Theory of homogeneous dynamical systems and their application

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Outline

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Place of homogeneous systems

- Different design and analysis problems:
  - estimation, control, optimization
  - various uncertainty models
    - parameters, signals, dynamics
  - performance specifications
    - energy, constraints,..., time
- Classes of plant models:
  - ODE, DI, TDS, PDE, *etc.*
  - linear *vs* nonlinear (canonical cases + constructive methods)
    - Lipschitz, LPV
    - Lurie, Persidskii,..., homogeneous

Interpretation and application of homogeneity

Homogeneity in mathematics is a kind of *symmetry*

- We are living in a heterogeneous world $\Rightarrow$ *symmetry* is rare
- Homogeneous object/system is an idealization
- Projection to an ideal $\Rightarrow$ more tools $\Rightarrow$ evaluation of the original
  - robustness is an important issue
  - advantageous performances

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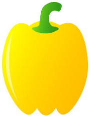


History of the subject

Pioneer works: Euler; Zubov, 1958; Rothschild & Stein, 1976

Applications:

- stability analysis (Zubov, 1958; Hermes, 1991a; 1991b; Rosier, 1992; Efimov et al., 2018);
- stabilization (Kawski, 1991; Sepulchre & Aeyels, 1996; Grüne, 2000; Levant, 2001; Bhat & Bernstein, 2005; Moulay & Perruquetti, 2006; Zimenko et al., 2020)
- observation (Levant, 2005; Andrieu et al., 2008; Perruquetti et al., 2008; Lopez-Ramirez et al., 2018a; 2018b)

Extensions:

- coordinate-free homogeneity (Khomenuk, 1961; Kawski, 1995)
- local homogeneity (Andrieu et al., 2008; Efimov & Perruquetti, 2010)
- TDS (Efimov & Perruquetti, 2011; Efimov et al., 2015; Zimenko et al., 2017; Efimov & Aleksandrov, 2020), DI (Filippov, 1988; Bernuau et al., 2013), PDE (Polyakov et al., 2016; Polyakov, 2018), TVS (Peuteman & Aeyels, 1999; Orlov, 2005; Ríos et al., 2016), DTS (Tuna & Teel, 2004; Sanchez et al., 2017; 2020)
- discretization (Efimov et al., 2017; 2019; Polyakov et al., 2019; 2022; Sanchez et al., 2020)

Mathematical definition of homogeneity

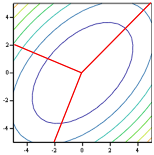
Definition For a function $f : \mathbb{R}^n \to \mathbb{R}$, if $\forall \lambda > 0$ and $\forall x \in \mathbb{R}^n$

$$f(\lambda x) = \lambda^{\nu} f(x),$$

then f is called *homogeneous* with degree ν.

Theorem *(Euler's theorem on homogeneous functions)*
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 homogeneous function of degree ν, then

$$\frac{df(x)}{dx} x = \nu f(x).$$



Examples of homogeneous functions for $x = [x_1 \ x_2]^T$

- A polynomial function of degree $\nu = 2$:

$$f(x) = x_1^2 + x_1 x_2 + x_2^2, \quad f(\lambda x) = \lambda^2 x_1^2 + \lambda^2 x_1 x_2 + \lambda^2 x_2^2 = \lambda^2 f(x),$$

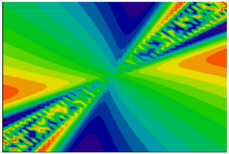
$$\frac{df(x)}{dx} x = (2x_1 + x_2)x_1 + (2x_2 + x_1)x_2 = 2(x_1^2 + x_1 x_2 + x_2^2) = 2f(x)$$

- Functions of degree $\nu = 0$: $\qquad f(x) = 1$

$$f(x) = \mathrm{sign}(x_1^2 - x_2^2), \ f(\lambda x) = \mathrm{sign}(\lambda^2 x_1^2 - \lambda^2 x_2^2) = \mathrm{sign}(x_1^2 - x_2^2) = f(x);$$

$$f(x) = \frac{x_1 + x_2}{x_1 - x_2}, \ f(\lambda x) = \frac{\lambda x_1 + \lambda x_2}{\lambda x_1 - \lambda x_2} = \frac{x_1 + x_2}{x_1 - x_2} = f(x)$$

- A combination of degree $\nu = 0.5$: $f(x) = \sin\left(\frac{x_1 + x_2}{x_1 - x_2}\right)\left(x_1^2 + x_1 x_2 + x_2^2\right)^{\frac{1}{4}}$



Homogeneity for dynamical systems

$$\dot{x} = f(x),\ x \in \mathbb{R}^n,\ f(0) = 0. \tag{1}$$

Definition For a function $f : \mathbb{R}^n \to \mathbb{R}^n$, if $\forall \lambda > 0$ and $\forall x \in \mathbb{R}^n$

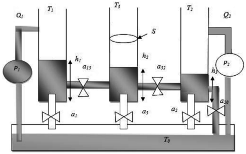
$$f(\lambda x) = \lambda^{\nu+1} f(x),$$

then the function f is called *homogeneous* with degree ν.

- Linear systems with degree $\nu = 0$:

$$f(x) = Ax, \quad f(\lambda x) = \lambda Ax = \lambda f(x).$$

- Nonlinear hydraulic three tank system with $\nu = -0.5$:



$$\begin{aligned}
\dot{h}_1 &= - \frac{a_{13}}{S} \lceil h_1 - h_3 \rfloor^{0.5} + \frac{1}{S} Q_1, \quad \lceil s \rfloor^{\alpha} = |s|^{\alpha} \mathrm{sign}(s), \\
\dot{h}_2 &= \frac{a_{32}}{S} \lceil h_3 - h_2 \rfloor^{0.5} - \frac{a_{20}}{S} \lceil h_2 \rfloor^{0.5} + \frac{1}{S} Q_2, \\
\dot{h}_3 &= \frac{a_{13}}{S} \lceil h_1 - h_3 \rfloor^{0.5} - \frac{a_{32}}{S} \lceil h_3 - h_2 \rfloor^{0.5}.
\end{aligned}$$

Stability definitions

Denote solution to (1) with initial condition $x_0 \in \mathbb{R}^n$ as $X(t, x_0)$, $0 \in \Omega \subset \mathbb{R}^n$.

Definition 1 At equilibrium $x = 0$ the system (1) is said to be

(*a*) *Lyapunov stable* (LS) if $\forall x_0 \in \Omega$ the solution $X(t, x_0)$ is defined $\forall t \geq 0$, and $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $\forall x_0 \in \Omega$: $\|x_0\| \leq \delta \Rightarrow \|X(t, x_0)\| \leq \epsilon\ \forall t \geq 0$;

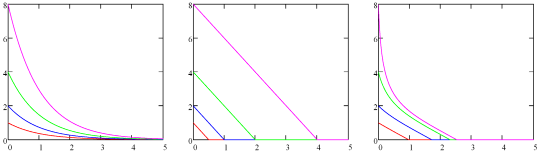
(*b*) *asymptotically stable* if it is LS and $\forall x_0 \in \Omega$: $\lim_{t \to +\infty} \|X(t, x_0)\| = 0$;

(*c*) *finite-time stable* if it is LS and finite-time converging from Ω: $\forall x_0 \in \Omega$ $\exists 0 \leq T_0(x_0) < +\infty$ s.t. $X(t, x_0) = 0\ \forall t \geq T_0(x_0)$;

(*d*) *fixed-time stable* if it is finite-time stable and $\sup_{x_0 \in \Omega} T_0(x_0) < +\infty$.

The set Ω is called the *domain* of stability/attraction.

If $\Omega = \mathbb{R}^n$, then these properties are called *global*.



Rate of convergence and homogeneity

$$\dot{x} = -a\lceil x\rfloor^{\alpha},\ x \in \mathbb{R},\ x(0) = x_0,\ a > 0,\ \alpha \geq 0.$$

- The system is Lyapunov stable: $V(x) = 0.5x^2$ and $\dot{V} = -a|x|^{\alpha+1} < 0$
- The system is homogeneous of degree $\nu = \alpha - 1$

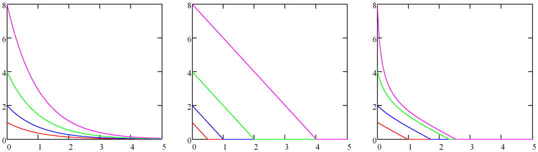
Solutions:

$$X(t, x_0) = \begin{cases} \sqrt[-\nu]{|x_0|^{-\nu} - \beta t}\,\mathrm{sign}(x_0) & \alpha < 1 \\ e^{-at}x_0 & \alpha = 1 \\ \frac{x_0}{\sqrt[\nu]{1 - |x_0|^{\nu}\beta t}} & \alpha > 1 \end{cases};$$

$$\beta = a(1-\alpha)$$

- finite-time stability for $\nu < 0$ with $T_0(x_0) = \frac{|x_0|^{-\nu}}{\beta}$
- exponential (asymptotic) stability for $\nu = 0$
- fixed-time stability with respect to a ball for $\nu > 0$:

$$T_1(x_0) = \frac{1 - |x_0|^{-\nu}}{|\beta|},\ \lim_{x_0 \to +\infty} T_1(x_0) = \frac{1}{|\beta|}$$

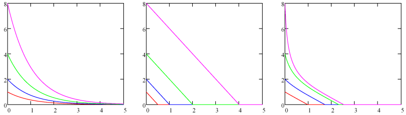


Scaling of trajectories

If $X(t, x_0)$ is a solution of (1) with initial condition $x_0 \in \mathbb{R}^n$, and (1) is homogeneous of degree ν, then $Y(t, y_0) = \lambda X(\lambda^\nu t, \lambda^{-1} y_0)$ for any $\lambda > 0$ is a solution of (1) for initial condition $y_0 = \lambda x_0$:

$$
\begin{aligned}
\frac{d}{dt} Y(t, y_0) &= \lambda \frac{d}{dt} X(\lambda^\nu t, \lambda^{-1} y_0) \\
&= \lambda^{\nu+1} \frac{d}{d\lambda^\nu t} X(\lambda^\nu t, \lambda^{-1} y_0) \\
&= \lambda^{\nu+1} f(X(\lambda^\nu t, \lambda^{-1} y_0)) = f(\lambda X(\lambda^\nu t, \lambda^{-1} y_0)) \\
&= f(Y(t, y_0)).
\end{aligned}
$$

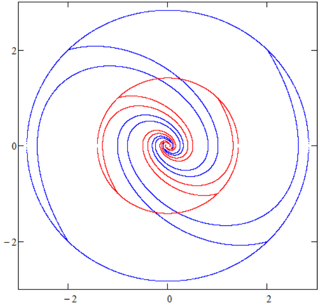
"Homogeneity" of solutions: $\quad X(t, \lambda x_0) = \lambda X(\lambda^\nu t, x_0)$.



Local $\Rightarrow$ Global

Denote $\mathbb{S} = \{x \in \mathbb{R}^n : \|x\| = 1\}$, then

$$\forall x \in \mathbb{R}^n \; \exists y \in \mathbb{S} : \; x = \lambda y, \; \lambda = \|x\|$$



- Behavior of all trajectories initiated on a sphere $\Leftrightarrow$ Behavior in $\mathbb{R}^n$
- Local stability $\Leftrightarrow$ Global stability
- Attractivity $\Rightarrow$ Stability

Lyapunov functions

Let (1) be homogeneous of degree ν + asymptotically stable $\Rightarrow \exists$ a Lyapunov function (LF) V:

$$f(\lambda x)=\lambda^{\nu+1}f(x),\quad V(x)>0,\quad \frac{\partial}{\partial x}V(x)f(x)<0.$$

- $\exists$ a LF $\iff \exists$ a homogeneous LF of degree $\mu>-\nu$:

$$V(\lambda x)=\lambda^{\mu}V(x)$$

- Properties of V:

$$c_1=\inf_{y\in\mathbb{S}}V(y),\ c_2=\sup_{y\in\mathbb{S}}V(y),\ V(x)=\|x\|^{\mu}V(y),$$

$$c_1>0,\ c_2>0\Longrightarrow c_1\|x\|^{\mu}\le V(x)\le c_2\|x\|^{\mu},$$

$$a=-\sup_{y\in\mathbb{S}}\frac{\partial}{\partial y}V(y)f(y),\ a>0$$

- Properties of $\dot V$: $\forall x\in\mathbb{R}^n\ \exists y\in\mathbb{S}$ s.t. $x=\lambda y$ with $\lambda=\|x\|$:

$$\frac{\partial}{\partial x}V(x)f(x)=\frac{\partial}{\partial\lambda y}V(\lambda y)f(\lambda y)=\lambda^{\nu+\mu}\frac{\partial}{\partial y}V(y)f(y)\le -a\|x\|^{\nu+\mu}\le-\frac{a}{c_2}V^{1+\frac{\nu}{\mu}}(x)$$

Role of homogeneity

Homogeneity is an algebraic property $\Rightarrow$ It can be easily checked

Linear systems $\subset$ Homogeneous systems $\subset$ Nonlinear systems:

Linear systems	*Homogeneous systems*	*Nonlinear systems*
Scalability of trajectories	Scalability of trajectories	?
Local = Global	Local = Global	Local $\neq$ Global
Attractivity $\Rightarrow$ Stability	Attractivity $\Rightarrow$ Stability	Attractivity $\not\Rightarrow$ Stability
Quadratic LF	Homogeneous LF	?
Exponential convergence	Degree dependent	?
0-GAS $\Rightarrow$ ISS	0-GAS $\Rightarrow$ ISS via degree	0-GAS $\not\Rightarrow$ ISS
Robustness to delay	Robustness to delay	?

Conventional (Euler) $\Rightarrow$ Weighted $\Rightarrow$ Local $\Rightarrow$ Geometric/Coordinate-free

Outline

Preliminaries

$$\dot{x} = f(x),\ x \in \mathbb{R}^n,\ f(0) = 0 \qquad (2)$$

- For $r_i > 0,\ i = \overline{1, n}$ define vector of weights $\mathrm{r} = [r_1 ... r_n]^T$
- For any r and $\lambda > 0$ define dilation matrix $\Lambda_r = \mathrm{diag}\{\lambda^{r_i}\}_{i=1}^n$
- A homogeneous norm:

$$|x|_r = \left(\sum_{i=1}^n |x_i|^{\rho/r_i}\right)^{1/\rho}, \rho > 0 \implies |\Lambda_r x|_r = \lambda |x|_r$$

For $x \in \mathbb{R}^n$, its Euclidean norm $|x|$ is related with $|x|_r$:

$$\underline{\sigma}_r(|x|_r) \leq |x| \leq \bar{\sigma}_r(|x|_r),\ \underline{\sigma}_r, \bar{\sigma}_r \in \mathcal{K}_\infty$$

- The homogeneous sphere $S_r = \{x \in \mathbb{R}^n : |x|_r = 1\}$

Weighted homogeneity

Definition 2

Function $g : \mathbb{R}^n \to \mathbb{R}$ *is called* r*-homogeneous if*

$$\exists d \geq 0 : g(\Lambda_r x) = \lambda^d g(x) \quad \forall x \in \mathbb{R}^n \ \forall \lambda > 0.$$

Function $f : \mathbb{R}^n \to \mathbb{R}^n$ *is called* r*-homogeneous if*

$$\exists d \geq -\min_{1 \leq i \leq n} r_i : f(\Lambda_r x) = \lambda^d \Lambda_r f(x) \quad \forall x \in \mathbb{R}^n \ \forall \lambda > 0.$$

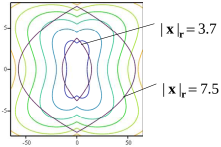
d is called *degree* *of homogeneity.*

$$\lambda \Longrightarrow \Lambda_r = \mathrm{diag}\{\lambda^{r_i}\}_{i=1}^n$$

Homogeneous function:

$$g(x_1, x_2) = \frac{x_1^2 + x_2^4}{|x_1| + |x_2|^2}, \ r = [2\ 1]^T, \ d = 2;$$

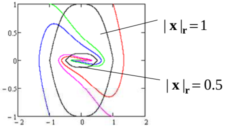
$$g(\mathbf{\Lambda_r x}) = g(\lambda x_1, \lambda x_2) = \lambda^2 g(x_1, x_2).$$



Homogeneous system:

$$\mathbf{f}(x_1, x_2) = [\sqrt[3]{x_2} - x_1^3 - x_2]^T, \ r = [1\ 3]^T, \ d = 0.$$

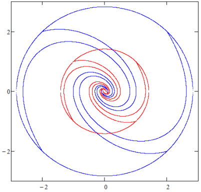
$$\mathbf{f}(\mathbf{\Lambda_r x}) = \mathbf{f}(\lambda x_1, \lambda^3 x_2) = \mathbf{\Lambda_r f(x)}.$$



Weighted homogeneity

Local attractivity $\Leftrightarrow$ Global asymptotic stability

$+$ Homogeneous LF



Proposition 1 Let (2) be r-homogeneous system with degree d and $x(t)$ be trajectory with initial condition x_0. The curve $t \mapsto \Lambda_r x(\lambda^d t)$ is trajectory of (2) with initial condition $\Lambda_r x_0$:

$$\frac{d}{dt}\left(\Lambda_r x(\lambda^d t)\right) = \lambda^d \Lambda_r f(x(\lambda^d t)) = f(\Lambda_r x(\lambda^d t)) \quad \forall \lambda > 0.$$

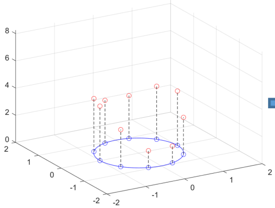
Theorem 1 Let f be C^0 and r-homogeneous with degree d. Assume (2) is GAS. Then $\forall k > \max(0, -d)$ $\exists$ r-homogeneous with degree k, proper, C^0 (C^1 on $\mathbb{R}^n \setminus \{0\}$) and positive definite function V s.t. $\dot{V} \le -aV^{\frac{d+k}{k}}$ for $a = -\max_{\{V=1\}} \dot{V}$, $a > 0$.

Numerical design of homogeneous LFs (Efimov et al., 2018)

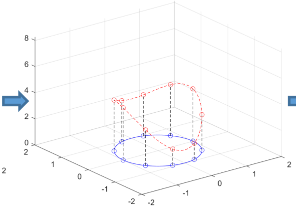
- Calculation of values of a homogeneous LF on the sphere S_r

$$U(x) = \int_0^{\overline{T}_q(\|x\|_r)} \|X(t,x)\|_r^{\mu} dt, \ \mu > \max\{1, \nu + \nu^2\}$$

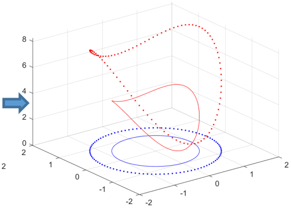
- Find suitable homogeneous approximation basis on S_r
- Approximation of LF on S_r in this basis
- Interpolation by homogeneity in $\mathbb{R}^n$



Probing



Approximation



Interpolation

Homogeneity & Heterogeneity

- Homogeneous systems have global behaviors:
  - no limit cycles
  - no isolated equilibria
- Homogeneous systems have many useful properties for
  - analysis
  - synthesis
- How to apply the theory of homogeneous systems in a heterogeneous world?

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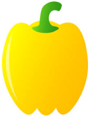


Local homogeneity

Definition 3 *Function* $g:\mathbb{R}^n\to\mathbb{R}$ *is* $(\mathrm{r},\lambda_0,g_0)$*-homogeneous* ($g_0:\mathbb{R}^n\to\mathbb{R}$) *if*

$$\exists d_0\geq 0\ :\ \lim_{\lambda\to\lambda_0}\lambda^{-d_0}g(\Lambda_r x)=g_0(x)\quad\forall x\in S_r.$$

Function $f:\mathbb{R}^n\to\mathbb{R}^n$ *is* $(\mathrm{r},\lambda_0,f_0)$*-homogeneous* ($f_0:\mathbb{R}^n\to\mathbb{R}^n$) *if*

$$\exists d_0\geq -\min_{1\leq i\leq n}r_i\ :\ \lim_{\lambda\to\lambda_0}\lambda^{-d_0}\Lambda_r^{-1}f(\Lambda_r x)=f_0(x)\quad\forall x\in S_r.$$

- Bi-limit homogeneity in (Andrieu, Praly, Astolfi, 2008) for $\lambda_0\in\{0,+\infty\}$ (the limit has to be uniform on S_r)
- The homogeneous functions g_0, f_0 for $0<\lambda_0<+\infty$ can be chosen explicitly:

$$g_0(x)=|x|_r^d\lambda_0^{-d_0}g(\Lambda_{r,0}\Lambda_{|x|}^{-1}x),\quad f_0(x)=|x|_r^d\lambda_0^{-d_0}\Lambda_{r,0}^{-1}f(\Lambda_{r,0}\Lambda_{|x|}^{-1}x),\tag{3}$$
$$\Lambda_{r,0}=\mathrm{diag}\{\lambda_0^{r_i}\}_{i=1}^n,\ \Lambda_{|x|}=\mathrm{diag}\{|x|_r^{r_i}\}_{i=1}^n$$

- Linearization $\neq$ Local homogeneity:

$$f(x)=-x^3+x^5\quad\Rightarrow\quad f_0(x)=-x^3$$

Stability analysis

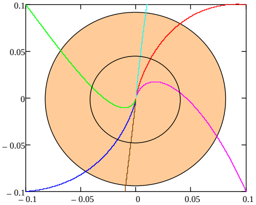
Relations between $\dot{x} = f(x)$ and $\dot{x} = f_0(x)$ (Zubov, 1958; Andrieu, Praly, Astolfi, 2008):

- f_0 is GAS for $\lambda_0 = 0 \Longrightarrow f$ is LAS at the origin;
- f_0 is GAS for $\lambda_0 = +\infty \Longrightarrow f$ is Lagrange stable.

$$\mathbf{f}(x_1, x_2) = \begin{bmatrix} -x_1 + x_1 x_2^4 - x_1^5 + 2x_1^2 - 2x_1^2 x_2 \\ -x_2 + x_1 - x_2^5 - x_1^2 x_2^3 - x_1 x_2^2 - x_2^2 \end{bmatrix}$$

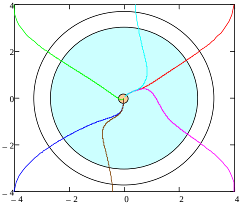
$\lambda_1 = 0,\ \mathbf{r}_1 = [1\ 1]^T,\ d_1 = 0$

$$\mathbf{f}_1(x_1, x_2) = \begin{bmatrix} -x_1 \\ -x_2 + x_1 \end{bmatrix}$$



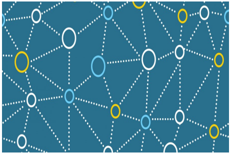
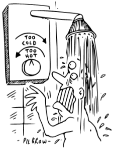
$\lambda_2 = +\infty,\ \mathbf{r}_2 = [1\ 1]^T,\ d_2 = 4$

$$\mathbf{f}_2(x_1, x_2) = \begin{bmatrix} x_1 x_2^4 - x_1^5 \\ -x_2^5 - x_1^2 x_2^3 \end{bmatrix}$$



Outline

Time-delay systems

- Functional differential equations (state dimension is infinite)
- Complexity of stability/robustness/performance analysis
  - Lyapunov-Krasovskii approach
  - Lyapunov-Razumikhin approach

Can symmetry/homogeneity help for analysis of time-delay systems?

Time-delay systems

Denote by $C^n[a,b]$, $0 \leq a < b \leq +\infty$ the Banach space of C^0 functions $\phi : [a,b] \to \mathbb{R}^n$ with the uniform norm $||\phi|| = \sup_{a \leq \varsigma \leq b} |\phi(\varsigma)|$

Autonomous functional differential equation of retarded type:

$$dx(t)/dt = f(x_t), \ t \geq 0, \tag{4}$$

- $x \in \mathbb{R}^n$ and $x_t \in C^n[-\tau, 0]$ is the state function
- $x_t(s) = x(t+s)$, $-\tau \leq s \leq 0$
- $f : C^n[-\tau, 0] \to \mathbb{R}^n$ is locally Lipschitz continuous, $f(0) = 0$

Weighted homogeneity

- For $r_i > 0,\ i = \overline{1,n}$ define vector of weights $\mathrm{r} = [r_1 ... r_n]^T$
- For any r and $\lambda > 0$ define dilation matrix $\Lambda_r = \mathrm{diag}\{\lambda^{r_i}\}_{i=1}^n$

- Homogeneous norms for $\rho > 0$:

$$x \in \mathbb{R}^n : |x|_r = \left(\sum_{i=1}^n |x_i|^{\rho/r_i}\right)^{1/\rho} \Longrightarrow |\Lambda_r x|_r = \lambda |x|_r;$$

$$\phi \in C^n[-\tau, 0] : ||\phi||_r = \left(\sum_{i=1}^n ||\phi_i||^{\rho/r_i}\right)^{1/\rho} \Longrightarrow ||\Lambda_r \phi||_r = \lambda ||\phi||_r$$

- Homogeneous spheres:

$$S_r = \{x \in \mathbb{R}^n : |x|_r = 1\},$$
$$\mathcal{S}_r = \{\phi \in C^n[-\tau, 0] : ||\phi||_r = 1\}$$

Definitions (Efimov&Perruquetti, 2011)

Definition 4 *The function* $g : C^n[-\tau, 0] \to \mathbb{R}$ *is called* r*-homogeneous if*

$$g(\Lambda_r \phi) = \lambda^d g(\phi)$$

$\forall \phi \in C^n[-\tau, 0]$, $\forall \lambda > 0$ *and some* $d \in R$.

The function $f : C^n[-\tau, 0] \to \mathbb{R}^n$ *is called* r*-homogeneous if*

$$f(\Lambda_r \phi) = \lambda^d \Lambda_r^1 f(\phi)$$

$\forall \phi \in C^n[-\tau, 0]$, $\forall \lambda > 0$ *and some* $d \geq -\min_{1 \leq i \leq n} r_i$.

The constant d is called the degree of homogeneity.

For $r = 1$ and $d = 0$:

$$g(\phi) = \frac{\phi(0)}{\max\{|\phi(0)|, |\phi(-\tau)|\}}$$

For $\mathrm{r} = [1\ 0.5]$ and $d = -0.5$:

$$f(\phi) = \begin{bmatrix} -\sqrt{|\phi_1(-\tau)|}\mathrm{sign}(\phi_1(-\tau)) + \phi_2(0) \\ -\mathrm{sign}(\phi_1(-\tau)) \end{bmatrix}$$

Local homogeneity (Efimov&Perruquetti, 2011)

A restriction of introduced homogeneity: the same behavior "globally"

Definition 5 $g : C_{[-\tau,0]} \to \mathbb{R}$ is called (r,λ_0,g_0)-homogeneous if

$$\lim_{\lambda\to\lambda_0} \lambda^{-d_0} g(\Lambda_r \phi) = g_0(\phi), \; d_0 \in \mathbb{R} \quad \forall \phi \in \mathcal{S}_r$$

is satisfied (uniformly on $\mathcal{S}_r$ for $\lambda_0 \in \{0, +\infty\}$).

The system (4) is called (r,λ_0,f_0)-homogeneous if

$$\lim_{\lambda\to\lambda_0} \lambda^{-d_0} \Lambda_r^{-1} f(\Lambda_r \phi) = f_0(\phi), \; d_0 \geq -\min_{1\leq i\leq n} r_i \quad \forall \phi \in \mathcal{S}_r$$

is satisfied (uniformly on $\mathcal{S}_r$ for $\lambda_0 \in \{0, +\infty\}$).

For a given λ_0, g_0 and f_0 are called approximating functions.

Scaling trajectories for $d = 0$ (Efimov et al., 2014)

Proposition 2 Let $x : \mathbb{R}_+ \to \mathbb{R}^n$ be a solution of the r-homogeneous system (4) with the degree $d = 0$ for an initial condition $x_0 \in C^n[-\tau, 0]$. For any $\lambda > 0$ define $y(t) = \Lambda_r x(t)$ for all $t \geq 0$, then $y(t)$ is also a solution of (4) with the initial condition $y_0 = \Lambda_r x_0$.

Local asymptotic stability + homogeneity $\Longrightarrow$ GAS.

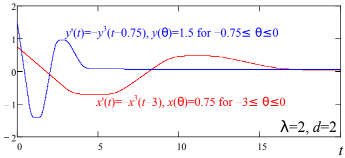
- **Lyapunov-Krasovskii approach** $\Longrightarrow$ derivative of a homogeneous functional is not homogeneous!
- **Lyapunov-Razumikhin approach** $\Longrightarrow$ stability/instability verification via *Lyapunov-Razumikhin Function* ($d \neq 0$)
- Local homogeneous approximations + stability/instability results (Yakubovich's oscillations) + ISS

Scaling trajectories for $d \neq 0$ (Efimov et al., 2015)

Proposition 3 Let $x(t, x_0)$ be a solution of r-homogeneous system (4) with degree d for $x_0 \in C_{[-\tau,0]}$, $\tau \in (0, +\infty)$. For any $\lambda > 0$, system

$$dy(t)/dt = f(y_t),\ t \geq 0$$

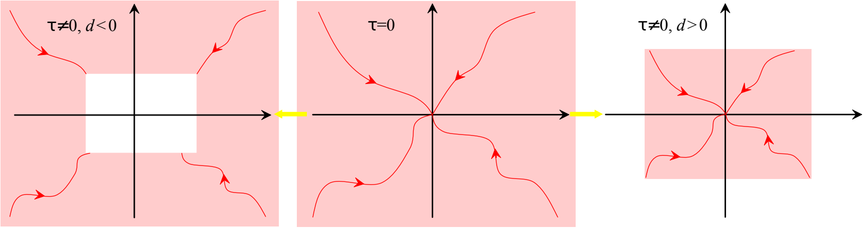
with $y_t \in C_{[-\lambda^{-d}\tau,0]}$, has a solution $y(t, y_0) = \Lambda_r x(\lambda^d t, x_0)$ $\forall t \geq 0$ with $y_0 \in C_{[-\lambda^{-d}\tau,0]}$, $y_0(s) = \Lambda_r x_0(\lambda^d s)$ for $s \in [-\lambda^{-d}\tau, 0]$.



Lemma 1 Let (4) be r-homogeneous with degree $d \neq 0$ and *GAS* for some $0 < \tau_0 < +\infty$, then it is *GAS* for any delay $0 < \tau < +\infty$ (*IOD*).

Robustness with respect to small delays (Efimov et al., 2015)

Lemma 2 Let $f(x_\tau) = F[x(t), x(t-\tau)]$ in (4) be r-homogeneous with degree $d > 0$ ($d < 0$) and *GAS* for $\tau = 0$, then $\forall \rho > 0$ $\exists 0 < \tau_0 < +\infty$ such that (4) is LAS in B_ρ^τ (GAS with respect to B_ρ^τ) $\forall 0 \leq \tau \leq \tau_0$.



Theorem 2 Let (4) be (r,$+\infty$,f_∞)-homogeneous with degree $d_\infty > 0$ and for the approximating system the set B_ρ^τ for some $0 < \rho < +\infty$ be *GAS* $\forall 0 \leq \tau \leq \tau_\infty < +\infty$, then (4) has bounded trajectories *IOD*.

Outline

Evolution PDE

- $\mathbb{B}$ is a real Banach space with a norm $\|\cdot\|$
- $S = \{u \in \mathbb{B} : \|u\| = 1\}$
- $\mathcal{L}(\mathbb{B})$ is the space of linear bounded operators $\mathbb{B} \to \mathbb{B}$

A nonlinear evolution equation

$$\dot{u}(t) = f(u(t)), \quad t \in \mathbb{R}_+,\ u(\cdot) \in \Omega \subset \mathbb{B},\ u(0) = \phi \in \Omega, \tag{5}$$

where operator $f : \Omega \to \mathbb{B}$ and $\Omega \subset \mathbb{B}$ is the domain of f

Definition 6 A map $\mathrm{d} : \mathbb{R} \to \mathcal{L}(\mathbb{B})$ is called dilation in $\mathbb{B}$ if
a) *semigroup* property: $\mathrm{d}(0) = I \in \mathcal{L}(\mathbb{B})$ and $\mathrm{d}(t+s) = \mathrm{d}(t)\mathrm{d}(s)\ \forall t, s \in \mathbb{R}$;
b) *strong continuity*: $\mathrm{d}(\cdot)u : \mathbb{R} \to \mathbb{B}$ is C^0 for any $u \in \mathbb{B}$;
c) *limit* property: $\lim\limits_{s\to-\infty} \|\mathrm{d}(s)u\| = 0$ and $\lim\limits_{s\to+\infty} \|\mathrm{d}(s)u\| = +\infty\ \forall u \in S$

Homogeneity definition (Polyakov et al., 2016)

Let $\mathrm{d} : \mathbb{R} \to \mathcal{L}(\mathbb{B})$ be a dilation in $\mathbb{B}$

Definition 7 A set $\Omega \neq \emptyset$ is d-homogeneous if $\mathrm{d}(s)\Omega = \Omega\ \forall s \in \mathbb{R}$

Example: $\Omega = \mathbb{R}^n$ or $\Omega = C_{[-\tau,0]}$ and $\mathrm{d}(s) = \mathrm{diag}\{e^{r_i s}\}_{i=1}^n$, $r_i > 0$

Definition 8 An operator $f : \Omega \to \mathbb{B}$ is d-homogeneous of degree $\nu \in \mathbb{R}$ on $\Omega \subset \mathbb{B}$ if Ω is d-homogeneous and

$$f(\mathrm{d}(s)u) = e^{\nu s}\mathrm{d}(s)f(u) \qquad \forall s \in \mathbb{R},\ u \in \Omega$$

- Korteweg-de Vries (KdV) equation:

$$f(u) = -u''' - uu',\ \Omega = C^3(\mathbb{R},\mathbb{R}),\ (\mathrm{d}(s)u)(x) = e^{2s}u(e^s x),\ \nu = 3$$

- Saint-Venant equation:

$$f(u) = \begin{bmatrix} -\frac{\partial}{\partial x}(u_1 u_2) \\ -\frac{\partial}{\partial x}(g u_1 + \frac{1}{2}u_2^2) \end{bmatrix},\ g > 0,\ \mathrm{d}(s) = \begin{bmatrix} e^{2s} & 0 \\ 0 & e^s \end{bmatrix},$$

$$\Omega = \{u \in C^1([0,1],\mathbb{R}_+) \times C^1([0,1],\mathbb{R}) : u_1(0)u_2(0) = 0, u_1(1)u_2(1) = u_1^{1.5}(1)\}$$

Scalability and FTS (Polyakov et al., 2016)

Theorem 3 Let $f : \Omega \to \mathbb{B}$ be d-homogeneous of degree $\nu \in \mathbb{R}$ and $u(\cdot, \phi) : [0, T) \to \Omega$ be a solution of (5). Then $u_s(\tau) = \mathrm{d}(s)u(e^{\nu s}\tau, \phi)$ is a solution of (5) for $u(0) = \mathrm{d}(s)\phi$ and $\forall s \in \mathbb{R}$

Corollary 1 Let $f : \Omega \to \mathbb{B}$ be d-homogeneous and $0 \in \Omega$, $f(0) = 0$. Zero solution in (5) is locally attractive/stable $\Rightarrow$ globally attractive/stable

Theorem 4 Let $f : \Omega \to \mathbb{B}$ be d-homogeneous of degree $\nu < 0$ and $0 \in \Omega$, $f(0) = 0$. Zero solution in (5) is uAS $\Rightarrow$ GFTS

- Fast Diffusion Equation:

$$f(u) = -\Delta(u^\alpha),\ \Omega = \mathbb{L}^1(M, \mathbb{R}_+),\ \mathrm{d}(s) = e^s,$$

  Δ is the Laplace operator and $M \in \mathbb{R}^n$ is a bounded connected domain with a regular boundary

Outline

A definition of homogeneity for discrete-time systems

$$x_{k+1} = f(x_k),\ k = 0, 1, \ldots, \tag{6}$$

where $x_k \in \mathbb{R}^n$ is the state, denote by $F(k, x_0)$ the corresponding solution

Definition 9 (Sanchez et al., 2017) Given vector of weights $\mathrm{r} = [r_1 \ldots r_n]^T$:

↑ a function $g : \mathbb{R}^n \to \mathbb{R}$ is r-homogeneous of degree $d \geq 0$ if

$$g(\Lambda_r x) = \lambda^d g(x) \quad \forall \lambda > 0, \forall x \in \mathbb{R}^n$$

↑ a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is r-homogeneous of degree $d \geq -\min_{1 \leq i \leq n} r_i$ if

$$f(\Lambda_r x) = \lambda^d \Lambda_r f(x) \quad \forall \lambda > 0, \forall x \in \mathbb{R}^n$$

↬ a map $f : \mathbb{R}^n \to \mathbb{R}^n$ is D_r-homogeneous of degree $d > 0$ if

$$f(\Lambda_r x) = \Lambda_r^d f(x) \quad \forall \lambda > 0, \forall x \in \mathbb{R}^n$$

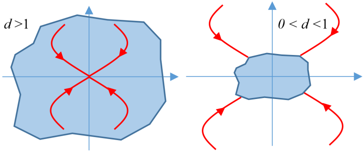
Scalability of solutions: $F(1, x_0) = f(x_0)$,$F(2, x_0) = f(f(x_0))$

- r-homogeneous: $F(2, \Lambda_r x_0) = \lambda^d \Lambda_r f\left(\lambda^d f(x_0)\right)$?
- D_r-homogeneous: $F(2, \Lambda_r x_0) = \Lambda_r^{d^2} f(f(x_0)) \Rightarrow F(k, \Lambda_r x_0) = \Lambda_r^{d^k} F(k, x_0)$!

Stability properties of D_r-homogeneous maps

Theorem 5 Let (6) be D_r-homogeneous of degree $d > 1$. If the origin is an isolated equilibrium of (6) then it is locally asymptotically stable.

Theorem 6 Let (6) be D_r-homogeneous of degree $d \in (0, 1)$. If the origin is the only equilibrium of (6) then it is globally practically asymptotically stable.



- *Any* positive definite r-homogeneous $V : \mathbb{R}^n \to \mathbb{R}_+$ is a LF
- Hyperexponential convergence rates
- Local approximations + Robustness
- The only condition is the degree of homogeneity

Stabilization of a planar system

Planar dynamics:

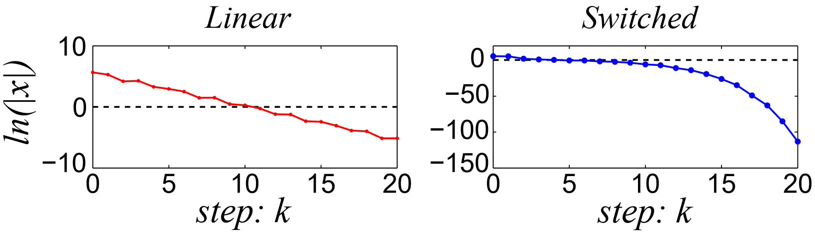
$$x_{1,k+1} = x_{2,k}, \quad x_{2,k+1} = u_k.$$

Nonlinear control:

$$u_k = -a_1 \lceil x_{1,k} \rfloor^{\nu_k^2} - a_2 \lceil x_{2,k} \rfloor^{\nu_k}, \; a_1, a_2 \in (0,1),$$

$$\nu_k = \begin{cases} \frac{2}{3} & V(x_k) \geq 1 \\ \frac{4}{3} & V(x_k) < 1 \end{cases}, \; V(x_k) = \frac{1}{2}|x_{1,k}| + |x_{2,k}|^{\frac{3}{2}}.$$

The closed-loop system is D_r-homogeneous and V is r-homogeneous for $\mathrm{r} = [3, 2]$.



Outline

Control for a chain of integrators IOD

Chain of integrators with a stabilizing controller (Bhat&Bernstein, 2005):

$$\dot{x}_i(t) = x_{i+1}(t),\ i = 1, \ldots, n-1,$$
$$\dot{x}_n(t) = u(t),$$

$$u(t) = -\sum_{i=1}^{n} k_i \lceil x_i(t) \rfloor^{\alpha_i},$$

where k_i form a Hurwitz polynomial and $\alpha_i \in \mathbb{R}_+$ provide homogeneity

Measurements with delays $\tau_i \in (0, \tau_{\max})$, $0 < \tau_{\max} < +\infty$:

$$\tilde{u}(t) = -\sum_{i=1}^{n} k_i \lceil x_i(t - \tau_i) \rfloor^{\alpha_i}$$

- LAS for some $0 < \tau_{\max} < +\infty$ if degree ≥ 0
- bounded trajectories for any $0 < \tau_{\max} < +\infty$ if degree < 0

Estimation for a chain of integrators IOD

$$\begin{aligned} \dot{x}_i(t) &= x_{i+1}(t),\ i=1,\dots,n; \\ \dot{x}_n(t) &= u(t), \\ y(t) &= x_1(t-\tau) \end{aligned}$$

Homogeneous observer (Perruquetti, Floquet, Moulay, 2008):

$$\begin{aligned} \dot{\hat{x}}_i(t) &= \hat{x}_{i+1}(t) - k_i \lceil \hat{x}_1(t) - y(t) \rfloor^{1+(i-1)\nu},\ i=1,\dots,n-1; \\ \dot{\hat{x}}_n(t) &= u(t) - k_n \lceil \hat{x}_1(t) - y(t) \rfloor^{1+(n-1)\nu}, \end{aligned}$$

k_i form a Hurwitz polynomial and $\nu \geq -\frac{1}{n-1}$ is the degree of homogeneity

GAS for $\tau = 0$ and $\nu < 0 \Longrightarrow \forall \tau > 0$ the estimation error stays bounded

Outline

- Verification of homogeneity: algebraic operations (weighted) or differentiation (geometric)
- An "*intermediate*" class of systems between linear and non-linear: local $\equiv$ global
- Local homogeneity: stability/instability in large $\Longleftrightarrow$ analysis at the origin of a simplified system
- Robustness: ISpS, ISS and iISS $\Longleftrightarrow$ GAS + degree constraints
- Robustness to delays