

$$\iint_\Omega w \frac{\partial u}{\partial t} \,\mathrm{d}x\,\mathrm{d}y = -\kappa \iint_\Omega \nabla w \cdot \nabla u \,\mathrm{d}x\,\mathrm{d}y + \iint_\Omega w f \,\mathrm{d}x\,\mathrm{d}y, \qquad u(0) = u_0.$$

for all $w \in V = \{w \in H^1(\Omega) \mid w|_{\partial\Omega_{\mathrm{D}}} = 0\}$ and almost all $t \in [0, T]$.

6.B Galerkin discretization in 2-D

Galerkin discretization

We thus arrive at the weak formulation of our problem, for example

$$\iint_\Omega w \frac{\partial u}{\partial t} \,\mathrm{d}x \,\mathrm{d}y = -\kappa \iint_\Omega \nabla w \cdot \kappa \nabla u \,\mathrm{d}x \,\mathrm{d}y + \iint_\Omega w f \,\mathrm{d}x \,\mathrm{d}y$$

for all $w \in V = \{w \in H^1(\Omega) \mid w|_{\partial\Omega_\mathrm{D}} = 0\}$ and almost all $t \in [0, T]$.

The basic idea for a Galerkin discretization:
Replace the infinite dimensional space V by an N-dimensional subspace $V_N \subset V$.
Note: V_N must be a subspace of V.

This thus leads to a solution $u_N \in L^2(0, T; V_N)$ which satisfies

$$\iint_\Omega w_N \frac{\partial u_N}{\partial t} \,\mathrm{d}x \,\mathrm{d}y = -\kappa \iint_\Omega \nabla w_N \cdot \kappa \nabla u_N \,\mathrm{d}x \,\mathrm{d}y + \iint_\Omega w_N f \,\mathrm{d}x \,\mathrm{d}y, \qquad u(0) = u_0,$$

for all $w_N \in V_N$.

Galerkin approximation: a basis for V_N

We want to find the function $u_N \in L^2(0,T;V_N)$ which satisfies

$$\iint_\Omega w_N \frac{\partial u_N}{\partial t}\,\mathrm{d}x\,\mathrm{d}y = -\kappa \iint_\Omega \nabla w_N \cdot \kappa \nabla u_N\,\mathrm{d}x\,\mathrm{d}y + \iint_\Omega w_N f\,\mathrm{d}x\,\mathrm{d}y, \qquad u(0) = u_0,$$

for all $w_N \in V_N$.

Choose a basis $\{\mathbf{N}_1(x,y), \mathbf{N}_2(x,y), \dots, \mathbf{N}_N(x,y)\}$ for $V_N \subset V$ and define the row-vector

$$\mathbf{N}(x,y) = \begin{bmatrix} \mathbf{N}_1(x,y) & \mathbf{N}_2(x,y) & \cdots & \mathbf{N}_N(x,y) \end{bmatrix}.$$

Because $u_N \in L^2(0,T;V_N)$ and $w_N \in V_N$, we can write

$$u_N(t,x,y) = \sum_{n=1}^{N} \mathbf{N}_n(x,y) u_n(t) = \mathbf{N}(x,y)\mathbf{u}(t),$$

$$w_N(x,y) = \mathbf{N}(x,y)\mathbf{w} = \mathbf{w}^\top \left(\mathbf{N}(x,y)\right)^\top,$$

where $\mathbf{u} \in L^2(0,T;\mathbb{R}^N)$ and $\mathbf{w} \in \mathbb{R}^N$ is a column vector.

Galerkin approximation: Mass and stiffness matrices

We want to find the function $\mathbf{u} \in L^2(0,T;V_N)$ which satisfies for all $\mathbf{w} \in \mathbb{R}^N$

$$\iint_\Omega w_N \frac{\partial u_N}{\partial t} \,\mathrm{d}x\,\mathrm{d}y = -\kappa \iint_\Omega \nabla w_N \cdot \kappa \nabla u_N \,\mathrm{d}x\,\mathrm{d}y + \iint_\Omega w_N f \,\mathrm{d}x\,\mathrm{d}y, \qquad u(0) = u_0,$$

$$u_N(t,x,y) = \mathbf{N}(x,y)\mathbf{u}(t), \qquad w_N(x,y) = \mathbf{w}^\top (\mathbf{N}(x,y))^\top,$$

Substitute the expressions for u_N and w_N into the above equation:

$$\iint_\Omega \mathbf{w}^\top \mathbf{N}^\top \mathbf{N} \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \,\mathrm{d}x\,\mathrm{d}y = -\kappa \iint_\Omega \mathbf{w}^\top \nabla \mathbf{N}^\top \cdot \nabla \mathbf{N} \mathbf{u} \,\mathrm{d}x\,\mathrm{d}y + \iint_\Omega \mathbf{w}^\top \mathbf{N}^\top f \,\mathrm{d}x\,\mathrm{d}y,$$

Which can be rewritten as

$$\mathbf{w}^\top \mathbf{E} \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}(t) = \mathbf{w}^\top \mathbf{A}\mathbf{u}(t) + \mathbf{w}^\top \mathbf{f}(t), \qquad \mathbf{u}(0) = \mathbf{u}_0,$$

where

$$\mathbf{E} = \iint_\Omega \mathbf{N}^\top \mathbf{N} \,\mathrm{d}x\,\mathrm{d}y, \;\; \mathbf{A} = -\kappa \iint_\Omega \nabla \mathbf{N}^\top \cdot \nabla \mathbf{N} \,\mathrm{d}x\,\mathrm{d}y, \;\; \mathbf{f}(t) = \iint_\Omega \mathbf{N}^\top f(t) \,\mathrm{d}x\,\mathrm{d}y$$

Because this equation should be satisfied for all $\mathbf{w} \in \mathbb{R}^N$, we conclude

$$\mathbf{E} \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}(t) = \mathbf{A}\mathbf{u}(t) + \mathbf{f}(t), \qquad \mathbf{u}(0) = \mathbf{u}_0,$$

Question 1

We take $\Omega = [0,1] \times [0,1]$ and consider two shape functions:

$$\mathbf{N}(x,y) = \begin{bmatrix} x & y \end{bmatrix}.$$

Compute

$$\mathbf{E} = \iint_{\Omega} \mathbf{N}^\top \mathbf{N} \, \mathrm{d}x \, \mathrm{d}y.$$

A) $\mathbf{E} = \begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} \end{bmatrix}$

B) $\mathbf{E} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$

C) $\mathbf{E} = \begin{bmatrix} \frac{1}{3} & 1 \\ 1 & \frac{1}{3} \end{bmatrix}$

D) $\mathbf{E} = \frac{2}{3}$

E) None of the above

Question 2

We take $\Omega = [0,1] \times [0,1]$ and consider two shape functions:

$$\mathbf{N}(x,y) = \begin{bmatrix} x & y \end{bmatrix}.$$

Compute

$$\mathbf{A} = \iint_\Omega \nabla \mathbf{N}^\top \nabla \mathbf{N} \, \mathrm{d}x \, \mathrm{d}y.$$

A) $\mathbf{A} = \begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} \end{bmatrix}$

B) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

C) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

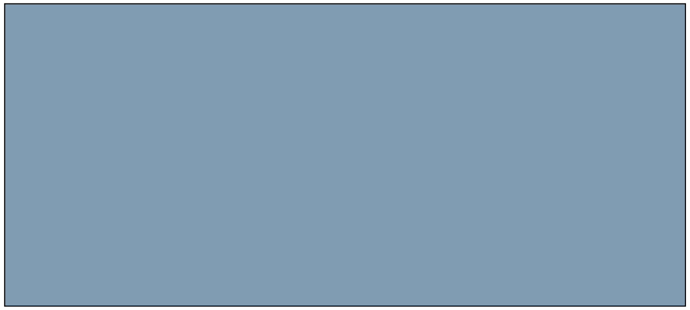
D) $\mathbf{A} = 2$

E) None of the above

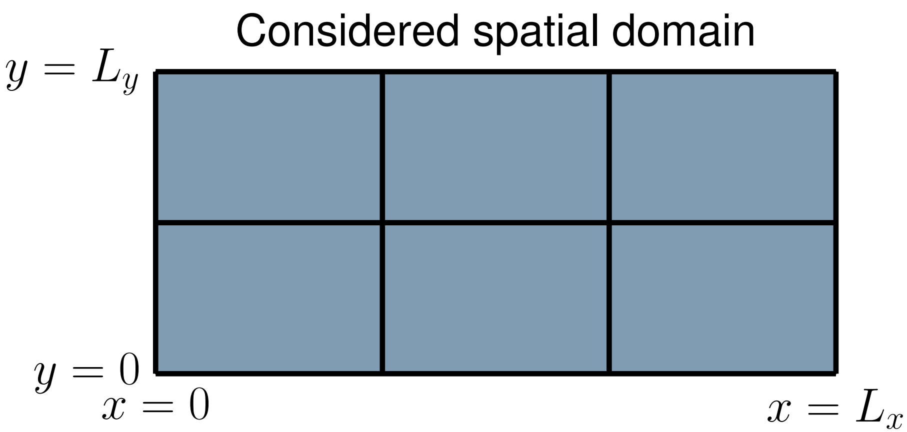
6.C Assembly procedure for 2-D Finite elements

2-D Finite Elements step-by-step

Considered spatial domain

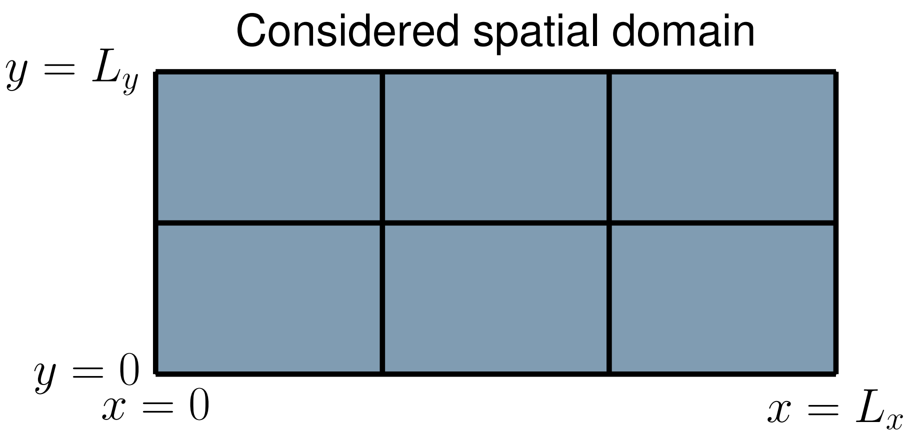


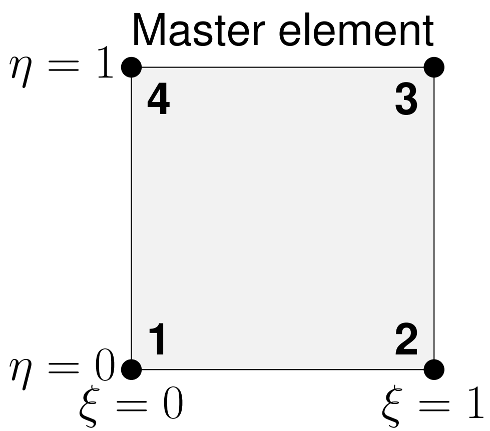
2-D Finite Elements step-by-step



STEP 1: Divide the domain $[0, L_x] \times [0, L_y]$ into M rectangular elements Ω^e.

2-D Finite Elements step-by-step

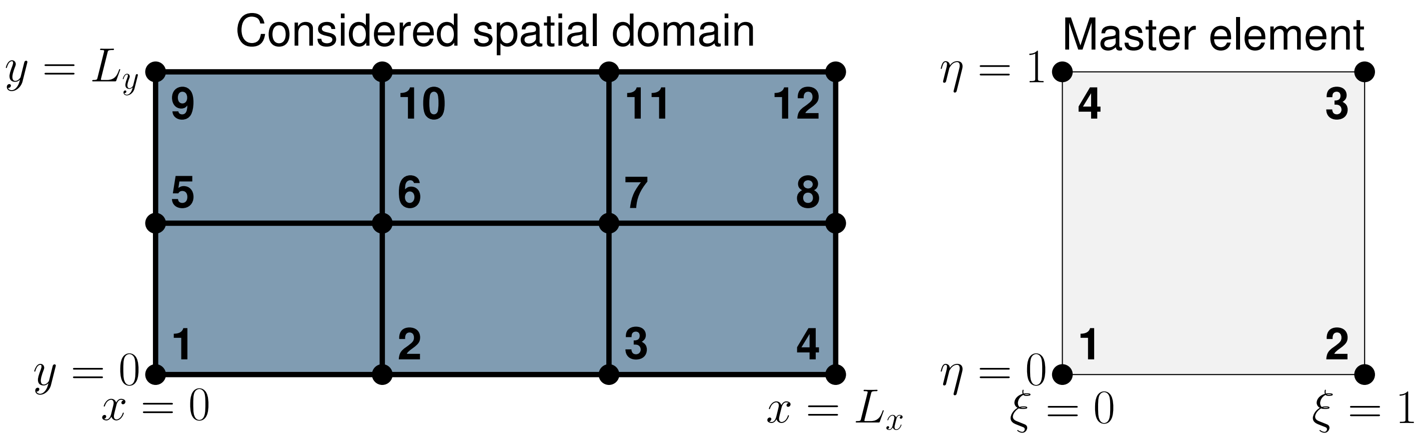




STEP 1: Divide the domain $[0, L_x] \times [0, L_y]$ into M rectangular elements Ω^e.
STEP 2: Choose shape functions for the master element

$$\mathbf{N}^e(\xi, \eta) = \begin{bmatrix} (1-\xi)(1-\eta) & \xi(1-\eta) & \xi\eta & (1-\xi)\eta \end{bmatrix}.$$

2-D Finite Elements step-by-step



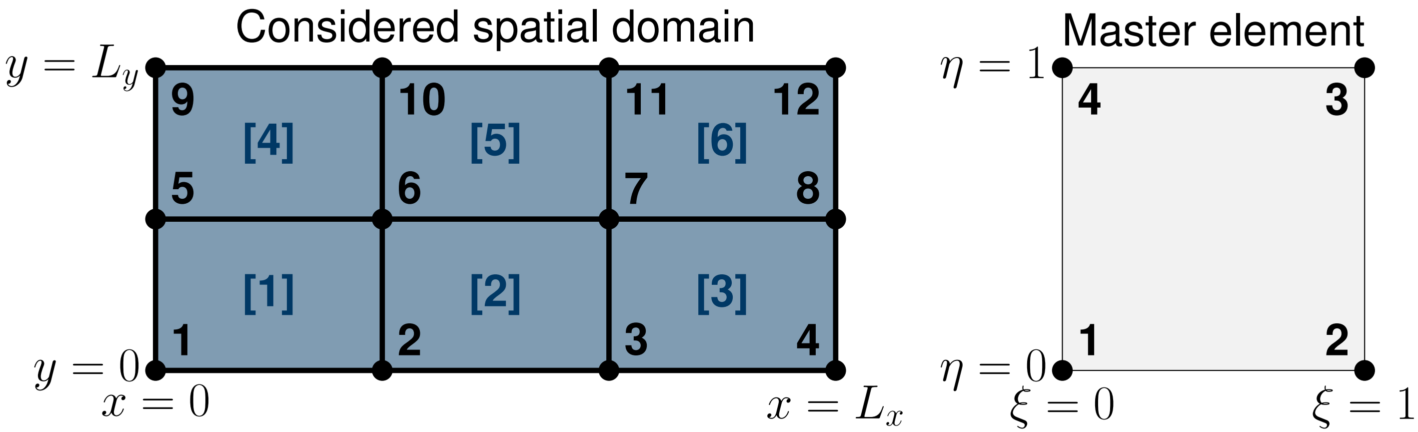
STEP 1: Divide the domain $[0, L_x] \times [0, L_y]$ into M rectangular elements Ω^e.
STEP 2: Choose shape functions for the master element

$$\mathbf{N}^e(\xi, \eta) = \begin{bmatrix} (1-\xi)(1-\eta) & \xi(1-\eta) & \xi\eta & (1-\xi)\eta \end{bmatrix}.$$

STEP 3: Define the nodes in the original domain based on the chosen master element. Assign a number to each node.

$$\texttt{node\_nmbrs} = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}.$$

The element list



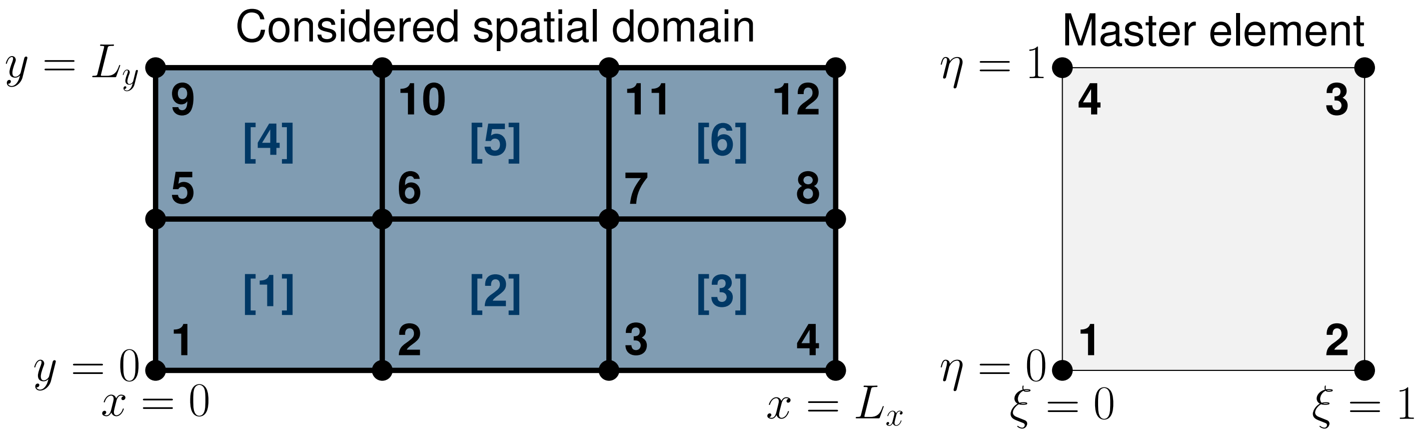
STEP 4: Build the element list.
The element list contains the numbers of the nodes in each each element.
The order in which elements are stored also assigns a number to each element.

$$\texttt{elem\_list} = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 2 & 3 & 7 & 6 \\ \vdots & \vdots & \vdots & \vdots \\ 7 & 8 & 12 & 11 \end{bmatrix}, \qquad \texttt{elem\_nmbrs} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Note: the ordering of the node numbers should match with the master element!

QUESTION 3: What is the fifth row in the matrix `elem_list`?

Element matrices (uniform mesh)



STEP 5: when all elements are of the same size $L_{e,x} \times L_{e,y}$, we can compute the contributions of one element directly:

$$\tilde{\mathbf{E}}^e = \int_0^{L_{e,x}} \int_0^{L_{e,y}} \left(\mathbf{N}^e\left(\tfrac{x}{L_{e,x}}, \tfrac{y}{L_{e,y}}\right)\right)^\top \mathbf{N}^e\left(\tfrac{x}{L_{e,x}}, \tfrac{y}{L_{e,y}}\right) \, \mathrm{d}x \, \mathrm{d}y$$

$$\tilde{\mathbf{A}}^e = \int_0^{L_{e,x}} \int_0^{L_{e,y}} \left(\frac{\partial \mathbf{N}^e\left(\tfrac{x}{L_{e,x}}, \tfrac{y}{L_{e,y}}\right)}{\partial x}\right)^\top \frac{\mathbf{N}^e\left(\tfrac{x}{L_{e,x}}, \tfrac{y}{L_{e,y}}\right)}{\partial x} + \left(\frac{\partial \mathbf{N}^e\left(\tfrac{x}{L_{e,x}}, \tfrac{y}{L_{e,y}}\right)}{\partial y}\right)^\top \frac{\mathbf{N}^e\left(\tfrac{x}{L_{e,x}}, \tfrac{y}{L_{e,y}}\right)}{\partial y} \, \mathrm{d}x \, \mathrm{d}y$$

$$\tilde{\mathbf{f}}^e = \int_0^{L_{e,x}} \int_0^{L_{e,y}} \left(\mathbf{N}^e\left(\tfrac{x}{L_{e,x}}, \tfrac{y}{L_{e,y}}\right)\right)^\top \, \mathrm{d}x \, \mathrm{d}y$$

Note: these formulas depend on the size of the elements $L_{e,x} \times L_{e,y}$!

Remark: relation to the standard element

Using the change of variables

$$(\xi, \eta) = \left(\frac{x}{L_{e,x}}, \frac{y}{L_{e,y}}\right), \qquad (x, y) = (L_{e,x}\xi, L_{e,y}\eta),$$

we can relate the integrals from the previous slide to the standard element $[0, 1]^2$

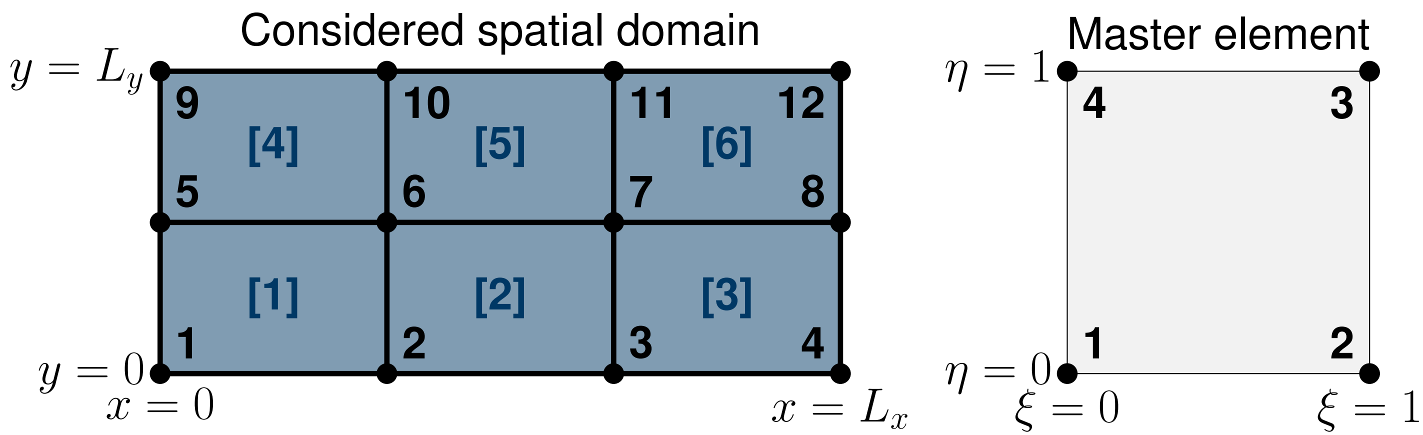
$$\tilde{\mathbf{E}}^e = L_{e,x}L_{e,y} \int_0^1 \int_0^1 (\mathbf{N}^e(\xi,\eta))^\top \mathbf{N}^e(\xi,\eta) \, \mathrm{d}\xi \, \mathrm{d}\eta =: L_{e,x}L_{e,y}\mathbf{E}^e,$$

$$\begin{aligned} \tilde{\mathbf{A}}^e = {} & \frac{L_{e,y}}{L_{e,x}} \int_0^1 \int_0^1 \left(\frac{\partial \mathbf{N}^e(\xi,\eta)}{\partial \xi}\right)^\top \frac{\mathbf{N}^e(\xi,\eta)}{\partial \xi} \, \mathrm{d}\xi \, \mathrm{d}\eta \\ & + \frac{L_{e,x}}{L_{e,y}} \int_0^1 \int_0^1 \left(\frac{\partial \mathbf{N}^e(\xi,\eta)}{\partial \eta}\right)^\top \frac{\mathbf{N}^e(\xi,\eta)}{\partial \eta} \, \mathrm{d}\xi \, \mathrm{d}\eta = \frac{L_{e,y}}{L_{e,x}} \mathbf{A}^e_{xx} + \frac{L_{e,x}}{L_{e,y}} \mathbf{A}^e_{yy}, \end{aligned}$$

$$\tilde{\mathbf{f}}^e = L_{e,x}L_{e,y} \int_0^1 \int_0^1 (\mathbf{N}^e(\xi,\eta))^\top \, \mathrm{d}\xi \, \mathrm{d}\eta =: L_{e,x}L_{e,y}\mathbf{f}^e.$$

Note: the two parts of $\tilde{\mathbf{A}}^e$ are scaled differently!

Assembly



STEP 6: Assemble the global mass and stiffness matrices $\mathbf{E}$ and $\mathbf{A}$ $(N \times N)$ and the global load vector $\mathbf{f}$ (length N) using the element list from STEP 4.

$$\texttt{elem\_list} = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 2 & 3 & 7 & 6 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Write the contribution of each element:

$$\mathbf{E}[[1,2,6,5],[1,2,6,5]] = \tilde{\mathbf{E}}^{e=1}, \qquad \mathbf{E}[[2,3,7,6],[2,3,7,6]] = \tilde{\mathbf{E}}^{e=2}, \qquad \ldots$$

Boundary conditions

STEP 7: Include the contributions of Robin boundary conditions.
Robin boundary conditions originate from terms like

STEP 8: Take into account (zero) Dirichlet boundary conditions by removing rows and columns corresponding to the constrained DOFs from $\mathbf{E}$, $\mathbf{A}$, and $\mathbf{f}$.

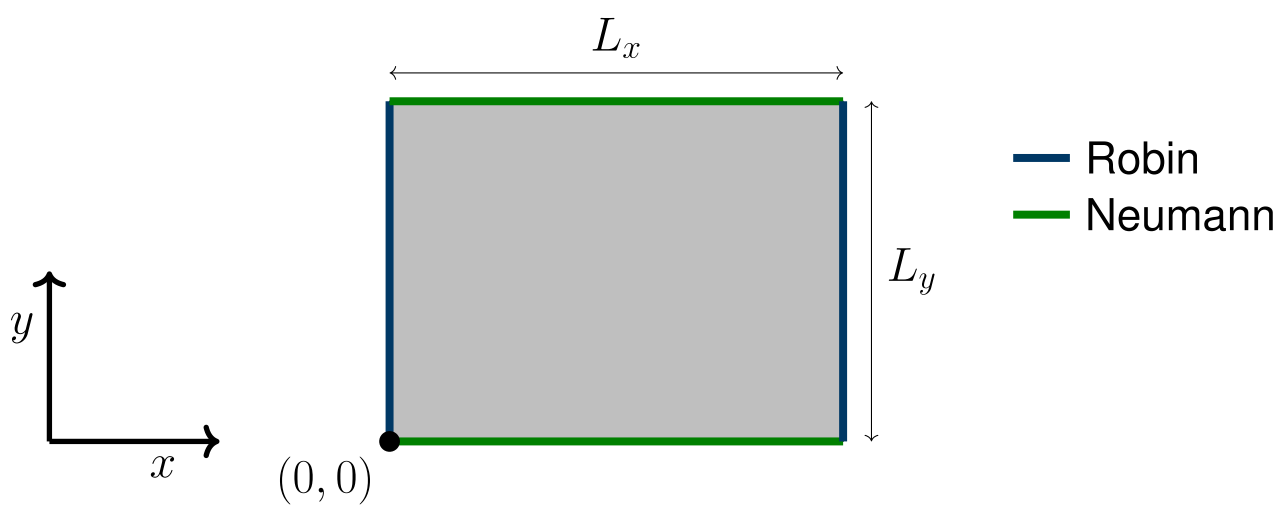
Example: boundary conditions

Consider the 2-D heat equation on $(x, y) \in [0, L_x] \times [0, L_y]$

$$0 = \kappa \left(\frac{\partial^2 u}{\partial x^2}(t, x, y) + \frac{\partial^2 u}{\partial y^2}(t, x, y) \right) + f(t, x, y),$$

$$u(t, 0, y) = u(t, L_x, y) = 0, \qquad \kappa \frac{\partial u}{\partial y}(t, x, 0) = hu(t, x, 0), \qquad -\kappa \frac{\partial u}{\partial y}(t, x, L_y) = hu(t, x, L_x)$$

$$u(0, x, y) = u_0(x, y).$$



Example: boundary conditions

We obtain the following weak form:

$$0 = -h \int_0^{L_x} \left([vu]_{y=0} + [vu]_{y=L_y} \right) \, \mathrm{d}x$$
$$- \kappa \int_0^{L_x} \int_x^{L_y} \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} \right) \, \mathrm{d}y \, \mathrm{d}x + \int_0^{L_x} \int_x^{L_y} vf \, \mathrm{d}y \, \mathrm{d}x.$$

To include the first two terms in the stiffness matrix $\mathbf{A}$, we need the element matrices

$$\tilde{\mathbf{E}}^e_{\mathrm{bot}} = \int_0^{L_{e,x}} \left(\mathbf{N}^e(\tfrac{x}{L_{e,x}}, 0) \right)^\top \mathbf{N}^e(\tfrac{x}{L_{e,x}}, 0) \, \mathrm{d}x,$$

$$\tilde{\mathbf{E}}^e_{\mathrm{top}} = \int_0^{L_{e,x}} \left(\mathbf{N}^e(\tfrac{x}{L_{e,x}}, 1) \right)^\top \mathbf{N}^e(\tfrac{x}{L_{e,x}}, 1) \, \mathrm{d}x.$$

For Robin BCs on the edges $x = 0$ and $x = L_{e,x}$ you would also need

$$\tilde{\mathbf{E}}^e_{\mathrm{left}} = \int_0^{L_{e,y}} \left(\mathbf{N}^e(0, \tfrac{y}{L_{e,y}}) \right)^\top \mathbf{N}^e(0, \tfrac{y}{L_{e,y}}) \, \mathrm{d}y,$$

$$\tilde{\mathbf{E}}^e_{\mathrm{right}} = \int_0^{L_{e,y}} \left(\mathbf{N}^e(1, \tfrac{x}{L_{e,y}}) \right)^\top \mathbf{N}^e(1, \tfrac{y}{L_{e,y}}) \, \mathrm{d}y.$$

Remark: relation to the standard element

Using the transformation

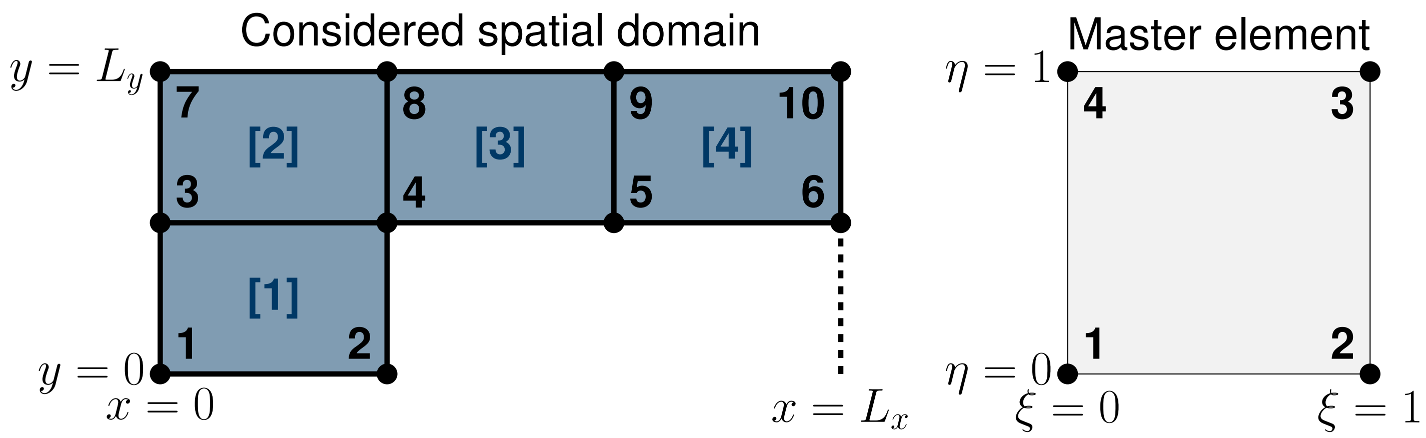
$$(\xi, \eta) = \left(\frac{x}{L_{e,x}}, \frac{y}{L_{e,y}} \right), \qquad (x, y) = (L_{e,x}\xi, L_{e,y}\eta),$$

we can relate the integrals from the previous slide to the standard element $[0, 1]^2$

$$\begin{aligned}
\tilde{\mathbf{E}}^e_{\text{bot}} &= L_{e,x} \int_0^1 \left(\mathbf{N}^e(\xi, 0)\right)^\top \mathbf{N}^e(\xi, 0)\, \mathrm{d}\xi = L_{e,x} \mathbf{E}^e_{\text{bot}}, \\
\tilde{\mathbf{E}}^e_{\text{top}} &= L_{e,x} \int_0^1 \left(\mathbf{N}^e(\xi, 1)\right)^\top \mathbf{N}^e(\xi, 1)\, \mathrm{d}\xi = L_{e,x} \mathbf{E}^e_{\text{top}}, \\
\tilde{\mathbf{E}}^e_{\text{left}} &= L_{e,y} \int_0^1 \left(\mathbf{N}^e(0, \eta)\right)^\top \mathbf{N}^e(0, \eta)\, \mathrm{d}\eta = L_{e,y} \mathbf{E}^e_{\text{left}}, \\
\tilde{\mathbf{E}}^e_{\text{right}} &= L_{e,y} \int_0^1 \left(\mathbf{N}^e(1, \eta)\right)^\top \mathbf{N}^e(1, \eta)\, \mathrm{d}\eta = L_{e,y} \mathbf{E}^e_{\text{right}}.
\end{aligned}$$

6.D Three final remarks

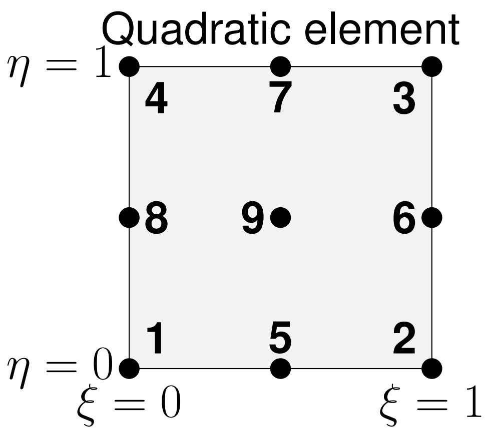
Remark 1/3: Domains that are not rectangular

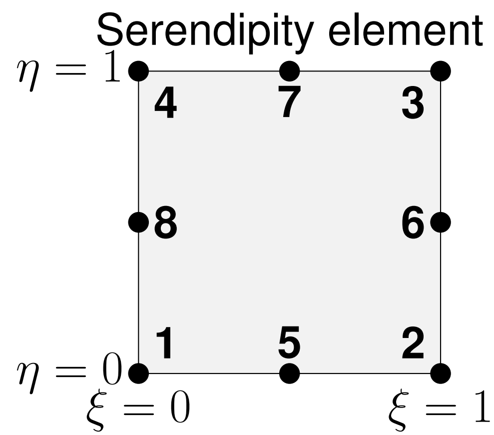


Only number the elements and nodes inside the considered domain.

$$\texttt{node\_nmbrs} = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 4 & 8 \\ 0 & 5 & 9 \\ 0 & 6 & 10 \end{bmatrix}, \qquad \texttt{elem\_list} = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 3 & 4 & 8 & 7 \\ 4 & 5 & 9 & 8 \\ 5 & 6 & 10 & 9 \end{bmatrix}$$

Remark 2/3: Second order elements





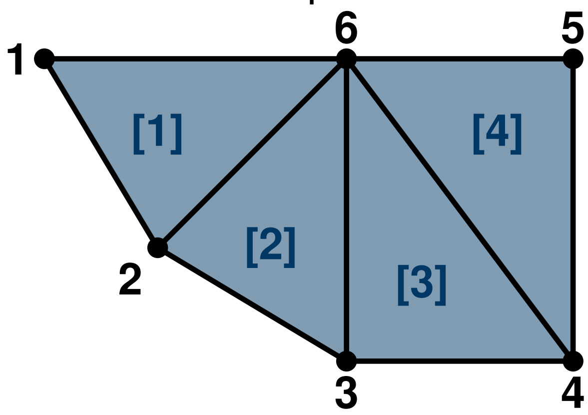
There are two commonly used quadratic shape functions on rectangular elements:

$$
\mathbf{N}^e(\xi,\eta) = \begin{bmatrix} p_0(\xi)p_0(\eta) \\ p_1(\xi)p_0(\eta) \\ p_1(\xi)p_1(\eta) \\ p_0(\xi)p_1(\eta) \\ p_{1/2}(\xi)p_0(\eta) \\ p_1(\xi)p_{1/2}(\eta) \\ p_{1/2}(\xi)p_1(\eta) \\ p_0(\xi)p_{1/2}(\eta) \\ p_{1/2}(\xi)p_{1/2}(\eta) \end{bmatrix}^\top, \qquad \mathbf{N}^e(\xi,\eta) = \begin{bmatrix} p_0(\xi)p_0(\eta) \\ p_1(\xi)p_0(\eta) \\ p_1(\xi)p_1(\eta) \\ p_0(\xi)p_1(\eta) \\ p_{1/2}(\xi)p_0(\eta) \\ p_1(\xi)p_{1/2}(\eta) \\ p_{1/2}(\xi)p_1(\eta) \\ p_0(\xi)p_{1/2}(\eta) \end{bmatrix}^\top,
$$

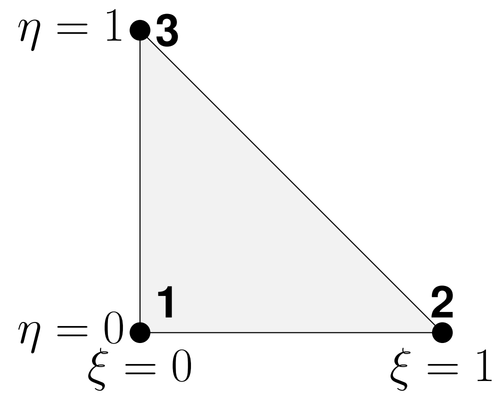
where $p_0(\xi) = (1-\xi)(1-2\xi)$, $p_{1/2}(\xi) = 4(1-\xi)\xi$, and $p_1(\xi) = (2\xi-1)\xi$.

Remark 3/3: Nonrectangular meshes

Considered spatial domain



Master element



We can no longer use the matrix `node_nmbrs` to assign numbers to the nodes. Instead we make a node list:

$$\texttt{node\_list} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_5 & y_5 \\ x_6 & y_6 \end{bmatrix}, \qquad \texttt{elem\_list} = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 3 & 6 \\ 3 & 4 & 6 \\ 4 & 5 & 6 \end{bmatrix},$$

where (x_i, y_i) is the position of node i.