Optimal Boundary Control for the Transport Equation under Uncertainty: A Turnpike Result

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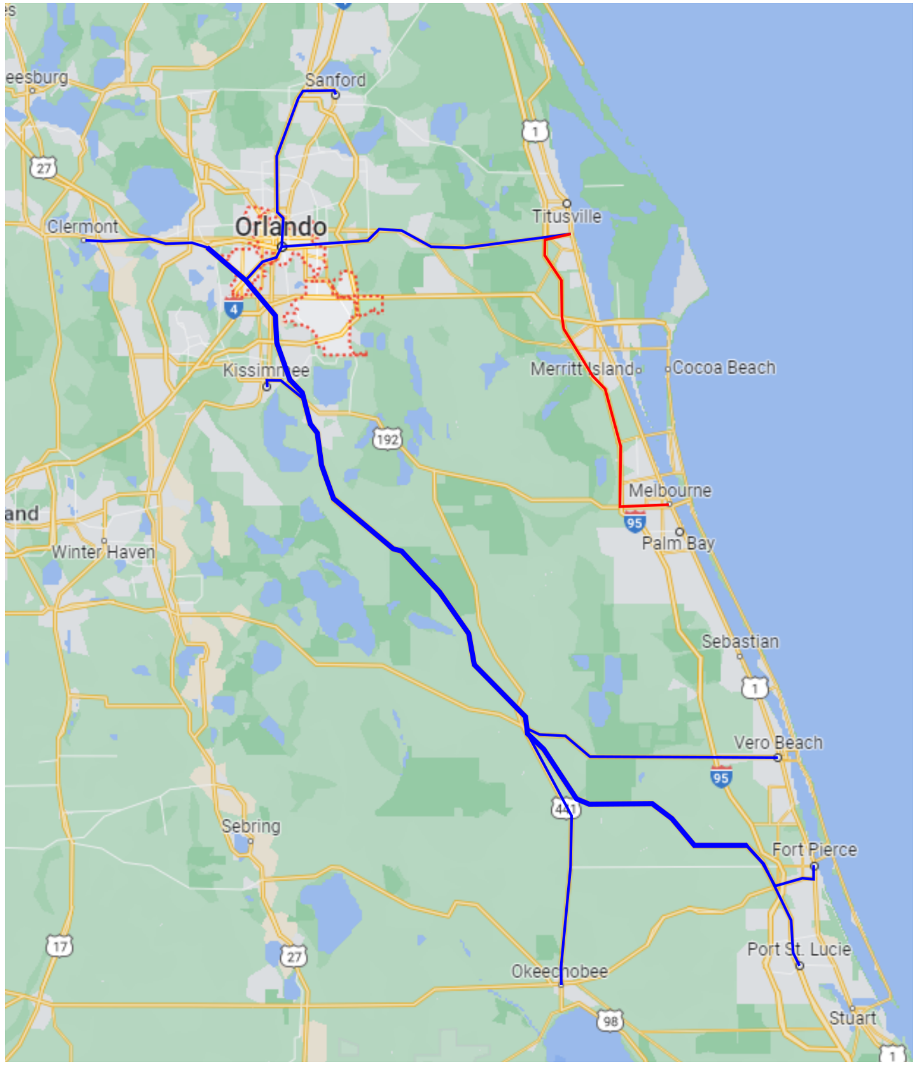
Motivation

The Turnpike Phenomenon



- There is a fastest route between any two points and if the origin and destination are close together and far from the turnpike, the best route may not touch the turnpike.
- But if origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end.

[Dorfman, Samuelson, Solow, 1958]: Linear Programming and Economic Analysis. *New York: McGraw-Hill*

Motivation

The Turnpike Phenomenon

Time dependent optimal control

$$\min_{u^\delta \in L^\infty(0,T;\mathbb{R}^m)} \int_0^T f_0(x(t), u^\delta(t))\, dt$$
$$\text{s.t.} \quad x'(t) = A\,x + B\,u^\delta$$
$$x(0) = x_0, \quad x(T) = x_1$$

Static state optimal control

$$\min_{u^\sigma \in \mathbb{R}^m} f_0(x, u^\sigma)$$
$$\text{s.t.} \quad A\,x + B\,u^\sigma = 0$$

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Static state optimal control

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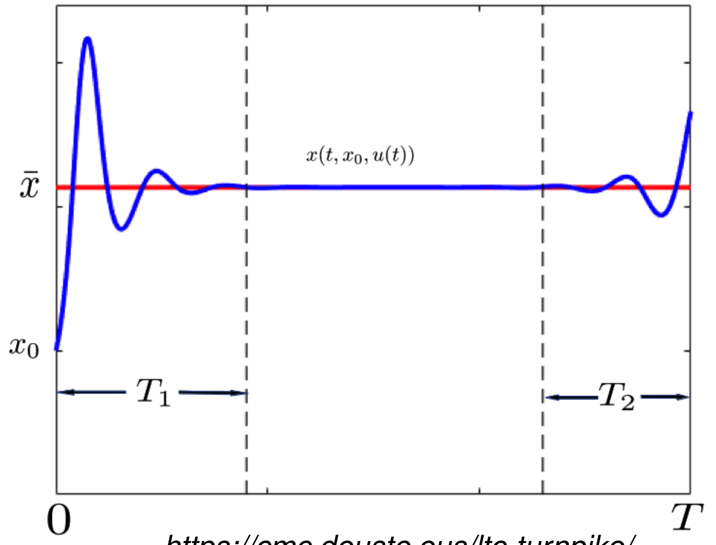
Theorem: Turnpike result

There exist constants $C_1 > 0$ and $C_2 > 0$, s.t. for every time $T > 0$ the time dependent optimal control problem has a unique solution $(u^{\delta,*}, x^{\delta,*})$, which satisfies

$$\|u^{\delta,*}(t) - u^{\sigma,*}\| + \|x^{\delta,*}(t) - x^{\sigma,*}\| \leq C_1 \Big(\exp\big(-C_2 t\big) + \exp\big(-C_2(T-t)\big)\Big),$$

where $(u^{\sigma,*}, x^{\sigma,*})$ is the (unique) solution of the static problem.



https://cmc.deusto.eus/ltc-turnpike/

A. Porretta, E. Zuazua (2013): *Long Time versus Steady State Optimal Control.* SIAM J. Control Optim. 51(6), 4242–4273

E. Trélat, E. Zuazua (2015): *The Turnpike Property in finite-dimensional nonlinear optimal control.* J. Differential Equations 258, pp. 81–114

A Turnpike Result for the Transport Eq.

Deterministic Optimal Control

For $c > 0$ consider the transport equation in one dimension

$$r_t(t,x) + c\, r_x(t,x) = m\, r(t,x),$$

with initial condition and boundary control

$$r(0,x) = r_{\text{ini}}(x) \qquad \text{and} \qquad r(t,0) = u(t).$$

For convex functions f and g consider the optimal control problems

Dynamic Optimal Control Problem

$$\begin{aligned} \min_{u \in L^2(0,T)} \quad & J_T(u) = \int_0^T f\big(u(t)\big) + g\big(r(t,L)\big)\, dt \\ \text{s.t.} \quad & r_t(t,x) + c r_x(t,x) = m\, r(t,x), \\ & r(0,x) = r_{\text{ini}}(x), \\ & r(t,0) = u(t). \end{aligned}$$

Static Optimal Control Problem

$$\begin{aligned} \min_{u \in \mathbb{R}} \quad & J(u) = f\big(u\big) + g\big(r(L)\big) \\ \text{s.t.} \quad & c r_x(x) = m\, r(x), \\ & r(0) = u. \end{aligned}$$

A Turnpike Result for the Transport Eq.

Deterministic Optimal Control

(A1) For $\varepsilon > 0$ let functions f and g satisfy

$$\big(f'(x_1) - f'(x_2)\big)\big(x_1 - x_2\big) + \big(g'(y_1) - g'(y_2)\big)\big(y_1 - y_2\big) \;\geq\; \varepsilon \, \|x_1 - x_2\|_2^2.$$

(A2) Let the derivative of g be Lipschitz continuous with Lipschitz constant L_k, i.e.,

$$\|g'(y_1) - g'(y_2)\|_2 \;\leq\; L_k \, \|y_1 - y_2\|_2.$$

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Theorem: Deterministic Turnpike

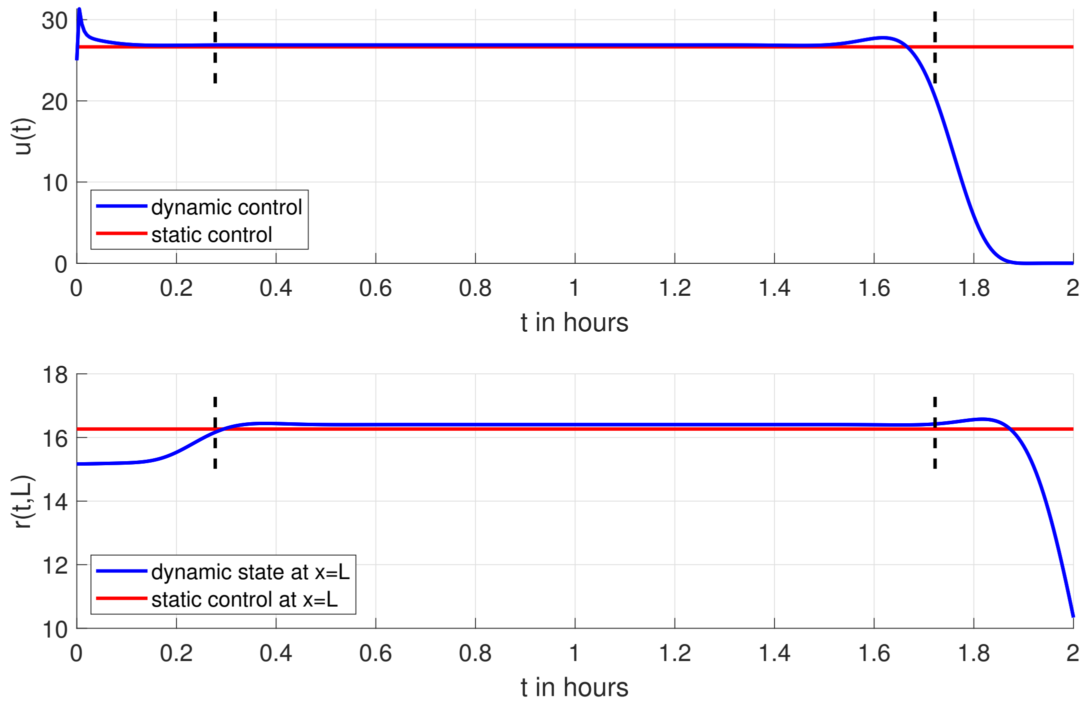
The optimal solution $u^\delta(t)$ of the dynamic optimal control problem

$$\min_{u \in L^2(0,T)} \quad J_T(u) = \int_0^T f\big(u(t)\big) + g\big(r(t,L)\big)\, dt$$

$$\text{s.t.} \quad r_t(t,x) + c r_x(t,x) = m\, r(t,x),$$
$$r(0,x) = r_{\text{ini}}(x),\; r(t,0) = u(t).$$

and the optimal solution u^σ of the corresponding static problem satisfy the integral turnpike property

$$\int_0^T \|u^\delta(t) - u^\sigma\|_2^2\, dt \;\leq\; \mathcal{C}.$$



A Turnpike Result for the Transport Eq.

Deterministic Optimal Control

Sketch of the proof.

Part I: The solution of the transport equation is given by

$$r(t,x) = \begin{cases} \exp(\, mt\,)\, r_{\text{ini}}(\, x - ct\,) & x > ct \\ \exp\left(\, m\frac{x}{c}\,\right) u\left(\, t - \frac{x}{c}\,\right) & x \leq ct \end{cases}$$

Part II: The derivative of the objective function is given by

$$J_T'(u) = f'\big(u(t)\big) + k\, g'\big(k\, u(t)\big)\, \psi(t),$$

with $k = \exp\left(m\frac{L}{c}\right)$ and $\psi(t) = \begin{cases} 1 & 0 < t < T - \frac{L}{c} \\ 0 & \text{else} \end{cases}$.

Part III: Let $u^{\delta}(t)$ and u^{σ} be the optimal dynamic and static solution. Then we have necessary optimality condition

$$f'\big(u^{\delta}(t)\big) - f'\big(u^{\sigma}\big) = k\, g'\big(k\, u^{\sigma}\big) - k\, g'\big(k\, u^{\delta}(t)\big)\, \psi(t).$$

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Sketch of the proof.

Part IV: Starting from assumption (A1) we have

$$\varepsilon \int_0^T \|u^\delta(t) - u^\sigma\|_2^2 \, dt \;\leq\; \int_0^T \Big(f'(u^\delta(t)) - f'(u^\sigma)\Big)\Big(u^\delta(t) - u^\sigma\Big) + \Big(g'(r^\delta(t,L)) - g'(r^\sigma(L))\Big)\Big(r^\delta(t,L) - r^\sigma(L)\Big) \, dt$$

⋮ (apply necessary optimality conditions, use integration by substitution) ⋮

$$\varepsilon \int_0^T \|u^\delta(t) - u^\sigma\|_2^2 \, dt \;\leq\; \int_{T-\frac{L}{c}}^T k\, g'(k\, u^\sigma)\Big(u^\delta(t) - u^\sigma\Big) \, dt$$
$$+ \int_0^{\frac{L}{c}} \Big(g'\big(\exp(mt) r_{\text{ini}}(L - ct)\big) - g'(ku^\sigma)\Big)\Big(\exp(mt) r_{\text{ini}}(L - ct) - ku^\sigma\Big) \, dt$$

⋮ (apply Cauchy-Schwarz inequality and (A2)) ⋮

$$\varepsilon \, \|u^\delta(t) - u^\sigma\|_{L^2(0,T)}^2 \;\leq\; z_1 \, \|u^\delta(t) - u^\sigma\|_{L^2(0,T)} + \frac{L_k}{c} z_2^2 \, \|r_{\text{ini}}(x)\|_{L^2(0,L)}^2 + z_3 \qquad \square$$

A Turnpike Result for the Transport Eq.

Optimal Control under Uncertainty

Assume that the initial data is now perturbed by a *Wiener process*, i.e., we have

$$r^{\omega}_{\text{ini}}(x) = r_{\text{ini}}(x) + W_x \qquad \text{with} \qquad W_x = \sqrt{2\,L}\,\sum_{k=1}^{\infty} \xi_k \,\frac{\sin\left(\left(k-\frac{1}{2}\right)\pi \frac{x}{L}\right)}{\left(k-\frac{1}{2}\right)\pi}.$$

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Theorem: TP with uncertain initial data

The optimal solution $u^{\delta}(t)$ of the dynamic optimal control problem

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$$\text{s.t.} \quad r_t(t,x) + c r_x(t,x) = m\, r(t,x),$$

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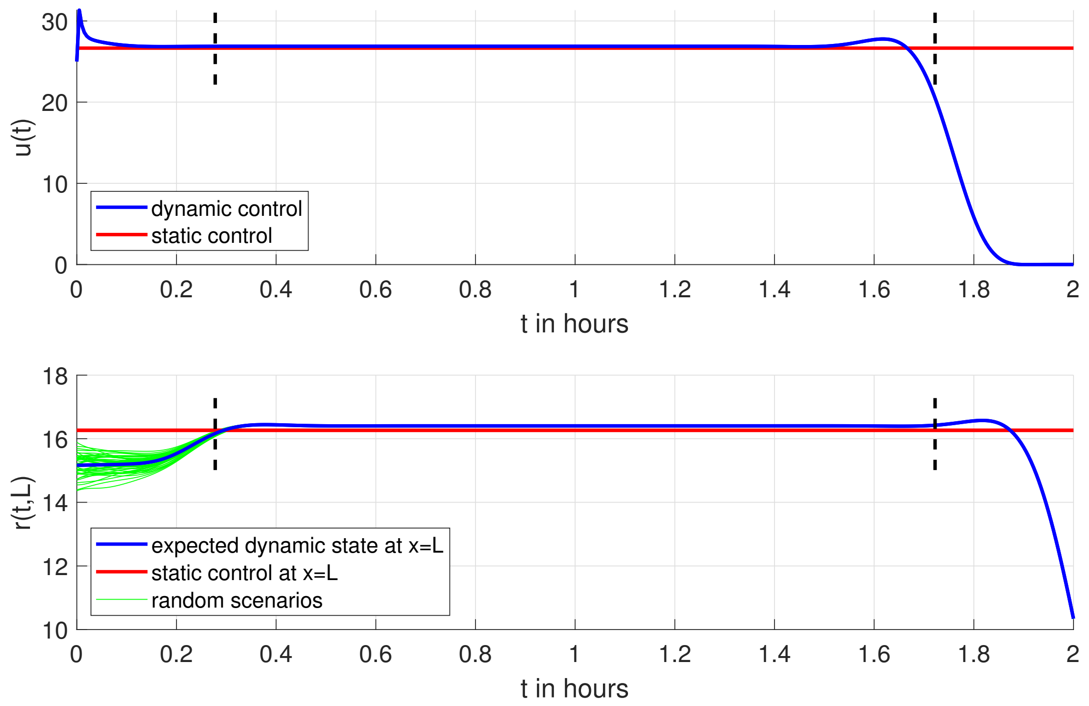
$$\int_0^T \|u^{\delta}(t) - u^{\sigma}\|_2^2 \, dt \;\leq\; \mathcal{C}.$$



A Turnpike Result for the Transport Eq.

Optimal Control under Uncertainty

We randomize the source term by a random variable ξ on an appropriate probability space: $m^\omega := \xi(\omega), \quad \omega \in \Omega$

(A3) Assume that $e_0(t)$ is uniformly bounded, where e_0 is defined as

$$e_0 : [0,T] \to \mathbb{R} \cup \{\pm\infty\} \quad t \mapsto \int_{-\infty}^{\infty} \exp(zt)\, \varrho_\xi(z)\, dz,$$

(A4) Assume that

$$\int_{-\infty}^{\infty} \exp\left(z\frac{L}{c}\right) \varrho_\xi(z)\, dz \quad < \infty$$

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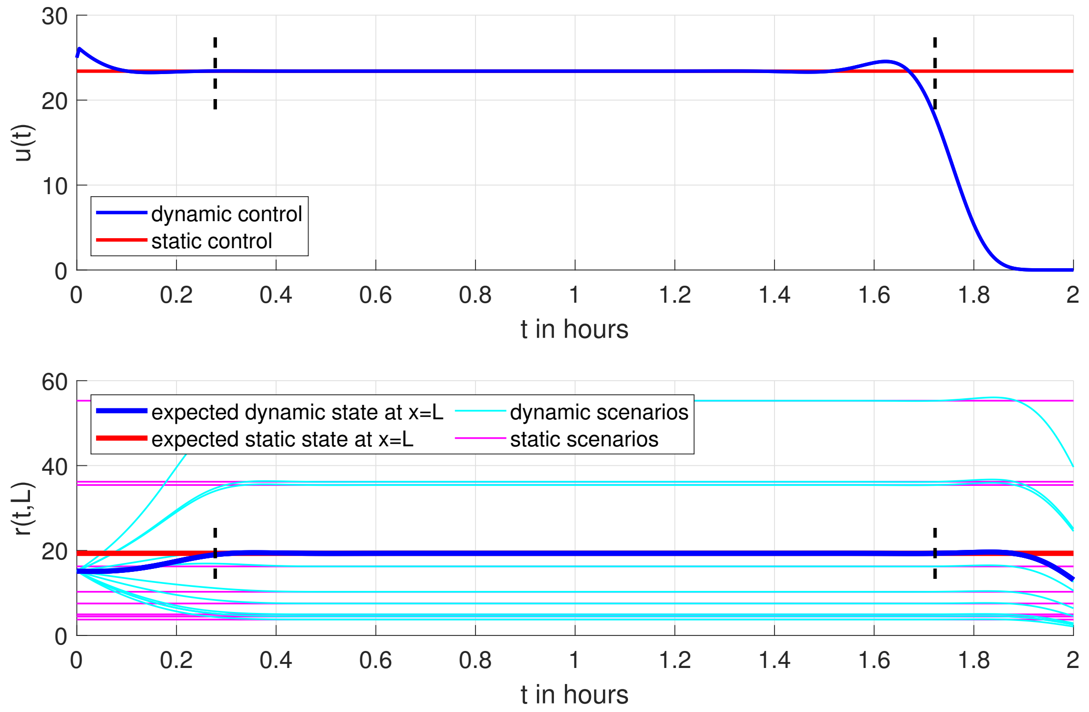

A Turnpike Result for the Transport Eq.

Semi-Linear Optimal Control and Future Work

Consider an optimal boundary control problem with the transport equation nonlinear source term:

$$
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A solution of the transport equation with nonlinear source term is given by

$$r(t,x) = \begin{cases} G^{-1}\Big(t + G\big(r_{\text{ini}}(x - ct) \big)\Big) & x > c\, t, \\ G^{-1}\Big(\frac{x}{c} + G\big(u(t - \frac{x}{c}) \big)\Big) & x \le c\, t, \end{cases}$$

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where G is an anti-derivative of $1/h$.

Under some assumptions we can compute the derivative

$$J'(u) = \frac{\delta F[u]}{\delta u(t)} = f'\big(u(t)\big) + g'\big(r(t,L,u(t))\big)\, \frac{\partial}{\partial u} r(t,x,u(t)),$$

and state necessary optimality conditions.

Lipschitz continuity of g' is not sufficient for a Turnpike anymore!

Optimal Boundary Control for the Transport Equation under Uncertainty

[Sakamoto and Schuster, 2023]: *A Turnpike Result for Optimal Boundary Control Problems with the Transport Equation under Uncertainty.* Preprint https://opus4.kobv.de/opus4-trr154/frontdoor/index/index/docId/509

[Sakamoto and Schuster, 2023]: *Optimal Boundary Control for the Transport Equation with Nonlinear Source Term under Uncertainty: A Turnpike Result.* Preprint