# Energy method for nonlocal Lévy operators and stability of weak solutions

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# Outline

1. Characterization of a symmetric Lévy operator
2. Nonlocal function spaces
3. Integro-Differential Equations (IDEs)
4. Convergence from elliptic IDEs to elliptic PDEs

# Part I:
# Characterization of a symmetric Levy operator

# Motivation

Let $\nu : \mathbb{R}^d \to [0,\infty]$ and $u : \mathbb{R}^d \to \mathbb{R}$ be measurable. Define the energy

$$\mathcal{Q}_\nu(u) = \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |u(x)-u(y)|^2 \nu(x-y)\mathrm{d}y\mathrm{d}x.$$

There are two observations governing the expression of $\mathcal{Q}_\nu(u)$.

- (**Symmetry**) Denote the symmetric of $\nu$ by $\nu^{sym}(h) = \frac{1}{2}\big(\nu(h)+\nu(-h)\big)$ then

$$\mathcal{Q}_\nu(u) = \mathcal{Q}_{\nu^{sym}}(u) = \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |u(x)-u(y)|^2 \nu^{sym}(x-y)\mathrm{d}y\mathrm{d}x.$$

- (**Consistency and Lévy integrability**)

**Theorem 1 (Lévy integrability).**

$$\mathcal{Q}_\nu(u) < \infty \quad \forall\ u \in C_c^\infty(\mathbb{R}^d) \quad \textit{if and only if} \quad \int_{\mathbb{R}^d} (1 \wedge |h|^2)\nu(h)\,\mathrm{d}h < \infty.$$

*where we denote* $\min(1,|h|^2) = 1 \wedge |h|^2$.

More generally we have the following characterization.

**Theorem 2 (Characterization of Lévy integrability).**

*Assume $\nu \geq 0$ is symmetric. Consider*

$$H_\nu(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) : \mathcal{Q}_\nu(u) < \infty\}.$$

*The following assertions are equivalent.*

(i) $\nu$ *is Lévy integrable, i.e.* $\nu \in L^1(\mathbb{R}^d, 1 \wedge |h|^2)$.

(ii) $H^1(\mathbb{R}^d) \hookrightarrow H_\nu(\mathbb{R}^d)$.

(iii) $\mathcal{Q}_\nu(u) < \infty$ *for all* $u \in H^1(\mathbb{R}^d)$.

(iv) $\mathcal{Q}_\nu(u) < \infty$ *for all* $u \in C_c^\infty(\mathbb{R}^d)$.

(v) $H_\nu(\mathbb{R}^d) \neq \{0\}$ ***(if in addition*** $\nu$ ***is radial)***.

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(v) $H_\nu(\mathbb{R}^d) \neq \{0\}$ ***(if in addition*** $\nu$ ***is radial)***.

**Definition 3 (Symmetric Lévy measure).**

The function $\nu : \mathbb{R}^d \setminus \{0\} \to [0, \infty)$, is called the density of a **symmetric Lévy measure** if,

$$\nu(h) = \nu(-h), \quad \forall \ h \neq 0 \text{ and } \quad \int_{\mathbb{R}^d} (1 \wedge |h|^2)\nu(h)\,\mathrm{d}h < \infty. \tag{L}$$

The $\mathcal{Q}_\nu(u)$ is the nonlocal energy associated with **the integrodifferential operator,**

$$Lu(x) := 2\,p.v. \int_{\mathbb{R}^d} (u(x) - u(y))\nu(x-y)\mathrm{d}y \tag{1}$$

$$= p.v. \int_{\mathbb{R}^d} (2u(x) - u(x+h) - u(x-h))\ \nu(h)\mathrm{d}h.. \tag{2}$$

Indeed, the first variation of $u \mapsto \frac{1}{2}\mathcal{Q}_\nu(u)$ on $H_\nu(\mathbb{R}^d)$ gives

$$\mathcal{Q}_\nu(u) = \langle Lu, u\rangle = \frac{1}{2}\mathcal{Q}'_\nu(u)(u), \qquad u \in C_c^\infty(\mathbb{R}^d). \tag{3}$$

**Definition 4 (Symmetric Lévy operator).**

- The integrodifferential operator $u \mapsto Lu$ is called to be **a nonlocal symmetric Lévy operator**.
- The condition ($L$) is known as **the Lévy condition** and the operator $L$ is a **Lévy operator** which arises as the generator of a **pure jumps Lévy process**.

Guy Foghem and Moritz Kassmann.
A General Framework For Nonlocal Neumann Problems.
*Preprint: Arxiv: 2204.06793*, 2022.

## Example: Fractional Laplacian

- It is readily seen that

$$\int_{\mathbb{R}^d} (1 \wedge |h|^2)|h|^{-d-2s} \mathrm{d}h < \infty \qquad \Longleftrightarrow \qquad s \in (0,1).$$

- For $\nu(h) = C_{d,s}|h|^{-d-2s}$, then the operator $L := (-\Delta)^s = (-\Delta)^{\alpha/2}$, $\alpha = 2s$ is the well-known **fractional Laplacian**:

$$(-\Delta)^s u(x) := C_{d,s}\, p.v. \int_{\mathbb{R}^d} \frac{(u(x)-u(y))}{|x-y|^{d+sp}} \mathrm{d}y, \tag{4}$$

where the normalizing $C_{d,s}$ is given by

$$C_{d,s} = \frac{\pi^{d/2}|\Gamma(-s)|}{2^{2s}\Gamma(\frac{d+2s}{2})}.$$

- Whereas $H_\nu(\mathbb{R}^d) = H^s(\mathbb{R}^d)$ is the classical **fractional Sobolev** space.
- It is straightforward to verify the following
  - $\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s}\widehat{u}(\xi)$ for all $u \in C_c^\infty(\mathbb{R}^d)$, where $\widehat{u}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x)\mathrm{d}x$.
  - $(-\Delta)^s u(x) \to -\Delta u(x)$ and $\|u\|_{H^s(\mathbb{R}^d)} \to \|u\|_{H^1(\mathbb{R}^d)}$ as $s \to 1$.
  - The following asymptotic behaviors hold

$$\lim_{s\to 0} \frac{C_{d,s}}{s(1-s)} = \frac{2}{|\mathbb{S}^{d-1}|} \qquad \text{and} \qquad \lim_{s\to 1} \frac{C_{d,s}}{s(1-s)} = \frac{4d}{|\mathbb{S}^{d-1}|}.$$

# Main Goal

Let $\Omega\subset\mathbb{R}^d$ be open bounded, $n(x)$ be the outward normal vector on $\partial\Omega$,

**Laplacce operator**

$$-\Delta u(x)=\operatorname{div}(\nabla u(x)),$$

**Lévy operator**

$$Lu(x)=2\,p.v.\int_{\mathbb{R}^d}(u(x)-u(y))\nu(x-y)\mathrm{d}y.$$

**local normal derivative**

$$\partial_n u(x)=\nabla u(x)\cdot n(x),$$

**nonlocal normal derivative**

$$\mathcal{N}u(x)=2\int_{\Omega}(u(x)-u(y))\nu(x-y)\mathrm{d}y.$$

We study the linear nonlocal Dirichlet and Neumann complement problems

$$Lu=f\quad\text{in }\Omega\quad\text{and}\quad\tau\mathcal{N}u+(1-\tau)u=g\quad\text{on }\Omega^c.\qquad (P_{\tau,\nu})$$

We show that, by rescaling argument one recovers **the local counterpart problems**

$$-\Delta u=f\quad\text{in }\Omega\quad\text{and}\quad\tau\partial_n u+(1-\tau)u=g\quad\text{on }\partial\Omega.\qquad (P_{\tau})$$

- The case $\tau=0$ corresponds to Dirichlet problems.
- The case $\tau=1$ corresponds to Neumann problems.
- The case $\tau=\frac{1}{2}$ corresponds to Robin problems.

# (Non)local Gauss-Green formula

Assume $\Omega \subset \mathbb{R}^d$ be open, bounded and Lipschitz. For $u, v \in C_c^\infty(\mathbb{R}^d)$

**Local Gauss-Green formula:**

$$-\int_\Omega \Delta u(x)\, v(x)\, \mathrm{d}x = \int_\Omega \nabla u(x) \cdot \nabla v(x) \mathrm{d}x - \int_{\partial\Omega} \partial_n u(x)\, v(x)\, \mathrm{d}\sigma(x). \qquad (G_0)$$

**Nonlocal Gauss-Green formula:**

$$\int_\Omega Lu(x) v(x) \mathrm{d}x = \mathcal{E}(u, v) - \int_{\Omega^c} \mathcal{N} u(y) v(y) \mathrm{d}y \qquad u, v \in C_b^2(\mathbb{R}^d). \qquad (G)$$

Here $\mathcal{E}(\cdot, \cdot)$ is **the nonlocal energy** given by

$$\mathcal{E}(u, v) = \iint\limits_{(\Omega^c \times \Omega^c)^c} (u(x) - u(y))(v(x) - v(y))\nu(x - y) \mathrm{d}x \mathrm{d}y. \qquad (E)$$

Note that $(\Omega^c \times \Omega^c)^c = (\mathbb{R}^d \times \mathbb{R}^d) \setminus (\Omega^c \times \Omega^c)$.

Serena Dipierro, Xavier Ros-Oton, and Enrico Valdinoci.
Nonlocal problems with Neumann boundary conditions.
*Rev. Mat. Iberoam.*, 33(2):377–416, 2017.

Guy Foghem and Moritz Kassmann.
A General Framework For Nonlocal Neumann Problems.
*Preprint: Arxiv: 2204.06793*, 2022.

# Part II:
# Nonlocal functions spaces

- Define the space $H_\nu(\Omega)$ by

$$H_\nu(\Omega) = \Big\{u \in L^2(\Omega)\ : \mathcal{E}_\Omega(u,u) := \iint\limits_{\Omega\Omega} \big|u(x)-u(y)\big|^2 \nu(x-y)\mathrm{d}y\mathrm{d}x < \infty\Big\}$$

equipped with the norm $\|u\|^2_{H_\nu(\Omega)} = \|u\|^2_{L^2(\Omega)} + \mathcal{E}_\Omega(u,u)$.

- We also define $V_\nu(\Omega|\,\mathbb{R}^d)$ and $V_{\nu,\Omega}(\Omega|\,\mathbb{R}^d)$ as follows:

$$V_\nu(\Omega|\,\mathbb{R}^d) = \Big\{u : \mathbb{R}^d \to \mathbb{R} \ \text{meas.}\ :\ u|_\Omega \in L^p(\Omega) \text{ and } \ |u|_{V_\nu(\Omega|\,\mathbb{R}^d)} < \infty\Big\}$$

$$\text{where} \qquad |u|^2_{V_\nu(\Omega|\,\mathbb{R}^d)} := \iint\limits_{\Omega\,\mathbb{R}^d} \big|u(x)-u(y)\big|^2 \nu(x-y)\mathrm{d}y\mathrm{d}x, \tag{5}$$

$$V_{\nu,\Omega}(\Omega|\,\mathbb{R}^d) = \Big\{u \in V_\nu(\Omega|\,\mathbb{R}^d)\ : u = 0 \ \text{ a.e on } \ \mathbb{R}^d \setminus \Omega\Big\}.$$

We equip $V_\nu(\Omega|\,\mathbb{R}^d)$ with the norm

$$\|u\|^2_{V_\nu(\Omega|\,\mathbb{R}^d)} = \|u\|^2_{L^2(\Omega)} + |u|^2_{V_\nu(\Omega|\,\mathbb{R}^d)}.$$

**Theorem 5.**

- *The spaces $H_\nu(\Omega)$ and $V_{\nu,\Omega}(\Omega|\,\mathbb{R}^d)$ are Hilbert spaces.*
- *If $\nu$ is of full support, then the space $V_\nu(\Omega|\,\mathbb{R}^d)$ is also a Hilbert space.*

# Important results

**Theorem 6 (Approximation by smooth functions).**

*Let $\nu$ satisfies ($L$) and let $\Omega \subset \mathbb{R}^d$ be open.*

- $C_c^\infty(\mathbb{R}^d)$ *is dense in* $H_\nu(\mathbb{R}^d)$.
- $C^\infty(\Omega) \cap H_\nu(\Omega)$ *is dense in* $H_\nu(\Omega)$ (***Meyer-Serrin type***).
- *If* $\partial\Omega$ *is compact and continuous, then* $C_c^\infty(\overline{\Omega})$ *is dense in* $H_\nu(\Omega)$.
- *If* $\partial\Omega$ *is compact and continuous, then* $C_c^\infty(\Omega)$ *is dense in* $V_{\nu,\Omega}(\Omega|\,\mathbb{R}^d)$.
- *If* $\partial\Omega$ *is compact and Lipschitz, then* $C_c^\infty(\mathbb{R}^d)$ *is dense in* $V_\nu(\Omega|\,\mathbb{R}^d)$.

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- *If* $\partial\Omega$ *is compact and continuous, then* $C_c^\infty(\Omega)$ *is dense in* $V_{\nu,\Omega}(\Omega|\,\mathbb{R}^d)$.
- *If* $\partial\Omega$ *is compact and Lipschitz, then* $C_c^\infty(\mathbb{R}^d)$ *is dense in* $V_\nu(\Omega|\,\mathbb{R}^d)$.

**Theorem 7 (Local Compactness).**

*Let $\nu$ satisfies ($L$) and let $\Omega \subset \mathbb{R}^d$ be open.*

- $\nu \notin L^1(\mathbb{R}^d) \iff H_\nu(\mathbb{R}^d) \hookrightarrow L^2_{loc}(\mathbb{R}^d)$ *is compact.*
- *If* $\nu \notin L^1(\mathbb{R}^d)$ *then* $H_\nu(\Omega) \hookrightarrow L^2_{loc}(\Omega)$ *is compact.*
- *If* $\nu \notin L^1(\mathbb{R}^d)$ *and* $\Omega$ *is* ***bounded*** *then* $V_{\nu,\Omega}(\Omega|\,\mathbb{R}^d) \hookrightarrow L^2(\Omega)$ *is compact.*

Guy Foghem.
$L^2$-Theory for nonlocal operators on domains
*PhD Thesis*, 2020.

## Traces spaces of $V_\nu(\Omega|\mathbb{R}^d)$

- Trace of $V_\nu(\Omega|\mathbb{R}^d)$ is defined by $T_\nu(\Omega^c) = \{u|_{\Omega^c} : u \in V_\nu(\Omega|\mathbb{R}^d)\}$ equipped with

$$\|u\|_{T_\nu(\Omega^c)} = \inf\{\|v\|_{V_\nu(\Omega|\mathbb{R}^d)} : u = v|_{\Omega^c}\}.$$

- Define the weights $\widetilde{\nu}, \overline{\nu} : \mathbb{R}^d \to [0,\infty)$ by

$$\widetilde{\nu}(x) = \int_\Omega (1 \wedge \nu(x-y))\mathrm{d}y \qquad \text{and} \qquad \overline{\nu}(x) = \operatorname*{ess\,inf}_{y\in\Omega} \nu(x-y).$$

**Theorem 8 (Nonlocal Trace Theorem I).**

*Let $\omega \in \{\widetilde{\nu}, \overline{\nu}\}$ then we have* $V_\nu(\Omega|\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d, \omega)$.

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**Theorem 9 (Nonlocal Trace Theorem II).**

- $\mathrm{Tr} : V_\nu(\Omega|\mathbb{R}^d) \to L^2(\Omega^c, \omega)$ *with* $\mathrm{Tr}\, u = u|_{\Omega^c}$, *is linear and continuous*
- $V_{\nu,\Omega}(\Omega|\mathbb{R}^d) \hookrightarrow V_\nu(\Omega|\mathbb{R}^d) \overset{\mathrm{Tr}}{\hookrightarrow} T_\nu(\Omega^c) \hookrightarrow L^2(\Omega^c, \omega)$.
- $\ker \mathrm{Tr} = V_{\nu,\Omega}(\Omega|\mathbb{R}^d)$ *and* $\mathrm{Im}\,\mathrm{Tr} = T_\nu(\Omega^c)$.

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- $V_{\nu,\Omega}(\Omega|\,\mathbb{R}^d) \hookrightarrow V_\nu(\Omega|\,\mathbb{R}^d) \overset{\mathrm{Tr}}{\hookrightarrow} T_\nu(\Omega^c) \hookrightarrow L^2(\Omega^c, \omega)$.
- $\ker \mathrm{Tr} = V_{\nu,\Omega}(\Omega|\,\mathbb{R}^d)$ *and* $Im\, \mathrm{Tr} = T_\nu(\Omega^c)$.

**Remark 1 ($\omega$ and $1 \wedge \nu$ are comparable).**

*If $\nu$ is **unimodal**, i.e., $\nu$ is radial and there is $c > 0$ such that $|x| \le |y| \implies \nu(y) \le c\nu(x)$, and $\nu$ is **doubling**, i.e. $\nu(2r) \le c'\nu(r)$ for all $r \ge 1$. Then we have*

$$\widetilde{\nu} \asymp \overline{\nu} \asymp 1 \wedge \nu. \tag{6}$$

# Part III
# Poincaré type inequalities and Well-posedness

**Theorem 10 (Local Poincaré inequality).**

*Let $\Omega$ be* ***open, bounded, connected with Lipschitz boundary****. There exists $C = C(d, p, \Omega) > 0$ such that*

$$\|u - ⨍_\Omega u\|^2_{L^2(\Omega)} \le C \int_\Omega |\nabla u(x)|^2 \mathrm{d}x \qquad \text{for all } u \in H^1(\Omega).$$

**Theorem 11 (Nonlocal Poincaré inequality).**

*Let $\Omega$ be* ***bounded*** *. Assume $\nu$ has full support and is unimodal, i.e., $\exists c > 0$ sch that $|y| \ge |x| \implies \nu(x) \ge c\nu(y)$. There exists $C = C(d, p, \Omega, \nu) > 0$*

$$\|u - ⨍_\Omega u\|^2_{L^2(\Omega)} \le C \iint_{\Omega\Omega} |u(x) - u(y)|^2 \nu(x-y) \mathrm{d}y\mathrm{d}x \quad \text{for all } u \in L^2(\Omega), \qquad (P_\Omega)$$

$$\|u - ⨍_\Omega u\|^2_{L^2(\Omega)} \le C\mathcal{E}(u, u) \qquad \text{for all } u \in V_\nu(\Omega|\mathbb{R}^d). \qquad (P)$$

*Inequality $(P_\Omega)$ remains true if $\Omega$ is* ***convex*** *and $\nu$* ***does not have full support****.*

*Note that $(P_\Omega) \implies (P)$.*

- Ex. Theorem 11 is true for $\nu(h) = (1-s)|h|^{-d-sp}$ or $\nu(h) = \mathbb{1}_{B_1(0)}(h)$.

- **Nonlocal Gauss-Green formula:** Recall that

$$\int_\Omega Lu(x)v(x)\mathrm{d}x = \mathcal{E}(u,v) - \int_{\Omega^c} \mathcal{N}u(y)v(y)\mathrm{d}y \qquad \text{for all} \quad u,v \in C_c^\infty(\mathbb{R}^d) \tag{7}$$

**Theorem 12 (Nonlocal Neumann problem).**

*Assume the* ***Poincaré inequality*** *holds,* $f \in L^2(\Omega)$ *and* $g \in L^2(\Omega^c, \omega^{-1})$.

- *There exists a unique* $u_* \in V_\nu(\Omega|\mathbb{R}^d)^\perp = \{v \in V_\nu(\Omega|\mathbb{R}^d) : ⨍_\Omega v = 0\}$ *such that*

$$\mathcal{E}(u,v) = \int_\Omega f(x)v(x)\mathrm{d}x + \int_{\Omega^c} g(y)v(y)\mathrm{d}y \quad \text{for all} \;\; v \in V_\nu(\Omega|\mathbb{R}^d)^\perp . \tag{$V'$}$$

- *The Neumann problem* $Lu = f$ *in* $\Omega$ *and* $\mathcal{N}u = g$ *on* $\Omega^c$ *has a weak solution* $u \in V_\nu(\Omega|\mathbb{R}^d)$, *i.e.,*

$$\mathcal{E}(u,v) = \int_\Omega f(x)v(x)\mathrm{d}x + \int_{\Omega^c} g(y)v(y)\mathrm{d}y, \quad \text{for all} \;\; v \in V_\nu(\Omega|\mathbb{R}^d), \tag{$V$}$$

*if and only if* $u = u_* + c$ *with* $c \in R$ *and* $f$ *and* $g$ *are* ***compatible***, *that is*

$$\int_\Omega f(x)\mathrm{d}x + \int_{\Omega^c} f(y)\mathrm{d}y = 0.$$

- *Moreover, every solution* $u$ *satisfies the weak regularity estimate*

$$\|u - ⨍_\Omega u\|_{V_\nu(\Omega|\mathbb{R}^d)} \le C(d,\Omega,\nu)\big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega^c,\omega^{-1})}\big) . \tag{8}$$

**Theorem 13 (Nonlocal Poincaré-Friedrichs inequality).**

*Assume $|\Omega| < \infty$ or $\Omega$ is* ***bounded in one direction*** *say $\Omega \subset (-R, R) \times \mathbb{R}^{d-1}$, $R > 0$.*

$$\nu \not\equiv 0 \iff \|u\|^2_{L^2(\Omega)} \leq C\mathcal{E}(u,u) \qquad \text{for all } u \in L^2_\Omega(\mathbb{R}^d). \tag{P-F}$$

*where $L^2_\Omega(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) : u = 0 \text{ a.e. on } \Omega^c\}$.*
*In other words $\nu \not\equiv 0$, i.e., $|\{\nu > 0\}| > 0$* ***if and only if*** *there is $C = C(d, R, \nu) > 0$ s.t.*

$$\|u\|^2_{L^2(\Omega)} \leq C\mathcal{E}(u,u) \qquad \text{for all } u \in L^2_\Omega(\mathbb{R}^d). \tag{P-F}$$

**Theorem 14 (Nonlocal Dirichlet problem).**

*Assume $\Omega$ is one direction or $|\Omega| < \infty$, $f \in L^2(\Omega)$ and $g \in T_\nu(\Omega^c)$.*

- *The Dirichlet problem $Lu = f$ in $\Omega$ and $u = g$ on $\Omega^c$ has a* ***unique weak*** *solution $u \in V_\nu(\Omega|\mathbb{R}^d)$, i.e.,*

$$u - g \in V_{\nu,\Omega}(\Omega|\mathbb{R}^d) \quad \text{and} \quad \mathcal{E}(u,v) = \int_\Omega f(x)v(x)\mathrm{d}x \quad \text{for all} \quad v \in V_{\nu,\Omega}(\Omega|\mathbb{R}^d). \tag{$V_0$}$$

- *Moreover, $u$ satisfies the weak regularity estimate*

$$\|u\|_{V_\nu(\Omega|\mathbb{R}^d)} \leq C(d,\Omega,\nu)\big(\|f\|_{L^2(\Omega)} + \|g\|_{T_\nu(\Omega^c)}\big). \tag{9}$$

# Part IV:
# Convergence of IDEs: From nonlocal to local

- Let $(\nu_\alpha)_{\alpha\in(0,2)}$ be a family of functions such that for every $\alpha$, and $\delta > 0$

$$\nu_\alpha \geq 0 \text{ is radial}, \quad \int_{\mathbb{R}^d} (1 \wedge |h|^2)\nu_\alpha(h)\mathrm{d}h = 1, \quad \lim_{\alpha\to 2} \int_{|h|>\delta} (1 \wedge |h|^2)\nu_\alpha(h)\mathrm{d}h = 0\,. \tag{10}$$

- For symmetric kernels $J^\alpha : \mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag} \to [0,\infty), \alpha \in (0,2)$, we set the conditions:

(E) **Elliptic condition:** There exists a constant $\Lambda \geq 1$ such that for every $\alpha \in (0,2)$ and all $x, y \in \mathbb{R}^d$, with $x \neq y$

$$\Lambda^{-1}\nu_\alpha(x-y) \leq J^\alpha(x,y) \leq \Lambda\nu_\alpha(x-y). \tag{E}$$

(I) For each $\alpha \in (0,2)$ the kernel $J^\alpha$ is **translation invariant**, i.e., for every $h \in \mathbb{R}^d$

$$J^\alpha(x+h, y+h) = J^\alpha(x,y). \tag{I}$$

- Introduce the Lévy type Integro-differential operator $L_\alpha$ and $\mathcal{N}_\alpha$ the nonlocal normal derivative across $\Omega$ defined by

$$L_\alpha u(x) = p.v.2\int_{\mathbb{R}^d} (u(x)-u(y))J^\alpha(x,y)\mathrm{d}y, \quad (x \in \mathbb{R}^d) \tag{11}$$

$$\mathcal{N}_\alpha u(x) = 2\int_\Omega (u(x)-u(y))J^\alpha(x,y)\mathrm{d}y, \qquad (x \in \Omega^c). \tag{12}$$

- We define **the symmetric matrix** $A(x) = (a_{ij}(x))_{1\le i,j\le d}$ by

$$a_{ij}(x) = \lim_{\alpha\to 2}\int_{B_\delta(0)} h_i h_j J^\alpha(x, x+h)dh \quad \text{for } x \in \mathbb{R}^d \text{ and } \delta > 0. \tag{13}$$

- The condition $(E)$, implies that the matrix $A(x) = (a_{ij}(x))_{1\le i,j\le d}$, is **elliptic** and,

$$d^{-1}\Lambda^{-1}|\xi|^2 \le A(x)\xi\cdot\xi \le d^{-1}\Lambda|\xi|^2 \quad \text{for every } x, \xi \in \mathbb{R}^d. \tag{14}$$

- Under the condition (I), the matrix $A$ is constant. In particular

• for $J^\alpha(x,y) = \nu_\alpha(x-y)$ we have $A(x) = \frac{1}{d}(\delta_{ij})_{ij} = \frac{1}{d}I_d$.

• for $J^\alpha(x,y) = \nu_\alpha(x-y) = C_{d,\alpha}|x-y|^{-d-\alpha}$ we have $A(x) = (\delta_{ij})_{ij} = I_d$ and $L_\alpha = (-\Delta)^{\alpha/2}$ is **the fractional Laplacian**

**Example 15 (Rescaled kernel $\nu$).**

Let $\nu : \mathbb{R}^d\setminus\{0\} \to (0,\infty)$ be radial and define $(\nu_\alpha)_\alpha$, $\nu_\alpha = \nu^{2-\alpha}$ and

$$\nu^\varepsilon(h) = \begin{cases} \varepsilon^{-d-2}\nu(h/\varepsilon) & \text{if } |h| \le \varepsilon \\ \varepsilon^{-d}|h|^{-2}\nu(h/\varepsilon) & \text{if } \varepsilon < |h| \le 1 \\ \varepsilon^{-d}\nu(h/\varepsilon) & \text{if } |h| > 1, \end{cases} \quad \text{provided} \quad \int_{\mathbb{R}^d}(1\wedge|h|^2)\,\nu(h)\mathrm{d}h = 1.$$

- **Nonlocal Gauss-Green formula:**

$$\int_{\Omega} L_\alpha u(x)v(x)\mathrm{d}x = \mathcal{E}^\alpha(u,v) - \int_{\Omega^c} \mathcal{N}_\alpha u(y)v(y)\mathrm{d}y \qquad u,v \in C_c^\infty(\mathbb{R}^d). \qquad (G_\alpha)$$

Recall that: $(\Omega^c \times \Omega^c)^c = (\mathbb{R}^d \times \mathbb{R}^d) \setminus (\Omega^c \times \Omega^c)$,

$$\mathcal{E}^\alpha(u,v) = \iint\limits_{(\Omega^c \times \Omega^c)^c} (u(x)-u(y))(v(x)-v(y))J^\alpha(x,y)\mathrm{d}y\mathrm{d}x \qquad (15)$$

$$L_\alpha u(x) = 2\,\mathrm{p.v.}\int_{\mathbb{R}^d} (u(x)-u(y))J^\alpha(x,y)\mathrm{d}y, \qquad (x \in \mathbb{R}^d) \qquad (16)$$

$$\mathcal{N}_\alpha u(x) = 2\int_{\Omega} (u(x)-u(y))J^\alpha(x,y)\mathrm{d}y, \qquad (x \in \Omega^c) \qquad (17)$$

- **Local Gauss-Green formula:**

$$-\int_{\Omega} \mathrm{div}(A(\cdot)\nabla u)v\,\mathrm{d}x = \mathcal{E}^A(u,v) - \int_{\partial\Omega} \frac{\partial u}{\partial n_A} v\,\mathrm{d}\sigma(x) \qquad u,v \in C^2(\mathbb{R}^d). \qquad (G_0)$$

here $\frac{\partial u(x)}{\partial n_A} = A(x)\nabla u(x)\cdot n(x)$ **is the outer normal derivative of** $u$ **on** $\partial\Omega$ **w.r.t.** $A$ and

$$\mathcal{E}^A(u,v) = \int_{\Omega} A(x)\nabla u(x)\cdot\nabla v(x)\mathrm{d}x. \qquad (18)$$

**Theorem 16 (Bourgain-Brezis-Mironescu type result).**

*Let $D \subset \mathbb{R}^d$ be an extension domain. For all $u \in L^2(D)$ we have*

$$\lim_{\alpha \to 2} \iint_{DD} (u(x) - u(y))^2 J^\alpha(x, y) \mathrm{d}y \mathrm{d}x = \int_D A(x) \nabla u(x) \cdot \nabla v(x) \mathrm{d}x,$$

$$\lim_{\alpha \to 2} \iint_{DD} (u(x) - u(y))^2 \nu_\alpha(x - y) \mathrm{d}y \mathrm{d}x = K_{d,2} \int_D |\nabla u(x)|^2 \mathrm{d}x \qquad K_{d,2} = \frac{1}{d}.$$

**Theorem 17 (Direct consequence).**

*Assume $\Omega \subset \mathbb{R}^d$ is bounded Lipschitz: Then $\mathcal{E}^\alpha(u, u) \to \mathcal{E}^A(u, u)$ as $\alpha \to 2$.*

**proof:** We have $(\Omega^c \times \Omega^c)^c = (\mathbb{R}^d \times \mathbb{R}^d) \setminus (\Omega^c \times \Omega^c)$ and $|\partial\Omega| = 0$ so that

$$\begin{aligned}\lim_{\alpha \to 2} \mathcal{E}^\alpha(u, u) &= \lim_{\alpha \to 2} \iint_{(\Omega^c \times \Omega^c)^c} (u(x) - u(y))^2 J^\alpha(x, y) \mathrm{d}y \mathrm{d}x \\ &= \lim_{\alpha \to 2} \Big[ \iint_{\mathbb{R}^d \mathbb{R}^d} - \iint_{\Omega^c \Omega^c} \Big] (u(x) - u(y))^2 J^\alpha(x, y) \mathrm{d}y \mathrm{d}x \\ &= \Big[ \int_{\mathbb{R}^d} - \int_{\Omega^c} \Big] A(x) \nabla u(x) \cdot \nabla v(x) \mathrm{d}x = \int_\Omega A(x) \nabla u(x) \cdot \nabla v(x) \mathrm{d}x.\end{aligned}$$

**Theorem 18 (BBM '01 & Augusto Ponce '04 & GF '20, Asymptotic compactness).**

*Assume* $\Omega \subset \mathbb{R}^d$ *is open, bounded and Lipschitz. Let the family* $(u_\alpha)_\alpha$ *such that*

$$\sup_{\alpha\in(0,2)} \Big( \|u_\alpha\|^2_{L^2(\Omega)} + \iint\limits_{\Omega\Omega} |u_\alpha(x) - u_\alpha(y)|^2 \nu_\alpha(x-y)\mathrm{d}y\mathrm{d}x \Big) < \infty.$$

*There is a subsequence* $\alpha_n \to 2$ *as* $n \to \infty$ *and* $u \in H^1(\Omega)$ *s.t.* $\|u_{\alpha_n} - u\|_{L^2(\Omega)} \xrightarrow{n\to\infty} 0.$

As a direct consequence of this we have.

**Theorem 19 (Robust Poincaré inequality).**

*There exist* $C > 0$ *and* $\alpha_0 \in (0,2)$ *such that, for all* $u \in L^2(\Omega)$ *and* $\alpha \in (\alpha_0, 2)$ *we have*

$$\|u - ⨍_\Omega u\|^2_{L^2(\Omega)} \le C \iint\limits_{\Omega\Omega} |u(x) - u(y)|^2 \nu_\alpha(x-y)\mathrm{d}y\mathrm{d}x,$$


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An estimate in the spirit of Poincaré's inequality.
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Some spin-offs:

- Therefore, for all $u \in H^1(\mathbb{R}^d)$,

$$\lim_{\alpha\to 2} \mathcal{E}^{\alpha}(u,u) = \mathcal{E}^{A}(u,u). \tag{19}$$

- If $(J^{\alpha})_{\alpha}$ satisfies thee translation invariant condition (I), one can show

$$\lim_{\alpha\to 2} L_{\alpha}\varphi(x) = -\operatorname{div}(A(x)\nabla\varphi(x)) \qquad \varphi \in C_b^2(\mathbb{R}^d). \tag{20}$$

- The above implies that for $v \in H^1(\mathbb{R}^d)$, $g_{\alpha} = \mathcal{N}_{\alpha}\varphi$ and $g = \frac{\partial\varphi}{\partial n_A}$,

$$\int_{\partial\Omega} g(x)v(x)\mathrm{d}\sigma(x) = \lim_{\alpha\to 2} \int_{\Omega^c} g_{\alpha}(y)v(y)\mathrm{d}y. \tag{21}$$

Indeed, combining **the local and the nonlocal Gauss-Green** formula gives

$$\begin{aligned}\int_{\partial\Omega} \frac{\partial\varphi(x)}{\partial n_A} v(x)\mathrm{d}\sigma(x) &= \mathcal{E}^{A}(\varphi, v) + \int_{\Omega} \operatorname{div}(A(x)\nabla\varphi(x))v(x)\mathrm{d}x \\ &= \lim_{\alpha\to 2} \mathcal{E}^{\alpha}(\varphi, u) - \int_{\Omega} L_{\alpha}\varphi(x)v(x)\mathrm{d}x \\ &= \lim_{\alpha\to 2} \int_{\Omega^c} \mathcal{N}_{\alpha}\varphi(y)v(y)\mathrm{d}y \qquad g_{\alpha} = \mathcal{N}_{\alpha}\varphi.\end{aligned}$$

- Taking $J^{\alpha}(x,y) = C_{d,\alpha}|x-y|^{-d-\alpha}$ gives $L_{\alpha} = (-\Delta)^{\alpha/2}$, $-\operatorname{div}(A\nabla) = -\Delta$ and

$$\frac{\partial\varphi(x)}{\partial n_A} = \frac{\partial\varphi(x)}{\partial n} = \nabla\varphi(x)\cdot n(x).$$

▶ $(G_\alpha) \to (G_0)$ as $\alpha \to 2$, i.e., letting $\alpha \to 2$ in the nonlocal Gauss-Green formula $(G_\alpha)$

$$\int_\Omega L_\alpha u(x)v(x)\mathrm{d}x = \mathcal{E}^\alpha(u,v) - \int_{\Omega^c} \mathcal{N}_\alpha u(y)v(y)\mathrm{d}y \tag{22}$$

one recovers the local local Gauss-Green formula $(G_0)$

$$-\int_\Omega \operatorname{div}(A(x)\nabla u(x))v(x)\mathrm{d}x = \mathcal{E}^A(u,v) - \int_{\partial\Omega} \frac{\partial u(x)}{\partial n_A} v(x)\mathrm{d}\sigma(x) \tag{23}$$

Furthermore we have the following more general convergence of the energies forms.

**Theorem 20 (GF/Kassmann/Voigt '19: Mosco convergence).**

*Define* $H_{\nu_\alpha}(\Omega) = \{u \in L^2(\Omega) : \mathcal{E}^\alpha_\Omega(u,u) < \infty\}$ *where*

$$\mathcal{E}^\alpha_\Omega(u,u) = \iint_{\Omega\Omega} (u(x)-u(y))^2 J^\alpha(x,y)\mathrm{d}y\mathrm{d}x \tag{24}$$

*Then, as* $\alpha \to 2$, *both nonlocal forms*

$$(\mathcal{E}^\alpha(\cdot,\cdot), V_{\nu_\alpha}(\Omega|\mathbb{R}^d))_\alpha, (\mathcal{E}^\alpha_\Omega(\cdot,\cdot), H_{\nu_\alpha}(\Omega))_\alpha \xrightarrow{\text{Mosco converge}} (\mathcal{E}^A(\cdot,\cdot), H^1(\Omega)).$$

▶ *Note that* ***Mosco convergence*** *implies the* ***Gamma convergence.***

## Theorem 21 (Convergence of Neumann problem).

*Let $\Omega$ be bounded Lipschitz and connected. Assume $(f_\alpha)_\alpha \rightharpoonup f$ (weakly) in $L^2(\Omega)$. Define $g_\alpha = \mathcal{N}_\alpha \varphi$ and $g = \frac{\partial \varphi}{\partial n_A}$ with $\varphi \in C_b^2(\mathbb{R}^d)$. Assume the elliptic condition ($E$), and $u_\alpha \in V_{\nu_\alpha}(\Omega|\mathbb{R}^d)^\perp$ satisfies (**weak solution**) the nonlocal Neumann problem*

$$L_\alpha u_\alpha = f_\alpha \quad \text{on} \quad \Omega \quad \text{and} \quad \mathcal{N}_\alpha u_\alpha = g_\alpha \quad \text{on} \quad \Omega^c. \tag{25}$$

*that is, $u_\alpha$ satisfies*

$$\mathcal{E}^\alpha(u_\alpha, v) = \int_\Omega f_\alpha(x) v(x) + \int_{\Omega^c} g_\alpha(x) v(x) \mathrm{d}x \quad \text{for all} \quad v \in V_{\nu_\alpha}(\Omega|\mathbb{R}^d)^\perp. \tag{26}$$

*Let $u \in H^1(\Omega)^\perp$ be the unique weak solution in $H^1(\Omega)^\perp$ of the Neumann problem*

$$-\operatorname{div}(A(\cdot)\nabla u) = f \quad \text{on} \quad \Omega \quad \text{and} \quad \frac{\partial u}{\partial n_A} = g \quad \text{on} \quad \partial\Omega. \tag{27}$$

*Assume that the condition (I) holds or that $g_\alpha = g = 0$ then $(u_\alpha)_\alpha$ converges to $u$ in $L^2(\Omega)$, i.e., $\|u_\alpha - u\|_{L^2(\Omega)} \xrightarrow{\alpha \to 2} 0$.*

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*Preprint: Arxiv: 2204.06793*, 2022.

**Sketch of the proof.**

▶ The **Robust Poincaré inequality** (Theorem 19) implies that, for $0 < \alpha_* < 2$ and $C > 0$,

$$\sup_{\alpha\in(\alpha_*,2)} \Big[\|u_\alpha\|^2_{L^2(\Omega)} + \mathcal{E}^\alpha(u_\alpha, u_\alpha)\Big] \leq C \tag{28}$$

▶ By the compactness (Theorem 18) there exists $\alpha_j \to 2$ and $u \in H^1(\Omega)^\perp$ such that

$$\|u_{\alpha_j} - u\|_{L^2(\Omega)} \xrightarrow{j\to\infty} 0$$

▶ In fact, for fixed $v \in H^1(\mathbb{R}^d)$ one can further establish the weak convergence

$$\mathcal{E}^{\alpha_j}(u_{\alpha_j}, v) \xrightarrow{\alpha_j\to 2} \mathcal{E}^A(u, v). \tag{29}$$

▶ We have previously shown that

$$\int_{\partial\Omega} g(x)v(x)\mathrm{d}\sigma(x) = \lim_{\alpha_j\to 2} \int_{\Omega^c} g_{\alpha_j}(y)v(y)\mathrm{d}y. \tag{30}$$

▶ By assumption $f_\alpha \rightharpoonup f$ in $L^2(\Omega)$.

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**Theorem 22 (Robust Poincaré-Friedrichs inequality).**

*Let $\Omega \subset \mathbb{R}^d$ be* ***bounded in one direction****. There exist $C > 0$ and $\alpha_0 \in (0, 2)$ such that, for all $u \in C_c^\infty(\Omega)$ and $\alpha \in (\alpha_0, 2)$ we have*

$$\|u\|^2_{L^2(\Omega)} \leq C \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |u(x) - u(y)|^2 \nu_\alpha(x - y)\mathrm{d}y\mathrm{d}x,$$

**Theorem 23 (Convergence of Dirichlet problem).**

*Assume $(f_\alpha)_\alpha$ weakly converges to some $f$ in $L^2(\Omega)$ as $\alpha \to 2$ and $g \in H^1(\mathbb{R}^d)$. Under the condition $(E)$, $u_\alpha \in V_{\nu_\alpha}(\Omega|\,\mathbb{R}^d)$ be the (****unique weak sense****) of nonlocal* ***Dirichlet*** *problem*

$$L_\alpha u_\alpha = f_\alpha \quad \text{on} \quad \Omega \quad \text{and} \quad u_\alpha = g \quad \text{on} \quad \Omega^c. \tag{31}$$

*that is we have*

$$u_\alpha - g \in V_{\nu_\alpha,\Omega}(\Omega|\,\mathbb{R}^d) \quad \text{and} \quad \mathcal{E}^\alpha(u_\alpha, v) = \int_\Omega f_\alpha(x)v(x) \quad \text{for all } v \in V_{\nu_\alpha,\Omega}(\Omega|\,\mathbb{R}^d).$$

*Let $u \in H^1(\Omega)$ be the unique* ***weak solution*** *in $H^1(\Omega)$ of the Dirichlet problem*

$$-\operatorname{div}(A\nabla u) = f \quad \text{on} \quad \Omega \quad \text{and} \quad u = g \quad \text{on} \quad \partial\Omega. \tag{32}$$

*Then $(u_\alpha)_\alpha$ converges to $u$ in $L^2_{loc}(\mathbb{R}^d)$.*

Thank You For Your Attention.

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