Geometrically exact beam model:
well-posedness, stabilization, networks

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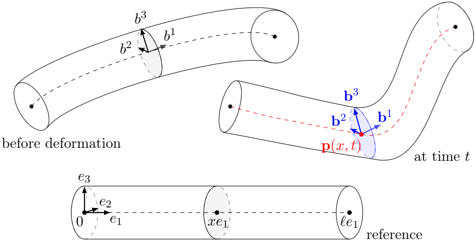
Presentation of the model

NASA Dryden Flight Research Center Photo Collection
http://www.dfrc.nasa.gov/gallery/photo/index.html
NASA Photo: ED01-0230-4 Date: August 13, 2001 Photo by: Carla Thomas
The Helios Prototype aircraft at approximately 10,000 feet flying above cloud cover northwest of Kauai, Hawaii.

Beam of length $\ell > 0$:

- Euler-Bernoulli
  spatial variable $x \in [0, \ell]$, time variable $t \geq 0$
  - Timoshenko
    $+$ cross sections not always perpendicular to centerline
    - Geometrically Exact Beam (GEB)
      "Geometrically exact Timoshenko beam", *"Geometric nonlinearity"*
      $+$ large motions (displacements, rotations of cross sections).

Presentation of the model



- Geometrically Exact Beam model (GEB)[1]
  - position $\mathbf{p} \in \mathbb{R}^3$ and rotation $\mathbf{R} \in \mathbb{R}^{3\times 3}$
  - fixed coordinate system
- **Intrinsic** GEB model (IGEB)[2]
  - linear velocity V, angular velocity W, internal forces Φ and internal moments Ψ, all in $\mathbb{R}^3$
  - moving coordinate system attached to the beam

[1] Reissner '81, Simo '85
[2] Hodges '03

Presentation of the model

Beam parameters: $\mathbf{M}$, $\mathbf{C}$, R.
Freely vibrating beam.

nonlinear transformation

$$(\mathbf{p}, \mathbf{R}) \overset{\mathcal{T}}{\longmapsto} \begin{bmatrix} V \\ W \\ \Phi \\ \Psi \end{bmatrix} = y$$

GEB

$$\partial_t \left(\begin{bmatrix} \mathbf{R} & 0 \\ 0 & \mathbf{R} \end{bmatrix} \mathbf{M} \begin{bmatrix} V \\ W \end{bmatrix} \right) = \partial_x \begin{bmatrix} \mathbf{R}\Phi \\ \mathbf{R}\Psi \end{bmatrix} + \begin{bmatrix} 0 \\ (\partial_x \mathbf{p}) \times (\mathbf{R}\Phi) \end{bmatrix}$$

IGEB

$$\partial_t y + A(x)\partial_x y + \overline{B}(x) y = \overline{g}(x, y)$$

first-order hyperbolic
semilinear
12 equations

Remark: $V, W, \Phi, \Psi \in \mathbb{R}^3$ are nonlinear functions of $\mathbf{p}, \mathbf{R}$ (we omit the formula here).

Single beam: well-posedness for IGEB

Notation: $y = \begin{bmatrix} v \\ z \end{bmatrix}, \quad v = \begin{bmatrix} V \\ W \end{bmatrix}, \ z = \begin{bmatrix} \Phi \\ \Psi \end{bmatrix}.$

We consider the IGEB model:

$$\begin{cases} \partial_t y + A(x)\partial_x y + \overline{B}(x) y = \overline{g}(x,y) & \text{in } (0,\ell)\times(0,T) \\ v = 0 & \text{on } \{\ell\}\times(0,T) \quad \text{clamped} \\ -z = -Kv & \text{on } \{0\}\times(0,T) \quad \text{velocity feedback control} \\ y = y^0 & \text{on } (0,\ell)\times\{0\} \end{cases}$$

With appropriate regularity of coefficients, eigenvalues/vectors of $A \Rightarrow$ assumptions on the beam parameters.

At least local **existence-uniqueness** results for **1-D first-order hyperbolic** systems:

- Bastin-Coron '16: in $C_t^0 H_x^2$ (H_x^2 data)
- Li '10: in $C^1_{x,t}$

Single beam: stabilization for IGEB

K symmetric positive definite

$$(1) \qquad \begin{cases} \partial_t y + A(x)\partial_x y + \overline{B}(x)y = \overline{g}(x,y) & \text{in } (0,\ell)\times(0,T) \\ v = 0 & \text{on } \{\ell\}\times(0,T) \\ -z = -Kv & \text{on } \{0\}\times(0,T) \\ y = y^0 & \text{on } (0,\ell)\times\{0\} \end{cases}$$

Theorem 1

If coefficients regular, the steady state $y \equiv 0$ of (1) is **locally H^2 exponentially stable**. Namely,
$\exists \varepsilon, \alpha, \eta > 0$ such that $\forall y^0 \in H^2(0,\ell;\mathbb{R}^{12})$ with $\|y^0\|_{H^2} \leq \varepsilon$ and fulfilling compatibility conditions, $\exists! y \in C^0([0,+\infty); H^2(0,\ell;\mathbb{R}^{12}))$ solution to (1), and

$$\|y(\cdot,t)\|_{H^2(0,\ell;\mathbb{R}^{12})} \leq \eta e^{-\alpha t}\|y^0\|_{H^2(0,\ell;\mathbb{R}^{12})}, \qquad \forall t.$$

- idea of the proof: quadratic Lyapunov functional
- a lot of work done in: Bastin-Coron '16
- possible because: first-order, A hyperbolic and no zero eigenvalue
- difficulty: $\overline{g}$ quadratic $+\ \overline{B}$ not small

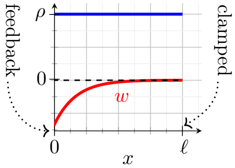
See: R.-Leugering arXiv:1912.02543v3 to appear in SICON (isotropic prismatic beams).

Single beam: stabilization for IGEB

Quadratic Lyapunov functional $\overline{\mathcal{L}}$:

easier to show exponential decay for $\overline{\mathcal{L}}$

$+$ $\overline{\mathcal{L}}$ equivalent to $\|y(\cdot,t)\|^2_{H^2(0,\ell;\mathbb{R}^{12})}$ when y is in some ball of $C^0_t C^1_x$



$$\overline{\mathcal{L}}(t) = \sum_{k=0}^{2} \int_0^\ell \left\langle \partial_t^k y \, , \left(\rho Q^{\mathcal{P}}(x) + w(x) \begin{bmatrix} \mathbf{0} & \mathbf{W}(x) \\ \mathbf{W}(x)^\intercal & \mathbf{0} \end{bmatrix} \right) \partial_t^k y \right\rangle dx.$$

Single beam: corresponding results for GEB

Corresponding GEB model:

$$(2) \quad \begin{cases} \partial_t \left(\left[\begin{smallmatrix} \mathbf{R} & 0 \\ 0 & \mathbf{R} \end{smallmatrix}\right] \mathbf{M} \left[\begin{smallmatrix} V \\ W \end{smallmatrix}\right] \right) = \partial_x \left[\begin{smallmatrix} \mathbf{R}\Phi \\ \mathbf{R}\Psi \end{smallmatrix}\right] + \left[\begin{smallmatrix} \mathbf{0} \\ (\partial_x \mathbf{p}) \times (\mathbf{R}\Phi) \end{smallmatrix}\right] & \text{in } (0,\ell) \times (0,T) \\ -\left[\begin{smallmatrix} \Phi \\ \Psi \end{smallmatrix}\right] = -K \left[\begin{smallmatrix} V \\ W \end{smallmatrix}\right] & \text{on } \{0\} \times (0,T) \\ (\mathbf{p}, \mathbf{R}) = (h^{\mathbf{p}}, h^{\mathbf{R}}) \quad \text{constant} & \text{on } \{\ell\} \times (0,T) \\ (\mathbf{p}, \mathbf{R}) = (\mathbf{p}^0, \mathbf{R}^0), \ (\partial_t \mathbf{p}, \mathbf{R}W) = (\mathbf{p}^1, w^0) & \text{on } (0,\ell) \times \{0\}. \end{cases}$$

Theorem 2

If coefficients regular and if data of both models fulfill compatibility conditions, then: $\exists!$ solution in $C^1_{x,t}$ to IGEB $\Rightarrow$ $\exists!$ solution in $C^2_{x,t}$ to GEB.

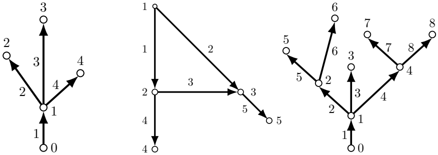
Idea of the proof: show that $\mathcal{T}$ is bijective, using: last six equations of IGEB as compatibility conditions + quaternions to parametrize rotations.

Corollary 1

Under assumptions of Theorems 1 and 2, where $y^0 = f(\mathbf{p}^0, \mathbf{R}^0, \mathbf{p}^1, w^0)$,
$\exists! (\mathbf{p}, \mathbf{R}) \in C^2([0,\ell] \times [0,+\infty); \mathbb{R}^3 \times \mathrm{SO}(3))$ solution to (2)
+ exponential decay of $\partial_t \mathbf{p}$, $\partial_t \mathbf{R}$ and Φ, Ψ.

See: R.-Leugering arXiv:1912.02543v3 to appear in SICON (isotropic prismatic beams).

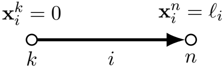
Networks of beams



- beams/edges indexed by $i \in \mathcal{I}$
- nodes indexed by $n \in \mathcal{N}$
- unknown state: $(\mathbf{p}_i, \mathbf{R}_i)_{i \in \mathcal{I}}$ or $(y_i)_{i \in \mathcal{I}}$

Notation: for any node n,

- $\mathcal{I}^n$ = indexes of edges incident to n
- $\mathbf{x}_i^n$ = end of the interval $[0, \ell_i]$ corresponding to n, for any $i \in \mathcal{I}^n$.



Networks: GEB and IGEB systems

At multiple nodes, the beams remain attached without rotating + balance of forces/moments.

$$
(3)\begin{cases}
\partial_t \left(\begin{bmatrix} \mathbf{R}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_i \end{bmatrix} \mathbf{M}_i \begin{bmatrix} V_i \\ W_i \end{bmatrix} \right) = \partial_x \begin{bmatrix} \mathbf{R}_i \Phi_i \\ \mathbf{R}_i \Psi_i \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ (\partial_x \mathbf{p}_i) \times (\mathbf{R}_i \Phi_i) \end{bmatrix} & \partial_t y_i + A_i(x) \partial_x y_i + \overline{B}_i(x) y_i = \overline{g}_i(x, y_i) & \text{in } (0,\ell_i) \times (0,T),\ i \in \mathcal{I} \\
\mathbf{p}_i(\mathbf{x}_i^n, t) = \mathbf{p}_j(\mathbf{x}_j^n, t), \quad (\mathbf{R}_i R_i^\intercal)(\mathbf{x}_i^n, t) = (\mathbf{R}_j R_j^\intercal)(\mathbf{x}_j^n, t) & (\overline{R}_i v_i)(\mathbf{x}_i^n, t) = (\overline{R}_j v_j)(\mathbf{x}_j^n, t) & t \in (0,T),\ i,j \in \mathcal{I}^n,\ n \in \mathcal{N}_M \\
\sum_{i \in \mathcal{I}^n} \tau_i^n \begin{bmatrix} \mathbf{R}_i \Phi_i \\ \mathbf{R}_i \Psi_i \end{bmatrix} (\mathbf{x}_i^n, t) = 0 & \sum_{i \in \mathcal{I}^n} \tau_i^n (\overline{R}_i z_i)(\mathbf{x}_i^n, t) = 0 & t \in (0,T),\ n \in \mathcal{N}_M \\
\tau_i^n \begin{bmatrix} \Phi_i \\ \Psi_i \end{bmatrix} = q_n & \tau_i^n z_i = q_n & \text{on } \{\mathbf{x}_i^n\} \times (0,T),\ i \in \mathcal{I}^n,\ n \in \mathcal{N}_S^z \\
(\mathbf{p}_i, \mathbf{R}_i) = (f_n^{\mathbf{p}}, f_n^{\mathbf{R}}) & v_i = q_n & \text{on } \{\mathbf{x}_i^n\} \times (0,T),\ i \in \mathcal{I}^n,\ n \in \mathcal{N}_S^v \\
(\mathbf{p}_i, \mathbf{R}_i) = (\mathbf{p}_i^0, \mathbf{R}_i^0), \quad (\partial_t \mathbf{p}_i, \mathbf{R}_i W_i) = (\mathbf{p}_i^1, w_i^0) & y_i = y_i^0 & \text{on } (0,\ell_i) \times \{0\},\ i \in \mathcal{I}
\end{cases}
$$

where $\overline{R}_i = \mathrm{diag}(R_i, R_i)$

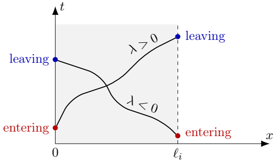
$\mathcal{N} = \mathcal{N}_M \cup \mathcal{N}_S^v \cup \mathcal{N}_S^z$

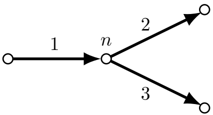
Networks: well-posedness for IGEB

We consider **the IGEB model**.

At least local in time **well-posedness**:

- any network
- rewrite as a single system → apply previous results (Bastin-Coron, Li)
- key step:
  - write system in diagonal form: new unknown state $r_i = \begin{bmatrix} r_i^- \\ r_i^+ \end{bmatrix}$
  - rule for each node n:
    components of r_i corresponding to characteristics *entering* $[0, \ell_i] \times [0, +\infty)$ at this node expressed as a function of the components of r_i corresponding to characteristics *leaving* $[0, \ell_i] \times [0, +\infty)$ at this node





$$\begin{pmatrix} r_1^-(\ell_1,t) \\ r_2^+(0,t) \\ r_3^+(0,t) \end{pmatrix} = f \begin{pmatrix} r_1^+(\ell_1,t) \\ r_2^-(0,t) \\ r_3^-(0,t) \end{pmatrix}$$

Networks: stabilization for IGEB

$\mathbf{H}_x^2 := \prod_{i=1}^N H^2(0,\ell_i;\mathbb{R}^{12})$.

Theorem 3

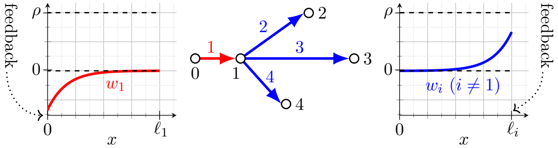
Star-shaped network, velocity feedback controls ($\tau_i^n z_i = -K_n v_i$, with K_n symmetric positive definite) at all simple nodes.
Then, the zero steady state of (3−IGEB) is locally H^2 exponentially stable:

$\exists \varepsilon, \beta, \eta \geq 1$ s.t. $\forall (y_i^0)_{i\in\mathcal{I}} \in \mathbf{H}_x^2$ with $\|y^0\|_{\mathbf{H}_x^2} \leq \varepsilon$ and compatibility conditions, $\exists!$ solution $y := (y_i)_{i\in\mathcal{I}} \in C^0([0,+\infty);\mathbf{H}_x^2)$ to (3−IGEB), and

$$\|y(\cdot,t)\|_{\mathbf{H}_x^2} \leq \eta e^{-\beta t}\|y^0\|_{\mathbf{H}_x^2}, \qquad \forall t.$$

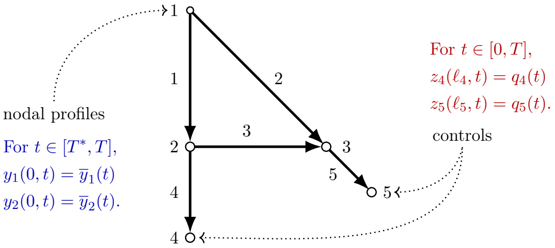
Quadratic Lyapunov functional:

$$\overline{\mathcal{L}} = \sum_{i\in\mathcal{I}} \sum_{k=0}^{2} \int_0^{\ell_i} \left\langle \partial_t^k y_i \,, \left(\rho\, Q_i^{\mathcal{P}} + w_i \begin{bmatrix} \mathbf{0} & \mathbf{W}_i \\ \mathbf{W}_i^\intercal & \mathbf{0} \end{bmatrix} \right) \partial_t^k y_i \right\rangle dx$$



See: R. arXiv:2009.07183 preprint.

Networks: local nodal profile controllability for IGEB



Theorem 4

Let

- $T^* = T^*(A_i, \ell_i) > 0$ large enough and $T > T^*$
- nodal profiles $\overline{y}_1, \overline{y}_2 \in C^1([T^*, T]; \mathbb{R}^{12})$ with small norm + transmission conditions

Then, $\forall (y_i^0)_{i \in \mathcal{I}} \in \prod_{i=1}^N C^1([0, \ell_i]; \mathbb{R}^{12})$ with small norm and compatibility conditions, $\exists q_4, q_5 \in C^1([0,T]; \mathbb{R}^6)$ controls with small norm s.t. the solution $(y_i)_{i \in \mathcal{I}} \in \prod_{i=1}^N C^1([0, \ell_i] \times [0,T]; \mathbb{R}^{12})$ to $(3-\mathrm{IGEB})$ has small norm and satisfies the nodal profiles.

- Zhuang and al. '18 (Saint-Venant)
- construct solution satisfying all conditions + substitute to obtain desired control
- possible because: first-order, hyperbolic, no zero eigenvalues

Networks: corresponding results for GEB

Then,

- invert the transformation between $(3{-}\mathrm{GEB})$ and $(3{-}\mathrm{IGEB})$
- deduce results corresponding to Theorem 4 (stabilization) and Theorem 5 (nodal profile control), for $(3{-}\mathrm{GEB})$

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Thank you for your attention! Questions?

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