Entropy Methods for Gas Dynamics on Networks

Yannick Holle

supervisors M. Herty and M. Westdickenberg

This work was funded by:
DFG Research Training Group
Energy, Entropy and Dissipative Dynamics

Mini-Workshop on Hyperbolic Problems
Friedrich-Alexander-UniversitÃďt Erlangen-NÃijrnberg
October 12, 2020

- Introduction to isentropic gas dynamics on networks
- A kinetic BGK model and its relaxation limit
- A maximum energy dissipation principle at the junction
- Comparison of the coupling conditions

Introduction to isentropic gas dynamics on networks

Applications

Many problems can be modeled by hyperbolic PDEs on networks:




Isentropic gas equations

$$\begin{cases} \partial_t \rho + \partial_x \rho u & = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + \kappa \rho^\gamma) & = 0 \end{cases} \quad \text{for } t, x,$$

with density $\rho \geq 0$, flow velocity $u \in \mathbb{R}$ and constants $\kappa > 0$, $1 < \gamma < 3$.

The system has several useful properties:

- a large class of entropy pairs (η, G) (η convex, $G' = \eta' F'$);
- globally defined Riemann invariants $\omega_{1,2} = u \pm a_\gamma \rho^{(\gamma-1)/2}$;
- a kinetic model.

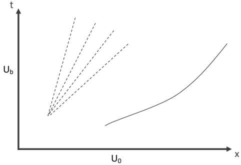
These properties allow us to use some tools which are not available for general hyperbolic systems.

Coupling conditions in the literature

We couple $i = 1, \dots, d$ weak solutions

$$\partial_t U^i + \partial_x F(U^i) = 0, \quad t > 0, x > 0,$$

at the junction $x = 0$ by a suitable condition.



Physically reasonable: conservation of mass at the junction

$$\sum_{i=1}^{d} A^i (\rho u)^i (t, 0) = 0, \quad \text{for a.e. } t > 0. \tag{M}$$

Additional conditions: (to obtain unique solutions)
Equality of dynamic pressure [R. M. Colombo, M. Garavello, 2006]:

$$(\rho u^2 + \kappa \rho^\gamma)^i (t, 0) = (\rho u^2 + \kappa \rho^\gamma)^j (t, 0) \quad \text{for a.e. } t > 0. \tag{P_D}$$

Equality of pressure [M. K. Banda, M. Herty, A. Klar, 2006]:

$$(\rho^\gamma)^i (t, 0) = (\rho^\gamma)^j (t, 0) \quad \text{for a.e. } t > 0. \tag{P}$$

Equality of stagnation enthalpy [G. A. Reigstad, 2015]:

$$\left(\frac{u^2}{2} + \frac{\gamma\kappa}{\gamma - 1} \rho^{\gamma - 1}\right)^i (t, 0) = \left(\frac{u^2}{2} + \frac{\gamma\kappa}{\gamma - 1} \rho^{\gamma - 1}\right)^j (t, 0) \quad \text{for a.e. } t > 0. \tag{H}$$

Known results and remarks

A general existence theorem for the generalized Cauchy problem based on **wave-front tracking** [R. M. Colombo, M. Herty, V. Sachers, 2008] ensures existence of solutions with every coupling condition on the last slide.

This result requires **subsonic** initial data which is **close to a stationary solution** and with **sufficiently small total variation**.

We will use an approach based on completely different methods:

- kinetic approach to approximate the solutions;
- compensated compactness to pass to the limit in the interior of the domain;
- formal derivation of a new coupling condition.

Therefore, we impose initial data with **finite total mass and energy** and an L^∞**-bound**.

A Kinetic BGK model and its relaxation limit

BGK model for isentropic gas dynamics (F. Bouchut, 1999)

Let

$$\begin{cases} \partial_t f_0^i + \xi \partial_x f_0^i = \frac{1}{\epsilon}(M_0[f^i] - f_0^i), \\ \partial_t f_1^i + \xi \partial_x f_1^i = \frac{1}{\epsilon}(M_1[f^i] - f_1^i), \end{cases} \qquad \text{for } t > 0,\ x > 0,$$

with $f^i = f^i(t, x, \xi) \in \mathbb{R}^2$, $f_0^i \geq 0$.

To define the Maxwellian $M[f]$, we introduce the macroscopic variables

$$\begin{pmatrix} \rho_f \\ \rho_f u_f \end{pmatrix} = \int_{\mathbb{R}} f(\xi)\, \mathrm{d}\xi.$$

We define

$$M[f](\xi) = M(\rho_f, u_f, \xi) = \begin{pmatrix} \chi(\rho_f, \xi - u_f) \\ ((1-\theta)u_f + \theta\xi)\chi(\rho_f, \xi - u_f) \end{pmatrix},$$

$$\chi(\rho, \xi) = c_{\gamma,\kappa}(a_\gamma^2 \rho^{\gamma-1} - \xi^2)_+^\lambda.$$

Convex kinetic entropies can be defined by the formula

$$H_S(f,\xi) = \int_{\mathbb{R}} \Phi(f,\xi,v)\, S(v)\,\mathrm{d}v, \quad \text{for convex } S.$$

(Φ is a positive kernel with an explicit formula).

We obtain a macroscopic entropy pair by

$$\eta_S(\rho,u) = \int_{\mathbb{R}} H_S(M(\rho,u,\xi),\xi)\,\mathrm{d}\xi,$$
$$G_S(\rho,u) = \int_{\mathbb{R}} \xi\, H_S(M(\rho,u,\xi),\xi)\,\mathrm{d}\xi.$$

Note: $S(v) = v^2/2$ leads to the physical energy and energy flux.

Kinetic coupling condition

We couple the kinetic equations by a coupling function

$$f^i(t, x = 0, \xi) = \Psi^i[f^j(t, x = 0, \mathbb{R}^-), j = 1, ..., d](\xi), \quad t > 0, \xi > 0.$$

It is physically reasonable to assume that **mass is conserved**

$$\sum_{i=1}^{d} A^i \left[\int_0^\infty \xi \Psi_0^i[g](\xi) \,\mathrm{d}\xi + \int_{-\infty}^0 \xi g_0^i(\xi) \,\mathrm{d}\xi \right] = 0,$$

and **energy is non-increasing**

$$\sum_{i=1}^{d} A^i \left[\int_0^\infty \xi H_{v^2/2}(\Psi^i[t, g](\xi), \xi) \,\mathrm{d}\xi + \int_{-\infty}^0 \xi H_{v^2/2}(g^i(\xi), \xi) \,\mathrm{d}\xi \right] \leq 0.$$

at the junction. Furthermore, we will need a continuity assumption on Ψ.

The existence of solutions to the kinetic model on networks can be shown if the inital total mass and energy are finite [Y. H., 2020].

Let f_ϵ^i solve the kinetic BGK model with $\epsilon > 0$. We aim to justify the limit $\epsilon \to 0$ by the **method of compensated compactness**.

Therefore, we use the additional assumption

$$\sum_{i=1}^{d} A^i \left[\int_0^\infty \xi H_S(\Psi^i[g](\xi), \xi) \, \mathrm{d}\xi + \int_{-\infty}^0 \xi H_S(g^i(\xi), \xi) \, \mathrm{d}\xi \right] \leq 0,$$

for every convex S with $S(v) = S(-v)$ and require that the kinetic Riemann invariants $\omega_{1,2}(\rho_f, u_f)(x = 0)$ of the initial data are bounded in L^∞.

This leads to the following result.

Theorem (Y. H., 2020)

$(\rho, \rho u)_\epsilon^i = \int_{\mathbb{R}} f_\epsilon^i \, \mathrm{d}\xi$ are uniformly bounded in $L^\infty_{t,x}$. After passing if necessary to a subsequence, $(\rho, \rho u)_\epsilon^i$ converge a.e. in $t, x > 0$ to an entropy solution $(\rho, \rho u)^i$.

The proof is based on the method of compensated compactness. See [P.-L. Lions, B. Perthame, P. E. Souganidis, 1996] and [F. Berthelin, F. Bouchut, 2002].

A maximum energy dissipation principle at the junction

Question: Which coupling condition is the physically correct one?

Our approach: We determine the coupling condition which conserves mass and dissipates as most energy as possible. For given data $g(\xi)$, $\xi < 0$, we solve

$$\begin{aligned}
&\inf && \sum_{i=1}^{d} A^i \int_0^\infty \xi H_{v^2/2}(\Psi^i(\xi), \xi)\,\mathrm{d}\xi \\
&\text{s.t.} && \sum_{i=1}^{d} A^i \left[\int_0^\infty \xi \Psi_0^i(\xi)\,\mathrm{d}\xi + \int_{-\infty}^0 \xi g_0^i(\xi)\,\mathrm{d}\xi \right] = 0
\end{aligned}$$

The solution of this optimization problem is given by

$$\Psi^i(\xi) = M(\rho_*, u_* = 0, \xi),$$

where $\rho_* > 0$ is the unique density which ensures conservation of mass.

Proof.

Using the sub-differential inequality

$$H_{v^2/2}(f,\xi) \geq H_{v^2/2}(M(\rho,u,\xi),\xi) + \eta'_{v^2/2}(\rho,u)\cdot(f - M(\rho,u,\xi))$$

leads to

$$\begin{aligned}
&\sum_{i=1}^{d} A^i \int_0^\infty \xi H_{v^2/2}(\Psi^i(\xi),\xi)\,\mathrm{d}\xi \\
&\geq \sum_{i=1}^{d} A^i \int_0^\infty \xi H_{v^2/2}(M(\rho_*,0,\xi),\xi)\,\mathrm{d}\xi \\
&\quad + \begin{pmatrix} (\eta'_{v^2/2}(\rho_*,0))_0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \sum_{i=1}^{d} A^i \int_0^\infty \xi \left(\Psi_1^i(\xi) - M_1(\rho_*,0,\xi)\right)\,\mathrm{d}\xi \end{pmatrix} \\
&= \sum_{i=1}^{d} A^i \int_0^\infty \xi H_{v^2/2}(M(\rho_*,0,\xi),\xi)\,\mathrm{d}\xi.
\end{aligned}$$

□

Using the sub-differential inequality for H_S and *assuming* that $\rho_*^\epsilon \to \rho_*$ in L^1_{loc} allows to prove

$$\overline{G_S(\rho^i, u^i)}(t,0) - G_S(\rho_*(t),0) - \eta_S'(\rho_*(t),0)\left(\overline{F(\rho^i, u^i)}(t,0) - F(\rho_*(t),0)\right) \leq 0,$$

for every convex S.

Compare this inequality with

Entropy formulation of boundary conditions [F. Dubois, P. LeFloch, 1988]

For boundary data $(\rho^b(t), u^b(t))$, we require for every convex S:

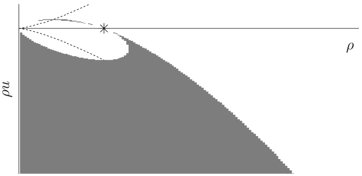
$$\overline{G_S(\rho, u)}(t,0) - G_S(\rho^b(t), u^b(t)) - \eta_S'(\rho^b(t), u^b(t))\left(\overline{F(\rho, u)}(t,0) - F(\rho^b(t), u^b(t))\right) \leq 0$$

Problem: Both conditions do not lead to uniqueness of self-similar Lax solutions to the generalized Riemann problem.

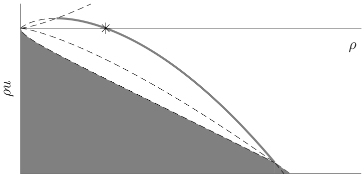
Possible ways to solve the problem

- boundary layer equations produced by different approximation techniques [K. T. Joseph, P. G. LeFloch, 1999],
- using the stronger Riemann problem formulation of boundary conditions.

y s



Superset of states satisfying the entropy formulation



Riemann problem formulation

We follow the Riemann problem approach and extend it to the network case.

A new coupling condition

We construct solutions to the **generalized Riemann problem** in the following way:

1. We consider the self-similar Lax solutions (ρ^i, u^i) to the Riemann problems with initial data

$$(\rho_0^i, u_0^i)(x) = \begin{cases} (\rho_*, 0)(x) & x < 0 \\ (\rho_0^i, u_0^i)(x) & x > 0 \end{cases}$$

2. The artificial density $\rho_* \geq 0$ is chosen such that

$$\sum_{i=1}^{d} A^i (\rho u)^i (t, 0+) = 0.$$

3. We restrict the obtained functions to $x > 0$ and obtain the desired solutions.

Theorem (Y. H., M. Herty, M. Westdickenberg, 2020)

For every $(\rho_0^i, u_0^i) \in \mathbb{R}^+ \times \mathbb{R}$ there exists a unique solution (ρ^i, u^i) to the generalized Riemann problem.

The proof is based on an extension of methods used by Reigstad (2015).

The idea is to leave out the conservation of mass first. We prove monotonicity of $\sum_{i=1}^{d} A^i (\rho u)^i (t, 0)$ w.r.t. ρ_* and conclude with the intermediate value theorem.

The generalized Riemann problem defines implicitly a condition on $(\rho^i, u^i)(t, 0)$, $i = 1, \ldots, d$. The implicit condition is used to define solutions to the **generalized Cauchy problem**.

Existence of solutions to the generalized Cauchy problem in the BV-setting with subsonic initial data can be shown by applying a general existence result.

The new coupling condition satisfies several (physical) properties:

- Existence and uniqueness of solutions to the generalized Riemann problem holds globally in state space.
- Energy is dissipated at the junction.
- A maximum principle on the Riemann invariants.
- Numerical results regarding the produced wave types (next slide).

Comparison of the coupling conditions

We consider the behavior of the energy at the junction

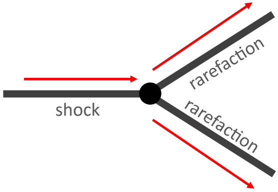
$$\sum_{i=1}^{d} A^{i} G_{v^2/2}(\rho^{i}, u^{i})(t, 0+) \begin{cases} \leq 0 & \text{energy dissipation,} \\ = 0 & \text{energy conservation,} \\ \geq 0 & \text{energy production,} \end{cases}$$

and obtain the following results for the different conditions

Equal density	energy dissipation and production possible
Equal momentum flux	energy dissipation and production possible
Equal stagnation enthalpy	conservation of energy
Equal artificial density	energy dissipation

We choose initial data which lead to a **stationary solution** to the equal density coupling condition (conservation of mass + equal density).

pipeline	$\rho_{0,k}$	$\rho_{0,k} u_{0,k}$
1	$+1.0000$	-1.0000
2	$+1.0000$	$+0.5000$
3	$+1.0000$	$+0.5000$



The different coupling conditions lead to the following **qualitative waves types**:

pipeline	Equal density	Equal momentum flux	Equal stagnation enthalpy	Equal artificial density
1	no waves	rarefaction wave	rarefaction wave	shock
2	no waves	shock	shock	rarefaction wave
3	no waves	shock	shock	rarefaction wave

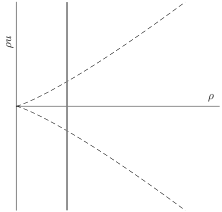
We obtain the following numerical results for the traces at the junction:

	Equal density		Equal momentum flux		Equal stagnation enthalpy		Equal artificial density	
pipeline	$\bar{\rho}_k$	$\bar{\rho}_k \bar{u}_k$	$\bar{\rho}_k$	$\bar{\rho}_k \bar{u}_k$	$\bar{\rho}_k$	$\bar{\rho}_k \bar{u}_k$	$\bar{\rho}_k$	$\bar{\rho}_k \bar{u}_k$
1	$+1.000$	-1.000	$+0.896$	-1.198	$+0.852$	-1.267	$+1.178$	-0.542
2	$+1.000$	$+0.500$	$+1.027$	$+0.600$	$+1.036$	$+0.634$	$+0.935$	$+0.271$
3	$+1.000$	$+0.500$	$+1.027$	$+0.600$	$+1.036$	$+0.634$	$+0.935$	$+0.271$
Energy production/ dissipation	-7.500×10^{-2}		-1.725×10^{-2}		≈ 0		-1.385×10^{-1}	

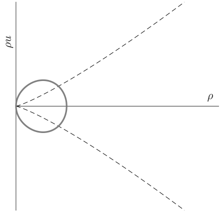
- F. Berthelin and F. Bouchut, Solution with finite energy to a BGK system relaxing to isentropic gas dynamics, *Ann. Fac. Sci. Toulouse*, **9** (2000), 605–630.
- F. Berthelin and F. Bouchut, Kinetic invariant domains and relaxation limit from a BGK model to isentropic gas dynamics, *Asymtotic Anal.*, **31** (2002), 153–176.
- G. A. Reigstad, Existence and uniqueness of solutions to the generalized Riemann problem for isentropic flow, *SIAM J. Appl. Math.*, **75** (2015), 679–702.
- Y. H., Kinetic relaxation to entropy based coupling conditions for isentropic flow on networks, *J. Differential Equations*, **269** (2020), 1192–1225.
- Y. H., M. Herty, and M. Westdickenberg, New coupling conditions for isentropic flow on networks, *Netw. Heterog. Media*, (2020), doi: 10.3934/nhm.2020016.

Thank you for your attention!

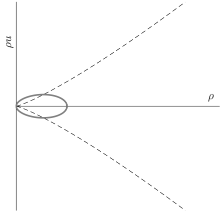
The different coupling conditions produce the following level-sets in the state space.



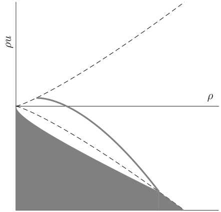
Equal pressure



Equal stagnation enthalpy



Equal momentum flux



Artificial density