Practical Course: Modeling, Simulation, Optimization

Week 8

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Contents

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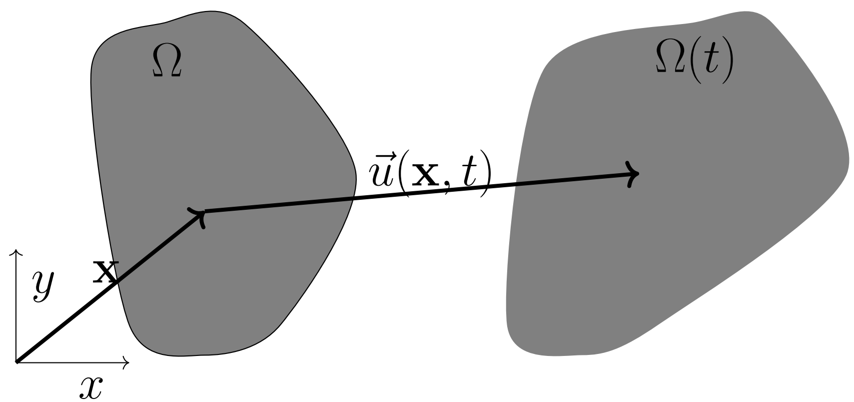
8.A The Euler-Bernoulli beam

Elasticity

Goal: Compute the deformation of a solid that is subjected to certain given forces.

The deformation of a solid is characterized by the displacement field

$$\vec{u}(x, y, t) = \begin{bmatrix} u_x(x, y, t) \\ u_y(x, y, t) \end{bmatrix}.$$

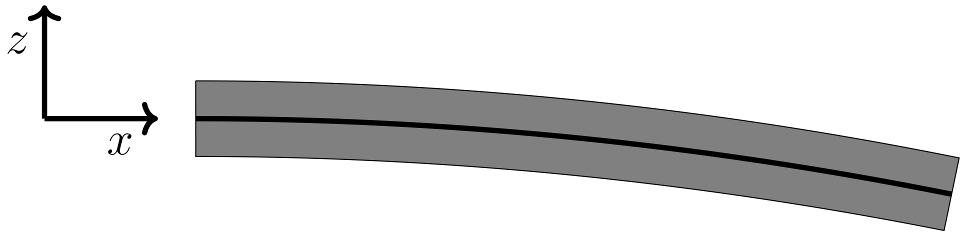


Goal: compute $\vec{u}(x, y, t)$.
In the previous lecture, we saw that a model in elasticity consists of three parts:

- Strain-displacement relations: $\varepsilon = \varepsilon(\vec{u})$.
- Material model: $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\varepsilon)$
- Force balance (conservation of momentum): $\mathbf{F}(\boldsymbol{\sigma}) = \mathbf{0}$.

Beam modeling

Idea: When we are modeling thin, long structures, we can use a 1-D model.



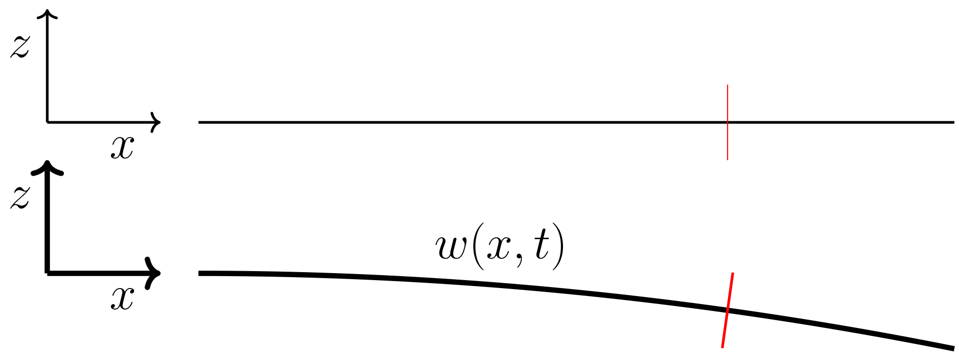
Instead of three components $u_x(x,y,z,t)$, $u_y(x,y,z,t)$, $u_z(x,y,z,t)$, the only unknown in the **Euler-Bernouilli beam model** is the displacement of the midplane in the z-direction $w(x,t)$.

The displacement of a point **on the midplane** $z = 0$ is approximated as:

$$u_x(x,y,0,t) = 0, \qquad u_y(x,y,0,t) = 0, \qquad u_z(x,y,0,t) = w(x,t).$$

But what about point that are not on the midplane $z = 0$?

Question 1



Assume that a cross sections remain perpendicular to the midplane.
What is the displacement in the x-direction u_x of a point that was at (x, z) in the undeformed configuration?

A) $z\frac{\partial w}{\partial x}(x,t)$

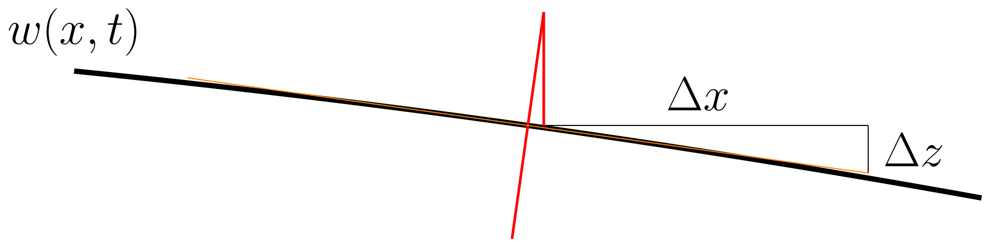
B) $-z\frac{\partial w}{\partial x}(x,t)$

C) $z\sin(\mathrm{atan}(-\frac{\partial w}{\partial x}))$

D) $z\cos(\mathrm{atan}(-\frac{\partial w}{\partial x}))$

E) None of the above.

Answer

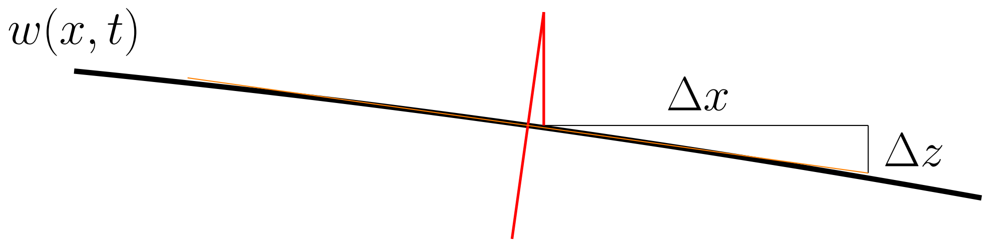


$$-\frac{\partial w}{\partial x}(x,t) = \frac{\Delta z}{\Delta x}.$$

$$\alpha = \text{atan}(-\frac{\partial w}{\partial x}(x,t)).$$

$$u_x(x,y,z,t) = z\sin(\alpha).$$

Answer



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$$\alpha = \operatorname{atan}(-\frac{\partial w}{\partial x}(x,t)).$$

$$u_x(x,y,z,t) = z\sin(\alpha).$$

We will use an approximation:

$$\sin(\operatorname{atan}(x)) = \frac{x}{\sqrt{1+x^2}} \approx x \qquad (x \approx 0).$$

Approximation of the displacement field for the Euler-Bernoulli beam:

$$u_x(x,y,z,t) = -z\frac{\partial w}{\partial x}(x,t), \qquad u_y(x,y,z,t) = 0, \qquad u_z(x,y,z,t) = w(x,t).$$

Strain and stress components

Approximation of the displacement field for the Euler-Bernoulli beam:

$$u_x(x,y,z,t) = -z\frac{\partial w}{\partial x}(x,t), \qquad u_y(x,y,z,t) = 0, \qquad u_z(x,y,z,t) = w(x,t).$$

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We find the strain components:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}, \qquad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = \frac{1}{2}\left(-\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x}\right) = 0.$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} = 0, \qquad \varepsilon_{xy} = \varepsilon_{yz} = 0.$$

Strain and stress components

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$$u_x(x,y,z,t) = -z\frac{\partial w}{\partial x}(x,t), \qquad u_y(x,y,z,t) = 0, \qquad u_z(x,y,z,t) = w(x,t).$$

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$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} = 0, \qquad \varepsilon_{xy} = \varepsilon_{yz} = 0.$$

We use the stress components:

$$\sigma_{xx} = E\varepsilon_{xx} = -zE\frac{\partial^2 w}{\partial x^2}.$$

Strain and stress components

Approximation of the displacement field for the Euler-Bernoulli beam:

$$u_x(x,y,z,t) = -z\frac{\partial w}{\partial x}(x,t), \qquad u_y(x,y,z,t) = 0, \qquad u_z(x,y,z,t) = w(x,t).$$

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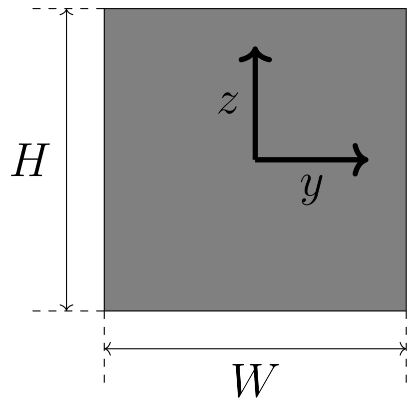
We use the stress components:

$$\sigma_{xx} = E\varepsilon_{xx} = -zE\frac{\partial^2 w}{\partial x^2}.$$

We can also define the bending moment:

$$M = \int_A z\sigma_{xx}\,\mathrm{d}A = -EI\frac{\partial^2 w}{\partial x^2}, \qquad I = \int_A z^2\,\mathrm{d}A$$

Question 2



Compute the second moment area $I = \int_A z^2 \mathrm{d}A$ for the cross section in the figure above.

A) $I = \frac{1}{24} W H^3$

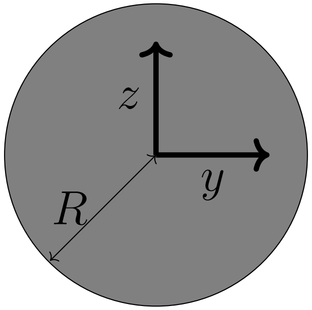
B) $I = \frac{1}{12} W H^3$

C) $I = \frac{1}{3} W H^3$

D) $I = \frac{1}{3} L H^3$

E) None of the above.

Question 3



Compute the second moment area $I = \int_A z^2 \mathrm{d}A$ for the cross section in the figure above.

A) $I = \frac{\pi}{3} R^3$

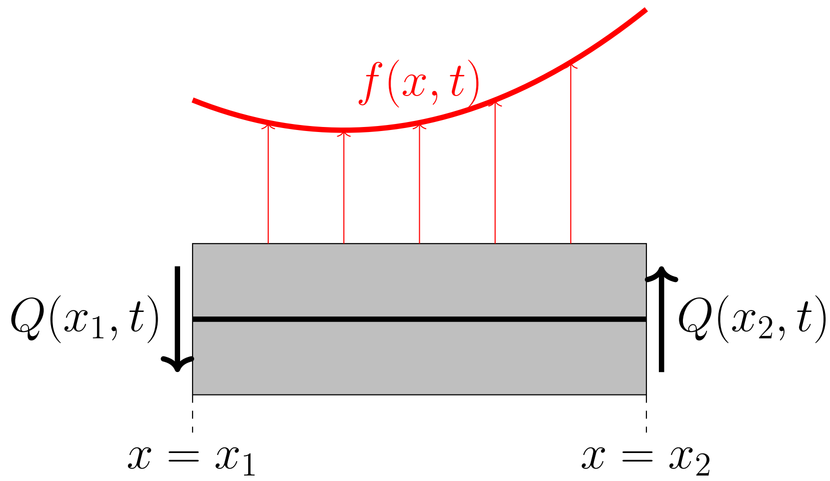
B) $I = \frac{\pi}{4} R^4$

C) $I = \frac{2\pi}{3} R^3$

D) $I = \frac{\pi}{2} R^4$

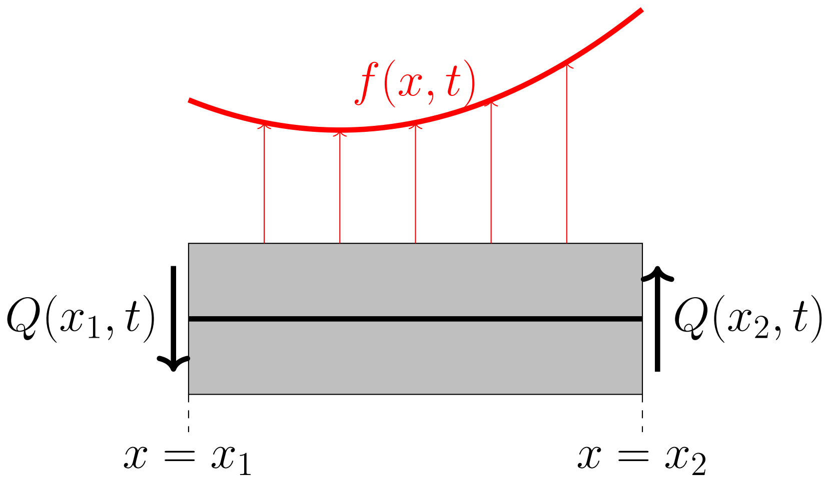
E) None of the above.

Force balance



$$\int_{x_1}^{x_2} \rho A \frac{\partial^2 w}{\partial t^2}(x,t)\,\mathrm{d}x = Q(x_2,t) - Q(x_1,t) + \int_{x_1}^{x_2} f(x,t)\,\mathrm{d}x,$$

Force balance

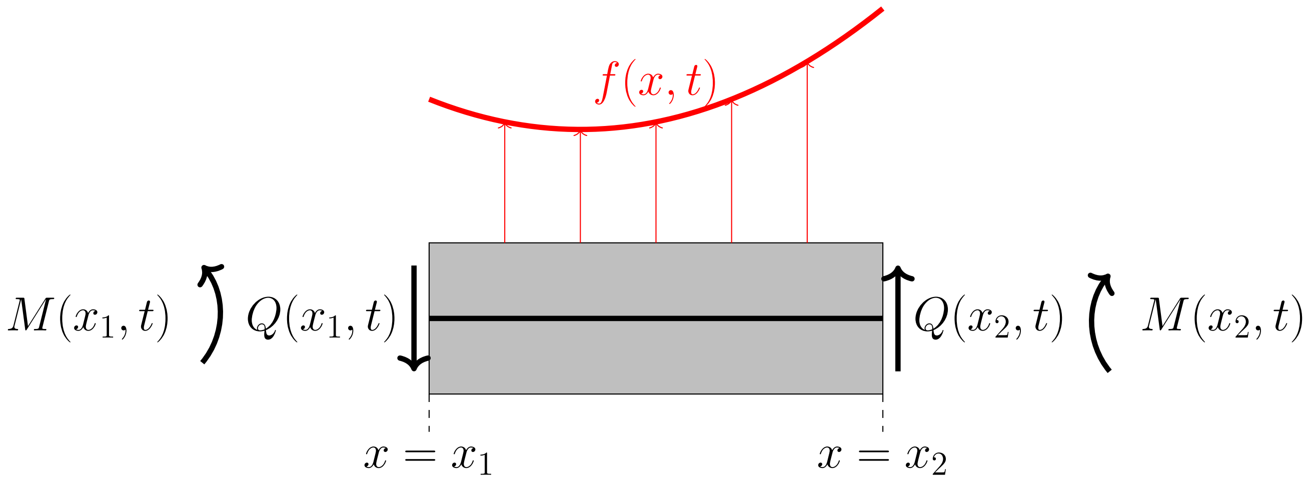


$$\int_{x_1}^{x_2} \rho A \frac{\partial^2 w}{\partial t^2}(x,t)\,\mathrm{d}x = Q(x_2,t) - Q(x_1,t) + \int_{x_1}^{x_2} f(x,t)\,\mathrm{d}x,$$

$$\int_{x_1}^{x_2} \rho A \frac{\partial^2 w}{\partial t^2}(x,t)\,\mathrm{d}x = \int_{x_1}^{x_2} \left(\frac{\partial Q}{\partial x}(x,t) + f(x,t)\right)\,\mathrm{d}x,$$

$$\rho A \frac{\partial^2 w}{\partial t^2}(x,t) = \frac{\partial Q}{\partial x}(x,t) + f(x,t).$$

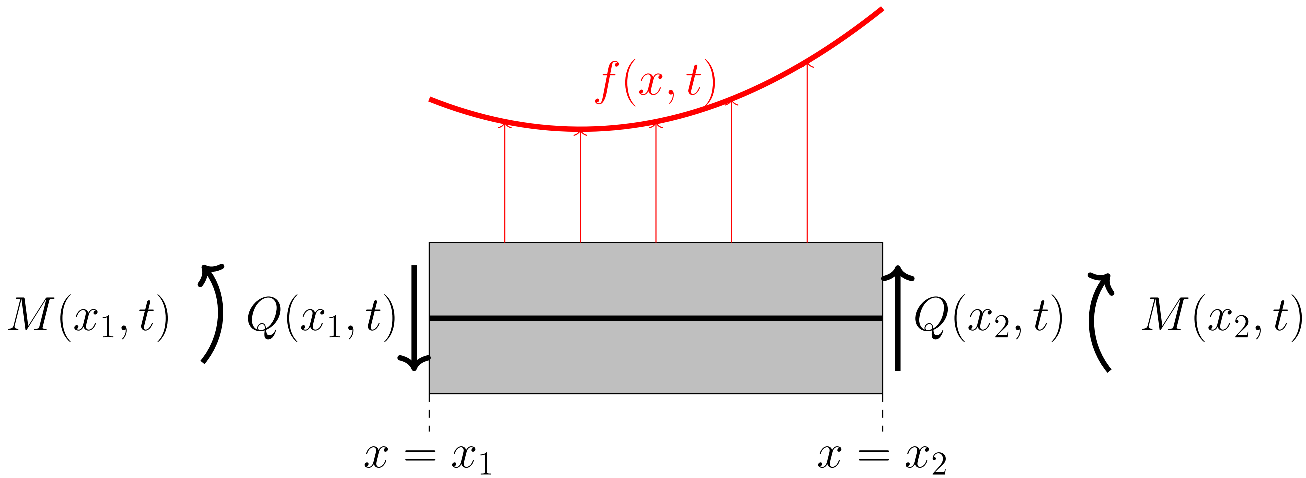
Moment balance



Moment balance (around $x = x_1$)

$$\int_{x_1}^{x_2} (x-x_1)\rho A \frac{\partial^2 w}{\partial t^2}(x,t)\,\mathrm{d}x = -M(x_2,t)+M(x_1,t)+(x_2-x_1)Q(x_2,t)+\int_{x_1}^{x_2}(x-x_1)f(x,t)\,\mathrm{d}x$$

Moment balance



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$$\int_{x_1}^{x_2} (x-x_1)\rho A \frac{\partial^2 w}{\partial t^2}(x,t)\,\mathrm{d}x = -M(x_2,t)+M(x_1,t)+(x_2-x_1)Q(x_2,t)+\int_{x_1}^{x_2}(x-x_1)f(x,t)\,\mathrm{d}x$$

$$\int_{x_1}^{x_2} (x-x_1)\rho A \frac{\partial^2 w}{\partial t^2}(x,t)\,\mathrm{d}x = \int_{x_1}^{x_2}\left(-\frac{\partial M}{\partial x}(x,t)+\frac{\partial}{\partial x}(x-x_1)Q(x,t)+(x-x_1)f(x,t)\right)\mathrm{d}x$$

$$(x-x_1)\rho A \frac{\partial^2 w}{\partial t^2}(x,t) = -\frac{\partial M}{\partial x}(x,t)+\frac{\partial}{\partial x}(x-x_1)Q(x,t)+(x-x_1)f(x,t).$$

Question 4

So far, we have

$$\rho A \frac{\partial^2 w}{\partial t^2}(x,t) = \frac{\partial Q}{\partial x}(x,t) + f(x,t).$$

$$(x - x_1)\rho A \frac{\partial^2 w}{\partial t^2}(x,t) = -\frac{\partial M}{\partial x}(x,t) + \frac{\partial}{\partial x}(x - x_1)Q(x,t) + (x - x_1)f(x,t).$$

Which of the following expressions is equal to $\frac{\partial M}{\partial x}(x,t)$?

A) $Q(x,t)$

B) $f(x,t) - \rho A \frac{\partial^2 w}{\partial t^2}(x,t)$

C) $-Q(x,t)$

D) $\rho A \frac{\partial^2 w}{\partial t^2}(x,t) - f(x,t)$

E) None of the above.

Question 5

So far, we have

$$\rho A \frac{\partial^2 w}{\partial t^2}(x,t) = \frac{\partial Q}{\partial x}(x,t) + f(x,t).$$

$$(x - x_1)\rho A \frac{\partial^2 w}{\partial t^2}(x,t) = -\frac{\partial M}{\partial x}(x,t) + \frac{\partial}{\partial x}(x - x_1)Q(x,t) + (x - x_1)f(x,t).$$

$$\frac{\partial M}{\partial x}(x,t) = Q(x,t).$$

Which of the following expressions is equal to $\frac{\partial^2 M}{\partial x^2}(x,t)$?

A) $Q(x,t)$

B) $f(x,t) - \rho A \frac{\partial^2 w}{\partial t^2}(x,t)$

C) $-Q(x,t)$

D) $\rho A \frac{\partial^2 w}{\partial t^2}(x,t) - f(x,t)$

E) None of the above.

Resulting beam equation

From the previous slides:

$$\frac{\partial^2 M}{\partial x^2}(x,t) = \rho A \frac{\partial^2 w}{\partial t^2}(x,t) - f(x,t), \qquad \frac{\partial M}{\partial x}(x,t) = Q(x,t),$$

$$M = -EI\frac{\partial^2 w}{\partial x^2}, \qquad I = \int_A z^2 \,\mathrm{d}A.$$

Resulting equation for the Euler-Bernoulli beam

$$\rho A \frac{\partial^2 w}{\partial t^2}(x,t) + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}(x,t)\right) = f(x,t).$$

For the boundary conditions, note that

- $w(x,t)$ is the transversal displacement
- $\frac{\partial w}{\partial x}(x,t)$ is the linearized rotation
- $-EI\frac{\partial^2 w}{\partial x^2}(x,t) = M(x)$ is the moment
- $-EI\frac{\partial^3 w}{\partial x^3}(x,t) = Q(x)$ is the force in the transversal direction.

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8.B Extensions

Extensions

- Timoshenko beam

$$u_x(x,y,z,t) = -z\varphi(x,t), \qquad u_y(x,y,z,t) = 0, \qquad u_z(x,y,z,t) = w(x,t).$$

Note that setting $\varphi(x,t) = \frac{\partial w}{\partial x}$ gives the Euler-Bernoulli beam.

$$\rho A \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(\kappa AG\left(\frac{\partial w}{\partial x} - \varphi\right)\right), \qquad \rho I \frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial}{\partial x}\left(EI\frac{\partial \varphi}{\partial x}\right) + \kappa AG\left(\frac{\partial w}{\partial x} - \varphi\right)$$

shear coefficient $\kappa = 5/6$, shear modulus $G = E/2(1+\nu)$.

Extensions

- Timoshenko beam

$$u_x(x,y,z,t) = -z\varphi(x,t), \qquad u_y(x,y,z,t) = 0, \qquad u_z(x,y,z,t) = w(x,t).$$

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shear coefficient $\kappa = 5/6$, shear modulus $G = E/2(1+\nu)$.

- Kirchhoff-love plate theory

$$u_x(x,y,z,t) = -z\frac{\partial w}{\partial x}(x,y,t), u_y(x,y,z,t) = -z\frac{\partial w}{\partial y}(x,y,t), u_z(x,y,z,t) = w(x,y,t).$$

$$\rho H \frac{\partial^2 w}{\partial t^2} + \frac{EH^3}{12(1-\nu^2)}\nabla^4 w = f$$

Extensions

- Timoshenko beam

$$u_x(x,y,z,t) = -z\varphi(x,t), \qquad u_y(x,y,z,t) = 0, \qquad u_z(x,y,z,t) = w(x,t).$$

Note that setting $\varphi(x,t) = \frac{\partial w}{\partial x}$ gives the Euler-Bernoulli beam.

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shear coefficient $\kappa = 5/6$, shear modulus $G = E/2(1+\nu)$.

- Kirchhoff-love plate theory

$$u_x(x,y,z,t) = -z\frac{\partial w}{\partial x}(x,y,t), u_y(x,y,z,t) = -z\frac{\partial w}{\partial y}(x,y,t), u_z(x,y,z,t) = w(x,y,t).$$

$$\rho H \frac{\partial^2 w}{\partial t^2} + \frac{EH^3}{12(1-\nu^2)}\nabla^4 w = f$$

- Mindlin-Reissner plate theory

$$u_x(x,y,z,t) = -z\varphi_x(x,y,t), u_y(x,y,z,t) = -z\varphi_y(x,y,t), u_z(x,y,z,t) = w(x,y,t).$$

8.C Finite element discretization for the Euler-Bernoulli beam

FE discretization

$$\rho A \frac{\partial^2 w}{\partial t^2}(x,t) + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2}(x,t) \right) = f(x,t).$$

Find the weak form: multiply by test function and use integration by parts **twice**!
When there are no external forces and moments, we find

$$\rho A \int_0^L v(x) \frac{\partial^2 w}{\partial t^2}(x,t) \mathrm{d}x + \int_0^L \frac{\partial^2 v}{\partial x^2}(x) EI \frac{\partial^2 w}{\partial x^2}(x,t) \, \mathrm{d}x = \int_0^L v(x) f(x,t) \, \mathrm{d}x.$$

Galerkin discretization ($w(x,t) = \mathbf{N}(x)\mathbf{w}(t)$ and $v(x) = \mathbf{v}^\top (\mathbf{N}(x))^\top$) gives

$$\mathbf{M}\ddot{\mathbf{w}}(t) + \mathbf{K}\mathbf{w}(t) = \mathbf{f}(t)$$

where

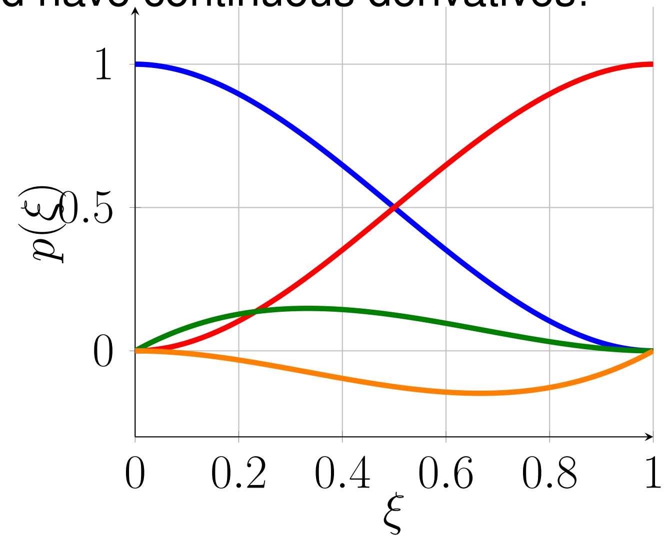
$$\mathbf{M} = \rho A \int_0^L (\mathbf{N}(x))^\top \mathbf{N}(x) \, \mathrm{d}x,$$

$$\mathbf{K} = EI \int_0^L \left(\frac{\mathrm{d}^2 \mathbf{N}}{\mathrm{d}x^2}(x) \right)^\top \frac{\mathrm{d}^2 \mathbf{N}}{\mathrm{d}x^2}(x) \, \mathrm{d}x,$$

$$\mathbf{f}(t) = \int_0^L (\mathbf{N}(x))^\top f(x,t) \, \mathrm{d}x.$$

FE shape functions in H^2

The shape functions should have continuous derivatives!



$$p_0^w(\xi) = 1 - 3\xi^2 + 2\xi^3,$$
$$p_0^\theta(\xi) = \xi - 2\xi^2 + \xi^3,$$
$$p_1^w(\xi) = 3\xi^2 - 2\xi^3,$$
$$p_1^\theta(\xi) = -\xi^2 + \xi^3,$$

Note that $p_0^w(\xi)$, $p_1^w(\xi)$, $p_0^\theta(\xi)$, and $p_1^\theta(\xi)$, are the unique 3rd order polynomials satisfying

$$p_0^w(0) = 1, \quad \frac{\partial p_0^w}{\partial \xi}(0) = 0, \quad p_0^w(1) = 0, \quad \frac{\partial p_0^w}{\partial \xi}(1) = 0,$$
$$p_0^\theta(0) = 0, \quad \frac{\partial p_0^\theta}{\partial \xi}(0) = 1, \quad p_0^\theta(1) = 0, \quad \frac{\partial p_0^\theta}{\partial \xi}(1) = 0,$$
$$p_1^w(0) = 0, \quad \frac{\partial p_1^w}{\partial \xi}(0) = 0, \quad p_1^w(1) = 1, \quad \frac{\partial p_1^w}{\partial \xi}(1) = 0,$$
$$p_1^\theta(0) = 0, \quad \frac{\partial p_1^\theta}{\partial \xi}(0) = 0, \quad p_1^\theta(1) = 0, \quad \frac{\partial p_1^\theta}{\partial \xi}(1) = 1.$$

(these are called Hermite interpolation polynomials)

Defining the element shape functions: option 1

Element shape functions (for element of unit length):

$$\mathbf{N}^e(\xi) = [p_0^w(\xi), \quad p_0^\theta(\xi), \quad p_1^w(\xi), \quad p_1^\theta(\xi)].$$

Defining the element shape functions: option 1

Element shape functions (for element of unit length):

$$\mathbf{N}^e(\xi) = [p_0^w(\xi), \quad p_0^\theta(\xi), \quad p_1^w(\xi), \quad p_1^\theta(\xi)].$$

Inside elements e and $e+1$ located at $[x_{e-1}, x_e]$ and $[x_e, x_{e+1}]$, we thus have

$$\mathbf{N}^e(\tfrac{x-x_{e-1}}{L_e})\mathbf{w}^e = p_0^w(\tfrac{x-x_{e-1}}{L_e})w_1^e + p_0^\theta(\tfrac{x-x_{e-1}}{L_e})w_2^e + p_1^w(\tfrac{x-x_{e-1}}{L_e})w_3^e + p_1^\theta(\tfrac{x-x_{e-1}}{L_e})w_4^e,$$

$$\mathbf{N}^{e+1}(\tfrac{x-x_e}{L_{e+1}})\mathbf{w}^{e+1} = p_0^w(\tfrac{x-x_e}{L_{e+1}})w_1^{e+1} + p_0^\theta(\tfrac{x-x_e}{L_{e+1}})w_2^{e+1} + p_1^w(\tfrac{x-x_e}{L_{e+1}})w_3^{e+1} + p_1^\theta(\tfrac{x-x_e}{L_{e+1}})w_4^{e+1}.$$

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$$\mathbf{N}^e(\xi) = [p_0^w(\xi), \quad p_0^\theta(\xi), \quad p_1^w(\xi), \quad p_1^\theta(\xi)].$$

Inside elements e and $e+1$ located at $[x_{e-1}, x_e]$ and $[x_e, x_{e+1}]$, we thus have

$$\mathbf{N}^e(\tfrac{x-x_{e-1}}{L_e})\mathbf{w}^e = p_0^w(\tfrac{x-x_{e-1}}{L_e})w_1^e + p_0^\theta(\tfrac{x-x_{e-1}}{L_e})w_2^e + p_1^w(\tfrac{x-x_{e-1}}{L_e})w_3^e + p_1^\theta(\tfrac{x-x_{e-1}}{L_e})w_4^e,$$

$$\mathbf{N}^{e+1}(\tfrac{x-x_e}{L_{e+1}})\mathbf{w}^{e+1} = p_0^w(\tfrac{x-x_e}{L_{e+1}})w_1^{e+1} + p_0^\theta(\tfrac{x-x_e}{L_{e+1}})w_2^{e+1} + p_1^w(\tfrac{x-x_e}{L_{e+1}})w_3^{e+1} + p_1^\theta(\tfrac{x-x_e}{L_{e+1}})w_4^{e+1}.$$

We need a C^1-function for the Galerkin approximation $\Rightarrow$
$\mathbf{N}^e(\tfrac{x-x_{e-1}}{L_e})\mathbf{w}^e$ and $\mathbf{N}^{e+1}(\tfrac{x-x_e}{L_e})\mathbf{w}^{e+1}$ and their derivatives should match at $x = x_e$.

$$\mathbf{N}^e(1)\mathbf{w}^e = w_3^e, \qquad \mathbf{N}^{e+1}(0)\mathbf{w}^{e+1} = w_1^{e+1}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{N}^e(\tfrac{x-x_{e-1}}{L_e})\mathbf{w}^e\right]_{x=x_e} = \frac{w_4^e}{L_e}, \qquad \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{N}^{e+1}(\tfrac{x-x_e}{L_{e+1}})\mathbf{w}^{e+1}\right]_{x=x_e} = \frac{w_2^{e+1}}{L_{e+1}}$$

When all elements are of the same size (i.e. $L_{e+1} = L_e$**)**, we need that

$$w_3^e = w_1^{e+1} = w(x_e), \qquad w_4^e = w_2^{e+1} = L_e\frac{\partial w}{\partial x}(x_e).$$

Defining the element shape functions: option 1

Element shape functions (for element of unit length):

$$\mathbf{N}^e(\xi) = [p_0^w(\xi), \quad p_0^\theta(\xi), \quad p_1^w(\xi), \quad p_1^\theta(\xi)].$$

Inside elements e and $e+1$ located at $[x_{e-1}, x_e]$ and $[x_e, x_{e+1}]$, we thus have

$$\mathbf{N}^e(\tfrac{x-x_{e-1}}{L_e})\mathbf{w}^e = p_0^w(\tfrac{x-x_{e-1}}{L_e})w_1^e + p_0^\theta(\tfrac{x-x_{e-1}}{L_e})w_2^e + p_1^w(\tfrac{x-x_{e-1}}{L_e})w_3^e + p_1^\theta(\tfrac{x-x_{e-1}}{L_e})w_4^e,$$

$$\mathbf{N}^{e+1}(\tfrac{x-x_e}{L_{e+1}})\mathbf{w}^{e+1} = p_0^w(\tfrac{x-x_e}{L_{e+1}})w_1^{e+1} + p_0^\theta(\tfrac{x-x_e}{L_{e+1}})w_2^{e+1} + p_1^w(\tfrac{x-x_e}{L_{e+1}})w_3^{e+1} + p_1^\theta(\tfrac{x-x_e}{L_{e+1}})w_4^{e+1}.$$

We need a C^1-function for the Galerkin approximation $\Rightarrow$
$\mathbf{N}^e(\frac{x-x_{e-1}}{L_e})\mathbf{w}^e$ and $\mathbf{N}^{e+1}(\frac{x-x_e}{L_e})\mathbf{w}^{e+1}$ and their derivatives should match at $x = x_e$.

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$$w_3^e = w_1^{e+1} = w(x_e), \qquad w_4^e = w_2^{e+1} = L_e \frac{\partial w}{\partial x}(x_e).$$

So in the total approximation $\mathbf{N}(x)\mathbf{w}(t)$, $\mathbf{w}(t)$ has the interpretation

$$\mathbf{w}(t) = \left[w(x_0,t), L_1\frac{\partial w}{\partial x}(x_0,t), w(x_1,t), L_1\frac{\partial w}{\partial x}(x_1,t), \ldots, w(x_M,t), L_1\frac{\partial w}{\partial x}(x_M,t)\right]^\top.$$

Defining the element shape functions: option 2 (preferred)

Element shape functions (for element of unit length):

$$\mathbf{N}^e(\xi) = [p_0^w(\xi), \quad L_e p_0^\theta(\xi), \quad p_1^w(\xi), \quad L_e p_1^\theta(\xi)].$$

Inside elements e and $e+1$ located at $[x_{e-1}, x_e]$ and $[x_e, x_{e+1}]$, we thus have

$$\mathbf{N}^e(\tfrac{x-x_{e-1}}{L_e})\mathbf{w}^e = p_0^w(\tfrac{x-x_{e-1}}{L_e})w_1^e + L_e p_0^\theta(\tfrac{x-x_{e-1}}{L_e})w_2^e + p_1^w(\tfrac{x-x_{e-1}}{L_e})w_3^e + L_e p_1^\theta(\tfrac{x-x_{e-1}}{L_e})w_4^e,$$

$$\mathbf{N}^{e+1}(\tfrac{x-x_e}{L_{e+1}})\mathbf{w}^{e+1} = p_0^w(\tfrac{x-x_e}{L_{e+1}})w_1^{e+1} + L_{e+1} p_0^\theta(\tfrac{x-x_e}{L_{e+1}})w_2^{e+1} + p_1^w(\tfrac{x-x_e}{L_{e+1}})w_3^{e+1} + L_{e+1} p_1^\theta(\tfrac{x-x_e}{L_{e+1}})w_4^{e+1}.$$

We need a C^1-function for the Galerkin approximation $\Rightarrow$
$\mathbf{N}^e(\tfrac{x-x_{e-1}}{L_e})\mathbf{w}^e$ and $\mathbf{N}^{e+1}(\tfrac{x-x_e}{L_e})\mathbf{w}^{e+1}$ and their derivatives should match at $x = x_e$.

$$\mathbf{N}^e(1)\mathbf{w}^e = w_3^e, \qquad \mathbf{N}^{e+1}(0)\mathbf{w}^{e+1} = w_1^{e+1}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{N}^e(\tfrac{x-x_{e-1}}{L_e})\mathbf{w}^e\right]_{x=x_e} = w_4^e, \qquad \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{N}^{e+1}(\tfrac{x-x_e}{L_{e+1}})\mathbf{w}^{e+1}\right]_{x=x_e} = w_2^{e+1}$$

When all elements are of the same size , we need that

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So in the total approximation $w_N(x,t) = \mathbf{N}(x)\mathbf{w}(t)$, $\mathbf{w}(t)$ has the interpretation

$$\mathbf{w}(t) = \left[w_N(x_0,t), \frac{\partial w_N}{\partial x}(x_0,t), w_N(x_1,t), \frac{\partial w_N}{\partial x}(x_1,t), \ldots, w_N(x_M,t), \frac{\partial w_N}{\partial x}(x_M,t)\right]^\top.$$

Assembly of the FE model (1/2)

Define the element shape function

$$\mathbf{N}^e(\xi) = [p_0^w(\xi), \quad L_e p_0^\theta(\xi), \quad p_1^w(\xi), \quad L_e p_1^\theta(\xi)].$$

$$p_0^w(\xi) = 1 - 3\xi^2 + 2\xi^3, \qquad p_1^w(\xi) = 3\xi^2 - 2\xi^3,$$
$$p_0^\theta(\xi) = \xi - 2\xi^2 + \xi^3, \qquad p_1^\theta(\xi) = -\xi^2 + \xi^3,$$

Compute the element mass and stiffness matrices:

$$\tilde{\mathbf{M}}^e = \rho A \int_0^{L_e} \left(\mathbf{N}^e(\tfrac{x}{L_e})\right)^\top \mathbf{N}^e(\tfrac{x}{L_e})\,\mathrm{d}x, \qquad \tilde{\mathbf{K}}^e = EI \int_0^{L_e} \left(\frac{\mathrm{d}^2\mathbf{N}^e(\frac{x}{L_e})}{\mathrm{d}x^2}\right)^\top \frac{\mathrm{d}^2\mathbf{N}^e(\frac{x}{L_e})}{\mathrm{d}x^2}\,\mathrm{d}x$$

Note: for a nonuniform mesh, $\tilde{\mathbf{M}}^e$ and $\tilde{\mathbf{K}}^e$ need to be computed for all element sizes that appear in the mesh.
Note: relating $\tilde{\mathbf{M}}^e$ and $\tilde{\mathbf{K}}^e$ to the matrices for an element of unit length $\mathbf{M}^e$ and $\mathbf{K}^e$ is tricky.

Assembly of the FE model (2/2)

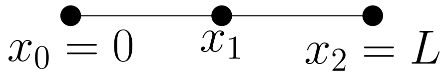
We define the vector of DOFs as

$$\mathbf{w}(t) = \left[w(x_0,t),\quad \frac{\partial w}{\partial x}(x_0,t),\quad w(x_1,t),\quad \frac{\partial w}{\partial x}(x_1,t),\ldots,\quad w(x_M,t),\quad \frac{\partial w}{\partial x}(x_M,t)\right]^\top.$$

Observe

- The first element only involves $w(x_0,t)$, $\frac{\partial w}{\partial x}(x_0,t)$, $w(x_1,t)$, $\frac{\partial w}{\partial x}(x_1,t)$. Write the contribution in the $([1,2,3,4],[1,2,3,4])$ parts of $\mathbf{M}$ and $\mathbf{K}$.
- The second element only involves $w(x_1,t)$, $\frac{\partial w}{\partial x}(x_1,t)$, $w(x_2,t)$, $\frac{\partial w}{\partial x}(x_2,t)$. Write the contribution in the $([3,4,5,6],[3,4,5,6])$ parts of $\mathbf{M}$ and $\mathbf{K}$.
- etc.
- The last element only involves $w(x_{M-1},t)$, $\frac{\partial w}{\partial x}(x_{M-1},t)$, $w(x_M,t)$, $\frac{\partial w}{\partial x}(x_M,t)$. Write the contribution in the $([2M-1,2M,2M+1,2M+2],[2M-1,2M,2M+1,2M+2])$ parts of $\mathbf{M}$ and $\mathbf{K}$.

Example

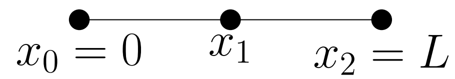


An example with $M = 2$ elements:

1. Determine the number of nodes $N = M + 1 = 3$.
2. Create zero matrices $\mathbf{M}$ and $\mathbf{K}$ of size $2N \times 2N = 6 \times 6$.

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

Example



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3. Compute the contributions $\tilde{\mathbf{M}}^1$ and $\tilde{\mathbf{K}}^1$ of element 1 and $\tilde{\mathbf{M}}^2$ and $\tilde{\mathbf{K}}^2$ of element 2.
4. Write the contributions of each element in the matrices $\mathbf{M}$ and $\mathbf{K}$

$$\mathbf{M} = \begin{bmatrix} m^1_{11} & m^1_{12} & m^1_{13} & m^1_{14} & 0 & 0 \\ m^1_{21} & m^1_{22} & m^1_{23} & m^1_{24} & 0 & 0 \\ m^1_{31} & m^1_{32} & m^1_{33} + m^2_{11} & m^1_{34} + m^2_{12} & m^2_{13} & m^2_{14} \\ m^1_{41} & m^1_{42} & m^1_{43} + m^2_{21} & m^1_{44} + m^2_{22} & m^2_{23} & m^2_{24} \\ 0 & 0 & m^2_{31} & m^2_{32} & m^2_{33} & m^2_{34} \\ 0 & 0 & m^2_{41} & m^2_{42} & m^2_{43} & m^2_{44} \end{bmatrix}, \qquad \mathbf{K} = \dots.$$

5. Remove rows and columns of nodes with zero Dirichlet boundary conditions.

(Undamped) eigenfrequencies and eigenmodes

After FE discretization (and without forcing, $f \equiv 0$) we obtain a system of ODEs

$$\mathbf{M}\ddot{\mathbf{w}}(t) + \mathbf{K}\mathbf{w}(t) = \mathbf{0}.$$

We are interested in free vibrating solutions of the form

$$\mathbf{w}(t) = \bar{\mathbf{w}} \sin(\omega t).$$

Insert this solution into the ODE:

$$\left(-\omega^2 \mathbf{M}\bar{\mathbf{w}} + \mathbf{K}\bar{\mathbf{w}}\right) \sin(\omega t) = \mathbf{0}.$$

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$$\left(-\omega^2 \mathbf{M}\bar{\mathbf{w}} + \mathbf{K}\bar{\mathbf{w}}\right) \sin(\omega t) = \mathbf{0}.$$

Write $\lambda = \omega^2$. We then see we need to solve the (generalized) eigenvalue problem

$$\mathbf{K}\mathbf{w}_k = \lambda_k \mathbf{M}\mathbf{w}_k$$

The eigenfrequencies in rad/s ω_k and the eigenfrequencies in Hz f_k are then

$$\omega_k = \sqrt{\lambda_k}, \qquad f_k = \frac{\omega_k}{2\pi} = \frac{\sqrt{\lambda_k}}{2\pi}.$$

The corresponding eigenmodes are $\mathbf{w}_k$.
Note that $\mathbf{w}_k$ contains both displacement ($w(x,t)$) and rotation information $\frac{\partial w}{\partial x}(x,t)$.

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8.D Convergence analysis for finite elements

Stability: Cea's lemma

Original infinite dimensional problem:
find $u \in V$ such that

$$a(u, w) = b(w), \qquad \forall w \in V$$

Galerkin approximation:
find $u_N \in V_N \subset V$ such that

$$a(u_N, w_N) = b(w_N), \qquad \forall w_N \in V_N$$

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$$a(u_N, w_N) = b(w_N), \qquad \forall w_N \in V_N$$

Assume that there are $m, M > 0$ such that for all $u, w \in V$

$$a(u, u) \geq m|u|^2, \qquad |a(u, w)| \leq M|u||w|.$$

Lemma (Cea)

$$|u - u_N| \leq \frac{M}{m} \inf_{w_N \in V_N} |u - w_N|$$

Proof: Because $w_N \in V$,

$$a(u - u_N, w_N) = a(u, w_N) - a(u_N, w_N) = b(w_N) - b(w_N) = 0.$$

Using this result, we can then compute

$$m|u - u_N|^2 \leq a(u - u_N, u - u_N) = a(u - u_N, u - w_N + \underbrace{w_N - u_N}_{\in V_N})$$

$$= a(u - u_N, u - w_N) \leq M|u - u_N||u - w_N|.$$

Consistency: convergence rates

Using Cea's lemma, we just need to compute

$$\inf_{w_N \in V_N} |u - w_N|.$$

Idea: we can choose a specific mapping $r_N : V \to V_N$ find a bound $|u - r_N u| \leq Ch^p$.
The operator r_N is typically chosen as the interpolation operator.
Using Cea's lemma, we then find that

$$|u - u_N| \leq \frac{M}{m} \inf_{w_N \in V_N} |u - w_N| \leq \frac{M}{m} |u - r_N u| \leq \frac{M}{m} Ch^p.$$

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For **linear 1-D elements**, we have (see e.g. Allaire Lemma 6.2.10)

$$|u - u_N|_{L^2} \leq Ch^2 |u''|_{L^2}, \qquad |u' - u_N'|_{L^2} \leq Ch |u''|_{L^2}.$$

For **quadratic 1-D elements**, we have (see e.g. Allaire Theorem 6.2.14)

$$|u - u_N|_{H^1} \leq Ch^2 |u'''|_{L^2}.$$

More general, for $\mathbb{P}_k$ **rectangular elements**, we have (see e.g. Allaire Theorem 6.3.27)

$$|u - u_N|_{H^1} \leq Ch^k |u|_{H^{k+1}}.$$