Quantitative rapid stabilization for some PDE models

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FAU–AvH Seminar

Outline of the presentation

1. Stabilization problems
2. Frequency Lyapunov for finite time stabilization
3. Fredholm backstepping for rapid stabilization

$$\dot{x} = Ax + Bu$$

Controllability

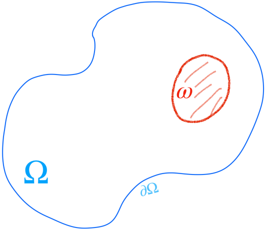
$\forall (x_0, x_1),\ \exists$ control $u(t)$ such that

$x(0) = x_0$ and $x(T) = x_1$

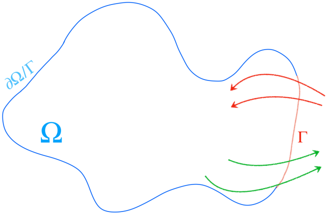
Stabilization

Construct feedback law $u = U(x)$ to
make the closed-loop system stable
(asymptotically, exponentially...)

Automatic, Constructive, Robustness



(Internal control)



(Boundary control)

Stabilization problems

Open channel (la Sambre)

Stabilization

Construct feedback law $u = U(x)$ to make the closed-loop system stable (asymptotically, exponentially...)

Automatic, Constructive, Robustness

Traffic road (La La Land)

Perseverance (Feb 18, 2021)

Rapid stabilization

Rapid Stabilization: for any $\lambda > 0$, construct feedback law such that

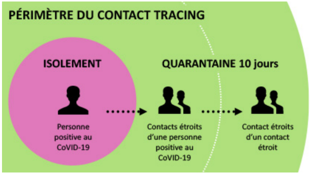
$$|x(t)| \leq C_\lambda e^{-\lambda t}|x(0)|, \quad \forall t > 0$$

Finite Time Stabilization:

- ★ solutions become 0 after a period of time T

Exoskeleton for walking (EPFL)

SpaceX landing



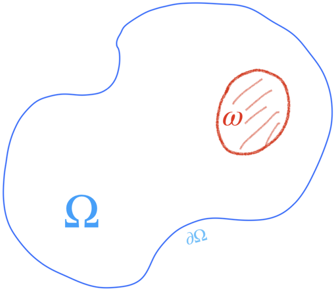
Stabilize Covid-19

Outline of the presentation

2) Frequency Lyapunov (local, multi-D)

Open problem: finite time stab. of the multi-D heat equation
(1D case answered by Coron–Nguyen, 2015)

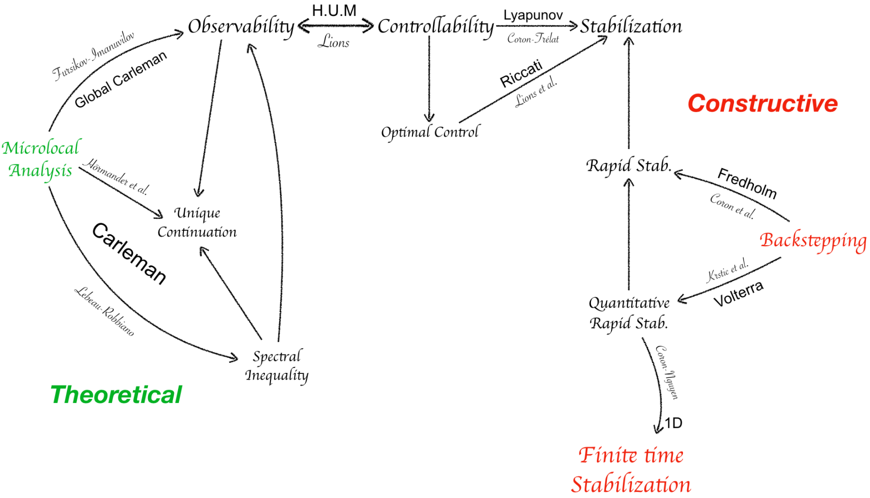
$$y_t - \Delta y = 1_\omega u \ \text{ in } \Omega$$



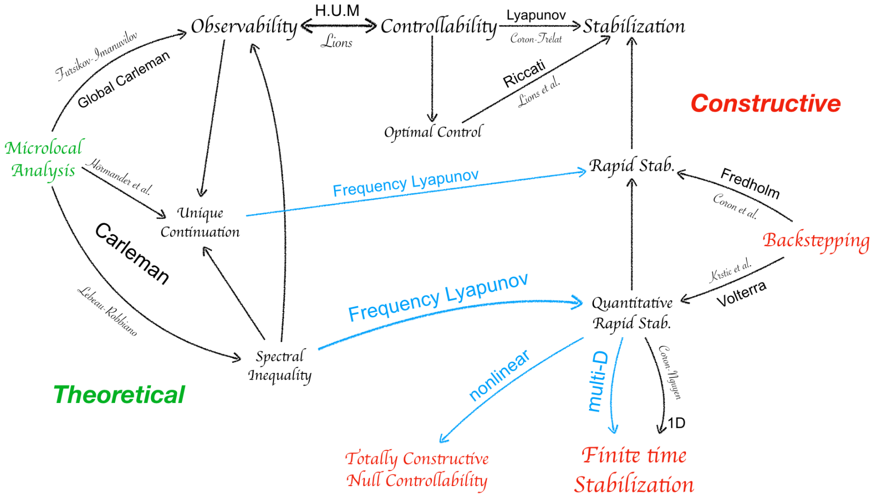
quantitative rapid stabilization

$\Downarrow$

finite time stabilization


Observability
H.U.M
Lions
Controllability
Lyapunov
Coron-Trélat
Stabilization
Fursikov-Imanuvilov
Global Carleman
Riccati
Lions et al.
Optimal Control
Constructive
Microlocal Analysis
Hörmander et al.
Unique Continuation
Carleman
Rapid Stab.
Fredholm
Coron et al.
Backstepping
Krstic et al.
Volterra
Quantitative Rapid Stab.
Lebeau-Robbiano
Spectral Inequality
Theoretical
Coron-Nguyen
1D
Finite time Stabilization

New connection



Obstruction on backstepping

Backstepping's success on 1D models

Schrödinger, viscous Burgers, KdV, water tank, parabolic, transport, hyperbolic of conservation laws, Kuramoto-Sivashinsky, degenerate operators, ODE-PDE...

Obstruction on backstepping

Backstepping's success on 1D models

Schrödinger, viscous Burgers, KdV, water tank, parabolic, transport, hyperbolic of conservation laws, Kuramoto-Sivashinsky, degenerate operators, ODE-PDE...

What about multi-D models?

- ☕ "We have tried a lot, but the kernel functions are quite complicated."

—— Coron

Riccati is complicated

Nonlinear algebraic Riccati equation to deal with

$$Q(y_0) = \min \left\{ \frac{1}{2} \int_0^\infty |(-\Delta)^{\frac{3}{4}} y(t)|^2 + |u(t)|^2 dt \right\}$$

$$2(Ay, Gy) + \sum_{1 \leq k \leq N} (\varphi_k, Gy)_\omega^2 = |(-\Delta)^{\frac{3}{4}} y|^2$$

$$(Gy, y) = 2Q(y)...$$

☕ " Not easy to get quantitative estimates. Regularity difficulty......"

—— My feeling

Communication with Trélat

☕ “Cher Shengquan,

par ailleurs, au cas où ça te serait utile, je t’envoie ce document...... C’est une idée qui remonte à loin : elle est bien expliquée par Russell 1978. Jean-Michel et moi avons utilisé cette idée, en la combinant à une homotopie, pour la contrôlabilité des paraboliques semi-linéaires 2004.

Amitiés Emmanuel"

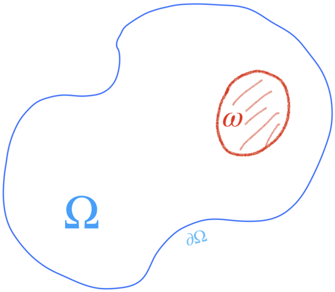
Attempt on the heat equation

The internal controlled heat equation:

$$\begin{cases} y_t - \Delta y = 1_\omega u(t,x), & (t,x) \in (0,T) \times \Omega, \\ y(0,x) = y_0(x), & x \in \Omega, \quad \omega \subset \Omega \subset \mathbb{R}^n. \end{cases} \tag{1}$$

Goal: $\forall \lambda > 0, \ \exists \ u$ and $C_\lambda > 0$

$$\|y(t)\|^2_{L^2(\Omega)} = \int_\Omega |y(t,x)|^2 dx \leq C_\lambda e^{-\lambda t} \|y_0\|^2_{L^2(\Omega)}$$



Spectral/Modal decomposition:

- $-\Delta \varphi_k = \tau_k \varphi_k$,
- $0 < \tau_1 \leq ... \leq \tau_N \leq \lambda < \tau_{N+1} ...$
- $\{\varphi_k\}_k$ orthonormal basis of $L^2(\Omega)$

Decompose

$$y(t,x) = \sum_i y_i(t)\varphi_i(x), \quad u(t,x) = \sum_i u_i(t)\varphi_i(x)$$

Then system (1) is equivalent to

$$\begin{aligned}\sum_{i\geq 1} \left(\dot{y}_i(t) + \tau_i y_i(t)\right)\varphi_i &= 1_\omega \sum_{j\geq 1} u_j(t)\varphi_j = \sum_{j\geq 1} u_j(t) 1_\omega \varphi_j \\ &= \sum_{j\geq 1} u_j(t) \Big(\sum_{i\geq 1} (1_\omega \varphi_j, \varphi_i)_{L^2(\Omega)} \varphi_i \Big) \\ &= \sum_{i\geq 1} \Big(\sum_{j\geq 1} (\varphi_j, \varphi_i)_{L^2(\omega)} u_j(t) \Big) \varphi_i\end{aligned}$$

thus, for $\forall i \geq 1$

$$\dot{y}_i(t) + \tau_i y_i(t) = \sum_{j\geq 1} (\varphi_j, \varphi_i)_{L^2(\omega)} u_j(t)$$

Notice that $u(t,x) = 0$ implies $y(t,x) = \sum e^{-\tau_i t} y_i(0)\varphi_i$

- Natural rapid decay for high frequency: $\lambda < \tau_{N+1} \leq \tau_{N+2} \leq ...$
- Remains to stabilize low frequency: $\tau_1 \leq ...\tau_N \leq \lambda$

Exponential stabilization

$$X := \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_N \end{pmatrix} \quad U := \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_N \end{pmatrix} \quad A := \begin{pmatrix} -\tau_1 & & & \\ & -\tau_2 & & \\ & & \dots & \\ & & & -\tau_N \end{pmatrix}$$

low frequency satisfy

$$\dot{X} = AX + J_N U, \quad (J_N)_{ij} = (e_i, e_j)_{L^2(\omega)}$$

Unique continuation $\Longrightarrow$ J_N invertible $\Longrightarrow$ (A, J_N) verify Kalman condition $\Longrightarrow$ Coron–Trélat method $\Longrightarrow$ Exponential stabilization

Quantitative rapid stabilization?

Limitation on Coron-Trélat

★ controllability, Lyapunov function ★ 1D, small decay, not quantitative

$$J_N = \big((e_i, e_j)_{L^2(\omega)}\big)_{i,j=1}^N : \text{ positive definite}$$

$$\dot{X} = AX + J_N U \quad \text{is controllable}$$

Inspired by Coron–Trélat

$\exists K \in \mathbb{R}^N$ and matrix Q s.t.

$$Q(A + J_N K) + (A + J_N K)^T Q = -I_N$$

Take $U := KX$ and

$$V(y) = \gamma X^T Q X - (y, \Delta y)_{L^2}$$

for some small ε

$$\dot{V}(z) \leq -\varepsilon V(z)$$

An attempt

$$J_N = \left((e_i, e_j)_{L^2(\omega)}\right)_{i,j=1}^N : \text{ positive definite}$$

$$\dot{X} = AX + J_N U \quad \text{is controllable}$$

A simple attempt

Choose $U(t) := -\gamma_\lambda X(t)$

$$\dot{X} = AX - \gamma_\lambda J_N X,$$

and

$$V(y) = \mu_\lambda X^T X + ||P_N^\perp y||_{L^2}^2$$

Thus

$$\frac{1}{2}\frac{d}{dt}X^T X = X^T AX - \gamma_\lambda X^T J_N X$$

Carleman for observability

Two methods for observability (90's)

- ★ Global Carleman
- ★ Lebeau–Robbiano

Spectral inequality

Let (e_i, τ_i) *be eigenfunctions of* $-\Delta$*, with* τ_i *increasing.*
There exists $C \geq 1$ *independent of* $\lambda > 0$ *such that*

$$||\sum_{\tau_i \leq \lambda} a_i e_i||^2_{L^2(\omega)} \geq C^{-1} e^{-C\sqrt{\lambda}} \sum_{\tau_i \leq \lambda} a_i^2$$

(Complex) Zygmund, Donnelly, Fefferman
(Real) Lebeau, Robbiano, Jerison, Zuazua, Phung, Buffe, Wang, Beauchard, Burq
(Zero measure) Logunov, Malinnikova, Nadirashvili

Quadratic form J_N

Quantitative estimate of $X^T J_N X$? looks like $(\sum a_i e_i)^2$...

Quadratic form J_N

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Spectral inequality

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Quadratic form J_N

Quantitative estimate of $X^T J_N X$?   looks like $(\sum a_i e_i)^2$...

Spectral inequality

$$||\sum_{\tau_i \leq \lambda} a_i e_i||^2_{L^2(\omega)} \geq C^{-1} e^{-C\sqrt{\lambda}} \sum_{\tau_i \leq \lambda} a_i^2$$

$$X^T J_N X \geq C^{-1} e^{-C\sqrt{\lambda}} ||X||_2^2$$

"Proof"

$$X^T J_N X = \sum_{1 \leq i,j \leq N} a_i (e_i, e_j)_{L^2(\omega)} a_j = \Big(\sum_{1 \leq i \leq N} a_i e_i, \sum_{1 \leq j \leq N} a_j e_j \Big)_{L^2(\omega)}$$

$$\geq C^{-1} e^{-C\sqrt{\lambda}} ||X||_2^2$$

Low frequency

$$2\mu_\lambda X^T \dot{X} \le -2\mu_\lambda \gamma_\lambda X^T J_N X \le -2\mu_\lambda\gamma_\lambda C^{-1} e^{-C\sqrt{\lambda}} ||X||_2^2$$

High frequency

$$2\left\langle P_N^{\perp} y, y_t \right\rangle \le -\lambda ||P_N^{\perp} y||_{L^2}^2 + \frac{\gamma_\lambda^2}{\lambda} ||X||_2^2$$

Choose

$$\gamma_\lambda := C e^{C\sqrt{\lambda}} \lambda, \quad \mu_\lambda := C^2 e^{2C\sqrt{\lambda}}$$

Quantitative rapid stabilization

$$\begin{aligned} \frac{d}{dt} V\left(y(t)\right) &\le -2\mu_\lambda\gamma_\lambda C^{-1} e^{-C\sqrt{\lambda}} ||X||_2^2 - \lambda ||P_N^{\perp} y||_{L^2}^2 + \frac{\gamma_\lambda^2}{\lambda} ||X||_2^2 \\ &= -\lambda V(y(t)) \end{aligned}$$

Quantitative rapid stabilization

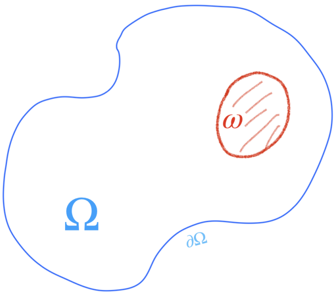
Xiang (2020, postdoc)

For $\lambda > 0$ the heat equation

$$y_t - \Delta y = -\gamma_\lambda 1_\omega P_\lambda y$$

satisfies

$$\|y(t)\| \leq C e^{C\sqrt{\lambda}} e^{-\lambda t} \|y(0)\|$$



Constructive physical feedback + *Theoretical microlocal analysis*

Frequency Lyapunov (Xiang)

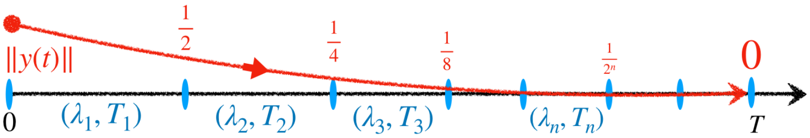
Stabilization

Finite time stabilization

Quantitative rapid stabilization: $\forall\lambda$, there is a feedback law such that

$$\|y(t)\| \leq Ce^{C\sqrt{\lambda}}e^{-\lambda t}\|y(0)\|$$

Finite time stabilization: find a sequence $\{(\lambda_k, T_k)\}$



- Multi-D heat equation [Xiang, 2020]
- Navier–Stokes equation [Xiang, Ann.IHP]
- Huge potential
  Stokes, fractional Laplace, degenerate operater, full domain, measurable set, etc.
  time dependent system, boundary control system
  observer design, delay systems

An advice from PhD defense

☕ "It would be great if you could further consider optimal cost, which means the smallest norm of the control leading to null controllability."

—— Fernández-Cara

Optimal cost for null control

Optimal cost

$$\|u\|_{L^2(0,T;L^2(\omega))} \leq Ce^{\frac{C}{T}}, \ \forall T \in (0,1)$$

Null controllability & Optimal cost (✓ ✗)

	Lebeau-Robbiano		Global Carleman		Frequency Lyapunov	
Constructive	semi				totally	
Nonlinearity	Liu-Takahashi-Tucsnak		nonlinear		nonlinear	
Heat	✓	T	✓	T	✓	T
Nonlinear heat			✓	T	✓	T
Stokes	✓	T	✗	T^4	✓	T
Navier—Stokes			✗	T^9	✓	T

Beyond classical results

- Exponential stabilization

- Null controllability with optimal costs

- Finite time stabilization of the 1D heat equation

Beyond classical results

- Exponential stabilization
  Quantitative rapid stabilization

- Null controllability with optimal costs
  Totally constructive approach

- Finite time stabilization of the 1D heat equation
  The multi-D heat equation, Navier–Stokes equation

Outline of the presentation

2) Backstepping

Backstepping: for $\lambda > 0$, find invertible transformation T

Original system | Target stable system

$$\begin{cases} y_t - y_{xx} = 0 \\ y(t,0) = 0 \\ y(t,L) = u(t) \end{cases} \xrightarrow{T\text{: backstepping}} \begin{cases} z_t - z_{xx} + \lambda z = 0 \\ z(t,0) = 0 \\ z(t,L) = 0 \end{cases}$$

$$\|y(t)\| \leq C_\lambda e^{-\lambda t} \|y(0)\|$$

Abstract version: construct (K, T) such that $u = Kx$ and $z = Tx$ verify

$$\dot{x} = Ax + Bu \xrightarrow{T\text{: backstepping}} \dot{z} = Az - \lambda z$$

Since $\dot{z} = T\dot{x} = T(Ax + BKx)$, it only requires the "Operator Equality":

$$\begin{cases} TA + BK = (A - \lambda I)T, \\ TB = B. \end{cases}$$

On the operator equality

$$\dot{x} = Ax + B{\color{red}u} \text{ controllable} \xrightarrow{\text{find } (T,K)} \begin{cases} TA + BK = (A - \lambda I)T & (*) \\ TB = B & (**) \end{cases}$$

Volterra transformation (Krstic et. al. 1992: the heat equation)

$$(Tf)(x) := f(x) - \int_0^x k(x,y)f(y)dy \text{ with kernel } k_{xx} - k_{yy} + l(k) = 0$$

Fredholm transformation

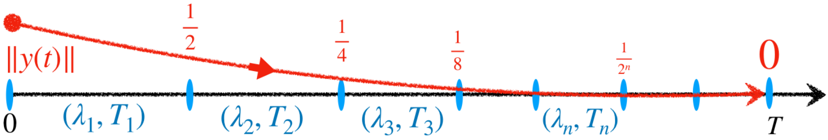
$$(Tf)(x) := f(x) - \int_0^L k(x,y)f(y)dy$$

- ★ Coron–Lü (2013): first consider Fredholm transformation based on KdV solve the kernel equations like $k_{xxx} + k_{yyy} + l(k) = 0$
- ★ Coron (2015): general ODE
- ★ Coron–Gagnon–Morancey (2018): adapt $(*) - (**)$ to solve the kernel equation in terms of eigenfunctions
- ★ Coron, Gagnpn, Hayat, Hu, Lissy, Lu, Marx, Morancey, Olive, Shang, Xiang, Zhang et. al.

Some of my contribution

Use Volterra transformation for finite time stabilization

- the linearized KdV [Xiang, SICON]
- the nonlinear KdV [Xiang, SCL]
- the global viscous Burgers equation [Coron-Xiang, JMPA]



Use Fredholm transformation for rapid stabilization

- the linear water tank [Coron-Hayat-Xiang-Zhang, ARMA]
- the 1D heat equation [Gagnon-Hayat-Xiang-Zhang, JFA]
- the linear water waves [Gagnon-Hayat-Xiang-Zhang, 2022]

$3/2$–threshold for Fredholm

$$\dot{x} = Ax + Bu \text{ controllable} \xrightarrow{\text{find } (T,K)} \begin{cases} TA + BK = (A - \lambda I)T & (*) \\ TB = B & (**) \end{cases}$$

Ideas of Fredholm backstepping:

$$z = T_\lambda x = (Id + \mathcal{K}_{comp})x$$

- ★ Abstract: eigenvalues of A behave as $\lambda_n \sim n^{\alpha}$.
- ★ Eigenmodes: $\forall K$, there is a unique T_K solving $(*)$

$$T\varphi_n = (-K(\varphi_n)) \sum_{p \in \mathbb{N}^*} \frac{b_p \varphi_p}{\lambda_n - \lambda_p + \lambda}.$$

- ★ Riesz basis: find a unique K such that T_K satisfying $(**)$
- ★ Fredholm: T is invertible
- ★ Threshold for $\alpha = 3/2$: Schrödinger ($\alpha = 2$), KdV ($\alpha = 3$), KS ($\alpha = 4$)

$$\sum_{n \in \mathbb{N}} \sum_{p \in \mathbb{N} \setminus \{n\}} \frac{1}{|\lambda_n - \lambda_p + \lambda|^2} < +\infty$$

Open problem (Coron, Collège de France, January 2017)

Fredholm backstppeing on the linearized water waves ($\alpha = 3/2$),

$$A = -i\left((g - \partial_x^2)|D_x| \tanh(h|D_x|)\right)^{1/2}$$

Compactness/duality method for $\alpha > 1$

Open problem: the linearized water waves ($\alpha = 3/2$)

$$A = -i\left((g - \partial_x^2)|D_x|\tanh(h|D_x|)\right)^{1/2}$$

Main difficulty: Riesz basis, sharp estimates

- transport equation [Zhang, 2019]
  linearized water tank [Coron-Hayat-X-Zhang, ARMA]
  (Riesz basis) $\alpha = 1$, explicit calculation and Hilbert transform
- the 1D heat equation [Gagnon-Hayat-X-Zhang, JFA]
  (Sharp estimates) optimal choice of spaces

Compactness/duality method (Gagnon-Hayat-X-Zhang, 2022)

- the linearized water waves ($\alpha = 3/2$)
- any Fourier multiplier operator ($\alpha > 1$)

Proposition (Gagnon–Hayat–X–Zhang, 2022)

Let $q_n = \sum_p \frac{\varphi_p}{\lambda_n - \lambda_p + \lambda}$. *For any* $r \in (-1,1)$, *the family* $(n^{-r} q_n)_{n\in\mathbb{N}^*}$ *is a Riesz basis of* H^r.

- Decompose $n^{-r} q_n$ and use compactness arguments, it suffices to show
  $(n^{-r} q_n)_{n\in\mathbb{N}^*}$ is ω-independent in H^r
- Case $r = 0$ with the help of the dual term $\overline{q_n} = \sum_p \frac{\varphi_p}{\lambda_p - \lambda_n + \lambda}$
  - (i) $(q_n)_{n\in\mathbb{N}^*}$ is either ω-independent in L^2 or L^2-dense.
  - (ii) $(\overline{q_n})_{n\in\mathbb{N}^*}$ is either ω-independent in L^2 or L^2-dense.
  - (iii) $(q_n)_{n\in\mathbb{N}^*}$ is ω-independent in L^2 $\Longleftrightarrow$ $(\overline{q_n})_{n\in\mathbb{N}^*}$ is ω-independent in L^2.
  - (iv) $(q_n)_{n\in\mathbb{N}^*}$ is L^2-dense $\Longleftrightarrow$ $(\overline{q_n})_{n\in\mathbb{N}^*}$ is L^2-dense.
  - (v) $(q_n)_{n\in\mathbb{N}^*}$ is L^2-dense $\Longleftrightarrow$ $(\overline{q_n})_{n\in\mathbb{N}^*}$ is ω-independent in L^2.
  - (vi) $(\overline{q_n})_{n\in\mathbb{N}^*}$ is L^2-dense $\Longleftrightarrow$ $(q_n)_{n\in\mathbb{N}^*}$ is ω-independent in L^2.
- Case $r > 0$ comes from the preceding case
- Case $r < 0$, use duality argument
  $(n^{-r} q_n)$ is ω-independent in H^r $\Longleftrightarrow$ $(n^r q_n)$ is dense in H^{-r}

The main result

Theorem (Gagnon–Hayat–X–Zhang, 2022)

Let $\alpha > 1$. *Let* $B \in H^{-3/4}$ *satisfying controllability condition. Let* $h(s)$ *a real valued-function satisfying*

- $|n_1 - n_2| n_1^{\alpha-1} \lesssim |h(n_1) - h(n_2)|$ *for any* $(n_1, n_2) \in \mathbb{N}^*$.
- $s^\alpha \lesssim |h(s)| \lesssim s^\alpha$ *for any* $s \in [1, +\infty)$.

Then, for any $\lambda > 0$, *there exists a bounded linear operator* $K \in \mathcal{L}(H^{3/4}; \mathbb{C}^2)$ *and an operator* T *such that* T *is an isomorphism from* $H^r(\mathbb{T})$ *to itself for any* $r \in (1/2 - \alpha, \alpha - 1/2)$ *and maps the system*

$$\partial_t u = i\, h(|D_x|)u + BK(u),\ (t,x) \in \mathbb{R}_+ \times \mathbb{T},$$

to

$$\partial_t v = i\, h(|D_x|)v - \lambda v,\ (t,x) \in \mathbb{R}_+ \times \mathbb{T}.$$

Consequently, the closed-loop system is exponentially stable in H^r *for* $r \in (1/2 - \alpha, \alpha - 1/2)$ *with decay rate* λ.

Thank you for your attention!