Discovering scaling laws for heat transport using convex optimization

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Joint work with:
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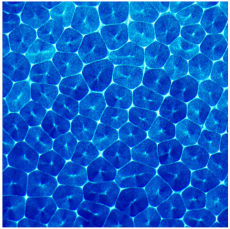
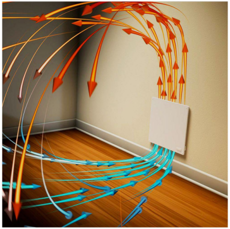
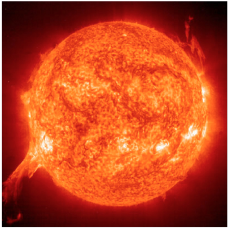
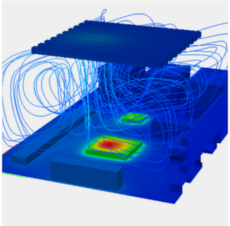
Anuj Kumar *(UC Santa Cruz)*

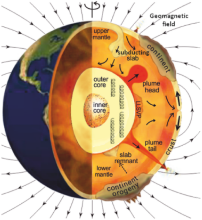
Why heat transport?









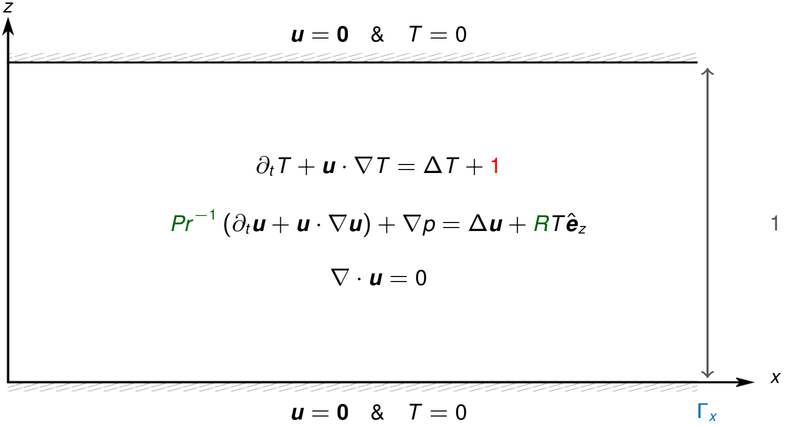


Surface tension

Boundary-driven

Internal heating

A periodic layer with cool and sticky boundaries

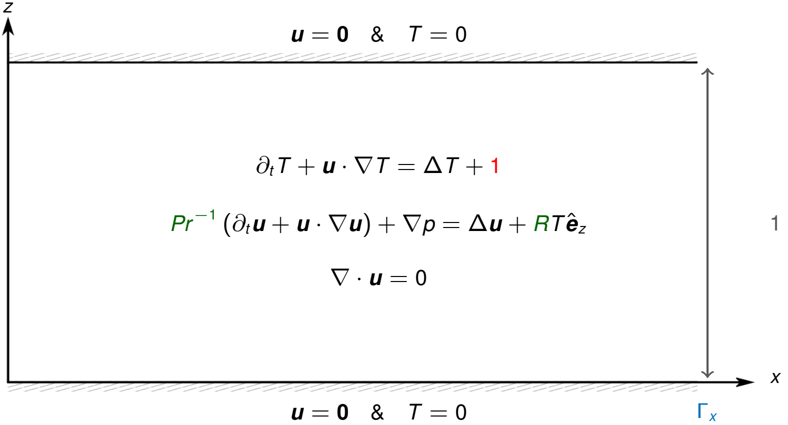


$$\partial_t T + \boldsymbol{u}\cdot\nabla T = \Delta T + 1$$

$$Pr^{-1}\left(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) + \nabla p = \Delta\boldsymbol{u} + R T\hat{\boldsymbol{e}}_z$$

$$\nabla\cdot\boldsymbol{u} = 0$$

A periodic layer with cool and sticky boundaries



$$\boldsymbol{u} = \boldsymbol{0} \quad \& \quad T = 0$$

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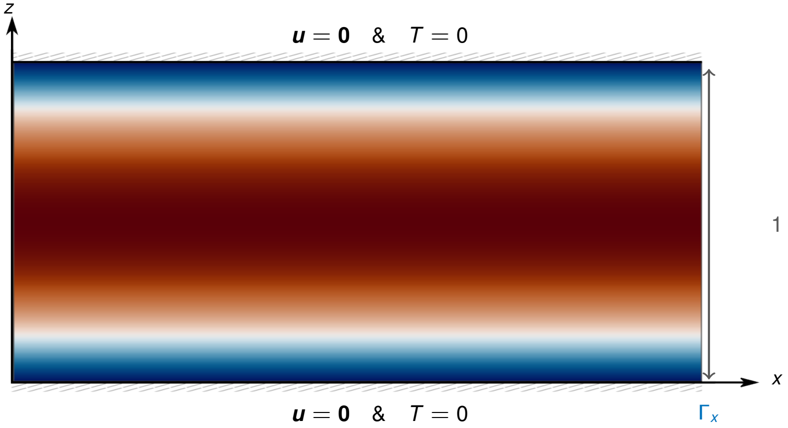
Which fraction of the heat input exists through the top vs the bottom?

How do these fractions depend on R (heating strength) and Pr (fluid's inertia)?

Weak heating (small R) = no flow



$$\mathcal{F}_{\text{top}} = \frac{1}{2} \qquad \mathcal{F}_{\text{bot}} = \frac{1}{2} \; (= 1 - \mathcal{F}_{\text{top}})$$

Strong heating (large R) = turbulence

Video courtesy of John Craske

$$\mathcal{F}_{\text{top}} > \frac{1}{2}$$

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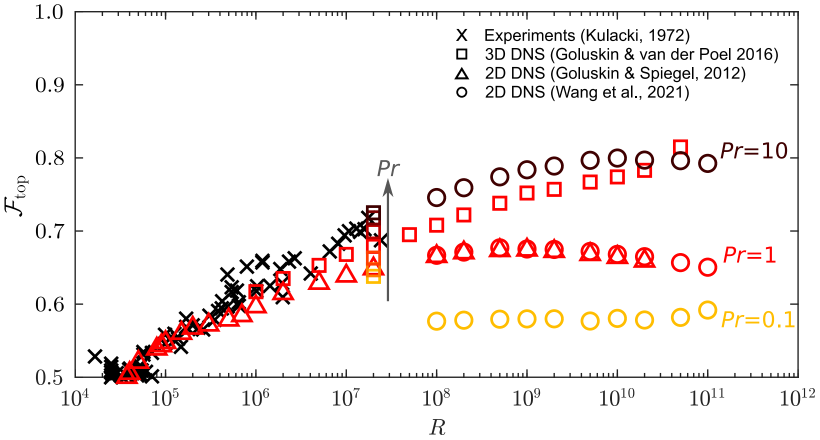
$$\mathcal{F}_{\text{top}} > \frac{1}{2}$$

Theorem (Goluskin & Spiegel, 2012): $\frac{1}{2} \leq \mathcal{F}_{\text{top}} \leq 1$ independently of R and Pr.

Simulations tell a different story!



Plan for today

1. Estimating $\mathcal{F}_{\text{top}}$ via infinite-dimensional convex optimization
2. Results from numerical approximation
3. From numerics to proofs
4. Extensions and open problems (depending on time!)

Basic identities

$$\mathcal{F}_{\text{top}} = \frac{1}{2} + \frac{1}{R}\overline{\langle|\nabla \boldsymbol{u}|^2\rangle} \qquad \sim \text{kinetic energy dissipation}$$

Notation:

w	$=$	vertical velocity of the fluid
T	$=$	temperature
$\langle\cdot\rangle$	$=$	space average
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Question: Can we prove that

$$\overline{\langle wT\rangle} \leq \frac{1}{2} - f(R, Pr)$$

for some positive function f?

Bounding time averages

Observation: If $\mathcal{V}\{\boldsymbol{u}, T\}$ remains uniformly bounded along solutions,

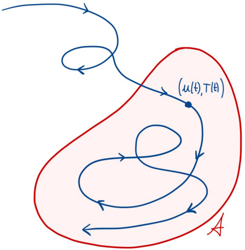
$$\overline{\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}\{\boldsymbol{u}(t), T(t)\}} = \limsup_{t\to\infty} \frac{\mathcal{V}\{\boldsymbol{u}(t), T(t)\} - \mathcal{V}\{\boldsymbol{u}_0, T_0\}}{t} = 0$$

Then,

$$\overline{\langle w(t)T(t)\rangle} = \overline{\langle w(t)T(t)\rangle + \tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}\{\boldsymbol{u}(t), T(t)\}}$$

$$= \overline{\langle w(t)T(t)\rangle + \mathscr{L}\mathcal{V}\{\boldsymbol{u}(t), T(t)\}}$$

$$\leq \inf_{\mathcal{V}} \sup_{(\boldsymbol{u},T)\in\mathscr{A}} \left\{ \langle wT\rangle + \mathscr{L}\mathcal{V}\{\boldsymbol{u}, T\} \right\}$$



Tobasco et al (2018) + Rosa & Temam (2020): The strategy is **sharp** for well-posed ODEs/PDEs with **compact** absorbing sets.

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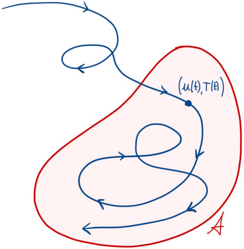

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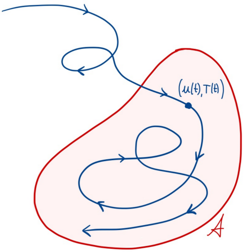

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How should one choose $\mathcal{V}$?

The Doering & Constantin way (1994+)

Consider

$$\mathcal{V}\{\boldsymbol{u}, T\} = \frac{\beta_1}{2 Pr\, R} \left\langle |\boldsymbol{u}|^2 \right\rangle + \frac{\beta_2}{2} \left\langle |T - \varphi(z)|^2 \right\rangle$$

where

$$\varphi(0) = 0 \qquad \varphi(1) = 0$$

Then,

$$\begin{aligned}\langle wT \rangle + \mathscr{L}\mathcal{V}\{\boldsymbol{u}, T\} = \tfrac{1}{2} &- \langle \tau(z) \rangle \\ &- \left\langle \left[\tau'(z) + 1 + \beta_2 z\right] T_z \right\rangle \\ &- \left\langle \beta_1 R^{-1} |\nabla \boldsymbol{u}|^2 + \beta_2 |\nabla T|^2 + \left[\tau'(z) - \beta_1\right] wT \right\rangle\end{aligned}$$

Maximize over the absorbing set $\mathscr{A}$ defined using

- Boundary conditions
- Incompressibility: $\nabla \cdot \boldsymbol{u} = 0$
- **Minimum principle:** at long times, $T \geq 0$ almost everywhere in the fluid domain
  $\rightsquigarrow$ introduce a "Lagrange multiplier" $\lambda(z)$

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$$\tau(0) = 1 \quad \tau(1) = 0$$

Then,

$$\langle wT \rangle + \mathscr{L}\mathcal{V}\{\boldsymbol{u}, T\} = \tfrac{1}{2} - \langle \tau(z) \rangle - \left\langle \left[\tau'(z) + 1 + \beta_2 z\right] T_z \right\rangle - \left\langle \beta_1 R^{-1} |\nabla \boldsymbol{u}|^2 + \beta_2 |\nabla T|^2 + \left[\tau'(z) - \beta_1\right] wT \right\rangle$$

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An explicit convex optimization problem

The mean vertical convective heat flux satisfies

$$\overline{\langle wT \rangle} \leq \frac{1}{2} - \int_0^1 \tau(z)\,\mathrm{d}z + \frac{1}{4\beta_2}\int_0^1 \left|\tau'(z) - \lambda(z) + \beta_2 z - \tfrac{1}{2}\beta_2\right|^2 \mathrm{d}z$$

provided that

$$\tau(0) = 1$$

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λ is non-decreasing

and, for all $\boldsymbol{u}$ and T satisfying $\nabla \cdot \boldsymbol{u} = 0$ and the flow's boundary condition,

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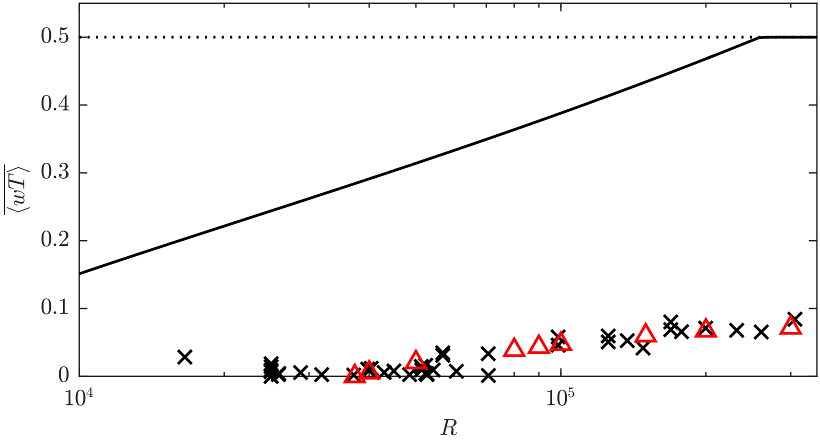
Numerically optimized bounds



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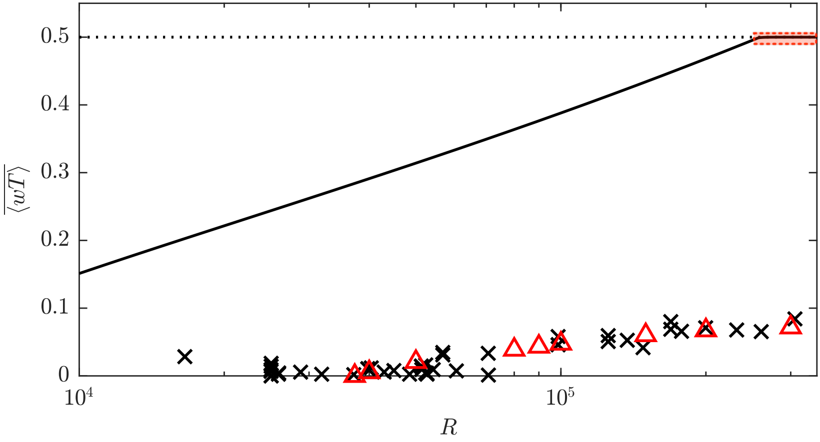

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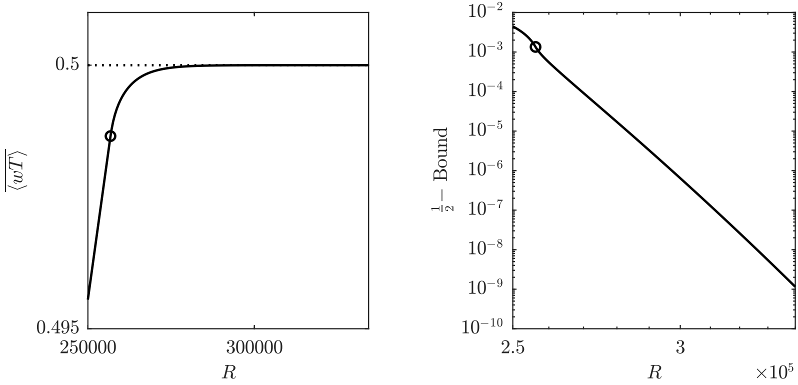
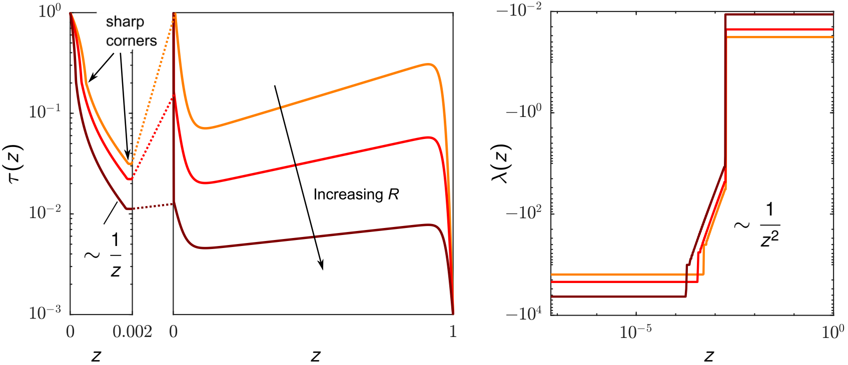


Optimizers


10^0
10^{-1}
10^{-2}
10^{-3}
sharp corners
$\sim \frac{1}{z}$
$\tau(z)$
0
0.002
0
1
z
z
Increasing R
-10^{-2}
-10^0
-10^2
-10^4
$\lambda(z)$
$\sim \frac{1}{z^2}$
10^{-5}
10^0
z

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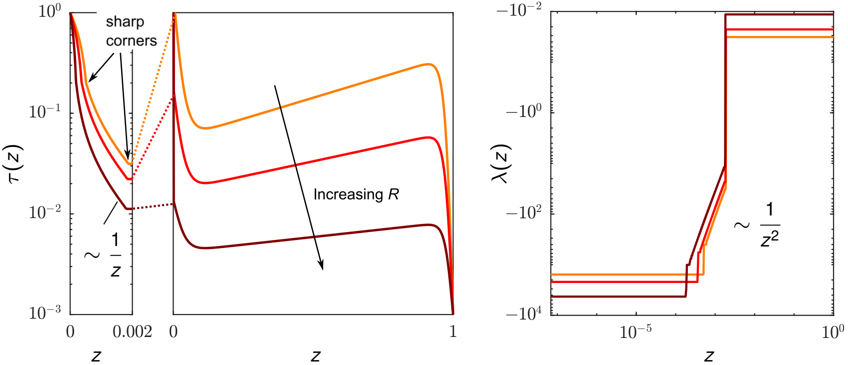
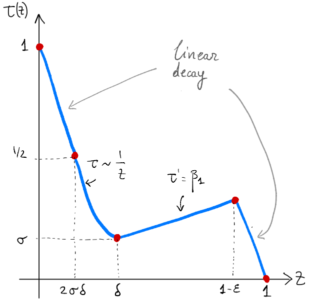
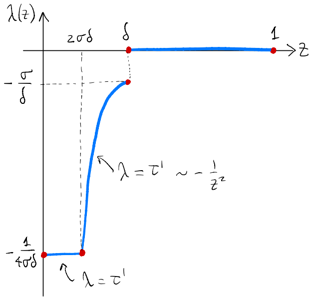


$\tau'(z) \approx \lambda(z)$ near $z = 0$

Turning numerical results into rigorous proofs





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Key estimate: $\langle (z + \sigma\delta)^{-2} wT \rangle \lesssim \langle |\nabla \boldsymbol{u}|^2 + |\nabla T|^2 \rangle$

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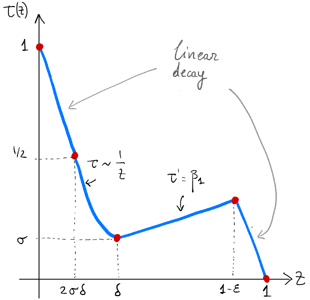

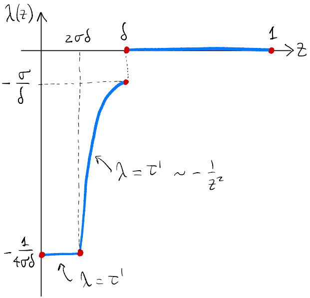

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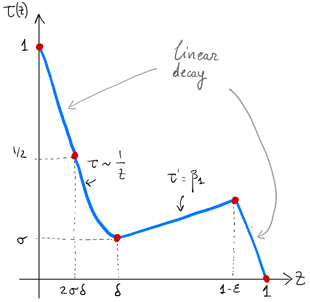
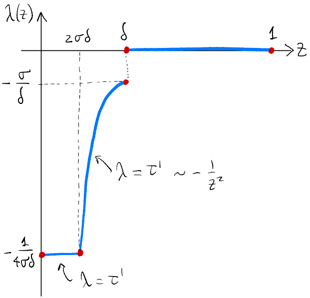
Turning numerical results into rigorous proofs





Theorem (Kumar, Arslan, F, Craske, Wynn, JFM2021)

There exist positive constants c_1 and c_2 such that, for any value of Pr and sufficiently large R,

$$\overline{\langle wT \rangle} \leq \frac{1}{2} - c_1 R^{\frac{1}{5}} \exp\left(-c_2 R^{\frac{3}{5}}\right)$$

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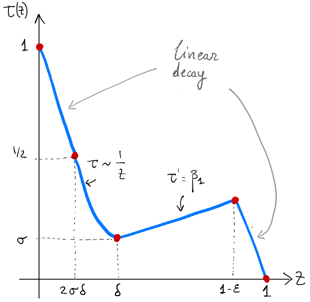
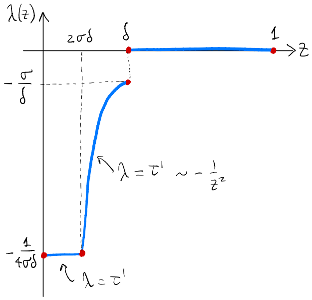




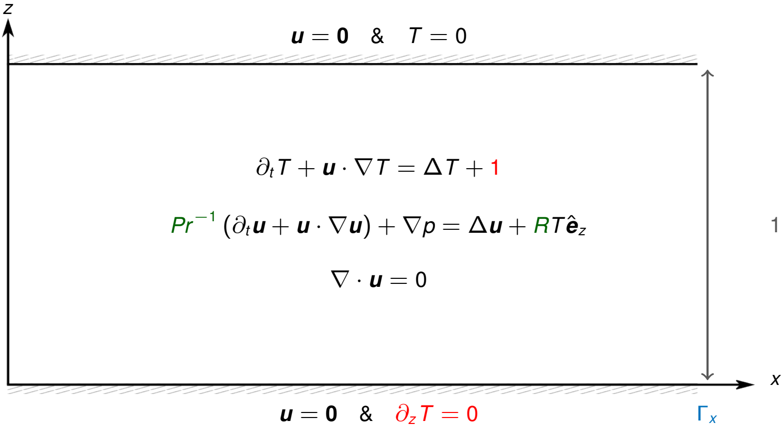
Theorem (Kumar, Arslan, F, Craske, Wynn, JFM2021)

There exist positive constants c_1 and c_2 such that, for any value of Pr and sufficiently large R,

$$\mathcal{F}_{top} \leq 1 - c_1 R^{\frac{1}{5}} \exp\left(-c_2 R^{\frac{3}{5}}\right)$$

Extensions

Extension 1: Insulating bottom boundary



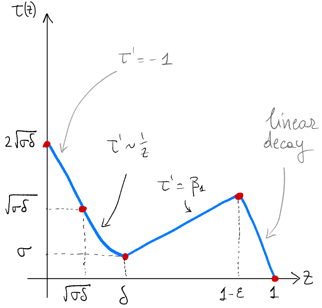
$$\partial_t T + \boldsymbol{u} \cdot \nabla T = \Delta T + 1$$

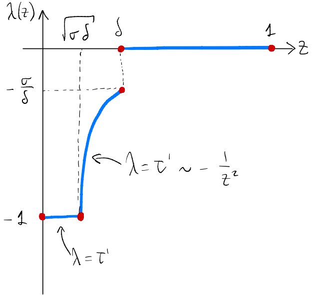
$$Pr^{-1} \left(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) + \nabla p = \Delta \boldsymbol{u} + R T \hat{\boldsymbol{e}}_z$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\overline{\langle wT \rangle} = R^{-1} \overline{\langle |\nabla \boldsymbol{u}|^2 \rangle} \sim \text{mean dissipation of kinetic energy}$$

Similar result, different exponent





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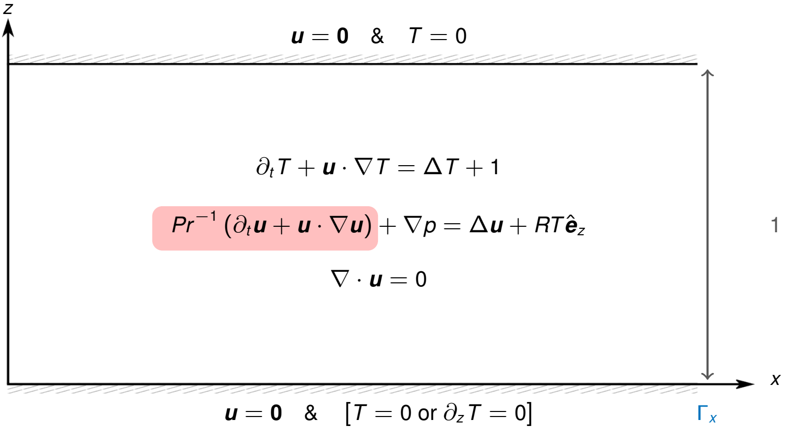
Extension 2: "Thick" fluids with $Pr = \infty$



$$\boldsymbol{u} = \boldsymbol{0} \quad \& \quad T = 0$$

$$\partial_t T + \boldsymbol{u}\cdot\nabla T = \Delta T + 1$$

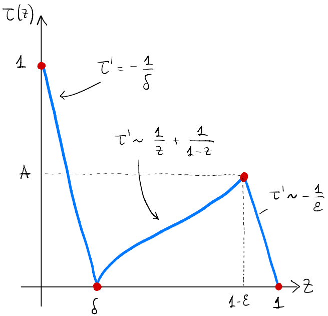
$$Pr^{-1}\left(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) + \nabla p = \Delta\boldsymbol{u} + RT\hat{\boldsymbol{e}}_z$$

$$\nabla\cdot\boldsymbol{u} = 0$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \& \quad [T = 0 \text{ or } \partial_z T = 0]$$

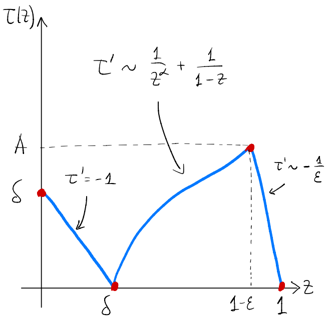
Much better bounds![1]

Cool bottom ($T = 0$)



Theorem: $\overline{\langle wT \rangle} \leq \frac{1}{2} - cR^{-2}$

Insulating bottom ($\partial_z T = 0$)



Theorem: $\overline{\langle wT \rangle} \leq \frac{1}{2} - cR^{-4}$

[1] Arslan, F, Craske & Wynn, arXiv:2205.03175 (2022)

Summary

Model	Arbitrary fluids	"Thick" fluids ($Pr = \infty$)
$T = 0$ / $T = 0$	$\overline{\langle wT \rangle} \leq \frac{1}{2} - c_1 R^{\frac{1}{5}} e^{-c_2 R^{\frac{3}{5}}}$	$\overline{\langle wT \rangle} \leq \frac{1}{2} - cR^{-2}$
$T = 0$ / $\partial_z T = 0$	$\overline{\langle wT \rangle} \leq \frac{1}{2} - c_1 R^{-\frac{1}{5}} e^{-c_2 R^{\frac{3}{5}}}$	$\overline{\langle wT \rangle} \leq \frac{1}{2} - cR^{-4}$

However:

Are the constructions leading to these bounds **optimal**?

Are the bounds **sharp** for convective flows?

Which flows optimize $\overline{\langle wT \rangle}$? Are they perhaps **steady**?

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$T = 0$ $\partial_z T = 0$	$\overline{\langle wT \rangle} \leq \frac{1}{2} - c_1 R^{-\frac{1}{5}} e^{-c_2 R^{\frac{3}{5}}}$	$\overline{\langle wT \rangle} \leq \frac{1}{2} - cR^{-4}$

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Summary

Model	Arbitrary fluids	"Thick" fluids ($Pr = \infty$)
$T = 0$ (top), $T = 0$ (bottom)	$\overline{\langle wT \rangle} \leq \frac{1}{2} - c_1 R^{\frac{1}{5}} e^{-c_2 R^{\frac{3}{5}}}$	$\overline{\langle wT \rangle} \leq \frac{1}{2} - cR^{-2}$
$T = 0$ (top), $\partial_z T = 0$ (bottom)	$\overline{\langle wT \rangle} \leq \frac{1}{2} - c_1 R^{-\frac{1}{5}} e^{-c_2 R^{\frac{3}{5}}}$	$\overline{\langle wT \rangle} \leq \frac{1}{2} - cR^{-4}$

However:

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