# Stabilization results for KdV equation

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# Outline

# KdV equation

The Korteweg-de Vries (KdV) equation $y_t + y_x + y_{xxx} + yy_x = 0$ was introduced by Korteweg and de Vries (1895) to model the propagation of long water waves in a channel.

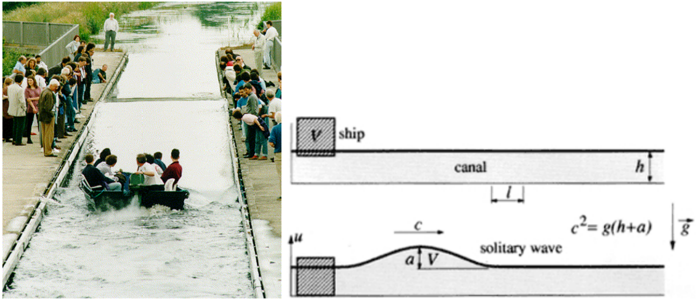


Figure: Solitary waves.

Widely studied in the last years, in particular its controllability and stabilization properties, see Cerpa (2014).

# KdV equation

It is well known that the length $L$ of the spatial domain plays an important role in the stabilization and controllability properties of the KdV equation.

- When $L = 2\pi$ it is possible to find a solution of the linearization around 0 of the KdV equation ($u(t,x) = 1 - \cos(x)$ ) that has a constant energy.
- More generally, if $L \in \mathcal{N}$, where $\mathcal{N}$ is the set of critical lengths defined by

$$\mathcal{N} = \left\{ 2\pi\sqrt{(k^2 + kl + l^2)/3}\ ;\ k, l \in \mathbb{N}^* \right\},$$

  we can find suitable initial data, such that the solution of the linear KdV equation has a constant energy.

Importance of control and stabilization problems.

# Problems addressed

In this talk, we focus on the following problems

- Stability results for the KdV equation with time-varying delay.
- Asymptotic behaviour of KdV in a star network with bounded and unbounded lengths.

# Why delay?

Time delay phenomena appear in many applications, for example in biology, mechanics or engineering. Delay terms are unavoidable in practice due to measurement lag, analysis time, or computation time.

Figure: Very hot water in a shower

# System studied

In this section, we are going to consider the following systems

$$
\begin{cases}
y_t + y_x + y_{xxx} + yy_x = 0, & t > 0,\ x \in (0, L), \\
y(0,t) = y(L,t) = 0, & t > 0, \\
y_x(L,t) = \alpha y_x(0,t) + \beta y_x(0, t - \tau(t)), & t > 0, \\
y(x,0) = y_0(x), & x \in (0, L), \\
y_x(t - \tau(0), 0) = z_0(t - \tau(0)), \quad 0 < t < \tau(0), &
\end{cases}
\tag{KdVd}
$$

We assume that the delay $\tau$ is a function of time $t$, which satisfies the following conditions

$$
0 < \tau_0 \leq \tau(t) \leq M, \qquad \forall t \geq 0, \tag{1}
$$

$$
\dot{\tau}(t) \leq d < 1, \qquad \forall t \geq 0, \tag{2}
$$

where $0 \leq d < 1$, and

$$
\tau \in W^{2,\infty}([0,T]), \qquad \forall T > 0. \tag{3}
$$

# Exponential stability

Assume that $\alpha$, $\beta$, $d$ in (KdVd) are real constants satisfying

$$\text{The matrix } \Phi_{\alpha,\beta} = \begin{pmatrix} \alpha^2 - 1 + |\beta| & \alpha\beta \\ \alpha\beta & \beta^2 + |\beta|(d-1) \end{pmatrix} \text{ is definite negative.} \tag{4}$$

**Theorem (P-Valein-Timimoun)**

*Suppose that* (1)-(4) *are satisfied and that* $L < \pi\sqrt{3}$. *Then, there exists* $r > 0$ *such that, for every* $(y_0, z_0) \in H$ *satisfying* $\|(y_0, z_0)\|_0 \leq r$, *we have* $E(t) \leq Ce^{-2\gamma t}E(0)$, $\forall t > 0$, *where, for* $\mu_1$ *and* $\mu_2$ *small enough,*

$$\gamma \leq \min\left\{ \frac{(9\pi^2 - 3L^2 - 2L^{3/2}r\pi^2)\mu_1}{3L^2(1+2L\mu_1)}, \frac{(1-d)\mu_2}{M(2\mu_2+|\beta|)} \right\}, \quad C \leq 1 + \max\left\{ L\mu_1, \frac{2\mu_2}{|\beta|} \right\}.$$

- In the no delay, and constant delay cases, it is also shown in Baudouin et al. (2019) the exponential stability for all $L \notin \mathcal{N}$. In our case, we can not obtain the same because our system is not invariant by translation in time.

# Well-posedness ideas

In this part, we put our focus in the study of linearization around 0 of (KdVd), that is

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \quad t > 0, \ x \in (0, L), \\ y(0,t) = y(L,t) = 0, \quad t > 0, \\ y_x(L,t) = \alpha y_x(0,t) + \beta y_x(0, t - \tau(t)), \quad t > 0, \\ y(x,0) = y_0(x), \quad x \in (0, L), \\ y_x(0, t - \tau(0)) = z_0(t - \tau(0)), \quad 0 < t < \tau(0). \end{cases} \tag{5}$$

Now, we introduce a new variable that takes into account the delay term (Nicaise et al. (2009)). Let $z(\rho, t) = y_x(0, t - \tau(t)\rho)$ for $\rho \in (0,1)$ and $t > 0$. Then, $z$ verifies the following transport equation

$$\begin{cases} \tau(t) z_t(\rho, t) + (1 - \dot{\tau}(t)\rho) z_\rho(\rho, t) = 0, \ t > 0, \ \rho \in (0,1), \\ z(0,t) = y_x(0,t), \quad t > 0, \\ z(\rho, 0) = z_0(-\tau(0)\rho), \quad \rho \in (0,1). \end{cases}$$

# Well-posedness ideas

Define $U = (y, z)^T$, then

$$\begin{cases} U_t(t) = \mathcal{A}(t)U(t), \qquad t > 0, \qquad \mathcal{A}(t)\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -y_x - y_{xxx} \\ \dfrac{\dot{\tau}(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix}, \\ U(0) = (y_0, z_0(-\tau(0)\cdot))^T =: U_0. \end{cases} \tag{6}$$

with

$$D(\mathcal{A}(t)) = \left\{(y, z) \in \left(H^3(0, L) \cap H_0^1(0, L)\right) \times H^1(0, 1),\ z(0) = y_x(0), \right. \\ \left. y_x(L) = \alpha y_x(0) + \beta z(1)\right\}.$$

Note that the domain of the operator $\mathcal{A}(t)$ is independent of time $t$, i.e., $D(\mathcal{A}(t)) = D(\mathcal{A}(0))$, $t > 0$. Constant domain system (CD-system). Introduce the Hilbert space $H = L^2(0, L) \times L^2(0, 1)$,

# Well-posedness ideas

The following theorem gives the existence and uniqueness results:

**Theorem (Kato (1970))**

*Assume that*

1. $\mathcal{Y} = D(\mathcal{A}(0))$ *is a dense subset of* $H$,
2. $D(\mathcal{A}(t)) = D(\mathcal{A}(0))$, *for all* $t > 0$,
3. *for all* $t \in [0, T]$, $\mathcal{A}(t)$ *generates a strongly continuous semigroup on* $H$ *and the family* $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$ *is stable with stability constants* $C$ *and* $m$ *independent of* $t$ *(i.e. the semigroup* $(S_t(s))_{s\geq 0}$ *generated by* $\mathcal{A}(t)$ *satisfies* $\|S_t(s)U\|_H \leq Ce^{ms}\|U\|_H$, *for all* $U \in H$ *and* $s \geq 0$*),*
4. $\partial_t \mathcal{A}(t)$ *belongs to* $L^\infty_*([0, T], B(\mathcal{Y}, H))$, *the space of equivalent classes of essentially bounded, strongly measure functions from* $[0, T]$ *into the set* $B(\mathcal{Y}, H)$ *of bounded operators from* $\mathcal{Y}$ *into* $H$.

*Then, problem* (6) *has a unique solution* $U \in C([0, T], \mathcal{Y}) \cap C^1([0, T], H)$ *for any initial datum in* $\mathcal{Y}$.

# Stability analysis

Consider the Lyapunov candidate

$$V(t) = E(t) + \mu_1 V_1(t) + \mu_2 V_2(t), \tag{7}$$

where $\mu_1$, $\mu_2 > 0$ and

$$E(t) = \frac{1}{2}\int_0^L y^2(x,t)dx + \frac{|\beta|}{2}\tau(t)\int_0^1 y_x^2(0, t-\tau(t)\rho)d\rho. \tag{8}$$

$$V_1(t) = \int_0^L x y^2(x,t)dx \tag{9}$$

$$V_2(t) = \tau(t)\int_0^1 (1-\rho) y_x^2(0, t-\tau(t)\rho)d\rho.$$

$E$ is the energy of the system, $V_1$ is classical for the KdV equation and $V_2$ comes from the delay term depending on time.

# Stability analysis

For $\mu_1$, $\mu_2 > 0$ small enough

$$\dot{V}(t) + 2\gamma V(t) \leq \left( \frac{L^2}{\pi^2}(\mu_1 + \gamma + 2L\mu_1\gamma) + \frac{2}{3}L^{3/2} r\mu_1 - 3\mu_1 \right) \int_0^L y_x^2 dx$$
$$+ (\gamma|\beta|M + 2\mu_2\gamma M - \mu_2(1-d)) \int_0^1 y_x^2(0, t - \tau(t)\rho) d\rho.$$

Now, following Baudouin et al. (2019), as $L < \pi\sqrt{3}$, it is possible to choose $r$ small enough to have $r < \frac{3(3\pi^2 - L^2)}{2L^{3/2}\pi^2}$. Then, we can choose $\gamma > 0$ such that $\dot{V}(t) + 2\gamma V(t) \leq 0$ and hence $V(t) \leq V(0)e^{-2\gamma t}$ for all $t > 0$. Using

$$E(t) \leq V(t) \leq \left( 1 + \max\left\{ L\mu_1, \frac{2\mu_2}{|\beta|} \right\} \right) E(t). \tag{10}$$

we obtain the exponential decay.

# Stability analysis: Numerical Simulations

Consider $L = 1$, $T = 10$, initial conditions $y_0(x) = 0.5(1 - \cos(2\pi x))$, $z_0(\rho) = -0.5\sin(2\pi\rho)$ and the delay is $\tau(t) = d(1.5 + \sin(t))$.

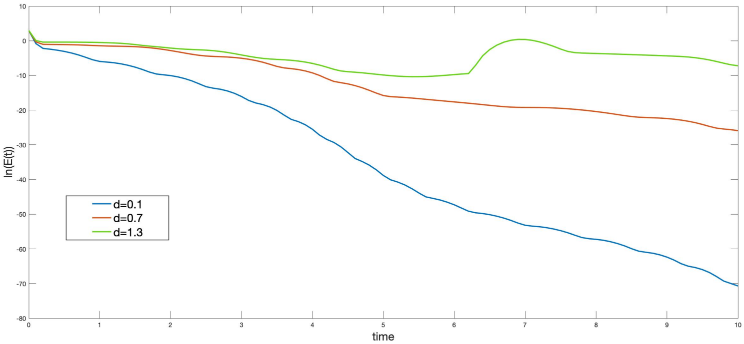


Figure: Time-evolution of $t \mapsto \ln(E(t))$ for different values of $d$.

# Stability analysis: Numerical Simulations

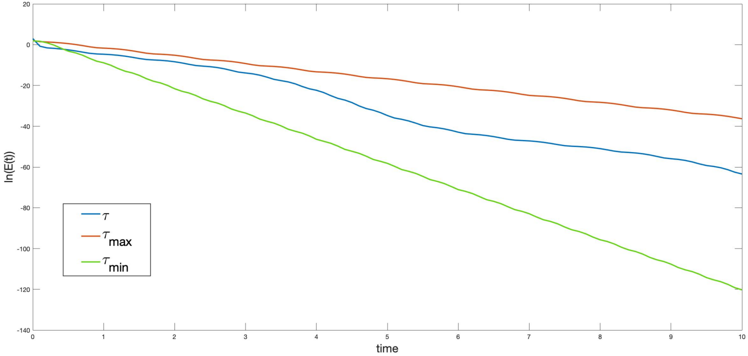


Figure: Time-evolution of $t \mapsto \ln(E(t))$ in the case of constant and varying boundary delay.

# Internal time-varying delay

Similar results can be obtained for the system

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x + a(x)y(x,t) \\ +b(x)y(x,t-\tau(t)) = 0, & t>0,\ x\in(0,L), \\ y(0,t) = y(L,t) = y_x(L,t) = 0, & t>0, \\ y(x,0) = y_0(x), & x\in(0,L), \\ y(x,t-\tau(0)) = z_0(x,t-\tau(0)), & 0<t<\tau(0),\ x\in(0,L), \end{cases} \tag{11}$$

# KdV Networks

We study the exponential stabilization problem of the KdV equation posed in a star shaped network where the branches mix finite intervals and half-lines.
Let $N = N_1 + N_2$ the number of edges of a network $\mathcal{T}$ described as the intervals $I_j$ for $j = 1, \dots N$, where $I_j = (0, \ell_j)$ with $\ell_j > 0$ for $j = 1, \dots N_1$ and $I_j = (0, \infty)$ for $j = N_1 + 1, \dots N$. The network is defined by $\mathcal{T} = \bigcup_{n=1}^{N} k_n$.

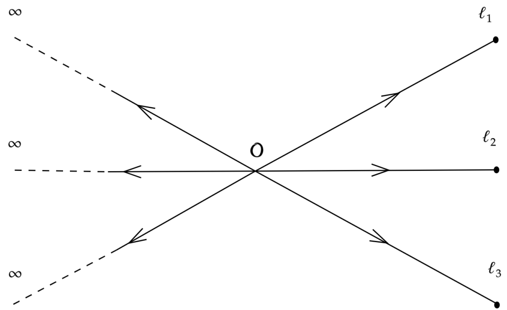


Figure: Star Shaped Network for $N_1 = 3$ and $N_2 = 3$.

# KdV Networks

Specifically, we consider the next evolution problems for the KdV equation

$$\begin{cases} \partial_t u_j + \partial_x u_j + u_j \partial_x u_j + \partial_x^3 u_j + a_j u_j = 0, & \forall x \in I_j,\ t > 0,\ j = 1, \dots, N, \\ u_j(t,0) = u_{j'}(t,0), & \forall j, j' = 1, \dots N, \\ \displaystyle\sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\alpha u_1(t,0) - \frac{N}{3}(u_1(t,0))^2, & t > 0, \\ u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0, & t > 0,\ j = 1, \dots, N_1, \\ u_j(0,x) = u_j^0(x), & x \in I_j, j = 1, \dots, N, \end{cases} \tag{KdV}$$

where $\alpha > \dfrac{N}{2}$, the damping terms $(a_j)_{j=1,\dots,N} \in \prod_{j=1}^{N} L^\infty(I_j)$ act locally on each branch. Let $I_{act} \subset \{1 \dots, N\}$ the set of acted index, then

$$\begin{cases} a_j = 0 \ \text{ for } \ j \in \{1, \dots, N\} \setminus I_{act}, \\ a_j(x) \geq c_j > 0 \ \text{ in an open nonempty set } \omega_j \text{ of } I_j,\ \text{ for all } j \in I_{act}, \\ \text{If } N_1 + 1 \leq j \leq N \in I_{act},\ \text{ then } \omega_j = (b_j, \infty) \text{ for } b_j > 0 \text{ given.} \end{cases} \tag{12}$$

# LKdV Networks

Consider first

$$\begin{cases} \partial_t u_j + \partial_x u_j + \partial_x^3 u_j + a_j u_j = 0, & \forall x \in I_j,\ t > 0,\ j = 1, \dots, N, \\ u_j(t,0) = u_{j'}(t,0), & \forall j, j' = 1, \dots N, \\ \sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\alpha u_1(t,0), & t > 0, \\ u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0, & t > 0,\ j = 1, \dots, N_1, \\ u_j(0,x) = u_j^0(x), & x \in I_j, j = 1, \dots, N, \end{cases} \tag{LKdV}$$

Well-posedness:

- (LKdV) is globally well-posed with initial data in $\mathbb{L}^2(\mathcal{T}) = \prod_{j=1}^{N} L^2(I_j)$.
- (KdV) is locally well-posed with initial data in $\mathbb{Y} = \prod_{j=1}^{N_1} L^2(I_j) \times \prod_{j=N_1+1}^{N} L^2_{(1+x^2)}(I_j)$.

# Observability inequality

We can check that, (KdV) or (LKdV) the energy satisfies

$$\dot{E}(t) = -\left(\alpha - \frac{N}{2}\right)|u_1(t,0)|^2 - \frac{1}{2}\sum_{j=1}^{N}|\partial_x u_j(t,0)|^2 - \sum_{j=1}^{N}\int_{I_j} a_j(x)|u_j(t,x)|^2 dx \leq 0. \tag{13}$$

Observe that, as $a_j \geq 0$, the term $a_j u_j$ provides dissipation to the energy. To prove the exponential stability, it is enough to show the following observability inequality

$$E(0) \leq C_{obs}\int_0^T \left(|u_1(t,0)|^2 + \sum_{j=1}^{N}|\partial_x u_j(t,0)|^2 + \sum_{j=1}^{N}\int_{I_j} a_j|u_j|^2 dx\right) dt. \tag{Obs}$$

In fact, using (Obs) and the dissipation law (13) we can show $E(t) \leq \gamma E(0)$ for $0 < \gamma < 1$, finally as (KdV) (or (LKdV)) is invariant by translation in time, we derive the exponential decay (Perla Menzala et al. (2002)).

# Exponential stability in $\mathbb{L}^2(\mathcal{T})$

Define $I_c = \{j \in \{1, \dots, N_1\};\ \ell_j \in \mathcal{N}\}$ and $I_\infty = \{j : N_1 + 1 \leq j \leq N\}$ . Then take $I_c^*$ (resp $I_\infty^*$ ) be the subset of $I_c$ (resp $I_\infty$ ) where we remove one index.

**Theorem**

*Let $I_c^* \cup I_\infty \subset I_{act}$ or $I_c \cup I_\infty^* \subset I_{act}$, assume that the damping terms $(a_j)_{j=1,\dots,N}$ satisfy (12). Then, there exist $C$, $\mu > 0$ such that for all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, the energy of any solution of* (LKdV) *satisfies $E(t) \leq CE(0)e^{-\mu t}$ for all $t > 0$.*

**Idea of the proof:** We follow a contradiction argument as in Rosier (1997). Suppose that (Obs) is false, then there exists $(\underline{u}^{0,n})_{n\in\mathbb{N}} \subset \mathbb{L}^2(\mathcal{T})$ such that $\|\underline{u}^{0,n}\|_{\mathbb{L}^2(\mathcal{T})} = 1$ and such that

$$\begin{aligned}
&\|u_1^n(t,0)\|_{L^2(0,T)} \to 0,\\
&\|\partial_x \underline{u}^n(t,0)\|_{L^2(0,T)} \to 0,\\
&\sum_{j=1}^{N} \int_0^T \int_{I_j} a_j |u_j^n|^2 dxdt \to 0.
\end{aligned}$$

# Exponential stability in $\mathbb{L}^2(\mathcal{T})$

After some compactness argument, we search $\underline{u}$ to solve the following problem

$$\begin{cases} \partial_t u_j + \partial_x u_j + \partial_x^3 u_j = 0, & \forall x \in I_j,\ t \in (0,T),\ j = 1,\dots,N, \\ u_j(t,0) = \partial_x u_j(t,0) = 0, & \forall j, = 1,\dots N, \\ \sum_{j=1}^{N} \partial_x^2 u_j(t,0) = 0, & t \in (0,T), \\ u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0, & t > 0,\ j = 1,\dots,N_1, \\ u_j \equiv 0 \text{ in } (0,T) \times \omega_j, & j \in I_{act} \\ u_j(0,x) = u_j^0(x), & x \in I_j, j = 1,\dots,N. \end{cases} \tag{14}$$

We want to show that $\underline{u} = \underline{0}$.Here we have two cases:

- $I_c^* \cup I_\infty \subset I_{act}$. In this case, for $j \in I_\infty$, $w = u_j$ solves

$$\begin{cases} \partial_t w + \partial_x w + \partial_x^3 w = 0, & \forall x \in (0,\infty),\ t \in (0,T), \\ w(t,0) = \partial_x w(t,0) = 0, t \in (0,T) & \\ w \equiv 0 \text{ in } (0,T) \times \omega_j. & \end{cases}$$

Then, by Holmgren's Theorem (see also (Linares and Pazoto, 2009, Theorem 1.1)), $w \equiv 0$ in $(0,\infty) \times (0,T)$.

# Exponential stability in $\mathbb{L}^2(\mathcal{T})$

Then, have the following problem

$$\begin{cases} \partial_t u_j + \partial_x u_j + \partial_x^3 u_j = 0, & \forall x \in (0, \ell_j),\ t \in (0,T),\ j = 1, \dots, N_1, \\ u_j(t,0) = \partial_x u_j(t,0) = 0, & \forall j, = 1, \dots N_1, \\ \sum_{j=1}^{N_1} \partial_x^2 u_j(t,0) = 0, & t \in (0,T), \\ u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0, & t > 0,\ j = 1, \dots, N_1, \\ u_j \equiv 0 \text{ in } (0,T) \times \omega_j, & j \in I_c^*. \end{cases}$$

The above system is exactly the same studied in Ammari and Crépeau (2018). Thus, by (Ammari and Crépeau, 2018, Theorem 3.1) as we are acting in $I_c^*$ we get $u_j \equiv 0$ for $j = 1, \dots, N_1$ and finally, $\underline{u} \equiv 0$.

- $I_c \cup I_\infty^* \subset I_{act}$. Follows similarly. $\square$

*This result is optimal in the sense that if we remove one index more, we can find solutions of* (LKdV) *with constant energy.*

# Exponential stability in $\mathbb{L}^2(\mathcal{T})$

### Theorem (Small amplitude solutions)

*Let $I_c^* \cup I_\infty \subset I_{act}$ or $I_c \cup I_\infty^* \subset I_{act}$ , assume that the damping terms $(a_j)_{j=1,\dots,N}$ satisfy* (12). *Then there exist $C,\mu,\epsilon > 0$ such that for all $\underline{u}^0 \in \mathbb{Y}$, with $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq \epsilon$ and $\sum_{j=N_1+1}^{N} \|u_j^0\|_{L^2_{(1+x^2)}(I_j)} \leq C\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$ , the energy of any solution of* (KdV) *satisfies $E(t) \leq CE(0)e^{-\mu t}$ for all $t > 0$.*

**Proof:** Let $\underline{u} = \underline{u}^1 + \underline{u}^2$ where $\underline{u}^1$ solves (LKdV) and $\underline{u}^2$ involves the nonlinear terms.

### Theorem (Semiglobal)

*Assume that the damping terms $(a_j)_{j=1,\dots,N}$ satisfy* (12) *and let $R > 0$ . If $I_{act} = \{1,\dots,N\}$, then, there exist $C(R) > 0$ and $\mu(R) > 0$ such that for all $\underline{u}^0 \in \mathbb{Y}$ with $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq R$, the energy of the solution of* (KdV) *satisfies $E(t) \leq CE(0)e^{-\mu t}$ for all $t > 0$.*

**Proof:** Similar as the linear case, estimation of the nonlinear terms and UCP for the nonlinear system.

# Exponential stability in $\mathbb{Y}$

We already have the exponential stability in $L^2(I_j)$ for $j = 1, \ldots, N$, thus we only have to prove the exponential stability in $L^2_{(1+x^2)}(I_j)$ for $j = N_1 + 1, \ldots, N$. Take $V_0(\underline{u}(t,\cdot)) = E(t)$, for $m = 1{,}2$ we define

$$V_m(\underline{u}) = \frac{1}{2}\sum_{j=1}^{N}\int_{I_j}(1+x^m)|u_j|^2dx + d_{m-1}V_{m-1}(\underline{u}), \tag{15}$$

where, $d_0$, $d_1 > 0$ and large enough.

### Theorem

*Assume that the damping terms $(a_j)_{j=1,\ldots,N}$ satisfy (12) and let $R > 0$. If $I_{act} = \{1, \ldots, N\}$, then, there exist $C(R) > 0$ and $\mu(R) > 0$ such that for all $\underline{u}^0 \in \mathbb{Y}$ with $\|\underline{u}^0\|_{\mathbb{Y}} \leq R$, we have for $\underline{u}$ solution of (KdV) $V_2(t) \leq CV_2(0)e^{-\mu t}$, for all $t > 0$.*

**Proof:**

- Exp stability of $V_0$ OK (Semiglobal result).
- Exp stability of $V_1$ using decay of $V_0$.
- Conclude for $V_2$ using decay of $V_0$ and $V_1$.

# Some perspectives

DELAYED SYSTEMS

- Take in our results $0 \leq \tau(t) \leq M$.
- Delay in the nonlinearity $y(t-h,x)y_x(t,x)$ or $y(t,x)y_x(t-h,x)$.
- Space-time varying delay $\tau(t,x)$.
- Other equations as Kawahara (fifth order KdV).

SYSTEM ON NETWORKS

- Global well-posedness of (KdV) $\mathbb{L}^2(\mathcal{T})$.
- Null controllability results for KdV on networks, zero-dispersion limit, Carleman estimates.
- Other systems on networks as Kuramoto-Sivashinksy, Benjamin-Bona-Mahony, Pulse propagation in optic fibers, etc.

Thank you for your attention

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