Relaxation approximation and asymptotic stability of stratified solutions to the Incompressible Porous Media equation

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FAU DCN-AvH Workshop
11 November 2022

Joint work with Roberta Bianchini and Marius Paicu

The Incompressible Porous Media (IPM) system in two space dimensions is an active scalar equation

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0, \\ \mathbf{u} = -\kappa \nabla P + \mathbf{g}\rho, \qquad \mathbf{g} = (0, -g)^T, \qquad \text{(Darcy law)} \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{IPM}$$

modelling the dynamics of a fluid of density $\rho = \rho(t, x, y) : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ through a porous medium according to the Darcy law, where $\kappa > 0$ and $g > 0$ are the permeability coefficient and the gravity acceleration respectively, which hereafter are assumed to be $\kappa = g = 1$.

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- Application to the transport of a dissolved contaminant in porous media where the contaminant is convected with the subsurface water. For instance, one could be interested in the time taken by the pollutant to reach the water table below.
- The incompressibility condition together with Darcy's law implies that

$$\mathbf{u} = (\mathcal{R}_1 \mathcal{R}_2 \rho, -\mathcal{R}_1^2 \rho)$$

  where $(\mathcal{R}_1, \mathcal{R}_2)$ is the two-dimensional homogeneous Riesz transform of order 0, i.e.

$$\mathcal{R}_1 = (-\Delta)^{-1/2} \partial_x, \qquad \mathcal{R}_2 = (-\Delta)^{-1/2} \partial_y.$$

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- For this system, Córdoba, Gancedo and Orive (07') proved the local well-posedness in Hölder space C^δ with $0 < \delta < 1$ by the particle-trajectory method.
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- Due to the form of the velocity, all the steady states of (IPM) are stratified i.e. constant in x.
- And among these steady states $\rho_{eq} = g(y)$, there are only some for which one can hope to stabilise the system around. Here we focus on the linear and stable ones: $\bar{\rho}_{eq}(y) = \rho_0 - y$ where $\rho_0 > 0$ is a constant averaged density.

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- The stability coming from the fact that $\bar{\rho}'_{eq}(y) < 0$ implies that the density of the fluid is proportional to the depth i.e. *the density of the fluid increases the deeper you go.*

- Concretely, such stabilisation mechanism can be seen directly in the equation. Indeed, the linearisation of (IPM) around $\bar{\rho}_{eq}(y)$ reads

$$\partial_t \rho - \mathcal{R}_1^2 \rho = (\mathcal{R}_2 \mathcal{R}_1 \rho, -\mathcal{R}_1^2 \rho) \cdot \nabla \rho. \qquad \text{(IPM-diss)}$$

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- In the opposite scenario, if the density is inversely proportional to the depth, then one would recover the opposite sign in front of $\mathcal{R}_1^2 \rho$ which should lead to instability. This can be related to the Rayleigh–Bénard convection instability occurring even in the presence of diffusion.

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- Here, in a sense, we rely on the fact that the stratification inherent in the model serves as a stabilising mechanism for solutions to derive global-in-time results.

- We refer to the work of Elgindi (17') about the justification of nonlinear asymptotic stability of (IPM) in the whole space $\mathbb{R}^2$ for initial data in $H^{20}(\mathbb{R}^2)$.
- The analogous result in the periodic finite channel $\mathbb{T} \times [-\pi, \pi]$ is due to Castro, Córdoba and Lear in $H^{10}(\mathbb{R}^2)$.

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Our contributions are:

- The nonlinear asymptotic stability of (IPM) in $\dot{H}^{1-\tau}(\mathbb{R}^2) \cap \dot{H}^s(\mathbb{R}^2)$ with $s > 3$ and for any $0 < \tau < 1$.
- A new relaxation approximation of (IPM) by a two-dimensional Boussinesq system with damped velocity.
- As a byproduct of the above two results, an existence result for the two-dimensional Boussinesq system with damped velocity in a very similar setting.

Theorem (Bianchini-CB-Paicu 2022)

For any $0 < \tau < 1$, *let* $s \geq 3 + \tau$. *For any initial datum* $\rho_{\rm in} \in \dot{H}^{1-\tau}(\mathbb{R}^2) \cap \dot{H}^s(\mathbb{R}^2)$, *there exists a constant value* $0 < \delta_0 \ll 1$ *such that, under the assumption*

$$\|\rho_{\rm in} - \bar{\rho}_{eq}\|_{\dot{H}^{1-\tau} \cap \dot{H}^s} \leq \delta_0,$$

there exists a unique global-in-time smooth solution ρ *to system* (IPM-diss) *satisfying the following inequality for all times* $t > 0$

$$\|\widetilde{\rho}\|_{L^\infty_T(\dot{H}^{1-\tau} \cap \dot{H}^s)} + \|\mathcal{R}_1 \widetilde{\rho}\|_{L^2_T(\dot{H}^{1-\tau} \cap \dot{H}^s)} + \|\nabla \mathcal{R}_1^2 \widetilde{\rho}\|_{L^1_T(L^\infty)} \lesssim \|\widetilde{\rho}_{\rm in}\|_{\dot{H}^{1-\tau} \cap \dot{H}^s},$$

where $\widetilde{\rho} = \rho - \bar{\rho}_{eq}$.

Idea of proof

Recall that the equation we are interested in reads:

$$\partial_t \rho - \mathcal{R}_1^2 \rho = (\mathcal{R}_2 \mathcal{R}_1 \rho, -\mathcal{R}_1^2 \rho) \cdot \nabla \rho. \tag{1}$$

To justify the global-in-time existence of this equation, one way is to recover the following bound

$$\int_0^t \|(\nabla \mathcal{R}_1 \mathcal{R}_2 \rho, \nabla \mathcal{R}_1^2 \rho)\|_{L^\infty} < \infty.$$

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But how can one retrieve such bound?

Let us investigate the toy-model:

$$\partial_t \rho - \mathcal{R}_1^2\rho = 0. \tag{2}$$

In a Sobolev framework, performing standard energy estimates leads to, for any $s\in\mathbb{R}$,

$$\|\rho\|_{L^\infty_T(H^s)} + \|\mathcal{R}_1\rho\|_{L^2_T(H^s)} \leq \|\rho_{in}\|_{L^2} \tag{3}$$

Issue: this only gives a L^2-in-time bound that is not enough to control the advection term (except if one assumes $s \geq 20$).

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- $\dot{\Delta}_j$ the blocks associated to the Littlewood-Paley decomposition in $|\xi|$;
- $\dot{\Delta}_q^h$ the blocks associated to the Littlewood-Paley decomposition in the direction ξ_1,

and we define the following *homogeneous* and *anisotropic* Besov semi-norms:

$$\|f\|_{\dot{B}^{s_1,s_2}} \triangleq \left\|2^{js_1}2^{qs_2}\|\dot{\Delta}_j\dot{\Delta}_q^h f\|_{L^2(\mathbb{R}^d)}\right\|_{\ell^1(j\in\mathbb{Z},k\in\mathbb{Z})}.$$

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Recall that $\dot{\Delta}_j$ localises the support of the Fourier transform of a distribution in an annulus and $\dot{\Delta}_q^h$ localise it in a stripe. Therefore applying $\dot{\Delta}_j\dot{\Delta}_q^h$ localise it in the intersection of an annulus and a stripe.

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Such approach has been used in the past by many authors, Chemin, Paicu, Zhang, Xin et al., in the context of anisotropic Navier-Stokes system, the MHD system..

The main motivation for that is that the Bernstein properties are now available in the directions $|\xi|$ and ξ_1. Indeed, when applying these localisation to the equation, we get

$$\frac{d}{dt}\|\dot{\Delta}_j\dot{\Delta}^h_q\rho\|^2_{L^2}+\|\mathcal{R}_1\dot{\Delta}_j\dot{\Delta}^h_q\rho\|^2_{L^2}=0$$

A new bound

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And now we can apply Gronwall-like inequality to "simplify the squares":

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$$\|\rho\|_{L^\infty_T(\dot{B}^{s_1,s_2})}+\|\rho\|_{L^1_T(\dot{B}^{s_1-2,s_2+2})}\lesssim\|\rho_{in}\|_{\dot{B}^{s_1,s_2}}\qquad(4)$$

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And using the embedding $\dot{B}^{\frac{3}{2},\frac{1}{2}}_{2,1} \hookrightarrow \dot{W}^{1,\infty}$: $\|\nabla\rho\|_{L^\infty} \leq \|\rho\|_{\dot{B}^{\frac{3}{2},\frac{1}{2}}_{2,1}}$, we can choose the regularity index s_1 and s_2 so $\int_0^t \|\nabla\rho\|_{L^\infty}$ is bounded by the initial data.

Additional difficulties

Additionally to obtaining the bound of $\int_0^t \|\nabla\rho\|_{L^\infty}$, the other difficulties we tackled are:

- We cannot totally close the estimates in the anisotropic Besov spaces, which would give a sharper result. This is due to the lack of commutator adapted to this double localisation that makes us lose a derivative.

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$\rightarrow$ Still a gap to fill

Relaxation approximation of (IPM)

The two-dimensional Boussinesq system reads

$$\begin{cases} \partial_t \eta + \mathbf{u} \cdot \nabla \eta = 0, \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \eta \mathbf{g}, \qquad \mathbf{g} = (0, -g), \\ \nabla \cdot \mathbf{u} = 0. \end{cases} \tag{E}$$

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Considering a damping in the equation of the vorticity and linearizing around the same linear and stable steady states as before, it is shown by Bianchini and Natalini that the system can be recast into

$$\begin{cases} \partial_t b - \mathcal{R}_1 \Omega = (\mathcal{R}_2 \Omega, -\mathcal{R}_1 \Omega) \cdot (\nabla b), \\ \partial_t \Omega - \mathcal{R}_1 b + \dfrac{\Omega}{\varepsilon} - = \Lambda^{-1}[(\mathcal{R}_2 \Omega, -\mathcal{R}_1 \Omega) \cdot (\nabla \Lambda \Omega)], \end{cases} \tag{2D-B}$$

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with $\Omega = \Lambda^{-1}\omega$ where ω is the vorticity.
For this system:

- Wan (19') proved the global well-posedness in H^s with $s \geq 5$.
- Bianchini and Natalini (21') derived time-decay estimates.

Let us have a closer look at the linear structure:

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Taking inspiration from the theory of partially dissipative systems, by applying the following "diffusive" scaling:

$$(\widetilde{b}^\varepsilon, \widetilde{\Omega}^\varepsilon)(\tau, x) \triangleq (b, \frac{\Omega}{\varepsilon})(t, x) \quad \text{with} \quad \tau = \varepsilon t. \tag{6}$$

The system (2D-B) in the scaled unknowns $(\widetilde{b}^\varepsilon, \widetilde{\Omega}^\varepsilon)$ reads as follows:

$$\begin{cases} \partial_t \widetilde{b}^\varepsilon - \mathcal{R}_1 \widetilde{\Omega}^\varepsilon = 0, \\ \varepsilon^2 \partial_t \widetilde{\Omega}^\varepsilon - \mathcal{R}_1 \widetilde{b}^\varepsilon + \widetilde{\Omega}^\varepsilon = 0. \end{cases} \tag{7}$$

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$$\begin{cases} \partial_t b - \mathcal{R}_1 \Omega = 0, \\ \partial_t \Omega - \mathcal{R}_1 b + \dfrac{\Omega}{\varepsilon} = 0. \end{cases} \tag{5}$$

Taking inspiration from the theory of partially dissipative systems, by applying the following "diffusive" scaling:

$$(\widetilde{b}^\varepsilon, \widetilde{\Omega}^\varepsilon)(\tau, x) \triangleq (b, \frac{\Omega}{\varepsilon})(t, x) \quad \text{with} \quad \tau = \varepsilon t. \tag{6}$$

The system (2D-B) in the scaled unknowns $(\widetilde{b}^\varepsilon, \widetilde{\Omega}^\varepsilon)$ reads as follows:

$$\begin{cases} \partial_t \widetilde{b}^\varepsilon - \mathcal{R}_1 \widetilde{\Omega}^\varepsilon = 0, \\ \varepsilon^2 \partial_t \widetilde{\Omega}^\varepsilon - \mathcal{R}_1 \widetilde{b}^\varepsilon + \widetilde{\Omega}^\varepsilon = 0. \end{cases} \tag{7}$$

Formally, as $\varepsilon \to 0$, the second equation tends to the Darcy's law $\widetilde{\Omega}^\varepsilon = \mathcal{R}_1 \widetilde{b}^\varepsilon$ and inserting it in the first one gives the linear part of the incompressible porous media equation: $\partial_t \widetilde{b}^\varepsilon - \mathcal{R}_1^2 \widetilde{b}^\varepsilon = 0$.

- The rigorous justification follows from arguments developed by CB and Danchin in the context of partially dissipative system such as the compressible Euler equations with damping and its convergence to the porous media equation.
- Contrary to the work about the compressible Euler equations, here we are not able to obtain an explicit convergence rate, again due to the lack of commutator in our anisotropic Besov framework.
- Our analysis reveals that (2DB) can be used as a relaxation of (IPM) and provides an existence result for the 2D-Boussinesq system similar to the one of (IPM).

Thank you for your attention.

Lemma (Embedding in Sobolev space)

Let $s_1, s_2, \tau_1, \tau_2 \in \mathbb{R}$ *such that* $\tau_1 < s_1 + s_2 < \tau_2$ *and* $s_2 > 0$*. If* $a \in \dot{H}^{\tau_1}(\mathbb{R}^2) \cap \dot{H}^{\tau_2}(\mathbb{R}^2)$ *and* $a \in B^{s_1,s_2}$*, then*

$$\|a\|_{B^{s_1,s_2}} \lesssim \|a\|_{B^{s_1+s_2}} \lesssim \|a\|_{\dot{H}^{\tau_1}} + \|a\|_{\dot{H}^{\tau_2}}.$$

It has a structure analogous to the linear damped Euler equations:

$$\begin{cases} \partial_t \rho + \operatorname{div} v = 0, \\ \partial_t v + \nabla P + \dfrac{v}{\varepsilon} = 0. \end{cases} \tag{8}$$

For this system, under diffusive scaling it is known that the solution converge to the porous media equation

$$\begin{cases} \partial_t \rho - \delta P(\rho) = 0, \\ v = \dfrac{\nabla P}{\rho}. \end{cases} \tag{9}$$