Controllability of Nodal Profile for a Network of Vibrating Strings

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Outline

Nodal Control Problems: Motivation by application

2010, 2014. **M. Gugat, etc.**

- The coupling of gas pipes.

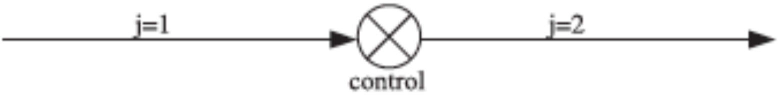

Figure: Illustration of the control of a compressor station with two pipes. The first pipe corresponds to the supplier and the second to the customer.

Nodal Control Problems: Motivation by application

2010, 2014. M. Gugat, etc.

- The coupling of gas pipes.
- One dimensional isothermal Euler equations.

For a single pipe,

$$\rho_t + q_x = 0,$$

$$q_t + \left(\frac{q^2}{\rho} + a^2\rho\right)_x = -f_g \frac{q|q|}{2D\rho},$$

$\rho(t,x)$:the density of the gas.
$q(t,x)$: the flux in the pipe.
$p = a^2\rho$: the pressure of the flow.

Nodal Control Problems: Motivation by application

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- Nodal Controls.

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$$\rho_t + q_x = 0,$$

$$q_t + \left(\frac{q^2}{\rho} + a^2\rho\right)_x = -f_g \frac{q|q|}{2D\rho},$$

The dynamics of the compressor :

$$q^{(1)}(t,0) = q^{(2)}(t,0),$$

$$u(t) = q^{(2)}(t,0)\left(\left(\frac{\rho^{(2)}(t,0)}{\rho^{(1)}(t,0)}\right)^{\kappa} - 1\right),$$

$u(t)$: the nodal control.

[0] Ehrhardt K, Steinbach M. Nonlinear gas optimization in gas networks. In Modeling, Simulation and Optimization of Complex Processes, Bock HG, Kostina E, Pu HX, Rannacher R (eds). Springer: Berlin, 2005.

Nodal Control Problems: Motivation by application

2010, 2014. M. Gugat, etc.

- The coupling of gas pipes.
- One dimensional isothermal Euler equations.
- Nodal Controls.
- Exact controllability of boundary traces.

(P): Given finite time $T_0 > \overline{T} > 0$, we choose $t^* \in (\overline{T}, T_0)$ such that after the finite time t^*,
can we find $u \in C^1([0, T_0])$ such that system has a classical solution that satisfies the given customer demand in density and flux by

$$(\rho^{(2)}, q^{(2)})\Big|_{x=L_2} = (\rho_B(t), q_B(t)) \qquad \text{for all } t \in [t^*, T_0],$$

where ρ_B and q_B are given C^1 functions?

Exact controllability of nodal profiles

Main Result: [2011. M. Gugat, etc.][1]

In a C^1-neighborhood of the stationary state $(\bar{\rho}^{(i)}, \bar{q}^{(i)})$ with $i = 1, 2$, we can construct a continuously differentiable compressor control u such that the demand is fulfilled exactly for all $t \in [t^*, T_0]$ with the lower bound of t^* greater than

$$\overline{T} = \max\left\{ \frac{L_2}{|\lambda_i^{(2)}(\bar{\rho}^{(2)}, \bar{q}^{(2)})|} \right\}, \quad i = 1, 2. \tag{1}$$

[1] M. Gugat, M. Herty, V. Schleper. Flow control in gas networks: exact controllability to a given demand, Math. Methods Appl. Sci., 34(7):745-757, 2011.

State of the art

- 2010, 2014. **M. Gugat, etc.** The coupling of gas pipes, one dimensional isothermal Euler equations, nodal controls.
- 2010. **Tatsien LI.** [2]1-D first order quasilinear hyperbolic systems.

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x = \mathbf{F}(\mathbf{u}),$$

where $\mathbf{u} = (u_1, ..., u_n)^{\mathrm{T}}, A(\mathbf{u}) = (a_{ij}(u)_{n\times n}), F(\mathbf{u}) = (f_1(u), ..., f_n(u))^{\mathrm{T}}$.

- 2010-2016. **Tatsien LI, Ke WANG, Qilong GU.** A complete theory on 1-D wave equations and unsteady flows on networks (without loops).
- 2019. **Yue Wang, Tatsien LI.** EBC of partial nodal profile.
- 2019. **Libin Wang, Ke Wang.** Asymptotic stability of the exact boundary controllability of nodal profile for 1-D quasilinear wave equations.

Remark

- Semi-global classical solution for nonlinear case.
- Local result in the C^1 (or C^2) neighborhood of the stationary state.

[2] Tatsien Li, Exact boundary controllability of nodal profile for quasilinear hyperbolic systems, Math. Methods Appl. Sci. 33(17), 2101–2106, 2010.

Outline

Definition

Definition 1: Degree

The **degree** $d(\geq 1)$ of the node E refers to the number of strings connected to it.
If $d = 1$, the node is **simple node**;
if $d > 1$, the node is **multiple node**.

Figure: Simple node $d = 1$

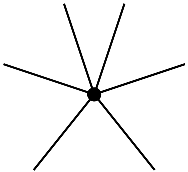
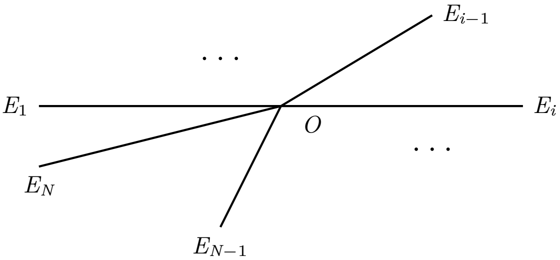
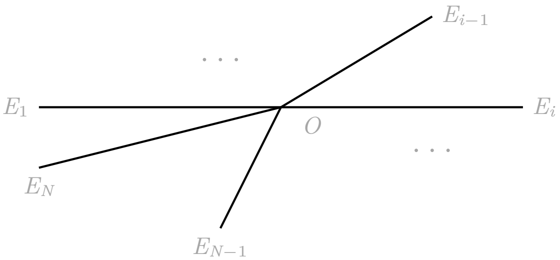


Figure: Multiple node $d > 1$

Star-shaped network

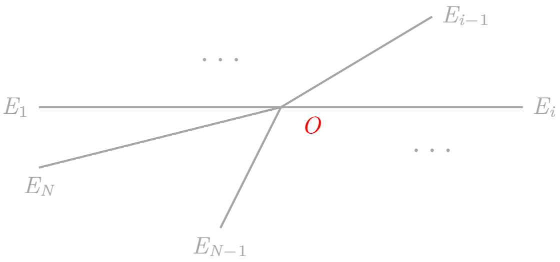
Consider a star-shaped network composed of N strings with a joint node O of degree N.
Let the coordinate of O be $x = 0$.
For $i = 1, \cdots, N$, E_i is the simple node of the i-th string with length L_i.
The i-th string can be parameterized as $x \in [0, L_i]$.





$$y^i_{tt} - a_i^2 y^i_{xx} = 0, \qquad 0 < x < L_i, t > 0 \qquad (i = 1, \cdots, N), \tag{2}$$

where y^i, the transversal displacement of the i-th string, is unknown C^2 function of (t, x) and $a_i > 0$ is given positive constant for each $i = 1, \cdots, N$.



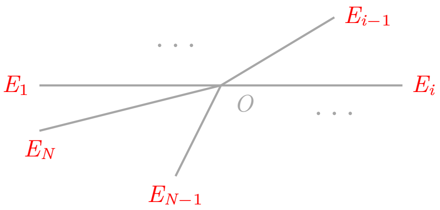
At the multiple node O, we prescribe N transmission conditions:
1 stress interface condition

$$\sum_{i=1}^{N} a_i^2 y_x^i = u_0(t) \tag{3}$$

and $(N-1)$ displacement interface conditions

$$y^i = y^1 \quad (i = 2, \cdots, N), \tag{4}$$

where $u_0(t)$ is a given function at O or a control function to be determined.



For $i = 1, \cdots, N$, at each simple node E_i, we prescribe any one of the following boundary conditions:

$$x = L_i : \quad y^i = u_i(t) \qquad \text{(Dirichlet type)}, \tag{5a}$$

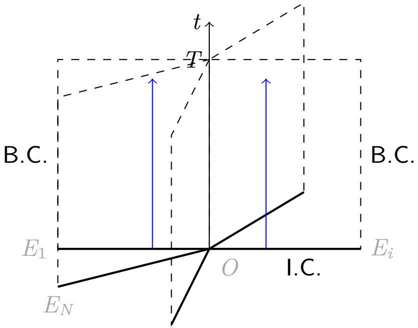
$$x = L_i : \quad y^i_x = u_i(t) \qquad \text{(Neumann type)}, \tag{5b}$$

$$x = L_i : \quad y^i_x + \theta_i y^i = u_i(t), \qquad \text{(Robin type)}, \tag{5c}$$

$$x = L_i : \quad y^i_x + \gamma_i y^i_t = u_i(t), \qquad \text{(Disspative type)}, \tag{5d}$$

where θ_i and γ_i are given positive constants, $u_i \in C^2$ (for (6a)) or $u_i \in C^1$ (for (6b)-(6d)) is a given boundary function or a control function to be determined.

Wellposedness



Let $T > 0$. With any given initial data

$$t = 0: \ (y^i, y^i_t) = (\phi_i, \psi_i), \quad 0 \le x \le L_i (i = 1, ..., N), \tag{6}$$

where $(\phi_i, \psi_i) \in C^2[0, L_i] \times C^1[0, L_i]$ compatible to the boundary data, the mixed initial-boundary valued problem (2)-(6) admits a unique piece-wise C^2 solution y on the domain $\mathcal{R}(T) = \cup_{i=1}^{N}\{(t, x)|0 \le t \le T, 0 \le x \le L_i\}$.

Exact Control Problem of Nodal Profile

Definition 2 & Notations

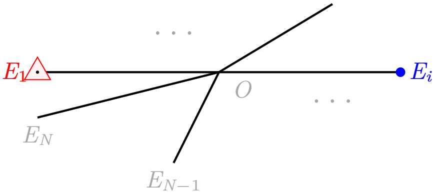
(Charged node):

—● with given (partial or total) nodal profiles.

(Controlled node):

—△ with boundary control to be determined.

(At most one boundary control on each node).



(P): Given finite time $T > \overline{T} > 0$. Can we find one

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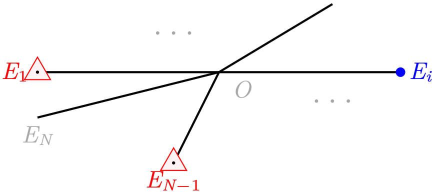
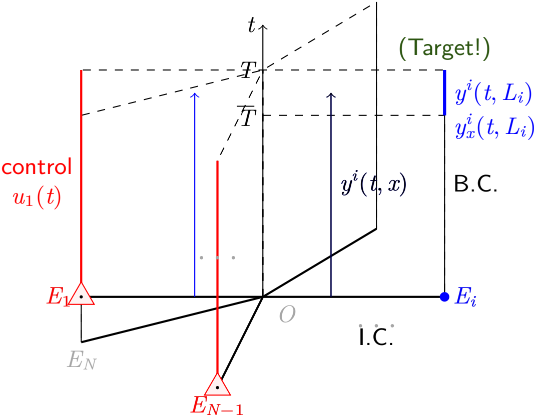


(P): Given finite time $T > \overline{T} > 0$. Can we find one or more boundary controls $u^j \in C^1[0, T]$ (or $\in C^2[0, T]$)

Exact Control Problem of Nodal Profile



(P): Given finite time $T > \overline{T} > 0$. Can we find one or more boundary controls $u^j \in C^1[0, T]$ (or $\in C^2[0, T]$) to drive the boundary trace of the solution to system (2)-(6) to desired values at certain node E_i after the given finite time $\overline{T}$:

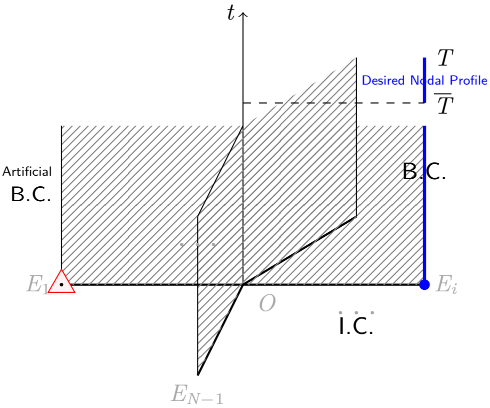
$$y^i(t, L_i) = \tilde{y}_i(t), \quad \overline{T} \leq t \leq T,$$

or

$$y^i_x(t, L_i) = \bar{y}_i(t), \quad \overline{T} \leq t \leq T,$$

where $\tilde{y}_i(t) \in C^2[\overline{T}, T]$ and $\bar{y}_i(t) \in C^1[\overline{T}, T]$ are given profile as functions of time and compatible to the B.C.

Exact Controllability of Nodal Profile: Constructive Method



1. Forward problem on the whole network.
2. Connect the trace. (not unique)

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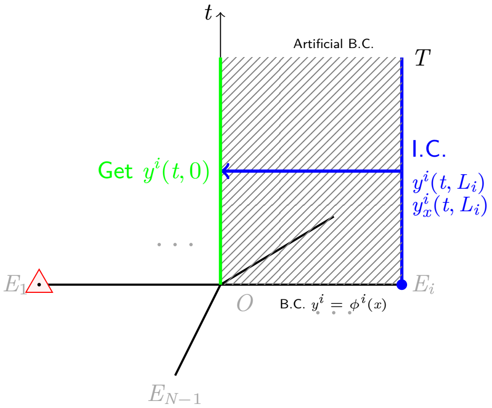

1. Forward problem on the whole network.
2. Connect the trace. (not unique)
3. Change the role of t and x. Sidewise problem from the charged node.

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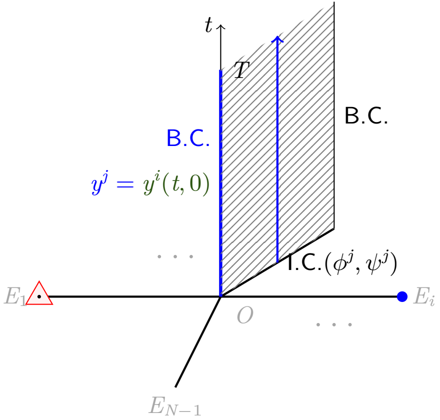

1. Forward problem on the whole network.
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3. Change the role of t and x. Sidewise problem from the charged node.
4. Solve forward problem on uncontrolled strings.

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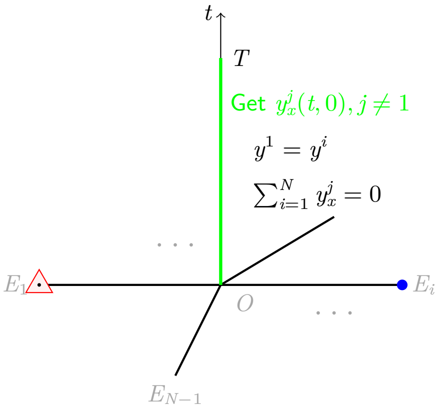

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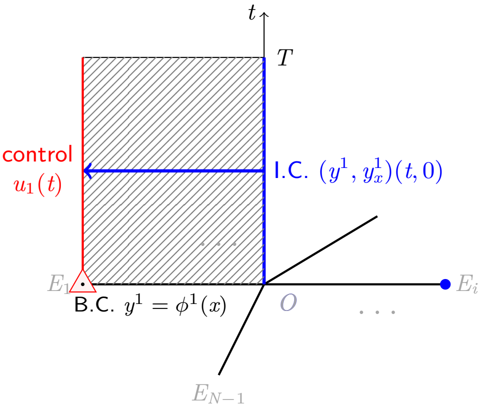

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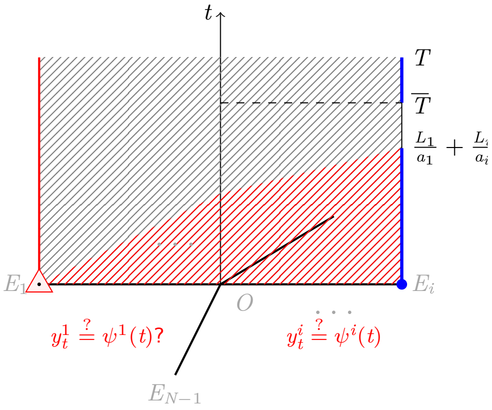
5. Solve sidewise problem from $x = 0$ to the controlled end
6. Determine the boundary control by the trace.

Exact Controllability of Nodal Profile: Constructive Method



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5. Solve sidewise problem from $x = 0$ to the controlled end
6. Determine the boundary control by the trace.
7. Verify I.C.; Uniqueness

Main Result: EBC of Nodal Profile

Theorem 1 [3]

For one charged node with given nodal profiles, there exist

- a finite time $\overline{T}$ (called controllability time),
- and $N(C)$ boundary controls

such that the system (2)-(5) is **exact controllable of nodal profile** after $\overline{T}$. Furthermore, the minimum number of desired controls

$$N(C) = \text{Freedom Degree of the charged node.}$$

Definition 3

Freedom Degree of the charged node is

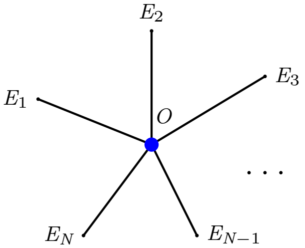
$$\text{The number of given nodal profiles} - \text{The number of related constraints} \,,$$

where the **constraints** refers to given boundary conditions or interface conditions that must be fitted for the given nodal profiles on the charged node.

[3] Yue Wang, Tatsien Li. Exact Boundary Controllability of Partial Nodal Profile for Wave Equations, Submitted, 2019.

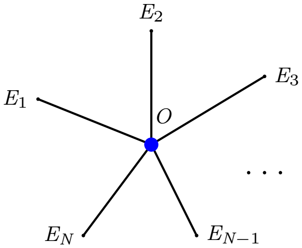
Case 1: Multiple node charged with N displacement nodal profiles

- Target: $(y^1, ..., y^N)(t, 0) = (\tilde{y}^1(t), ..., \tilde{y}^N(t)), t \in [\overline{T}, T]$.



Case 1: Multiple node charged with N displacement nodal profiles

- Target: $(y^1, ..., y^N)(t, 0) = (\tilde{y}^1(t), ..., \tilde{y}^N(t)), t \in [\overline{T}, T]$.
- $(N-1)$ interface conditions $y^j = y^1, j \neq 1$.
- Freedom Degree: $N(O) = 1$.



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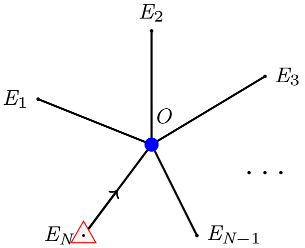
- $(N-1)$ interface conditions $y^j = y^1, j \neq 1$.
- Freedom Degree: $N(O) = 1$.
- Number of desired controls: 1.

- simple node:
  controllability time: $\overline{T} > \frac{L_N}{a_N}$.



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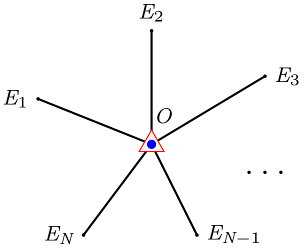
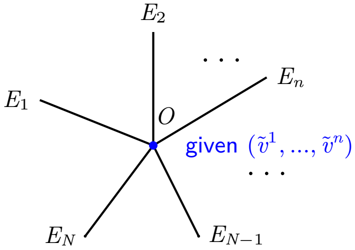
- Freedom Degree: $N(O) = 1$.
- Number of desired controls: 1.

- simple node:
  controllability time: $\overline{T} > \frac{L_N}{a_N}$.
- multiple O:
  controllability time: $\overline{T} > 0$.



Case 2: Multiple node charged with partial nodal profiles

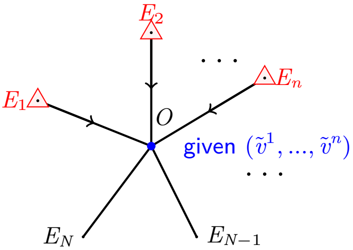
- Target: $(y_x^1, ..., y_x^n)(t, 0) = (\tilde{y}^1(t), ..., \tilde{y}^n(t)), t \in [\overline{T}, T]$.
- 1 transmission condition $\sum_i^N a^i y_x^i = 0$.
- If $n = N$, freedom Degree: $N(O) = N - 1$.



Case 2: Multiple node charged with partial nodal profiles

- Target: $(y_x^1, ..., y_x^n)(t, 0) = (\tilde{y}^1(t), ..., \tilde{y}^n(t)), t \in [\overline{T}, T]$.
- If $n < N$, freedom Degree: $N(O) = n$.
- Number of controls: n.

- n simple nodes: $\overline{T} > \max_{1 \le j \le n} \frac{L_j}{a_j}$.



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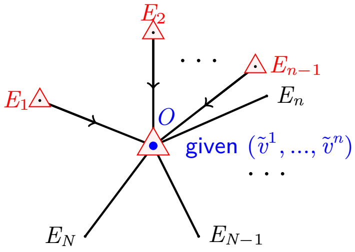
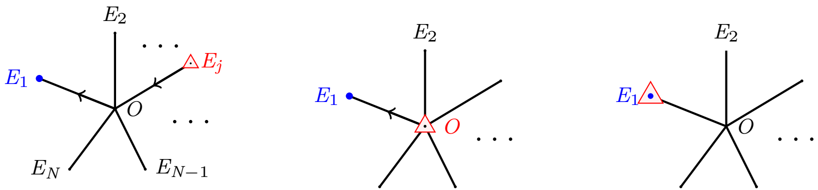
- If $n < N$, freedom Degree: $N(O) = n$.
- Number of controls: n.

- n simple nodes:
  $\overline{T} > \max_{1 \le j \le n} \frac{L_j}{a_j}$.
- multiple O and $n-1$ simple nodes:
  $\overline{T} > \max_{1 \le j \le n-1} \frac{L_j}{a_j}$.



Case 3: Simple node charged with partial nodal profiles



(a) Simple controlled node $E_j (j \neq 1)$ (b) Multiple controlled node O (c) In-situ controlled node

The charged node E_1 is always of freedom degree 1.

Fig.	Charged node	Controlled node	Controllability Time T
(a)	E_1	$E_j (j \neq 1)$	$\overline{T} > \frac{L_1}{a_1} + \frac{L_j}{a_j}$
(b)	E_1	O	$\overline{T} > \frac{L_1}{a_1}$
(c)	E_1	E_1(in-situ)	$\overline{T} > 0$

Summary

In summary, we find that for one charged node,

- Minimum number of desired controls= Freedom degree of the charged node.
- Controllability time = the transmission time from the controlled node to the charged node.

The proof is based on

- Existence and Uniqueness Theory on global piecewise C^2 solution to the mixed problem on the network.
- Method: Constructive Method.

Some Remarks

Remark1: Planar Tree-like Network

The previous results can be generalized to the tree-like network satisfying some requirements of number and position of charged nodes and controlled nodes.

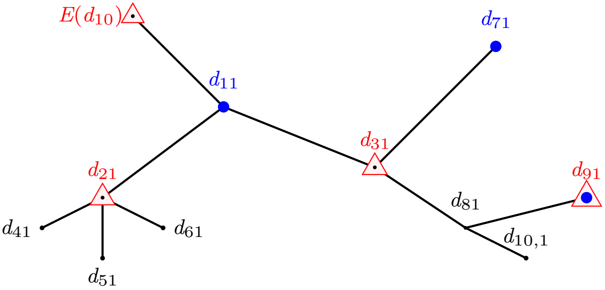


Figure: Charged nodes d_{11}, d_{71} and d_{91}, of which the degree of freedom is 2, 1 and 1.

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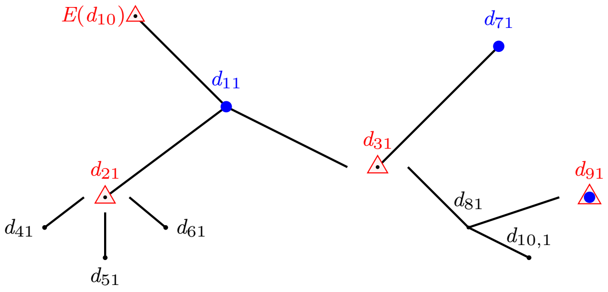


Figure: Spilt nodes d_{21}, d_{31} and d_{91}.

Some Remarks

Remark2: Quasilinear Case

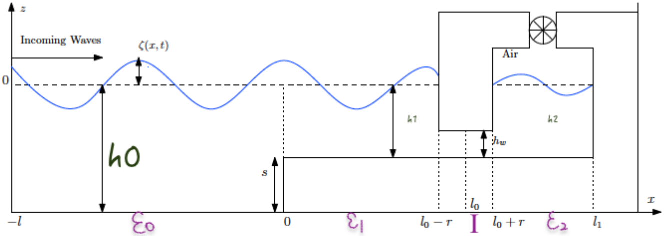
The previous results can be generalized to the corresponding quasilinear case .

$$u^i_{tt} - (K^i(u^i, u^i_x))_x = F^i(u^i, u^i_x, u^i_t), \qquad t \geq 0, d_{i0} \leq x \leq d_{i1}, \tag{7}$$

- In C^2 neighborhood of stationary state.
- Semi-global C^2 solution with C^2 norm.
- Construct a solution.

Remark3: Harvesting Model

This method can be used to other one-dimension hyperbolic system in the framework of classical solutions.



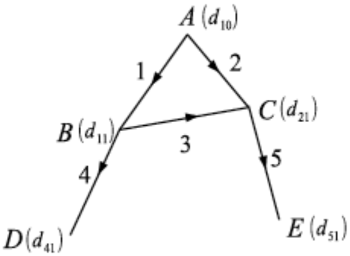
1D nonlinear shallow water equations [4].

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x(\frac{q^2}{h}) + gh\partial_x \zeta = 0 \end{cases} \tag{8}$$

[4] E. Bocchi, J. He and G. Vergara-Hermosilla, Modelling and simulation of a wave energy converter, ESAIM: PROCEEDINGS AND SURVEYS, 1-10, 2019.

Remark4: Saint-Venant system on a network with loops

- The Saint-Venant system has no exact boundary controllability on a network with loops in general. **[1994, J.E. Lagnese, G. Leugering, E. J. P. G. Schmidt].**
- BUT it is possible to get the exact boundary controllability of nodal profile for Saint-Venant system on certain networks with loops.**[2019, Kaili Zhuang, G. Leugering, Tatsien Li].**



Outline

Introduction

- Working space: Hilbert space.
- Linear control problem of nodal profile.
- Duality method: duality between controllability and observability .
- observability inequalities.
- Exact controllability problem (Final data at T): J.-L. Lions[5], E. Zuazua[6], J. M.Coron.[7].

[5] J. -L. Lions, Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués, Vol. I, Masson, (1988).

[6] E. Zuazua, Exact controllability for semilinear wave equations in one space dimension, Annales de l'I. H. P., section C, tome 10, no 1 (1993), p. 109-129.

[7] J. M. Coron, Control and Nonlinearity, Mathematical Surveys and Monographs Vol.136, American Mathematical Society, 2007.

Star-shaped Network: The Three String Network with Two Controlled Nodes

We consider the simplest non-trivial network of strings that cannot be reduced to a single string: the three string network.

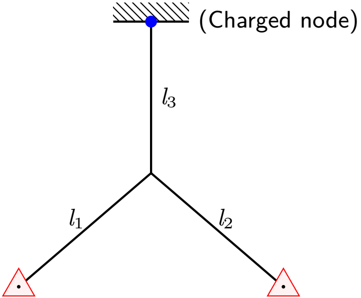


Figure: Three string Network with two controls

Equations of Motion of the Network

Let T, l_1, l_2, l_3 be positive numbers. We consider the following non-homogeneous system

$$\begin{cases} y^i_{tt} - y^i_{xx} = 0, \quad (t,x) \in (0,T) \times (0,l_i), \quad i = 1,2,3, \\ x = 0 : y^1(t,0) = y^2(t,0) = y^3(t,0), \quad t \in (0,T), \\ \qquad y^0_x(t,0) + y^1_x(t,0) + y^2_x(t,0) = 0, \quad t \in (0,T), \\ x = l_3 : y^3(t,l_3) = 0, \quad t \in (0,T), \\ x = l_i : y^i_x(t,l_i) = u^i(t), \quad t \in (0,T), \quad i = 1,2, \end{cases} \tag{9}$$

with any given initial data

$$(y^i, y^i_t)(0,x) = (y^i_0, y^i_1)(x), \quad 0 \le x \le l_i, \quad i = 1,2,3, \tag{10}$$

which models the vibrations of a network formed by three elastic strings with the lengths l_1, l_2, l_3 coupled at one of their extremes at $x = 0$. The functions $y^i = y^i(t,x) : [0,l_i] \to \mathbb{R}, i = 1,2,3,$ represent the transversal displacements of the strings. Here we take Neumann controls u^1 and u^2.

We introduce the working spaces

$$\begin{aligned} V &= \left\{ \psi \in \prod_{i=1}^{3} H^1(0, l_i) \middle| \psi^1(0) = \psi^2(0) = \psi^3(0), \psi^3(l_3) = 0 \right\}, \\ H &= \prod_{i=1}^{3} L^2(0, l_i) \end{aligned} \tag{11}$$

with the inner product

$$\begin{aligned} (\vec{\phi}, \vec{\psi})_V &= \sum_{i=1}^{3} \frac{1}{2} \int_0^{l_i} \phi_x^i \cdot \psi_x^i \mathrm{d}x, \\ (\vec{\phi}, \vec{\psi})_H &= \sum_{i=1}^{3} \frac{1}{2} \int_0^{l_i} \phi^i \cdot \psi^i \mathrm{d}x. \end{aligned} \tag{12}$$

In the following, $\vec{y} = (y^1, y^2, y^3)$, and $\vec{y}_0, \vec{y}_1$ denotes the corresponding initial displacement and velocity, respectively.

Theorem 2.1 (Wellposedness)

For any given initial data $(\vec{y}_0, \vec{y}_1) \in V \times H$ and boundary functions $u^1, u^2 \in L^2(0, T)$, the corresponding compatibility conditions are satisfied at the nodes $(0, l_1)$ and $(0, l_2)$, then system (9)-(10) admits a unique solution [8]:

$$\vec{y} \in C([0, T]; V) \bigcap C^1([0, T]; H). \tag{13}$$

Furthermore, we can get a Neumann trace hidden regularity result at the end $x = l_3$ with homogenous Dirichlet boundary condition that: the map

$$\{\vec{y}_0, \vec{y}_1\} \mapsto y^3_x\big|_{x=l_3} : V \times H \to L^2\big(\{x = l_3\} \times (0, T)\big) \tag{14}$$

is continuous.

[8] J.E. Lagnese, G. Leugering, Schmidt, Modeling, Analysis and Control of Dynamic Elastic Multi-Link Structures, 1994

Boundary Control Problem of Nodal Profile

(P): Let $T > \overline{T} > 0$. For given desired profile function $\bar{y}_3(t) \in L^2(\overline{T}, T)$ to find boundary controls $u^1, u^2 \in L^2(0, T)$ so that

$$\mathcal{F}(u^1, u^2) = \bar{y}_3(t).$$

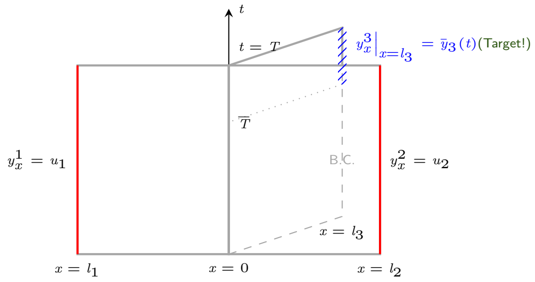


Figure: Control Problem: y-system

Control Operator

Step 1: Define a mapping to describe the control problem

For given $T > \overline{T} > 0$, define a mapping

$$\boxed{\begin{aligned} \mathcal{F} : U &\longrightarrow L^2(\overline{T}, T) \\ (u^1, u^2) &\mapsto y_x^3(t, l_3) \end{aligned}} \tag{15}$$

where $U = L^2(0, T) \times L^2(0, T)$, $y_x^3(t, l_3)$ is the Neumann trace at the end $x = l_3$ of the corresponding solution $\vec{y} \in C^0([0, T]; V) \cap C^1([0, T]; H)$ to the system (9) with null initial data $(\vec{y}_0, \vec{y}_1)(x) = (0, 0)$ and boundary controls u^1, u^2 at $x = l_1$ and $x = l_2$. Then

Definition 2.2 [Boundary Controllability of Nodal Profile]

The system (9) is controllable of nodal profile at charged node $x = l_3$ in time $t \in (\overline{T}, T)$ if the control operator $\mathcal{F}$ is onto.

Main Result

Our main controllability result is

Theorem 2.3

Let

$$\overline{T} = l_3 + \max\{l_1, l_2\} \tag{16}$$

and $T > \overline{T}$. Then the system (9) is controllable of nodal profile at $x = l_3$ in time $t \in (\overline{T}, T)$.

Method: Duality between controllability and observability

Framework:

- Controllability for a linear control system

[10] Rudin, Functional Analysis, 1973, P97;

[10] Coron, Control and Nonlinearity, P35

Method: Duality between controllability and observability

Framework:

- Controllability for a linear control system
- $\Longleftrightarrow$ the surjectivity of a certain linear map $\mathcal{F}$ from a Hilbert space H_1 to another Hilbert space H_2.

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Framework:

- Controllability for a linear control system
- $\Longleftrightarrow$ the surjectivity of a certain linear map $\mathcal{F}$ from a Hilbert space H_1 to another Hilbert space H_2.
- $\Longleftrightarrow$ the existence of $c > 0$ such that

$$\|\mathcal{F}^*(x_2)\|_{H_1} \geq c\|x_2\|_{H_2}, \quad \forall x_2 \in H_2, \tag{17}$$

where $\mathcal{F}^* : H_2 \to H_1$ is the adjoint of $\mathcal{F}$. This inequality (17) is called the observability inequality. [9][10]

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- compute $\mathcal{F}^*$
- and then prove (17).

[10] Rudin, Functional Analysis, 1973, P97;

[10] Coron, Control and Nonlinearity, P35

Dual System; Observability

Step 2: Compute adjoint operator $\mathcal{F}^*$. We introduce the following adjoint system:

$$\begin{cases} \phi^i_{tt} - \phi^i_{xx} = 0 \qquad (i = 1, 2, 3), \\ x = 0 : \displaystyle\sum_{i=1}^{3} \phi^i_x = 0, \\ \qquad\quad \phi^1 = \phi^2 = \phi^3, \\ x = l_i : \phi^i_x = 0 \qquad (i = 1, 2), \\ x = l_3 : \phi^3 = \begin{cases} 0, & t \in [0, \overline{T}], \\ \varphi_3(t), & t \in [\overline{T}, T], \end{cases} \\ t = T : (\phi^i, \phi^i_t) = (0, 0) \qquad (i = 1, 2, 3), \end{cases} \tag{18}$$

where $\varphi_3(t) \in L^2(\overline{T}, T)$.

Lemma 2.4

$\mathcal{F}$ is an onto mapping if and only if for the adjoint mapping $\mathcal{F}^*$, there exists a constant $c > 0$ such that

$$\|\mathcal{F}^*(\varphi_3)\|_U \geq c\|\varphi_3\|_{L^2[\overline{T},T]}, \qquad \forall \varphi_3 \in L^2(\overline{T},T). \tag{19}$$

$$\boxed{\mathcal{F}^* : \ L^2(\overline{T},T) \longrightarrow L^2(0,T) \times L^2(0,T)} \tag{20}$$

we are going to take a dense set in $H_0^1(\overline{T},T) \subset L^2(\overline{T},T)$ to define how $\mathcal{F}^*$ operates., and prove the corresponding observability inequality. Then, by a classical density argument and extension by continuity, the definition and inequality can be extended to the whole space $L^2(\overline{T},T)$.

Adjoint Operator

Lemma 2.5

$\mathcal{F}^*$, the adjoint operator of $\mathcal{F}$, is densely defined on $H_0^1(\overline{T}, T)$ as:

$$\mathcal{F}^*(-\varphi_3(t)) = (\phi^1(\cdot, l_1), \phi^2(t, l_2)) \quad \forall \varphi_3(t) \in H_0^1(\overline{T}, T),$$

where $(\phi^1(t, l_1), \phi^2(t, l_2)$ are the trajectories of the solution $\vec{\phi} := \vec{\phi}(t, x)$ to the adjoint system (18). Moreover, we have the following dual identity

$$\big\langle (u^1, u^2), \big(\phi^1(\cdot, l_1), \phi^2(\cdot, l_2)\big) \big\rangle_{L^2(0,T)\times L^2(0,T)} = \langle \mathcal{F}(u^1, u^2), -\varphi_3 \rangle_{L^2(\overline{T},T)}. \quad (21)$$

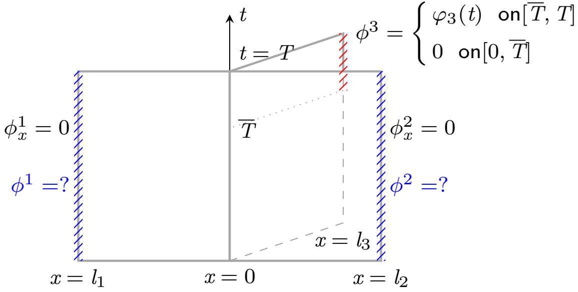
Observability Inequality

Theorem 2.6

Let $\overline{T} > l_3 + \max\{l_1, l_2\}$ and $T > \overline{T}$. There exists a constant $c > 0$, such that the observability inequality the adjoint problem (18)that

$$\| \left(\phi^1(\cdot, l_1), \phi^2(\cdot, l_2)\right) \|_{L^2(0,T)\times L^2(0,T)} \geq c\|\varphi_3(\cdot)\|_{L^2(\overline{T},T)}, \forall \varphi_3 \in H_0^1(\overline{T}, T). \tag{22}$$

So far, we shift the control problem of y-system (9) to the dual problem of ϕ -system (18)(shown in Fig.) .



Prove the Observability Inequality

Step3: The observability inequality. From the densely defined of $\mathcal{F}^*$, here we only prove the observability inequality for $\varphi_3 \in H_0^1(\overline{T}, T)$.
Two methods:

- the explicit solution of the adjoint problem (18), side-wise D' Alembert Formula[11]. (Dáger, Zuazua)
- the multiplier method. (Alabau)

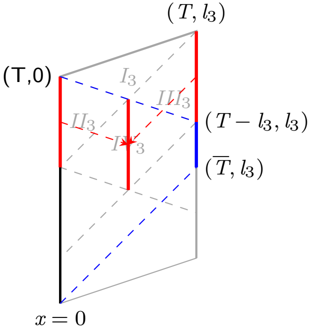
[11] René Dáger, Enrique Zuazua. Wave Progagation, Observation and Control in 1− d Flexible Multi-Structures

Proof of Observability Inequality

Step3.1(Backward Problem):

- Consider the backward mixed initial boundary value problem (18).
- $\Longrightarrow$ Unique solution $\vec{\phi} = (\phi^1(t,x), \phi^2(t,x), \phi^3(t,x)) \in C^0([0,T];\bar{V}) \times C^1([0,T];H)$, where $\bar{V} = \left\{\phi \in \prod_{i=1}^3 H^1(0,l_3) | \phi^1(0) = \phi^2(0) = \phi^3(0), \phi^1_x(l_1) = \phi^2_x(l_2) = 0\right\}$..
- $\Longrightarrow$ At the ends $x = l_1$ and $x = l_2$ we get the traces of the solution $\phi^1(t,l_1), \phi^2(t,l_2) \in L^2(0,T)$.

Step3.2 (Estimation for ϕ^3):



It is easy to check that

$$\phi^3(t,x)=\begin{cases}0, & (t,x)\in I_3,\\ \phi^3(t+x,0), & (t,x)\in II_3,\\ \phi^3(t+l_3-x,l_3), & (t,x)\in III_3,\\ \phi^3(t+l_3-x,l_3)+\phi^3(t+x,0), & (t,x)\in IV_3,\end{cases} \tag{23}$$

$$\|\varphi_3\|^2_{L^2(T-l_3,T)}=\int_{T-l_3}^{T}\left[\phi^3(t-\frac{l_3}{2},\frac{l_3}{2})-\phi^3(t,0)\right]^2\,\mathrm{d}t \tag{24}$$

$$\|\varphi_3\|^2_{L^2(\overline{T},T-l_3)}=\frac{1}{4}\int_{\overline{T}}^{T-l_3}\left(\phi^3(t+l_3,0)+\phi^3(t-l_3,0)+\int_{t-l_3}^{t+l_3}\phi^3_x(\tau,0)\,\mathrm{d}\tau\right)^2\,\mathrm{d}t \tag{25}$$

If $\bar{T} - l_3 > 0$, we get

$$\begin{aligned}\|\varphi_3\|^2_{L^2(\overline{T},T)} \leq & 6\|\phi^3(\cdot,0)\|^2_{L^2(\overline{T}-l_3,T)} + \int_{T-l_3}^{T}\left[\int_{t-l_3}^{t}\phi_x^3(\tau,0)\,\mathrm{d}\tau\right]^2\,\mathrm{d}t \\ & + \frac{1}{2}\int_{\overline{T}}^{T-l_3}\left[\int_{t-l_3}^{t+l_3}\phi_x^3(\tau,0)\,\mathrm{d}\tau\right]^2\,\mathrm{d}t \\ := & I_1 + I_2 + I_3\end{aligned} \tag{26}$$

Moreover, in account of the transmission conditions at the adjoint node $x = 0$, we have

$$\phi^3(t,0) = \phi^1(t,0) = \phi^2(t,0),$$

$$-\phi_x^3(t,0) = \phi_x^1(t,0) + \phi_x^2(t,0),$$

then

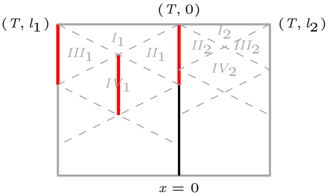
$$I_1 = 6\|\phi^i(\cdot,0)\|^2_{L^2(\overline{T}-l_3,T)}, \qquad i = 1,2, \tag{27}$$

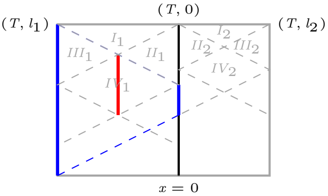
$$I_2 \leq 2\sum_{i=1,2}\int_{T-l_3}^{T}\left[\int_{t-l_3}^{t}\phi_x^i(\tau,0)\,\mathrm{d}\tau\right]^2\,\mathrm{d}t, \tag{28}$$

$$I_3 \leq \sum_{i=1,2}\int_{\overline{T}}^{T-l_3}\left[\int_{t-l_3}^{t+l_3}\phi_x^i(\tau,0)\,\mathrm{d}\tau\right]^2\,\mathrm{d}t. \tag{29}$$

Step3.3 (Estimation for ϕ^i and ϕ^i_x, $i = 1, 2$):

(Step 3.3.1. Tools) Explicit Solution. Sidewise D' Alermbert Formula.





Step3.3 (Estimation for ϕ^i and ϕ^i_x, $i = 1, 2$):

(Step 3.3.1. Tools) Explicit Solution. Sidewise D' Alermbert Formula.

(Step 3.3.2. Estimation for (27)**)**

$$\|\phi^1(\cdot,0)\|^2_{L^2(\overline{T}-l_3,T)} \le 2\|\phi^1(\cdot,l_1)\|^2_{L^2(\overline{T}-l_3-l_1,T)}. \quad (30)$$

(Step 3.3.3. Estimation for (28)**)**

$$\begin{aligned}\int_{T-l_3}^{T}\left(\int_{t-l_3}^{t}\phi^1_x(\tau,0)\,\mathrm{d}\tau\right)^2 \mathrm{d}t &\le c\|\phi^1(\cdot,l_1)\|^2_{L^2(T-l_1-2l_3,T)} \\ &\le c\|\phi^1(\cdot,l_1)\|^2_{L^2(\overline{T}-l_1-l_3,T)}.\end{aligned} \quad (31)$$

(Step 3.4. Estimation for (29)**)**

$$\int_{\overline{T}}^{T-l_3}\left[\int_{t-l_3}^{t+l_3}\phi^1_x(\tau,0)\,\mathrm{d}\tau\right]^2 \mathrm{d}t \le c\|\phi^1(\cdot,l_1)\|^2_{L^2(\overline{T}-l_3-l_1,T)}. \quad (32)$$

The corresponding estimation for ϕ^2 is similar. Finally, substituting all estimations into (27)-(29), we get

$$\|\varphi_3\|^2_{L^2(\overline{T},T)} \leq c\left(\|\phi^1(\cdot,l_1)\|^2_{L^2(\overline{T}-l_3-l_1,T)} + \|\phi^2(\cdot,l_2)\|^2_{L^2(\overline{T}-l_3-l_2,T)}\right), \quad (33)$$

which implies the observability inequality (22) provided with $\overline{T} > l_3 + \max\{l_1, l_2\}$.

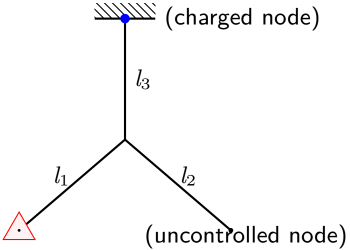
So far, we establish the duality between controllability and observability and prove the observability inequality (22), so we give a positive answer to the control problem as

Theorem 2.7 (Controllability)

Let

$$\overline{T} = l_3 + \max\{l_1, l_2\} \tag{34}$$

and $T > \overline{T}$. Then the control problem (9) is controllable of nodal profile at $x = l_3$ in time $t \in (\overline{T}, T)$.



$$
\begin{aligned}
x = l_3 &: y^3(t, l_3) = 0, && t \in (0, T), \\
x = l_2 &: y^2_x(t, l_2) = 0, && t \in (0, T), \\
x = l_1 &: y^1_x(t, l_1) = u^1(t), && t \in (0, T),
\end{aligned} \tag{35}
$$

Theorem 2.8

Let

$$\overline{T} = l_3 + l_1 \tag{36}$$

and $T > \overline{T}$. Then the system with (35) is controllable of nodal profile at $x = l_3$ in time $t \in (\overline{T}, T)$.

$$
\boxed{\begin{aligned} \mathcal{F}_1 : L^2(0, T) &\longrightarrow L^2(\overline{T}, T) \\ u^1 &\mapsto y^3_x(t, l_3) \end{aligned}} \tag{37}
$$

proposition 2.9

Let $\overline{T} > l_3 + l_1$ and $T > \overline{T}$. There exists a constant $c > 0$, such that the observability inequality

$$\|\phi^1(\cdot, l_1)\|_{L^2(0,T)} \geq c\|\varphi_3(\cdot)\|_{L^2(\overline{T},T)}, \forall \varphi_3 \in H_0^1(\overline{T}, T) \tag{38}$$

holds.

Key.
If we were able to prove that there exists $c > 0$ such that

$$\|\phi^1(\cdot, 0)\|_{L^2(\overline{T}-l_3,T)} \leq c\|\phi^1(\cdot, l_1)\|_{L^2(0,T)}, \tag{39}$$

$$\int_{T-l_3}^{T}\left[\int_{t-l_3}^{t}\phi_x^1(\tau,0)\,\mathrm{d}\tau\right]^2\mathrm{d}t + \int_{\overline{T}}^{T-l_3}\left[\int_{t-l_3}^{t+l_3}\phi_x^1(\tau,0)\,\mathrm{d}\tau\right]^2\mathrm{d}t \leq c\|\phi^1(\cdot, l_1)\|_{L^2(0,T)}, \tag{40}$$

and

$$\int_{T-l_3}^{T}\left[\int_{t-l_3}^{t}\phi_x^2(\tau,0)\,\mathrm{d}\tau\right]^2\mathrm{d}t + \int_{\overline{T}}^{T-l_3}\left[\int_{t-l_3}^{t+l_3}\phi_x^2(\tau,0)\,\mathrm{d}\tau\right]^2\mathrm{d}t \leq c\|\phi^1(\cdot, l_1)\|_{L^2(0,T)}, \tag{41}$$

we would obtain the observability inequality .
However, (41) does not obviously hold.

$$\int_a^b \left[\int_{t_1}^{t_2} \phi_x^2(\tau,0)\,\mathrm{d}\tau\right]^2 \mathrm{d}t = \int_a^b \left[2\phi^2\left(\frac{t_1+t_2}{2},\frac{t_2-t_1}{2}\right) - \phi^2(t_1,0) - \phi^2(t_2,0)\right]^2 \mathrm{d}t \quad (42)$$

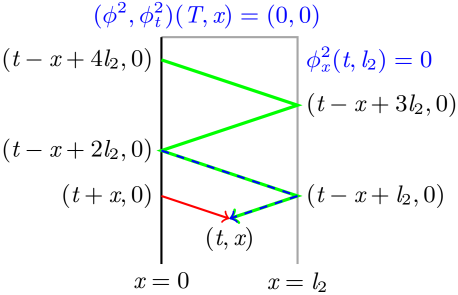


Figure: Waves going from $x=0$ and reflecting at $x=l_2$

$$\phi^2\left(\frac{t_1+t_2}{2},\frac{t_2-t_1}{2}\right) = \phi^2(t_2,0) + \sum_{\{n\in\mathbb{N}^+ \mid t_1+2nl_2\in[0,T]\}} \phi^2(t_1+2nl_2,0). \quad (43)$$

$$\int_a^b \left[\int_{t_1}^{t_2} \phi_x^2(\tau,0)\,\mathrm{d}\tau\right]^2 \mathrm{d}t \le c\|\phi^2(\cdot,0)\|_{L^2(t_1(a),t_2(b))}. \quad (44)$$

$$\begin{aligned}
&\int_{T-l_3}^{T}\left[\int_{t-l_3}^{t}\phi_x^2(\tau,0)\,\mathrm{d}\tau\right]^2\mathrm{d}t+\int_{\overline{T}}^{T-l_3}\left[\int_{t-l_3}^{t+l_3}\phi_x^2(\tau,0)\,\mathrm{d}\tau\right]^2\mathrm{d}t\\
\le& c\|\phi^2(\cdot,0)\|^2_{L^2(T-2l_3,T)}+c\|\phi^2(\cdot,0)\|^2_{L^2(\overline{T}-l_3,T)}\\
\le& c\|\phi^1(\cdot,0)\|^2_{L^2(\overline{T}-l_3,T)}\\
\le& c\|\phi^1(\cdot,l_1)\|^2_{L^2(\overline{T}-l_3-l_1,T)},
\end{aligned} \tag{45}$$

where the constants c are different from line to line. The third inequality (41) is proved.

In summary, the observability inequality (38) holds provide with $\overline{T}-l_3-l_1>0$. □.

Nodal Control Problem. (to do).

- Optimal L^2 control. HUM control(J.-L.-Lions 1988).
- Nonlinear Case. Lack of duality. (Implicit duality[12]. Exact Observability Problem for quasilinear hyperbolic problem.)

[12] Tatsien Li. Controllability and Observability for Quasilinear Hyperbolic Systems. 2010.

Thanks.