Chair
DYNAMICS, CONTROL
AND NUMERICS
FAU

Department of
DATA SCIENCE

FRIEDRICH-ALEXANDER
UNIVERSITÄT
ERLANGEN-NÜRNBERG
NATURWISSENSCHAFTLICHE
FAKULTÄT

Practical Course: Modeling, Simulation, Optimization

Week 7

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Chair in Dynamics, Control, and Numerics, Friedrich-Alexander-University Erlangen-Nürnberg

Contents

7.A Elasticity

Elasticity

Goal: Compute the deformation of a solid that is subjected to certain given forces.

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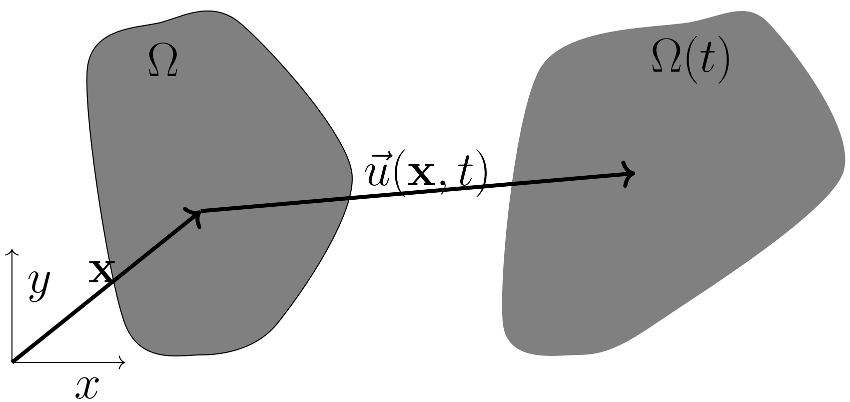
The deformation of a solid is characterized by the displacement field

$$\vec{u}(x, y, t) = \begin{bmatrix} u_x(x, y, t) \\ u_y(x, y, t) \end{bmatrix}.$$



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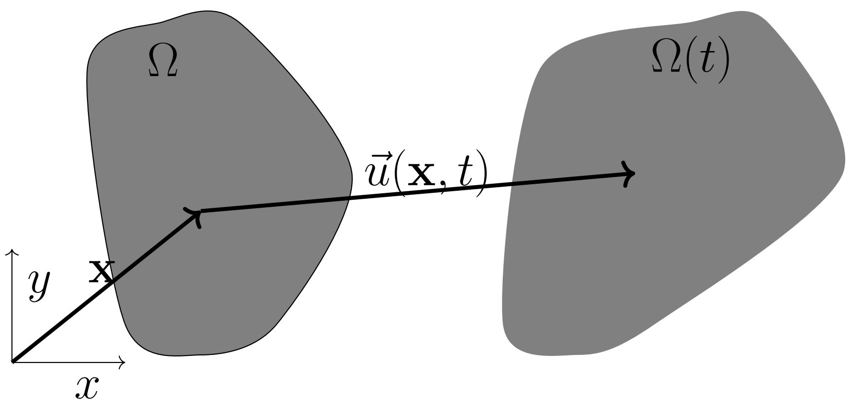


Goal: compute $\vec{u}(x, y, t)$.
Equations essentially follow from Newton's second law

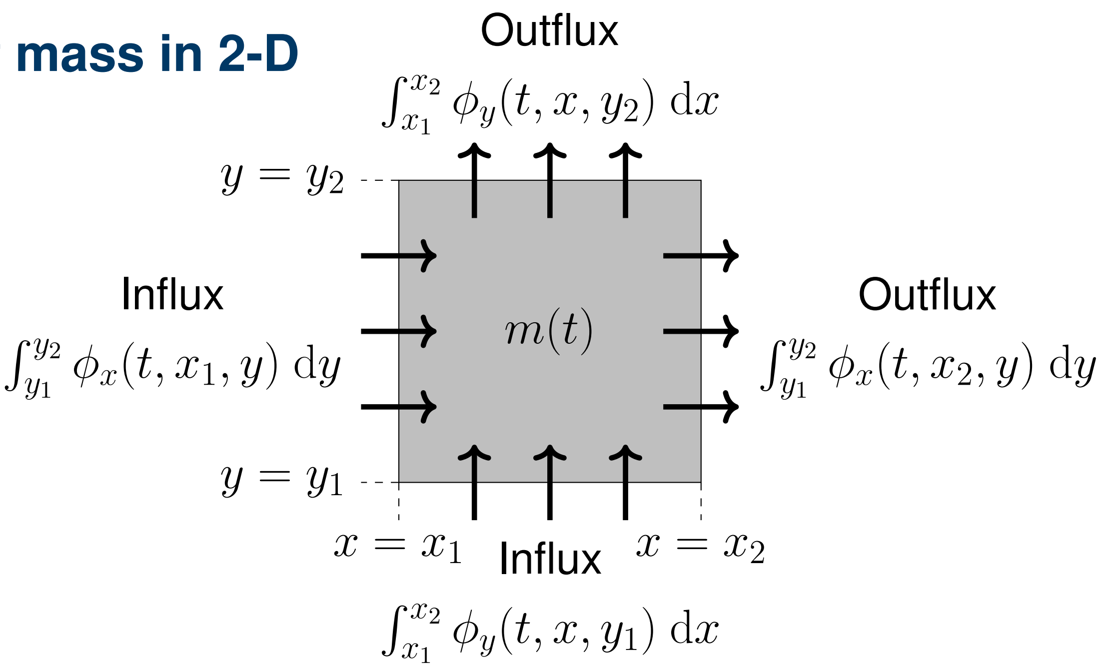
$$\mathbf{F} = m\mathbf{a}, \qquad \begin{bmatrix} F_x \\ F_y \end{bmatrix} = m \begin{bmatrix} a_x \\ a_y \end{bmatrix}.$$

For static problems, the acceleration $\mathbf{a}$ is zero

$$\mathbf{F} = \mathbf{0}, \qquad \begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

7.B The force balance

Recall: Conservation of mass in 2-D



$$\frac{\partial m}{\partial t}(t,x,y) = \int_{y_1}^{y_2} \left(\phi_x(t,x_1,y) - \phi_x(t,x_2,y)\right)\, \mathrm{d}y + \int_{x_1}^{x_2} \left(\phi_y(t,x,y_1) - \phi_y(t,x,y_2)\right)\, \mathrm{d}x$$
$$= -\int_{y_1}^{y_2} \int_{x_1}^{x_2} \left(\frac{\partial \phi_x}{\partial x}(t,x,y) + \frac{\partial \phi_y}{\partial y}(t,x,y) \right) \mathrm{d}x\, \mathrm{d}y$$

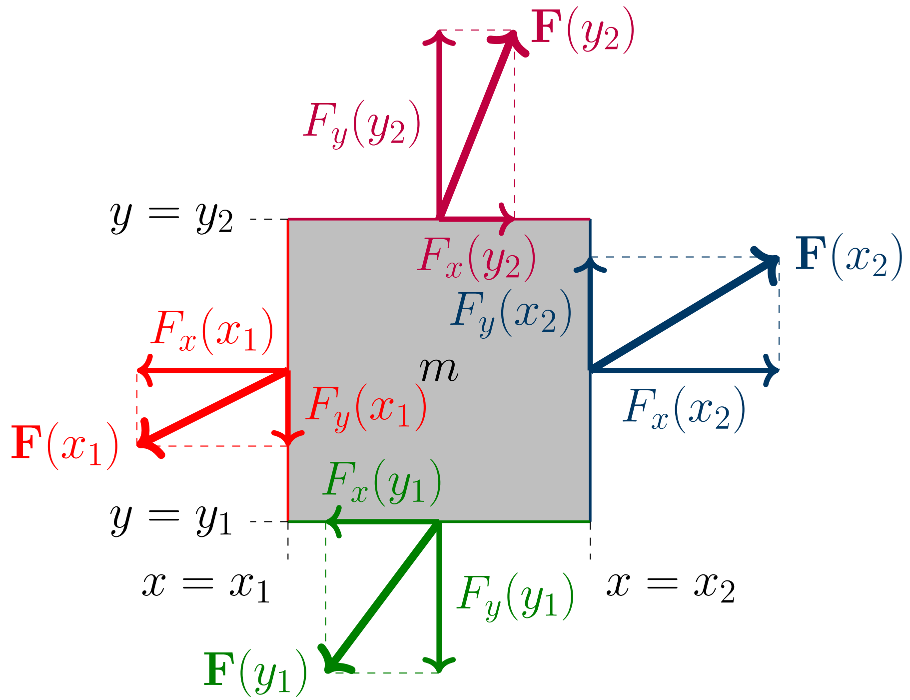
Because $m(t) = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \rho(t,x,y)\, \mathrm{d}x\, \mathrm{d}y$ and $[x_1,x_2] \times [y_1,y_2]$ is arbitrary:

Conservation of mass in 2-D

$$\frac{\partial \rho}{\partial t}(t,x,y) = -\frac{\partial \phi_x}{\partial x}(t,x,y) - \frac{\partial \phi_y}{\partial y}(t,x,y).$$

The force balance (1/2)

Consider an arbitrary square inside the domain of interest.



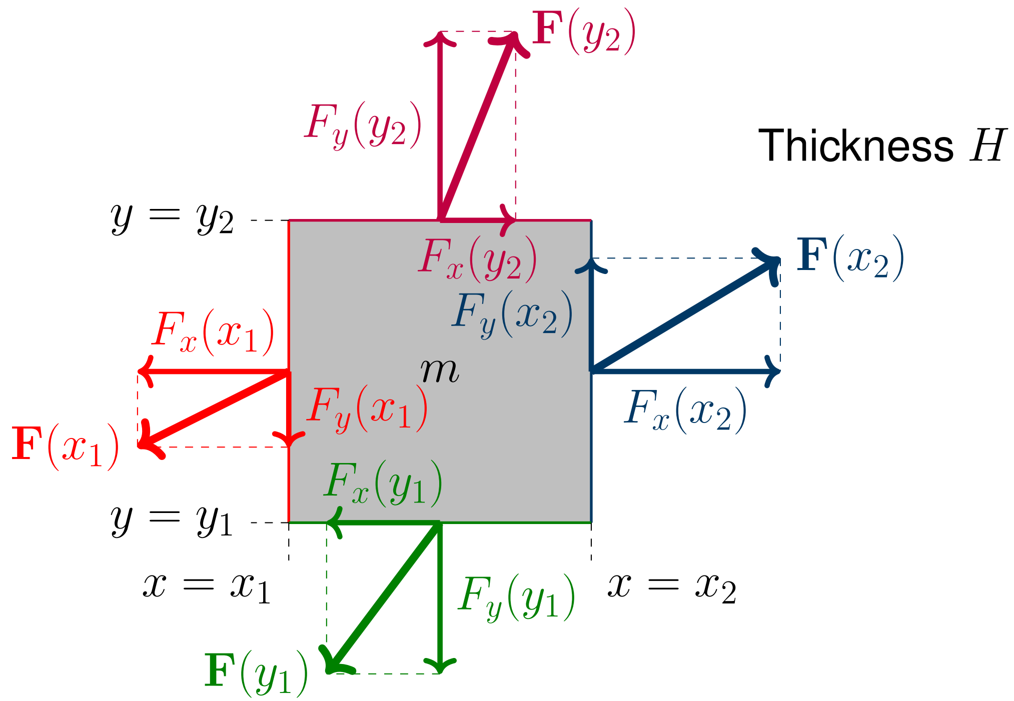
Force balance:

$$F_x(x_2) - F_x(x_1) + F_x(y_2) - F_x(y_1) = ma_x$$
$$F_y(x_2) - F_y(x_1) + F_y(y_2) - F_y(y_1) = ma_y,$$

or in vector form

$$\mathbf{F}_x(x_2) - \mathbf{F}_x(x_1) + \mathbf{F}_x(y_2) - \mathbf{F}_x(y_1) = m\mathbf{a}$$

The stress tensor σ



A famous theorem by Cauchy asserts that there exist functions $\sigma_{xx}(x,y)$, $\sigma_{xy}(x,y)$, $\sigma_{yx}(x,y)$, and $\sigma_{yy}(x,y)$ such that

$$\mathbf{F}(x_2) = H\int_{y_1}^{y_2} \begin{bmatrix} \sigma_{xx}(x_2,y) \\ \sigma_{yx}(x_2,y) \end{bmatrix} \mathrm{d}y, \qquad \mathbf{F}(x_1) = H\int_{y_1}^{y_2} \begin{bmatrix} \sigma_{xx}(x_1,y) \\ \sigma_{yx}(x_1,y) \end{bmatrix} \mathrm{d}y,$$

$$\mathbf{F}(y_2) = H\int_{x_1}^{x_2} \begin{bmatrix} \sigma_{xy}(x,y_2) \\ \sigma_{yy}(x,y_2) \end{bmatrix} \mathrm{d}x, \qquad \mathbf{F}(y_1) = H\int_{x_1}^{x_2} \begin{bmatrix} \sigma_{xy}(x,y_1) \\ \sigma_{yy}(x,y_1) \end{bmatrix} \mathrm{d}x.$$

Question 1

$$\mathbf{F}(x_2) = H \int_{y_1}^{y_2} \begin{bmatrix} \sigma_{xx}(x_2, y) \\ \sigma_{yx}(x_2, y) \end{bmatrix} \, \mathrm{d}y, \qquad \mathbf{F}(x_1) = H \int_{y_1}^{y_2} \begin{bmatrix} \sigma_{xx}(x_1, y) \\ \sigma_{yx}(x_1, y) \end{bmatrix} \, \mathrm{d}y,$$

$$\mathbf{F}(y_2) = H \int_{x_1}^{x_2} \begin{bmatrix} \sigma_{xy}(x, y_2) \\ \sigma_{yy}(x, y_2) \end{bmatrix} \, \mathrm{d}x, \qquad \mathbf{F}(y_1) = H \int_{x_1}^{x_2} \begin{bmatrix} \sigma_{xy}(x, y_1) \\ \sigma_{yy}(x, y_1) \end{bmatrix} \, \mathrm{d}x.$$

What is the unit of the stress components $\sigma_{xx}(x, y)$, $\sigma_{xy}(x, y)$, $\sigma_{yx}(x, y)$, and $\sigma_{yy}(x, y)$?

A) Nm

B) N

C) N/m

D) N/m^2

E) None of the above.

An important remark $\boldsymbol{\sigma}$

A famous theorem by Cauchy asserts that
there exist functions $\sigma_{xx}(x,y)$, $\sigma_{xy}(x,y)$, $\sigma_{yx}(x,y)$, and $\sigma_{yy}(x,y)$ such that

$$\mathbf{F}(x_2) = H\int_{y_1}^{y_2} \begin{bmatrix}\sigma_{xx}(x_2,y)\\ \sigma_{yx}(x_2,y)\end{bmatrix} \mathrm{d}y, \qquad \mathbf{F}(x_1) = H\int_{y_1}^{y_2} \begin{bmatrix}\sigma_{xx}(x_1,y)\\ \sigma_{yx}(x_1,y)\end{bmatrix} \mathrm{d}y,$$

$$\mathbf{F}(y_2) = H\int_{x_1}^{x_2} \begin{bmatrix}\sigma_{xy}(x,y_2)\\ \sigma_{yy}(x,y_2)\end{bmatrix} \mathrm{d}x, \qquad \mathbf{F}(y_1) = H\int_{x_1}^{x_2} \begin{bmatrix}\sigma_{xy}(x,y_1)\\ \sigma_{yy}(x,y_1)\end{bmatrix} \mathrm{d}x.$$

By introducing the matrix (stress tensor)

$$\boldsymbol{\sigma}(x,y) = \begin{bmatrix}\sigma_{xx}(x,y) & \sigma_{xy}(x,y)\\ \sigma_{yx}(x,y) & \sigma_{yy}(x,y)\end{bmatrix}.$$

we can rewrite these equations as

$$\mathbf{F}(x_2) = H\int_{y_1}^{y_2} \boldsymbol{\sigma}(x_2,y)\mathbf{n}\,\mathrm{d}y, \qquad \mathbf{F}(x_1) = -H\int_{y_1}^{y_2} \boldsymbol{\sigma}(x_1,y)\mathbf{n}\,\mathrm{d}y,$$

$$\mathbf{F}(y_2) = H\int_{x_1}^{x_2} \boldsymbol{\sigma}(x,y_2)\mathbf{n}\,\mathrm{d}x, \qquad \mathbf{F}(y_1) = -H\int_{x_1}^{x_2} \boldsymbol{\sigma}(x,y_1)\mathbf{n}\,\mathrm{d}x.$$

Conservation of angular momentum shows that $\boldsymbol{\sigma}(x,y)$ must be symmetric,
i.e. that $\sigma_{xy}(x,y) = \sigma_{yx}(x,y)$

The force balance (2/2)

From the previous slides, we have

$$\mathbf{F}(x_2) - \mathbf{F}(x_1) + \mathbf{F}(y_2) - \mathbf{F}(y_1) = m\mathbf{a}$$

$$\mathbf{F}(x_2) = H \int_{y_1}^{y_2} \begin{bmatrix} \sigma_{xx}(x_2, y) \\ \sigma_{yx}(x_2, y) \end{bmatrix} \, \mathrm{d}y, \qquad \mathbf{F}(x_1) = H \int_{y_1}^{y_2} \begin{bmatrix} \sigma_{xx}(x_1, y) \\ \sigma_{yx}(x_1, y) \end{bmatrix} \, \mathrm{d}y,$$

$$\mathbf{F}(y_2) = H \int_{x_1}^{x_2} \begin{bmatrix} \sigma_{xy}(x, y_2) \\ \sigma_{yy}(x, y_2) \end{bmatrix} \, \mathrm{d}x, \qquad \mathbf{F}(y_1) = H \int_{x_1}^{x_2} \begin{bmatrix} \sigma_{xy}(x, y_1) \\ \sigma_{yy}(x, y_1) \end{bmatrix} \, \mathrm{d}x.$$

Using the fundamental theorem of calculus

$$\mathbf{F}_x(x_2) - \mathbf{F}_x(x_1) = H \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{\partial}{\partial x} \begin{bmatrix} \sigma_{xx}(x, y) \\ \sigma_{yx}(x, y) \end{bmatrix} \, \mathrm{d}x \, \mathrm{d}y$$

$$\mathbf{F}_x(y_2) - \mathbf{F}_x(y_1) = H \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{\partial}{\partial y} \begin{bmatrix} \sigma_{xy}(x, y) \\ \sigma_{yy}(x, y) \end{bmatrix} \, \mathrm{d}y \, \mathrm{d}x.$$

We thus find the following equations for the **static force balance**

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \qquad \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0.$$

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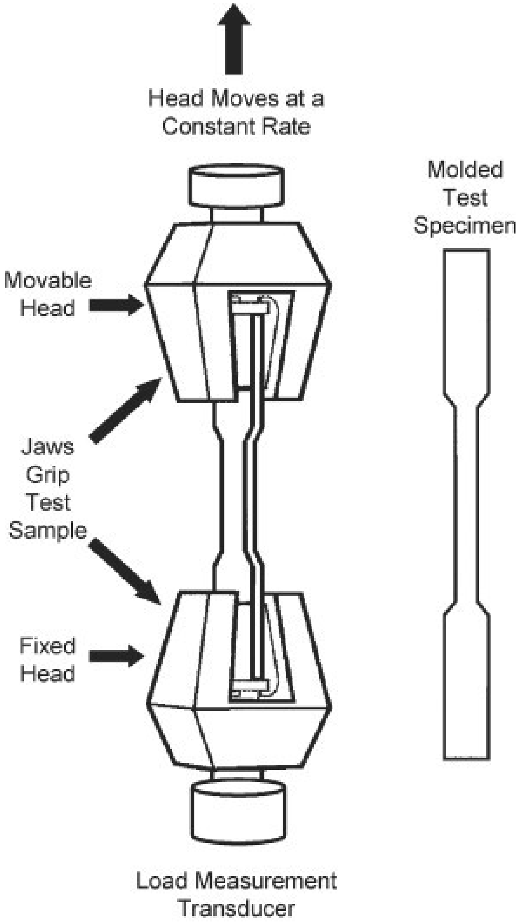
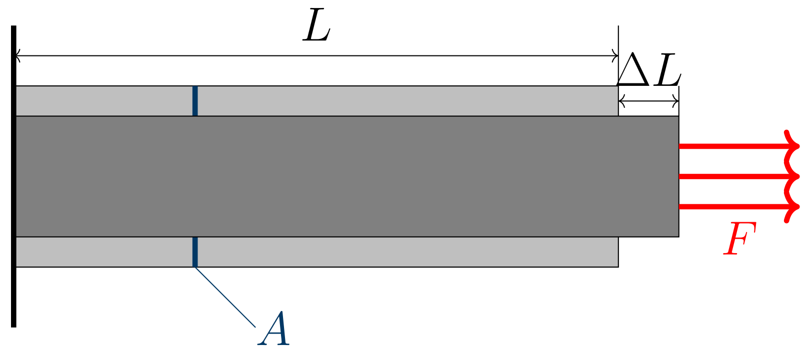
7.C Material models for linear elasticity

Tensile test

Measure the force F required to extend the specimen at a constant rate $\frac{d}{dt}\Delta L$.



Stress and strain



- When the cross section of the specimen A is twice as high,
  the force F (required for the same extension ΔL) will also be twice as high.

  Conclusion: The extension ΔL of the rod depends on the **stress**

$$\sigma_{xx} = \frac{F}{A}.$$

- When the length of the specimen L is twice as big,
  the extension of the specimen ΔL (for the same force F) will also be twice as high.

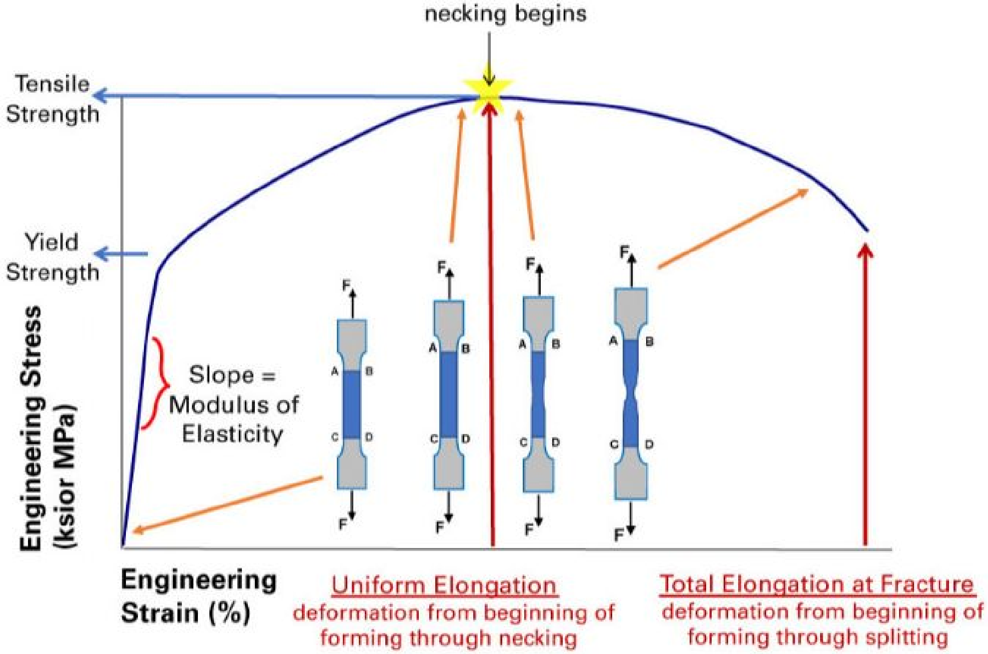
  Conclusion: The required force F depends on the **strain**

$$\varepsilon_{xx} = \frac{\Delta L}{L}.$$

Combining these two ideas we conclude:

We in fact measure the relation between stress and strain in the tensile test

Typical measurement in a tensile test



In linear elasticity (small deformations, stress below the yield stress):

$$\sigma_{xx} = E\varepsilon_{xx},$$

where E is the **Young's modulus** (also sometimes the modulus of elasticity).
Note: E only depends on the used material!

Question 2

$$\sigma_{xx} = \frac{F}{A}, \qquad \varepsilon_{xx} = \frac{\Delta L}{L}, \qquad \sigma_{xx} = E\varepsilon_{xx}.$$

What is the unit of the Young's modulus E?

A) N

B) N / m

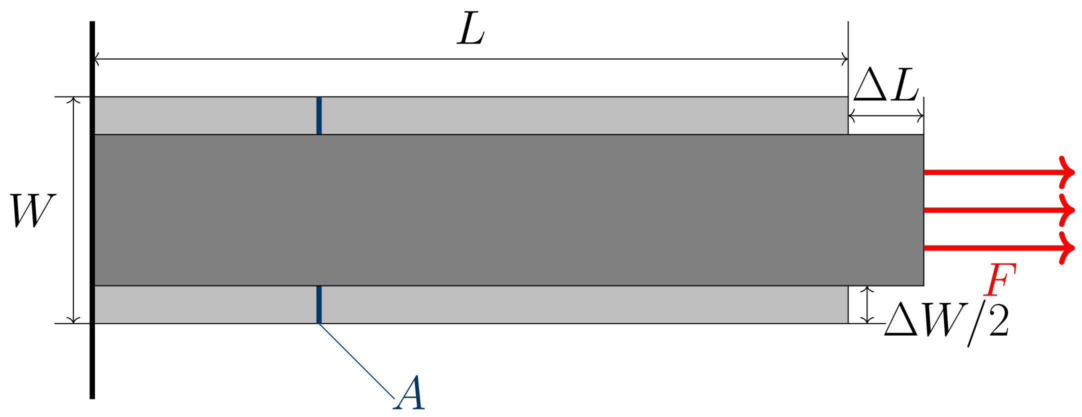
C) N / m^2

D) N / m^3

E) None of the above.

Poisson's ratio

When we pull the specimen, also the width changes!



- When the width of the specimen W is twice as big, the change in width ΔW (for the same force F) will also be twice as high.

  Conclusion: The required force F depends on the **strain**

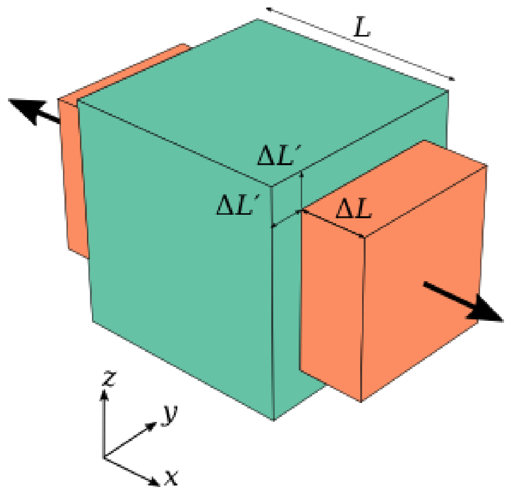
$$\varepsilon_{yy} = \frac{-\Delta W}{W}.$$

We can now define **Poisson's ratio**

$$\nu = \frac{-\varepsilon_{yy}}{\varepsilon_{xx}} = \frac{\Delta W}{\Delta L}\frac{L}{W}$$

Note: ν only depends on the used material!

Question 3



In the undeformed situation, the cube is $L \times W \times W$ with volume $V_0 = LW^2$.
Write $\Delta W/2$ for the distance $\Delta L'$ in the picture.
Let V_1 denote the volume of the cube in the deformed situation?

A) $V_1/V_0 = (1 + \varepsilon_{xx})(1 + 2\nu\varepsilon_{xx})$

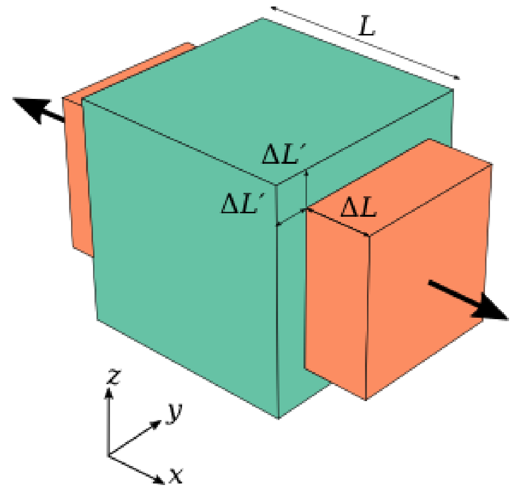
B) $V_1/V_0 = (1 + \varepsilon_{xx})(1 - 2\nu\varepsilon_{xx})$

C) $V_1/V_0 = (1 + \varepsilon_{xx})(1 + \nu\varepsilon_{xx})^2$

D) $V_1/V_0 = (1 + \varepsilon_{xx})(1 - \nu\varepsilon_{xx})^2$

E) None of the above.

Physical limits for Poisson's ratio



Volume in the deformed situation:

$$V_1 = (L + \Delta L)(W - \Delta W)^2 = L(1 + \varepsilon_{xx})W^2(1 - \varepsilon_{yy})^2 = V_0(1 + \varepsilon_{xx})(1 - \nu\varepsilon_{xx})^2.$$

Make a Taylor series expansion around $\varepsilon_{xx} = 0$,

$$V_1 = V_0(1 + (1 - 2\nu)\varepsilon_{xx}) + O(\varepsilon_{xx}^2).$$

Physical insight: we cannot have that $V_1 < V_0$ when $\varepsilon_{xx} > 0$.

Physical limits for Poisson's ratio

$$0 \leq \nu \leq \tfrac{1}{2}$$

The material is called **incompressible** when $\nu = \frac{1}{2}$.

Hooke's law for an isotropic material (1/2)

A **linear isotropic material** behaves the same in all directions.
The behavior of a linear isotropic material is completely characterized by E and ν.

Derivations on the previous slides were considering only loading in the x-direction:
$\sigma_{xx} = F/A$, $\sigma_{yy} = \sigma_{zz} = 0$, for which we found $\varepsilon_{xx} = \sigma_{xx}/E$ and $\varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx}$.

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Isotropic material $\Rightarrow$ similar relations for loading in the y- and z-directions. Linear material $\Rightarrow$ strains for loading in different directions can be added up.

Hooke's law (1/2)

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{bmatrix}.$$

Or, after inverting the matrix

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{bmatrix}.$$

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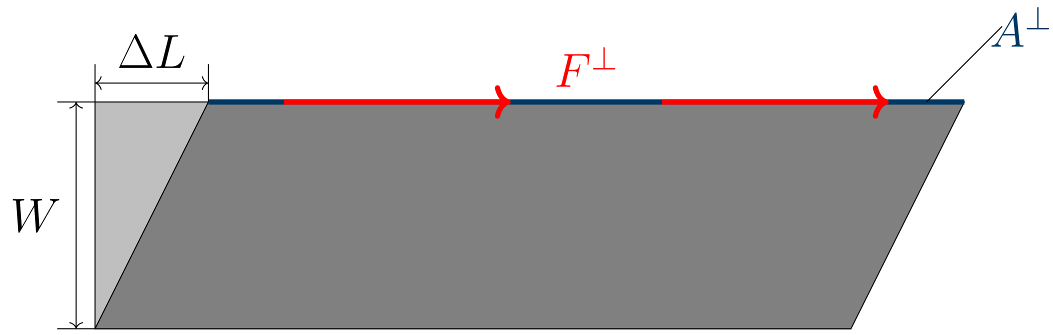
$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{bmatrix}.$$

A second experiment



- When the cross section of the specimen $A^{\perp}$ is twice as high,
  the force $F^{\perp}$ (required for the same extension ΔL) will also be twice as high.

  Conclusion: The extension ΔL of the rod depends on the **shear stress**

$$\sigma_{xy} = \frac{F^{\perp}}{A^{\perp}}.$$

- When the width of the specimen W is twice as big,
  the extension of the specimen ΔL (for the same force $F^{\perp}$) will also be twice as high.

  Conclusion: The required force $F^{\perp}$ depends on the **shear strain**

$$\varepsilon_{xy} = \frac{1}{2}\frac{\Delta L}{W}.$$

Hooke's law for an isotropic material (2/2)

The relation between σ_{xy} and ε_{xy} is given by the shear modulus G

$$\sigma_{xy} = G\varepsilon_{xy}.$$

Isotropic material $\Rightarrow$ we also have

$$\sigma_{yz} = G\varepsilon_{yz}, \qquad \sigma_{zx} = G\varepsilon_{zx}.$$

Note: we use that $\sigma_{xy} = \sigma_{yx}$, $\sigma_{yz} = \sigma_{zy}$, and $\sigma_{zx} = \sigma_{xz}$.
It can also be shown that $G = E/(1+\nu)$.

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Note: we use that $\sigma_{xy} = \sigma_{yx}$, $\sigma_{yz} = \sigma_{zy}$, and $\sigma_{zx} = \sigma_{xz}$.
It can also be shown that $G = E/(1+\nu)$.

We also have the following relations for the shear stresses and shear strains

$$\begin{bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix}.$$

Inverting this relation we also find

$$\begin{bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \frac{E}{1+\nu} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix}.$$

Summary: Material model for an isotropic material (Hooke's law)

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix}$$

2D simplifications: Plane stress and plane strain

The **plane stress** for *thin* plates follow by setting $\sigma_{zz} = \sigma_{yz} = \sigma_{zx} = 0$.

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix}$$

The **plane strain** for *thick* plates follow by setting $\varepsilon_{zz} = \varepsilon_{yz} = \varepsilon_{zx} = 0$.

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

7.D Equations for linear elasticity

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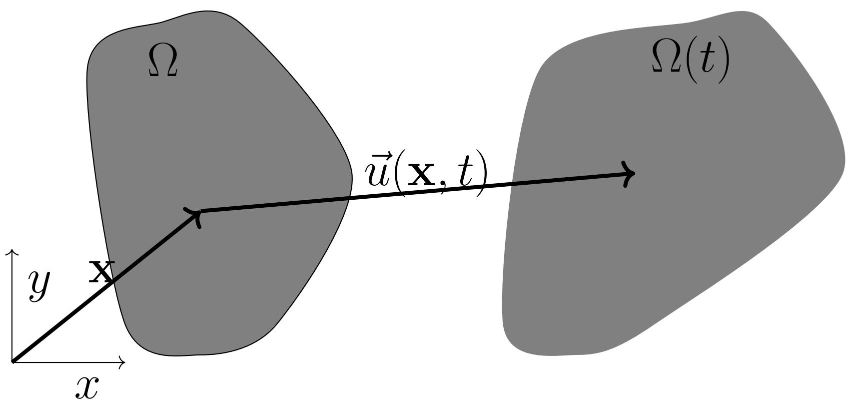
$$\vec{u}(x,y,t) = \begin{bmatrix} u_x(x,y,t) \\ u_y(x,y,t) \end{bmatrix}.$$



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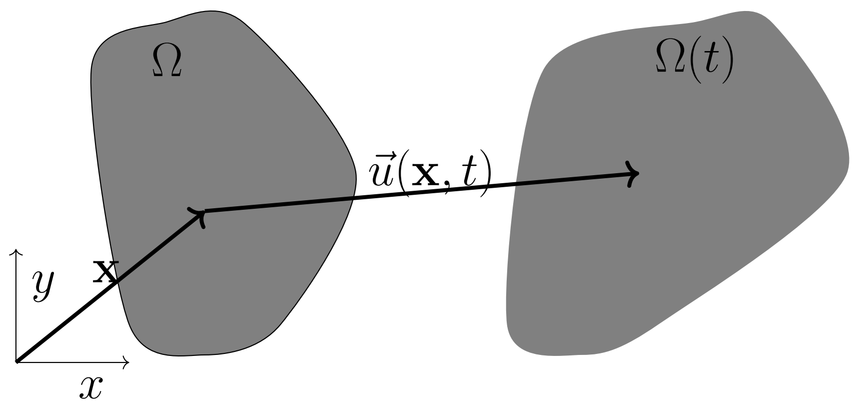

Goal: compute $\vec{u}(x,y,t)$.
Equations essentially follow from Newton's second law

$$\mathbf{F} = m\mathbf{a}, \qquad \begin{bmatrix} F_x \\ F_y \end{bmatrix} = m \begin{bmatrix} a_x \\ a_y \end{bmatrix}.$$

For static problems, the acceleration $\mathbf{a}$ is zero

$$\mathbf{F} = \mathbf{0}, \qquad \begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Strain-displacement relations

We need one more ingredient to close the model.

The strain is related to the gradient of the displacement field.

$$\nabla \vec{u} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{bmatrix}.$$

The linear strain is just the symmetric part of $\nabla \vec{u}$, i.e.

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} = \frac{1}{2}\left(\nabla \vec{u} + (\nabla \vec{u})^\top\right) = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{\partial u_y}{\partial y} \end{bmatrix}.$$

Side remark: linear strain is only valid for small deformations, i.e. when $\nabla \vec{u}$ is small. Otherwise, we should use

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left((\mathbf{I} + \nabla \vec{u})^\top (\mathbf{I} + \nabla \vec{u}) - \mathbf{I}\right) = \frac{1}{2}\left(\nabla \vec{u} + (\nabla \vec{u})^\top + (\nabla \vec{u})^\top \nabla \vec{u}\right)$$

Resulting equations for a plane-stress problem

- Strain-displacement relations

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{\partial u_y}{\partial y} \end{bmatrix}.$$

- Stress-strain relations (material model)

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix}$$

- Force balance

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \qquad \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0.$$

Note: the structure is similar for all problems elasticity
e.g. geometric nonlinearities, plasticity, visco-elasticity, etc.

Tip 1: for a FE discretization

Derive the weak form of the force balance as follows:

$$\iint_\Omega v_x \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} \right) \, \mathrm{d}\mathbf{x} = 0, \qquad \iint_\Omega v_y \left(\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} \right) \, \mathrm{d}\mathbf{x} = 0.$$

Note the test function $\mathbf{v}$ now also has two components v_x and v_y.

Integration by parts / Green identities now give that

$$\int_{\partial\Omega} v_x \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \end{bmatrix} \mathbf{n} \, \mathrm{d}S - \iint_\Omega \left(\frac{\partial v_x}{\partial x} \sigma_{xx} + \frac{\partial v_x}{\partial y} \sigma_{xy} \right) \, \mathrm{d}\mathbf{x} = 0,$$
$$\int_{\partial\Omega} v_y \begin{bmatrix} \sigma_{yx} \\ \sigma_{yy} \end{bmatrix} \mathbf{n} \, \mathrm{d}S - \iint_\Omega \left(\frac{\partial v_y}{\partial x} \sigma_{yx} + \frac{\partial v_y}{\partial y} \sigma_{yy} \right) \, \mathrm{d}\mathbf{x} = 0.$$

Important observation:
the boundary terms now contain the forces applied at the boundary!

So if an edge is ‘free’, i.e. no force is applied, the corresponding boundary term vanishes.

If we have a prescribed force at the boundary, we can insert it in the weak form.

Tip 2: Galerkin discretization of vector-valued functions

For a scalar-valued function $u(x,y)$, we can use the approximation

$$u(x,y) = \mathbf{N}(x,y)\mathbf{u}.$$

Two options for a vector valued case:

- Option 1:

$$\vec{u}(x,y) = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix} = \begin{bmatrix} \mathbf{N}(x,y)\mathbf{u}_x \\ \mathbf{N}(x,y)\mathbf{u}_y \end{bmatrix} = \begin{bmatrix} \mathbf{N}(x,y) & 0 \\ 0 & \mathbf{N}(x,y) \end{bmatrix} \begin{bmatrix} \mathbf{u}_x \\ \mathbf{u}_y \end{bmatrix}.$$

$$\vec{u}(x,y) = \begin{bmatrix} \mathbf{N}_1(x,y) & \cdots & \mathbf{N}_N(x,y) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \mathbf{N}_1(x,y) & \cdots & \mathbf{N}_N(x,y) \end{bmatrix} \begin{bmatrix} u_{x1} \\ \vdots \\ u_{xN} \\ u_{y1} \\ \vdots \\ u_{yN} \end{bmatrix}$$

- Option 2:

$$\vec{u}(x,y) = \begin{bmatrix} \mathbf{N}_1(x,y) & 0 & \mathbf{N}_2(x,y) & 0 & \cdots & \mathbf{N}_N(x,y) & 0 \\ 0 & \mathbf{N}_1(x,y) & 0 & \mathbf{N}_2(x,y) & \cdots & 0 & \mathbf{N}_N(x,y) \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ \vdots \\ u_{xN} \\ u_{yN} \end{bmatrix}$$

The second option is preferred from a numerical point of view, because nonzero elements in the matrices are closer to the diagonal.