Some conclusions on fluid-structure interactions

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Overview

1. Well-posedness and input-output stability

2. Asymptotic behaviour

Well-posedness and input-output stability

Some background

The space $\mathcal{H}^\infty(\mathbb{C}_0, W)$ consist of all the analytic functions $G : \mathbb{C}_0 \to Z$ for which

$$\sup_{s\in\mathbb{C}_0} ||G(s)||_W < \infty.$$

The condition $G \in \mathcal{H}^\infty(\mathbb{C}_0, L(U,Y))$ is equivalent to the fact that if $u \in L^2([0,\infty); U)$, then $y \in L^2([0,\infty); Y)$.

This property is also called input-output stability.

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A system of viscous fluid-structure

We consider the linear model

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0, \quad (x \in \mathcal{E}), \tag{1}$$

$$\frac{\partial q}{\partial t} + \frac{\partial h}{\partial x} - \mu \frac{\partial^2 q}{\partial x^2} = 0, \quad (x \in \mathcal{E}), \tag{2}$$

$$\dot{h}_S(t) + \frac{\partial q}{\partial x} = 0 \qquad (x \in \mathcal{I}), \tag{3}$$

$$\frac{\partial q}{\partial t} + \frac{\partial p}{\partial x} = 0 \qquad (x \in \mathcal{I}), \tag{4}$$

$$h(t,a^-) - \mu\frac{\partial q}{\partial x}(t,a^-) = p(t,a^+) + h_S(t) - \mu\frac{\partial q}{\partial x}(t,a^+), \tag{5}$$

$$h(t,b^+) - \mu\frac{\partial q}{\partial x}(t,b^+) = p(t,b^-) + h_S(t) - \mu\frac{\partial q}{\partial x}(t,b^-), \tag{6}$$

$$\ddot{h}_S(t) = \int_a^b p(t,x)dx + u(t) \qquad (t>0). \tag{7}$$

Here, we consider a output given

$$y(t) = h_S(t) \qquad (t \geq 0). \tag{8}$$

Our first main result is a following reformulation of the system.
Set

$$X := \mathbb{C} \times H^1(\mathcal{E}) \times L^2(\mathcal{E}) \times \mathbb{C} \times \mathbb{C}. \tag{9}$$

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First main result

Theorem

Equations (1)-(8) can be recast as

$$\begin{aligned} \dot{z} &= Az + Bu \\ y &= Cz, \end{aligned} \tag{10}$$

where the components of the vector $z(t)$ *are* $h_S(t)$, $h(t,\cdot)$, $q(t,\cdot)$, $q(t,a)$ *and* $q(t,b)$, B *is in* $\mathcal{L}(\mathbb{C}, X)$, C *is in* $\mathcal{L}(X, \mathbb{C})$ *and* A *is a generator of an analytic semigroup on* X.

Corollary

Equations (1)-(8) *define a well-posed linear system with state space* X *defined in* (9) *and input and output spaces* $U = Y = \mathbb{C}$.

Informally, this means: on any time interval $[T, t]$, for any initial state x_0 and any input function u, it has a unique state trajectory x and a unique output function y, both defined on $[T, t]$.

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Some ideas about the first Theorem

For $t \geq 0$, we set $q_a(t) := q(t,a)$ and $q_b(t) := q(t,b)$. Since (3) implies that q is a linear function of x on $\mathcal{I}$, for every $t \geq 0$ and $x \in \mathcal{I}$,

$$\dot{h}_S(t) = -\frac{q_b(t) - q_a(t)}{b-a}, \tag{11}$$

$$q(t,x) = q_a(t)\left(\frac{x-b}{a-b}\right) + q_b(t)\left(\frac{x-a}{b-a}\right), \tag{12}$$

$$\frac{\partial q}{\partial x}(t,x) = \frac{q_b(t) - q_a(t)}{b-a}. \tag{13}$$

We differentiate (4) with respect to x and use (5)-(3) to arrive at

$$\begin{cases} \frac{\partial^2 p}{\partial x^2}(t,x) = \ddot{h}_S(t) \quad (x \in \mathcal{I}), \\ p\left(t, a^+\right) = p_a(t),\ p\left(t, b^-\right) = p_b(t), \end{cases} \tag{14}$$

where

$$p_a(t) := h\left(t, a^-\right) - \mu \frac{\partial q}{\partial x}\left(t, a^-\right) - h_{sol}(t) - \mu \dot{h}_{sol}(t), \tag{15}$$

$$p_b(t) := h\left(t, b^+\right) - \mu \frac{\partial q}{\partial x}\left(t, b^+\right) - h_{sol}(t) - \mu \dot{h}_{sol}(t). \tag{16}$$

The first equation in (14) implies that, for every $t \geq 0$, $p(t,x)$ is a second order polynomial in x so that

$$\begin{aligned}\int_a^b p(t,x)\ dx &= p(t,a)l - \dot{q}_a(t)\frac{l^2}{3} - \dot{q}_b(t)\frac{l^2}{6}\\ &= p(t,b)l + \dot{q}_a(t)\frac{l^2}{6} + \dot{q}_b(t)\frac{l^2}{3},\end{aligned}$$

where we set $l := b - a$.

Combining this with (7) and (11) we deduce that

$$\left[1+\frac{l^3}{3}\right]\dot{q}_a(t)-\left[1-\frac{l^3}{6}\right]\dot{q}_b(t)=p(t,a)l^2+lu(t),$$
$$-\left[1-\frac{l^3}{6}\right]\dot{q}_a(t)+\left[1+\frac{l^3}{3}\right]\dot{q}_b(t)=-p(t,b)l^2-lu(t).$$

Inverting the above linear system, we get

$$\begin{bmatrix}\dot{q}_a(t)\\ \dot{q}_b(t)\end{bmatrix}=M\begin{bmatrix}p(t,a)\\ -p(t,b)\end{bmatrix}+\frac{1}{l}M\begin{bmatrix}u(t)\\ -u(t)\end{bmatrix}. \qquad (17)$$

Considering equation (5)-(6) together with (13) we deduce

$$p(t,a) = h(t,a^-) - \mu \frac{\partial q}{\partial x}(t,a^-) - h_S(t) + \mu \frac{q_b - q_a}{b-a}, \qquad (18)$$

and

$$p(t,b) = h(t,b^+) - \mu \frac{\partial q}{\partial x}(t,b^+) - h_S(t) + \mu \frac{q_b - q_a}{b-a}. \qquad (19)$$

Let X be defined by (9), set

$$W := \mathbb{C} \times H^1(\mathcal{E}) \times H^2(\mathcal{E}) \times \mathbb{C} \times \mathbb{C},$$

and denote by $z := \begin{bmatrix} h_S & h & q & q_a & q_b \end{bmatrix}^T$ a generic element of X. Consider the operator $A : \mathcal{D}(A) \to X$ defined by

$$\mathcal{D}(A) := \{z \in W \mid q(a) = q_a, \, q(b) = q_b\}, \tag{20}$$

$$Az := \begin{bmatrix} -\frac{q(b)-q(a)}{b-a} \\ -\frac{dq}{dx} \\ -\frac{dh}{dx} + \mu\frac{d^2q}{dx^2} \\ R_1 z \\ R_2 z \end{bmatrix}, \tag{21}$$

where

$$\begin{bmatrix} R_1 z \\ R_2 z \end{bmatrix} := M \begin{bmatrix} h(a^-) - \mu\frac{dq}{dx}(a^-) - h_S + \mu\frac{q_b-q_a}{b-a} \\ -h(b^+) - \mu\frac{dq}{dx}(b^+) - h_S + \mu\frac{q_b-q_a}{b-a} \end{bmatrix}.$$

We set

$$Bu := [0,0,0,\frac{lu}{2\left(1+\frac{l^3}{12}\right)},-\frac{lu}{2\left(1+\frac{l^3}{12}\right)}]^T \text{ and } Cz := h_S, \qquad (22)$$

and we observe that $B \in \mathcal{L}(\mathbb{C}, X)$ and $C \in \mathcal{L}(X, \mathbb{C})$.

Ideas about the second Theorem

We have the following result where $\mathbb{C}_0$ denotes the open right-half plane

$$\mathbb{C}_0 := \{s \in \mathbb{C} \;:\; \mathrm{Re}\, s > 0\}. \tag{23}$$

Proposition

The resolvent set $\rho(A)$ *contains* $\mathbb{C}_0$*.*

Proposition

The transfer function of the system (1)-(8) is given by

$$G(s) := \frac{1}{\left(1+\frac{l^3}{12}\right)s^2 + \frac{l^2}{2}s\sqrt{1+\mu s} + \mu l s + l} \qquad (s \in \mathbb{C}_0). \quad (24)$$

Lemma

Let F be the function defined by

$$F(s) = \left(1+\frac{l^3}{12}\right)s^2 + \frac{l^2}{2}s\sqrt{1+\mu s} + \mu l s + l, \quad (25)$$

and let $\mathbb{C}_0$ be the open right-half plane, as defined in (23). Then there exists a neighborhood $\mathcal{O}$ of $\overline{\mathbb{C}_0}$ such that F is holomorphic on $\mathcal{O}$. Moreover, F does not vanish on $\overline{\mathbb{C}_0}$.

Asymptotic behaviour

In this work we study the correct version of this model for vertical displacements of a floating structure, which now reads:

$$\left(1+\frac{(b-a)^3}{12}\right)\ddot{h}_S(t) = -\frac{(b-a)^2}{2}F*\dot{h}_S(t) - \mu(b-a)\,\dot{h}_S(t) - (b-a)\,h_S(t), \quad (26)$$

where μ is the viscosity coefficient of the fluid, $(b-a)$ is the width of the interval $\mathcal{I}=[a,b]$ obtained by projecting the floating object (supposed symmetric around the axis $x=\frac{1}{2}(a+b)$) on the flat horizontal bottom, and $\mathcal{E}=\mathbb{R}\setminus[a,b]$ denotes the viscous fluid domain. Moreover, F is the causal distribution with Laplace transform $\widehat{F}(s)=\sqrt{1+\mu\,s}$.

Diffusive representation

Consider the original system, set $\dot{h} := v$ and $z := F * \dot{h}_S$, it can then be viewed as a coupled system

$$\begin{cases} \left(1 + \frac{l^3}{12}\right) \ddot{h}_S + z(t) + \dot{h}_S + l\mu h_S = 0 \\ v(t) = \dot{h}_S(t) \\ \partial_t \varphi(t,\xi) = -\xi\varphi(t,\xi) + v(t); \;\; \varphi(0,\xi) = 0 \\ z(t) = \int_{\mu^{-1}}^{\infty} g(\xi)\partial_t\varphi(t,\xi)\,\mathrm{d}\xi + 1v(t). \end{cases} \tag{27}$$

where

$$g(\xi) \;\; := \;\; \frac{1}{\pi}\frac{\sqrt{\mu\xi - 1}}{\mu\xi}\,, \quad \textit{for } \xi > 1/\mu\,. \tag{28}$$

Proposition

For all $(h_{S,0}, \omega_0) \in \mathbb{C}^2$, *the solution of the coupled system* (27), *with initial condition* $(h_{S,0}, \omega_0, 0)$, *satisfies*

$$(h_S, \dot{h}_S, \varphi)(t) \to_{t\to\infty} 0 \quad \text{in } \mathbb{C}^2 \times \tilde{H},$$

where

$$\tilde{H} = \left\{ \varphi \in L^2_{\mathrm{loc}}\left(\mathbb{R}^+, \mathrm{d}g\right), \int_0^\infty \xi |\varphi|^2 \mathrm{d}g(\xi) < \infty \right\}.$$

Asymptotic behaviour (special case)

Proposition

If all the poles s_k of the transfer function lie in the left halfplane $\Re(s) < -\frac{1}{\mu}$, then the asymptotic behaviour of the solution h_S of the system (27) *reads*

$$h_S(t) \sim K\, e^{-\frac{t}{\mu}}\, t^{-3/2}\,, \quad \text{as} \quad t \to +\infty\,.$$

Analytical solution

Case $\mu = 0$:

$$h_S(t) = \left(C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)\right) e^{-\delta t}, \tag{29}$$

when $B^2 < 4AI$, where

$$\delta = \frac{B}{2A}, \quad \omega_0 = \sqrt{\frac{I}{A}}, \quad \omega_d = \sqrt{\omega_0^2 - \delta^2} = \frac{\sqrt{4AI - B^2}}{2A}, \tag{30}$$

are the damping coefficient, the undamped natural angular frequency and the damped angular frequency, respectively. The constants C_1 and C_2, are given by

$$C_1 = h_0, \quad C_2 = \frac{\dot{h}_0 + h_0\delta}{\omega_d} = \frac{h_0 B + 2\dot{h}_0 A}{\sqrt{4AI - B^2}}. \tag{31}$$

Case $\mu > 0$

Theorem

The solution of the GFDE is given by

$$h_S(t) = \exp(-\varepsilon\, t) \left(\sum_{i=1}^{4} \Theta_i\, \mathcal{E}_{\frac{1}{2}}(\lambda_i, t) \right), \tag{32}$$

with constants $\Theta_i := r_i\, h_0 + \dot{r}_i\, \dot{h}_0$.

Theorem (Matignon 1996)

We have the following asymptotic equivalents for $\mathcal{E}_\alpha(\lambda, t)$ as t reaches $+\infty$:

- *for* $|\arg(\lambda)| \leq \alpha\frac{\pi}{2}$,

$$\mathcal{E}_\alpha(\lambda, t) \sim \frac{1}{\alpha}\lambda^{\frac{1}{\alpha}-1} e^{\lambda^{\frac{1}{\alpha}} t}, \tag{33}$$

- *for* $|\arg(\lambda)| > \alpha\frac{\pi}{2}$,

$$\mathcal{E}_\alpha(\lambda, t) \sim \frac{\alpha}{\Gamma(1-\alpha)}\lambda^{-2} t^{-1-\alpha}. \tag{34}$$

- The case of Proposition 4 is recovered as a special case, which occurs if and only if *all* the roots λ_i fulfill $|\arg(\lambda_i)| > \frac{\pi}{4}$.
- Otherwise, if *but one* λ_0 lies in the sector $|\arg(\lambda)| < \frac{\pi}{4}$, then a very different asymptotic behaviour is to be found, namely a purely exponentially decaying one, with decay rate $\delta := \varepsilon - \Re(\lambda^2) > 0$ (it must be positive indeed, since asymptotic stability has already been proved in Proposition 3).

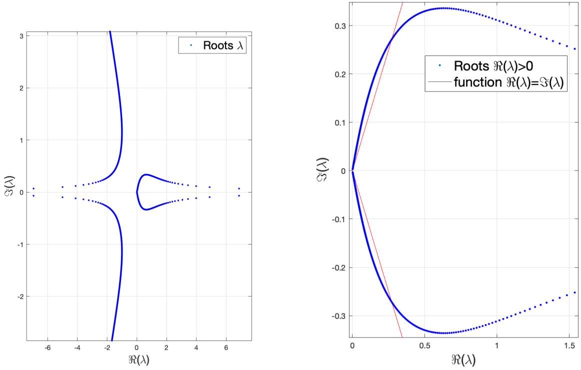
Theorem

- *If there is at least one root with* $\Re(\lambda_j) > |\Im(\lambda_j)|$ *then the asymptotics is of exponential type, with rate* $\delta(\mu) := \frac{1}{\mu} - \Re(\lambda^2) > 0$

$$h_S(t) \sim \sum_j C_j \exp((\lambda_j^2 - \frac{1}{\mu})\, t), \tag{35}$$

- *or all the four roots lie in* $|\arg(\lambda)| > \frac{\pi}{4}$*, then the asymptotics is of mixed type,*

$$h_S(t) \sim C\, t^{-\frac{3}{2}} \exp(-\frac{1}{\mu}\, t)\,. \tag{36}$$



(a) (b)

Figure: Evolution of the four roots λ_i in the σ-plane, as a function of μ. (a): global picture with 4 trajectories. (b): zoom in the right-half plane $\Re(\sigma) > 0$, 2 trajectories crossing the segment $\Re(\lambda) = |\Im(\lambda)|$ for a critical value μ^c of the viscosity.

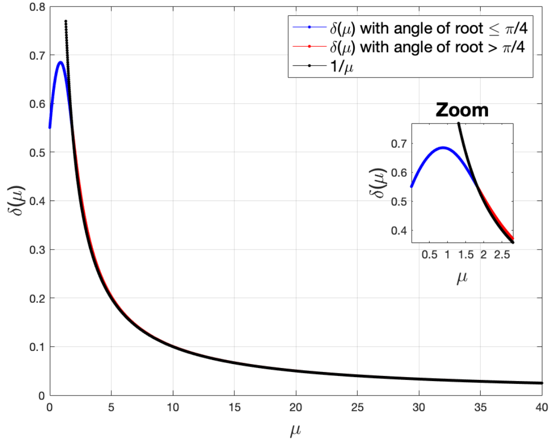


Figure: Damping rate $\delta(\mu) = \Re(\lambda^2) - \frac{1}{\mu}$ as a function of viscosity μ

Thank you !

Day of indigenous resistance

Figure: Lautaro, leader of the Mapuche resistance.