Large-amplitude modulation of periodic traveling waves

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I. Introduction

TOPICS:

• Fascinating connection between modulation of periodic waves and quasilinear hyperbolic systems.

• Pseudo-differential analysis with rapidly-varying coefficients.

• Shock wave (Kreiss symmetrizer) techniques applied to oscillatory solutions of reaction diffusion systems.

• Emergent behavior in Turing-type pattern formation.

• Celebration of long-running collaboration with Guy, retired from U. Bordeaux I and now associate mayor in Pyrenees.

II. Context/History

(a)  (b) 

Figure: Belousov-Zhabatinski clock reaction (BZ) 1951, mimicking biological (e.g., Krebs) cycles: oscillatory behavior, contradicting then thermodynamics- published finally in nonreviewed journal, 1959. http://nonlinear.s.chiba-u.jp/ kitahata/bz.html

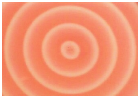
(a)  (b) 

Figure: front-type patterns (different regime): a) spiral. b) target.

Latter predicted by Turing 1952 via abstract considerations.

Formal modulation approximation

Basic planar periodic patterns predicted by Turing through (small-amplitude) Turing bifurcation from uniform state.

Q: How to understand large-scale front-type patterns?
("emergent," or large-scale dynamic behavior.)

A: As locally planar **modulation** of planar periodic waves...
([Howard-Kopell 1976].)

Modulation, e.g. voice modulation: variation in parameters (pitch, volume, tone) of periodic signal, in this case sound wave. Also familiar are frequency (FM) or amplitude (AM) modulation of electromagnetic waves. **Problem**: Given initial modulation, determine approximate (space-time) evolution of parameters.

Reaction diffusion (RD) à la Howard-Kopell

Consider general RD system, small-wavelength limit $\epsilon \to 0$:

$$\epsilon \mathcal{D}_t u + f(u) = \epsilon^2 \Delta_x u, \quad x \in \mathbb{R}^d,\ u, f \in \mathbb{R}^n. \qquad (1)$$

(Or, $\mathcal{D}_t u + f(u) = \Delta_x u$ under rescaling $(x,t) \to (x/\epsilon, t/\epsilon)$.)

Assumption

There is a (nontrivial) smooth function $p(k,\theta)$ of $k \in \mathcal{K} \subset \mathbb{R}^d$, $\mathcal{K}$ bounded and bounded away from the origin, and θ, 2π- periodic in θ, and a smooth function $\omega(k)$ such that

$$\omega(k)\mathcal{D}_\theta p + f(p) = |k|^2 \mathcal{D}_\theta^2 p. \qquad (2)$$

(periodic traveling waves $p(k \cdot x + \omega(k)t)$ in $(x/\epsilon, t/\epsilon)$ coords.)

Planar periodic waves and modulations

Assumption 0.1 implies that for all $k \in \mathcal{K}$,

$$\bar{u}(t,x) = p(k, \psi_k(t,x)/\epsilon), \qquad \psi_k(t,x) = k \cdot x + \omega(k)t \tag{3}$$

is an exact solution of (1). Consider now a smooth modulation

$$u^{\epsilon,a}(t,x) = p(k, \psi/\epsilon), \tag{4}$$

k and ψ smooth functions of (t,x), with truncation error

$$R^\epsilon := \epsilon \mathcal{D}_t u + f(u) - \epsilon^2 \Delta u. \tag{5}$$

(modulation in phase ψ and wave number k.)

Computation of the residual

Computing

$$\begin{aligned}
\epsilon \mathcal{D}_t u^{\epsilon,a} &= \mathcal{D}_t \psi \mathcal{D}_\theta p + \epsilon \sum_l \mathcal{D}_t k_l \mathcal{D}_{k_l} p, \\
\epsilon \mathcal{D}_{x_j} u^{\epsilon,a} &= \mathcal{D}_{x_j} \psi \mathcal{D}_\theta p + \sum_l \epsilon \mathcal{D}_{x_j} k_l \mathcal{D}_{k_l} p, \\
\epsilon^2 \Delta_x u^{\epsilon,a} &= |\mathcal{D}_x \psi|^2 \mathcal{D}_\theta^2 p + \epsilon \Delta \psi \mathcal{D}_\theta p + 2\epsilon \sum_{j,l} \mathcal{D}_{x_j} \psi \mathcal{D}_{x_j} k_l \mathcal{D}^2_{\theta,k_l} p \\
&\quad + \epsilon^2 \sum_{j,l,m} \mathcal{D}_{x_j} k_l \mathcal{D}_{x_j} k_m \mathcal{D}^2_{k_l,k_m} p,
\end{aligned} \tag{6}$$

and combining, we have that the main ($O(\epsilon^0)$) term in R^ϵ is

$$\mathcal{D}_t \psi \mathcal{D}_\theta p + f(p) - |\mathcal{D}_x \psi|^2 \mathcal{D}_\theta^2 p = (\mathcal{D}_t \psi - \omega(k)) \mathcal{D}_\theta p + (|k|^2 - |\mathcal{D}_x \psi|^2) \mathcal{D}_\theta^2 p;$$

$\mathcal{D}_t \psi = \omega(\mathcal{D}_x \psi)$, $k = \mathcal{D}_x \psi \Rightarrow$ valid to $O(\epsilon)$. **Hamilton-Jacobi!**

Comments

Scalar hyperbolic! In 1D, $\to$ Burgers eq. $\psi_t + \omega(\psi_x) = 0$. Inspired by dispersive analyses in optics, EM, KdV (Lax, Joly-M-Rauch, Whitham, ...), $\to$ systems, **weakly nonlinear geometric optics.**

Remark

For f smooth, $k \neq 0$, and p, ω of (2) transversal, Assumption 0.1 holds local to (k, ω, p) (IFT+smooth dep. of solns. of ODE). Excluded limit $k \to 0$ is singular for (2), moreover period $X = 1/k \to \infty$. Condition $\mathcal{D}_x \psi = k \neq 0$ quite restrictive on geometry of wave fronts of $u^{\epsilon,a}$ = level sets of ψ, $\Rightarrow$ foliation of $\mathbb{R}^d$. In particular, no closed level surface, so doesn't (quite) explain motivating target patterns; behavior at center not understood.

Rigorous verification: 1D small-amplitude case

Proposition (Doelman-Sandstede-Scheel-Schneider 2009)

In 1D, for modulations Ψ *close to a single periodic wave, corresponding to data* $\Psi_x - k_* \in H^s_{ul}$*, there exists an exact solution of* (1) *remaining* ϵ*-close to modulation approximation* (4) *up to time* $0 \le t \le T$*, for some* $T > 0$ *depending on the size of the initial data* $\Psi_x - k_*$ *(time* $0 \le t \le 1/\epsilon$ *in unscaled coordinates).*

Interpretation: $\sim \infty$-dimensional center manifold: model reduction for original RD. $t \le T$ optimal (singularity formation).

Limitations: analysis based on Bloch decomposition methods of [Schneider 1996] (* important breakthrough) for exact periodic waves. Thus, very much tied to 1D, small-amplitude modulations from a single wave... (Only result to our knowledge.)

III. Description of main results: higher-order expansion

Continuing, we seek a general *multi-scale expansion* ([JMR,HK])

$$u^{\epsilon,m}(t,x) = U_{\epsilon,m}(t,x,\frac{1}{\epsilon}\psi(x,t) + \varphi_{\epsilon,m-1}(t,x)) \qquad (7)$$

(with convention $\varphi_{\epsilon,-1} = 0$) with

$$U_{\epsilon,m}(t,x,\theta) = \sum_{n=0}^{m} \epsilon^n U_n(t,x,\theta), \qquad \varphi_{\epsilon,m}(x,t) = \sum_{n=0}^{m} \epsilon^n \varphi_n(x,t), \qquad (8)$$

satisfying the consistency condition

$$R^{\epsilon,m} := \epsilon \mathcal{D}_t u^{\epsilon,m} + f(u^{\epsilon,m}) - \epsilon^2 \Delta_x u^{\epsilon,m} = O(\epsilon^{m+1}) \qquad (9)$$

on the residual (truncation error) $R^{\epsilon,m}$ of approx. soln. $u^{\epsilon,m}$.

Transversality assumption on profiles in fast variable

Assumption

For $k \in \mathcal{K}$, consider the linearization

$$L(k, \mathcal{D}_\theta) = -\omega(k)\mathcal{D}_\theta - f'(p(k,\theta)) + |k|^2\mathcal{D}_\theta^2, \tag{10}$$

of profile equation (2) about p, satisfying (by translation invariance)

$$L(k, \mathcal{D}_\theta)D_\theta p = 0. \tag{11}$$

Assume that $\mathcal{D}_\theta p(k, \cdot)$ is a simple eigenvalue of $L(k, \mathcal{D}_\theta)$ in $L^2(\mathbb{T})$.

Assumption 0.2 is equivalent to *transversality* of solutions p of the profile ODE (2), which may be recognized as a slightly strengthened, linear version of Assumption 0.1 (cf. Remark 1).

Existence of all-orders approximations

Let H^s denote usual Sobolev spaces in x, t, and $\mathcal{H}^s_\epsilon$ def'd by $\|h\|^2_{\mathcal{H}^s_\epsilon} := \sum_{j=0}^s \epsilon^{2j}\|\mathcal{D}^j_{x,t}h\|^2_{L^2}$. By $\epsilon^{-1/2}\|\cdot\|_{\mathcal{H}^s_\epsilon} \sim H^s(x/\epsilon, t/\epsilon)$, and Sobolev embedding, $\|h\|_{L^\infty} \le C\epsilon^{-1/2}\|h\|_{\mathcal{H}^s_\epsilon}$ for $s \ge [d/2]+1$.

Theorem

Under Assumption 0.2, if ψ is a smooth solution of $\mathcal{D}_t\psi = \omega(\mathcal{D}_x\psi)$ on $[0,T]\times\mathbb{R}^d$, and $f \in C^\infty$, then there are asymptotic solutions (7), (9) at all orders, with $U_0 = p(k(t,x),\theta)$, $k = \mathcal{D}_x\psi$, and $\|(U_n,\varphi_{n-1})\|_{L^\infty} \lesssim \|\mathcal{D}_{x,t}\psi\|_{C^{2n}}$, $\|R^{m,\epsilon}\|_{\mathcal{H}^s_\epsilon} \lesssim \epsilon^{m+1}\|\mathcal{D}^2_{x,t}\psi\|_{H^{s+2(m+1)}}$.

Proof.

U's determined by lin'd profile eqn., Ψ's by Fredholm Alt. □

$|\psi_x| = |k| \le C$ by assumption. $\|\mathcal{D}^2_x\psi\|_{\mathcal{H}^s_\epsilon} = \|k_x\|_{\mathcal{H}^s_\epsilon} \le C \Rightarrow$ geom. constraint, eg.: (i) no target for $d \ge 2$, (ii) $k_x \to 0$ at ∞ (Sobolev).

Convergence error

Given approximate solution $u^{\epsilon,m}$, define convergence error

$$v^{\epsilon,m} := u - u^{\epsilon,m}, \quad v^{\epsilon,m}|_{t=0} = 0, \tag{12}$$

u an exact solution. The equation for $v^{\epsilon,a}$ may be expressed as

$$\epsilon \mathcal{P}_{u^{\epsilon,m}} v^{\epsilon,m} = -R^{\epsilon,m} + Q(u^{\epsilon,m}, v^{\epsilon,m}) := \epsilon e^{\epsilon,m}, \tag{13}$$

where $\epsilon \mathcal{P}_{u^{\epsilon,m}} := \epsilon \mathcal{D}_t + g^\epsilon - \epsilon^2 \Delta_x$ is the linearization of (1) about $u^{\epsilon,m}$ and Q is quadratic in v for $\|v\|_{L^\infty} \leq C$.

Note: $\mathcal{P}_{u^{\epsilon,m}} = \mathcal{D}_t + \epsilon^{-1} g^\epsilon - \epsilon \Delta_x$ singular, loss $e^{\epsilon,m} = \epsilon^{-1} R^{\epsilon,m}$.

Diffusive stability assumption on profiles in fast variable

Letting $\xi \in [-1,1)$ denote Floquet number and $\eta_2, \dots, \eta_d \in \mathbb{R}^{d-1}$ Fourier frequencies in directions transverse to k, and $\sigma(L)$ denote spectrum of L, define the Bloch-Fourier operator

$$L_{\xi,\eta}(k, \mathcal{D}_\theta) = -\omega(k)(\mathcal{D}_\theta + i\xi) - f'(p(k,\theta)) + |k|^2(\mathcal{D}_\theta + i\xi)^2 - |\eta|^2$$

associated with linearized operator (10) (more later).

Assumption

For $k \in \mathcal{K}$, $\Re\sigma\big(L_{\xi,\eta}(k, \mathcal{D}_\theta)\big) \le -c|(\xi,\eta)|^2$ for some $c > 0$.

Assumption 0.4 may be recognized as the *diffusive stability* condition of Schneider [CIMP96], sufficient for linearized and nonlinear stability of component planar periodic waves (3).

Linear estimates

Our central result is the following linear estimate.

Theorem

Under Assumption 0.4, for $f \in C^{s+1+2(m+1)}$, $\mathcal{D}_x^2\psi \in H^{s+2(m+1)}$, and $h \in \mathcal{H}_\epsilon^s$, $\mathcal{P}_{u^{\epsilon,m}} v = h$, $v|_{t=0} \equiv 0$ has a unique solution $v \in \mathcal{H}_\epsilon^{s+1}(\mathbb{R}^d \times [0,T])$, with $\|v\|_{\mathcal{H}_\epsilon^{s+1}} \lesssim \|h\|_{\mathcal{H}_\epsilon^s}$.

(Sharp in case $k \equiv \mathrm{constant}$ of an exactly periodic planar wave.)

Proof.

Discussed below (main analysis and main point). □

With $e = \epsilon^{-1} R^{\epsilon,m}$, $\Rightarrow \|v\|_{\mathcal{H}_\epsilon^{s+1}} \lesssim \epsilon^{-1}\|R^{\epsilon,m}\|_{\mathcal{H}_\epsilon^s} \le \epsilon^m$, loss of ϵ^{-1}.

Nonlinear validation

Corollary

Under Assumption 0.4, for $f \in C^{s+1+2(m+1)}$, $\mathcal{D}_x^2\psi \in H^{s+2(m+1)}$, $m \geq 2$, $s > [d/2]$, there is a unique exact solution u of (1) such that $u|_{t=0} = u^{\epsilon,m}|_{t=0}$ and $\|u - u^{\epsilon,m}\|_{\mathcal{H}_\epsilon^{s+1}} \lesssim \epsilon^m \|\mathcal{D}_{x,t}^2\psi\|_{H^{s+2(m+1)}}$.

Proof, prepared data. (initial layer analysis for general case).

Defining $\bar{v}^{\epsilon,m} = \epsilon^{-m} v^{\epsilon,m}$ and $\bar{Q}(u^{\epsilon,m}, \bar{v}) = \epsilon^{-2m} Q(u^{\epsilon,m}, \epsilon^m \bar{v})$, get

$$\bar{v} = \mathcal{T}\bar{v} := \mathcal{P}_{u^{\epsilon,m}}^{-1}\Big(-\epsilon^{-(m+1)} R^{\epsilon,m} + \epsilon^{m-1} \bar{Q}(u^{\epsilon,m}, \bar{v})\Big),$$

$|\mathcal{P}_{u^{\epsilon,m}}^{-1}|_{\mathcal{H}_\epsilon^s}$, $\|\epsilon^{-(m+1)} R^{\epsilon,m}\|_{\mathcal{H}_\epsilon^s}$ bounded, $\bar{Q}$ quadratic. For $m \geq 2$, $\mathcal{T}$ is contractive in $\|\cdot\|_{\mathcal{H}_\epsilon^s}$ on $B(0, 2\|\mathcal{P}_{u^{\epsilon,m}}^{-1}\epsilon^{-(m+1)} R^{\epsilon,m}\|_{\mathcal{H}_\epsilon^s})$, Lip. constant (Moser's inequality) $\lesssim \epsilon^{m-1}\|v\|_{L^\infty} \lesssim \epsilon^{m-3/2}\|v\|_{\mathcal{H}_\epsilon^s}$. □

Comments

• Fixed point argument rather standard. Note that fixed loss in ϵ (in this case ϵ^{-1} in linear solution operator is harmless ($\sim$ "tame estimate"), can be overcome by high-enough order expansion, by quadratic order of nonlinear term. See work of Gùes, Grenier, 90's.
• Formal mth order remainder $O(\epsilon^{m+1})$ yields convergence error ϵ^m, due to ϵ^{-1} loss. However, expanding further recovers sharp ϵ^{m+1} estimate. (Special quasi-1D structure overcomes singularity in general linear solution operator). Standard- but interesting!
• Same estimates as obtained by Doelman and all [DSSS09] for 1D. (Their results stated in rescaled coordinates, $\Rightarrow$ time $t \leq 1/\epsilon$.)

REMAINING TO VERIFY: LINEAR ESTIMATE...

IV. Linear stability analysis

Recall stability analysis of (exact) planar periodic waves:

Linearized operator (periodic-coeffient) Fourier transformed in transverse directions ($\sim$ frequencies $(\eta_2, \dots, \eta_d)$), on line:

$$L_\eta = -\omega(k)\mathcal{D}_\theta - f'(p(k,\theta)) + |k|^2\mathcal{D}_\theta^2 - |\eta|^2 = L_0 - |\eta|^2. \quad (14)$$

Associated Bloch-Fourier operator $(\mathcal{D}_\theta \to (\mathcal{D}_\theta + i\xi))$:

$$L_{\xi,\eta}(k, \mathcal{D}_\theta) = -\omega(k)(\mathcal{D}_\theta + i\xi) - f'(p(k,\theta)) + |k|^2(\mathcal{D}_\theta + i\xi)^2 - |\eta|^2,$$

on interval $[0, 2\pi)$, periodic B.C. ($\Leftrightarrow$ (14) w/ $u(2\pi) = e^{2\pi i\xi}u(0)$.)

Inverse Bloch-Fourier transform formula:

$$\big(e^{Lt}f\big)(x_1, \tilde{x}, t) = \Big(\int e^{i(\xi x_1 + \eta\cdot\tilde{x})} e^{L_{\xi,\eta}t}\hat{f}(\xi, \eta, \cdot)d\eta d\xi\Big)(x_1), \quad (15)$$

$\hat{f}$ = BFT (unitary wrt H^s), gen'n of IFT. By (14), effectively 1D.

Mode filters, diffusive stability condition, and linear bounds

ADVANTAGE: $L_{\xi,\eta}$ acts on bdd interval, has discrete spectra. So, decomposes into a scalar critical mode $\lambda(\xi,\eta)$ branching from the simple zero e-value at $(\xi,\eta)=(0,0)$ (Assumption 0.2), analytically depending on (ξ,η), plus rest enjoying uniform spectral gap. Thus, $e^{L_{\xi,\eta}t}$ decomposes into $e^{\lambda(\xi,\eta)t}P(\xi,\eta)$, where P is critical e-projection, or mode filter, plus unif. exp.-decaying remainder.

• Bound $|\mathcal{P}^{-1}_{u^{\epsilon},m}|_{\mathcal{H}^s_\epsilon} \le C$ thus straightforward... (scalar exponentiation plus generalized Parseval identity), assuming diffusive stab. cond. $\Re\lambda(\xi,\eta)\le -c|(\xi,\eta)|^2$, $c>0$ [Schneider96].

BUT- These tools not available in fully variable-coefficient case!

Pseudodifferential analysis with rapidly varying coefficients

INSTEAD: Adapt techniques of Métivier-Z [AMS Mem.2005] for inviscid ($\sim$ small-wavelength) limit of viscous shock waves. (Traveling fronts vs. periodic waves, but hyperbolic limit.)

- In many ways, more natural here! (periodic setting)

IDEA: mimic FT for var. coeffs. by semiclassical ($\epsilon\mathcal{D}_y \to i\eta$, $\epsilon\mathcal{D}_t \to i\tau$) pseudodiff. ("frozen-coeff.") analysis/paradiff. calc. of Bony, reducing again to "quasi-1D" problem in fast variable.

RULES: energy estimates only are allowed. (Adjoints, inner products, composition of symbols valid up to lower order commutators.) Bloch decomp., stationary phase, etc. unavailable.

Paralinearization/reduction to planar form

Local to any $(\underline{x}, \underline{t})$, can coordinatize $z = \Psi(t,x)$, $y = (y_2, \dots, y_d)$ (here using $k = \mathcal{D}_x \neq 0$), whence $\mathcal{D}_x \cdot \nabla\psi/|\nabla\psi| \to k\mathcal{D}_z$ and $\mathcal{D}_t \to \mathcal{D}_t + \mathcal{D}_t\Psi\mathcal{D}_z = \mathcal{D}_t + \omega(k)\mathcal{D}_z + \epsilon\mathcal{D}_t\varphi\mathcal{D}_z$, giving lin'd eq. matching planar case (14), $z/\epsilon \sim$ fast var. θ. (used (HJ) here!)

Paralinearizing in slow vars. $\epsilon\mathcal{D}_t \to i\tau$, $\epsilon\mathcal{D}_y \to i\eta$, and weighting $v \to ve^{-(\gamma/\epsilon)t}$, we reduce mod $H.O.T.$ to planar resolvent equation

$$(\lambda - L_\eta)v = \epsilon e, \quad \lambda := \gamma + i\tau$$

assoc. with (14) (here ODE in fast spatial variable $\theta = z/\epsilon$).

(we cannot paralinearize in z as we do not have uniform Lip. bd.)

General strategy

GOAL: "Semigroup-type" resolvent bound (fast time)

$$\|(\lambda - L_\eta)^{-1}\|_{L^2} \lesssim \frac{1}{\gamma} = \frac{1}{\Re\lambda}, \quad \gamma \geq \bar{\gamma}\epsilon,$$

$\bar{\gamma}$ suff. large, $\Rightarrow$ (Parseval)

$$\|e^{-\bar{\gamma}t} v\|_{L^2} \lesssim (1/\gamma)\|e^{-\bar{\gamma}t}\epsilon e\|_{L^2} = (1/\bar{\gamma})\|e^{-\bar{\gamma}t} e\|_{L^2}.$$

time-globally (slow time), and thus, on bdd (slow) time,

$$\|v\|_{L^2} \lesssim \|e\|_{L^2}$$

. (same countour as C_0 semigroup, $\gamma + i\tau$, γ fixed...)

RULES: Energy est's. only- no BT, no semigp. machinery!

Floquet transformation and spectral structure

Mimicking [MZ05], express as 1st-order: $\mathcal{D}_\theta \mathcal{V} = \mathcal{M}\mathcal{V} + \epsilon \mathcal{E}$,

$$\begin{aligned} \mathcal{M}(\eta,\lambda) &= \frac{1}{k}\begin{pmatrix} 0 & 1 \\ (\lambda + \epsilon|\eta|^2) + f'(p(k,z)) & \omega(k) \end{pmatrix} + H.O.T., \\ \mathcal{V} &= \begin{pmatrix} v \\ \epsilon k \mathcal{D}_z v \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} 0 \\ e \end{pmatrix}. \end{aligned} \tag{16}$$

By partition of unity argument, reduce to (x,t) near $(\underline{x},\underline{t})$, arbitrary finite decomposition of frequency. High-freq. std [MZ05].

Bdd. freq.: Here, use Floquet's Lemma: smooth periodic change of coordinates reducing to $\mathcal{M}_1(\lambda,\eta) \equiv \text{constant}$, const.-coeff.!

KEY OBS.: Bloch-Floquet spectrum encoded in eigenstructure of $\mathcal{M}_1$: e-values λ for L_η $\Leftrightarrow$ pure imaginary e-values for $\mathcal{M}_1(\eta,\lambda)$.

Kreiss symmetrizer estimates (last step)

Proposition

For $\lambda + |\eta|^2$ bounded, $\lambda = \gamma + i\tau$, there exist locally smooth symmetrizers for $\mathcal{M}_1$, that is, matrices $\mathcal{S}(k, \lambda, \eta)$, C^1 in (λ, k, η) such that $\mathcal{S} = \mathcal{S}^*$, $|\mathcal{S}|$ uniformly bounded, and

$$\Re \mathcal{S}(k, \lambda, \eta)\mathcal{M}_1(k, \lambda, \eta) \geq (\gamma + \epsilon|\eta|^2) \tag{17}$$

Cor.: $(\gamma + \epsilon|\eta|^2)\|\mathcal{V}_I\|_{L^2} \lesssim \epsilon(\|\mathcal{E}\|_{L^2} + \|\mathcal{V}_1\|_{L^2})$, $\bar{\gamma} \gg 1$, $\Rightarrow$ **DONE**.

Proof. Energy est. mod. $\epsilon\|\mathcal{V}_1\|^2$, integrating $\mathcal{S}\mathcal{V}_1$ vs. eqn.:

$$\epsilon \Re\big(\mathcal{S}\mathcal{E}, \mathcal{V}_1\big)_{L^2} \sim \Re\big(\mathcal{S}\mathcal{M}_1\mathcal{V}_1, \mathcal{V}_1\big)_{L^2} \gtrsim (\gamma + \epsilon|\eta|^2)\|\mathcal{V}_1\|^2_{L^2}.$$

("Window" $\gamma = \bar{\gamma}\epsilon \gg \epsilon$, $\Rightarrow$ $O(\epsilon)$ commutators absorb [MZ05].)

Construction of symmetrizers/proof of Proposition

Medium freq.: No $\sigma(L_\eta)$, by diss. stab., hence $\mathcal{M}_1$ has spectral gap. Existence of symmetrizer thus follows from Lyapunov's Lemma for ODE, uniformity by compactness of frequency range.

Small freq.: At the origin, $\mathcal{M}_1$ has 1-dim. kernel corresponding to simple e-value of L_0, with spectral gap on complementary e-spaces. Near origin, construct $\mathcal{S}$ separately on critical and comp. e-spaces. Complementary space immediate (Lyapunov Lemma). So, problem reduces to construction for critical subspace (key point).

- Spirit of Lyapunov/Kreiss: issue, smoothness, Jordan block.

Model case (in fact canonical...)

1D modulation eqn. $k_t + \omega(k)_\theta = dk_{\theta\theta}$ (WLOG $d = 1$), lin'd about exact periodic $k \equiv \text{constant}$, $\Rightarrow$ e-value eqn. $u_{\theta\theta} = \lambda u + \omega'(k) u_\theta$, or $\mathcal{D}_\theta U = m(\lambda, k) U$, with $U = (u, u_\theta)^T$, $m = \begin{pmatrix} 0 & 1 \\ \lambda & \omega'(k) \end{pmatrix}$.

Symmetrizer: Case (i) $\omega'(k) \neq 0$. Decouples to single critical mode, straightforward (don't discuss). Case (ii) $\omega'(k_*) = 0$, $\Rightarrow$ $\omega'(k) = f\kappa$, $\kappa := k - k_*$: Jordan block at $\lambda, \kappa = 0$. Seek $s = \begin{pmatrix} \alpha & 1 + i\sigma \\ 1 - i\sigma & \beta \end{pmatrix}$, with $\Re sm \gtrsim \gamma$. Computing,

$$\Re(sm) = \begin{pmatrix} (1+i\sigma)\lambda & \alpha + (1+i\sigma)f\kappa \\ \beta\lambda & (1-i\sigma) + \beta f\kappa \end{pmatrix} = \begin{pmatrix} \gamma - \sigma\tau & X \\ \bar{X} & 1 + o(1) \end{pmatrix},$$

$X = O(\gamma) + \alpha + f\kappa + i\sigma f\kappa - \beta i\tau$. Choose $\sigma, \beta = 0$, $\alpha = -f\kappa$, $\Rightarrow$

(Continued)

cancel dangerous κ, τ terms, giving (Sylvester's criterion)

$$\Re sm = \begin{pmatrix} \gamma & O(\gamma) \\ O(\gamma) & 1 + o(1) \end{pmatrix} \gtrsim \gamma. \qquad \text{(DONE)}$$

Unique choice of (α, β, σ): General case involves additional λ, κ error terms, still unique solution canceling "bad" terms.

Analogous to Kreiss' "glancing", Jordan block due to char. ∂.

V. Discussion and open problems

• $\sim$ "1/2" Kreiss construction (no B.C.) Suggests wider appl's.
• MultiD $\Leftrightarrow$ 1D, since η enters as $\lambda + \epsilon|\eta|^2$ (Laplacian).
• "Glancing" phenomena appear here already in 1D, whereas for hyperbolic BVP for $d \geq 2$. Consistent with intuition of "1 1/2 D" in periodic case, due to addition of Bloch number ξ.
• Extension to general diffusion interesting, appears do-able.
• Likewise, extension to relaxation systems, formal modulation eqns. nonscalar. See, e.g., 1-D [Noble-Rodrigues, IUMJ 2013], thin-film flow. Modern biomorphology eqns. (Murray-Oster, etc.) similar. Natural extension or Turing's original investigations.
• Floquet transf. replaces conjugation lemma of [MZ1] ($\sim$ exp. const.-coeff. eqns. on half-line). In many ways more natural here!

THANKS FOR YOUR ATTENTION

*Ref.: [MZ] DCDS-S Vo1 5-Iss9: September 2022 special issue.

https://www.aimsciences.org/journal/1937-1632/2022/15/9