Shape optimization problems in thermal insulation with thin insulating layers

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Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set, and let $h \colon \partial\Omega \to \mathbb{R}$ be a non-negative function. Denoting by ν the exterior unit normal to the boundary of Ω, we define

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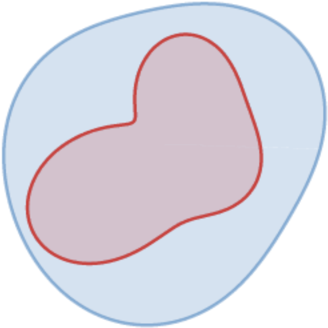

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where $\mathcal{K}, \mathcal{H}$ are suitable class of functions, and

$$\mathcal{F}_\varepsilon(v,h) = \int_\Omega g(v,\nabla v)\,dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla v|^2\,dx + \int_{\partial\Omega_\varepsilon} q(v)\,d\sigma.$$

We will study this problems by means of Γ-limit.

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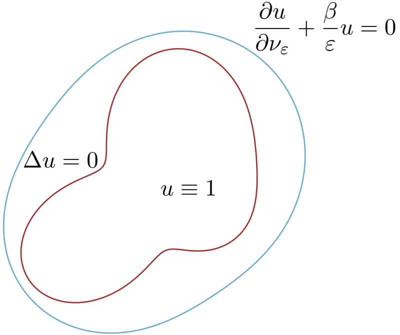

We will see some techniques that we can use in order to prove that for small $\varepsilon > 0$, we can write

$$\min_{v\in\mathcal{K}} \mathcal{F}_\varepsilon(v,h) = \mathcal{F}_0(h) + \varepsilon\mathcal{F}^{(1)}(h) + R(\Omega,h,\varepsilon),$$

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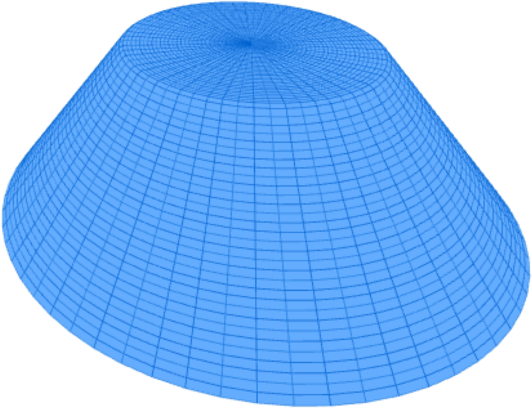
Finally, we will show that, as the intuition suggests, if ε is small enough then the optimal configuration for the insulating layer concentrates near the points of $\partial\Omega$ where the mean curvature is relatively small.

We will also see that the "linear" situation (to be intended with respect to the distance from the boundary $\partial\Omega$) gives hints about the way to approach the problem.



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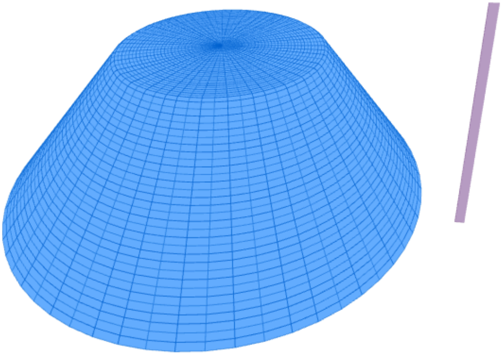

Reinforcement problem

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where $f \in L^2(\Omega)$ is fixed, and $v \in W_0^{1,2}(\Omega_\varepsilon)$.

If we let u_ε be the unique minimizer of $\mathcal{F}_\varepsilon$, then u_ε solves in the weak sense

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega, \\ \Delta u_\varepsilon = 0 & \text{in } \Sigma_\varepsilon, \\ \dfrac{\partial u_\varepsilon^-}{\partial \nu} = \varepsilon \dfrac{\partial u_\varepsilon^+}{\partial \nu} & \text{on } \partial\Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

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If we study the Γ-limit of the functional $\mathcal{F}_\varepsilon$ as $\varepsilon \to 0^+$, we can prove that u_ε converges in $L^2(\Omega)$ to a function u_0 solving

$$\begin{cases} -\Delta u_0 = f & \text{in } \Omega, \\ \gamma(x, u_0) = 0 & \text{on } \partial\Omega, \end{cases}$$

where γ depends on the choice of h_ε.

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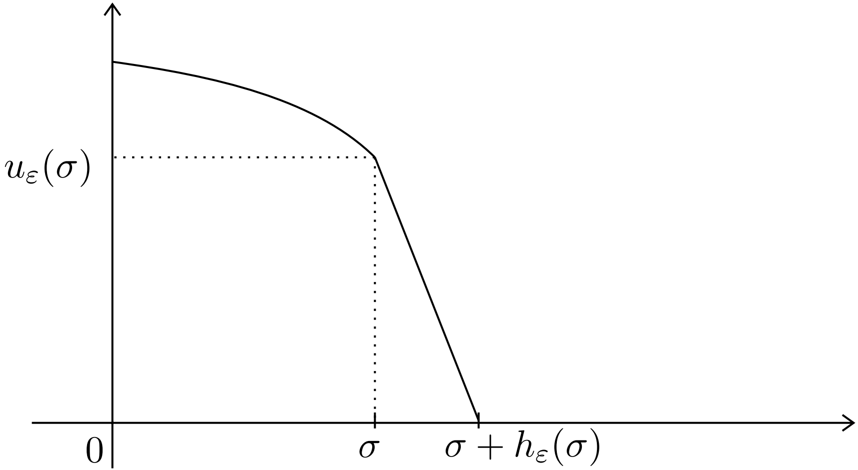
To understand why this happens, let us assume that u_ε is a linear function of the distance from $\partial\Omega$ in Σ_ε, namely: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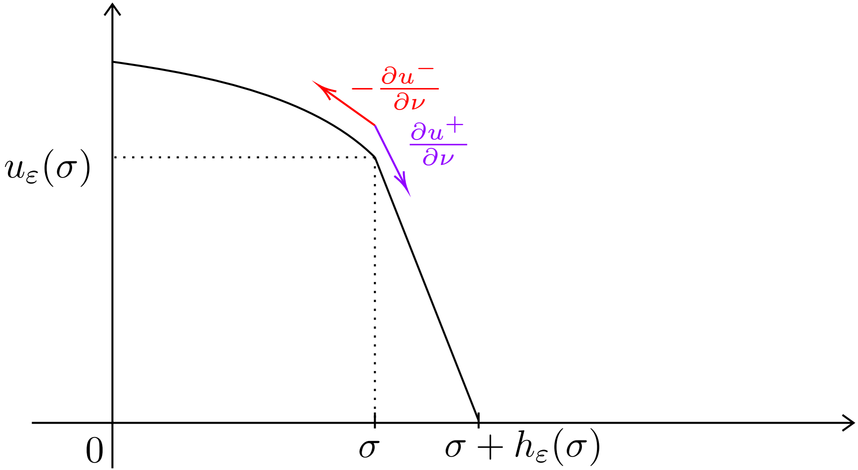
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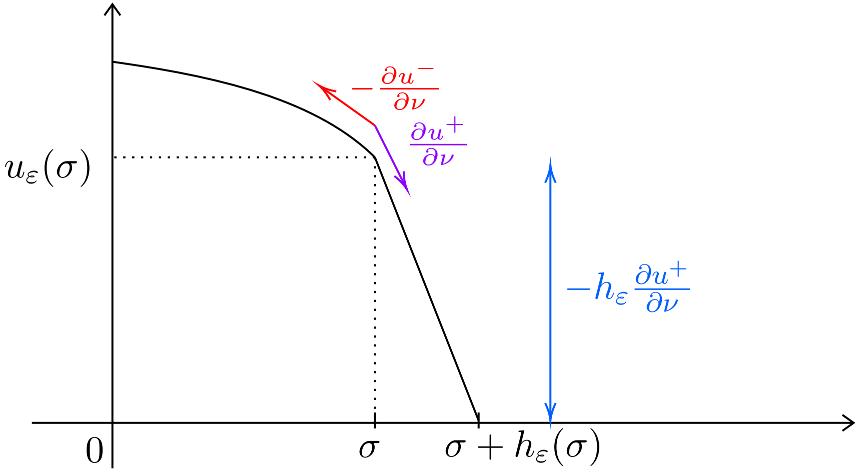
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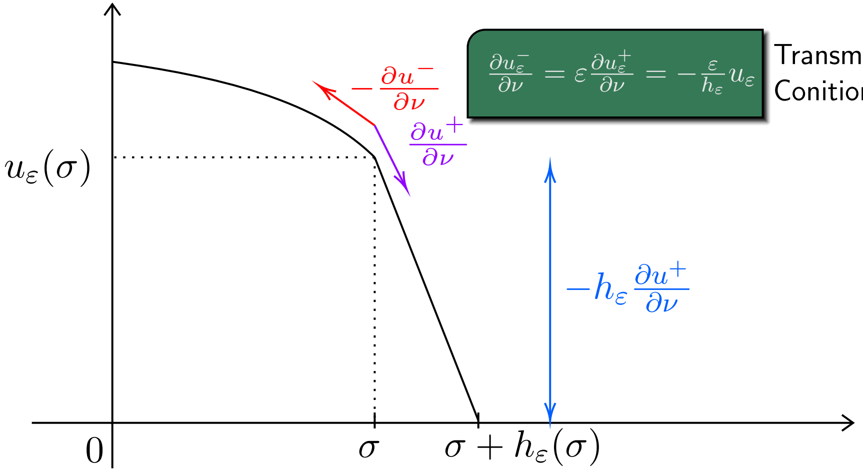

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From now on, we will focus our interest in the case $h_\varepsilon(x) = \varepsilon h(x)$.

Theorem (Acerbi, Buttazzo, *Ann. Inst. H. Poincaré Anal. Non Linéaire* - 1986)

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive Lispchitz function $h \colon \partial\Omega \to \mathbb{R}$.

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It is worth mentioning that this theorem turns to be a particular case of the more general results in E. Acerbi and G. Buttazzo (1986) ["Reinforcement problems in the calculus of variations", *Annales de l'I.H.P. Analyse non linéaire.*]

This kind of problem is often referred to as reinforcement problem because if $f \equiv 1$, we have that

$$\min \left\{ \mathcal{F}_\varepsilon(v,h) \,\middle|\, v \in W_0^{1,2}(\Omega_\varepsilon), h \in \mathcal{H} \right\}$$

is equivalent to maximize the torsional rigidity with a thin reinforcement layer with density $\varepsilon h(x)$:

$$\max_{h \in \mathcal{H}} T(\Omega_\varepsilon)$$

Thermal Insulation problem

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where $f \in L^2(\Omega)$ is fixed, and $v \in W^{1,2}(\Omega_\varepsilon)$.

As before, if we let u_ε be the unique minimizer of $\mathcal{F}_\varepsilon$, then u_ε solves in the weak sense

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Again, we can prove that u_ε converges in $L^2(\Omega)$ to a function u_0 solving

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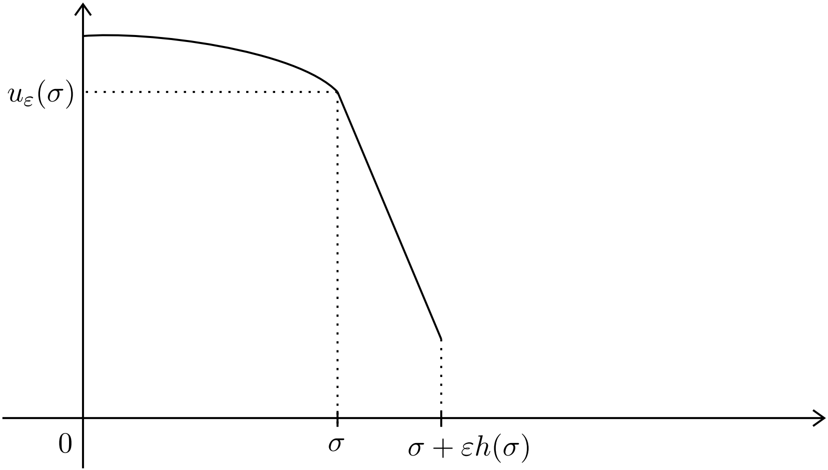
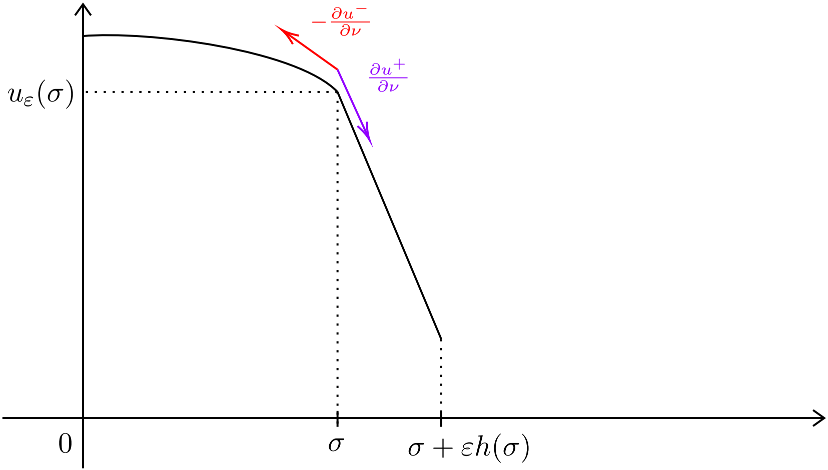
We can gather information by the linear approximation in this case as well: let us assume that for $x \in \Sigma_\varepsilon$, and approximate $\nu_\varepsilon \sim \nu$ (where ν_ε is the outer unit normal to Ω_ε)



$u_\varepsilon(\sigma)$
0
σ
$\sigma + \varepsilon h(\sigma)$



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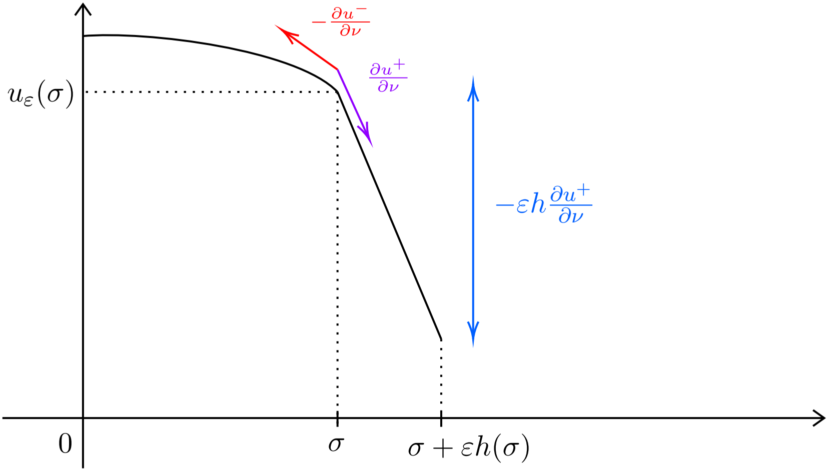
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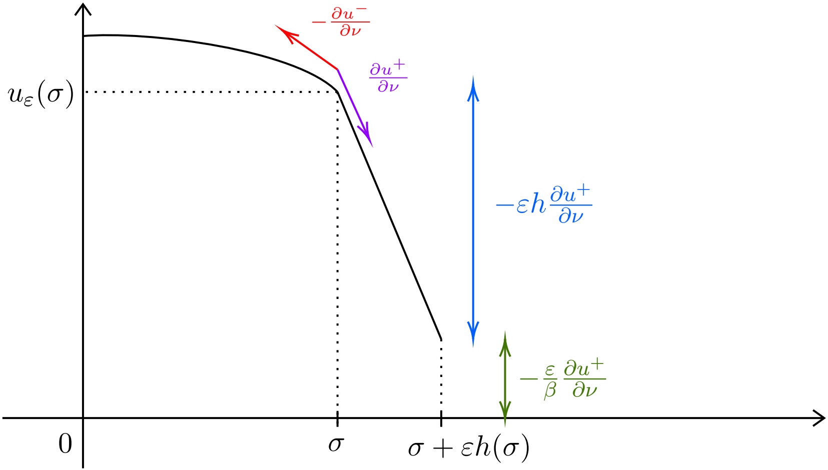

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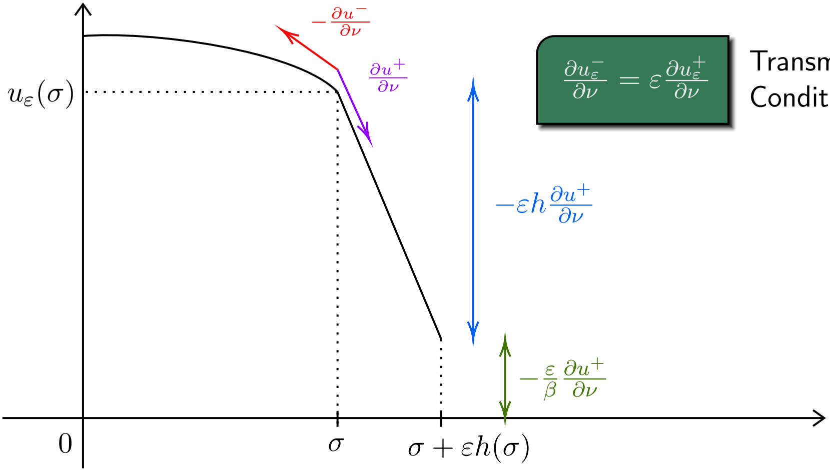

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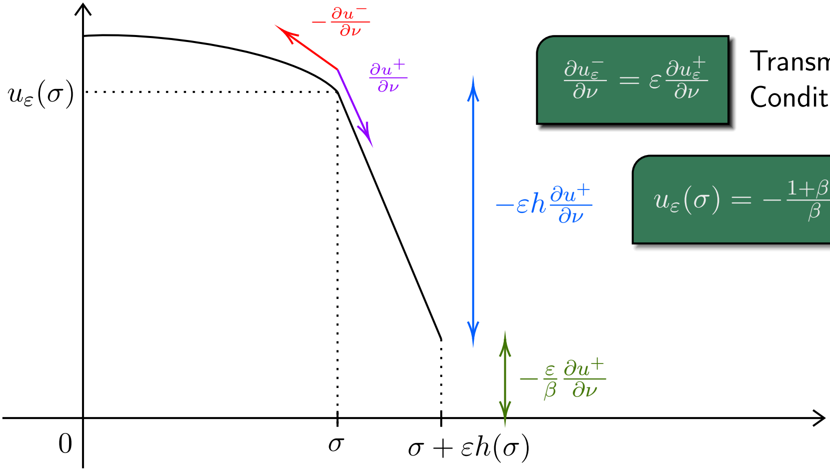
Transmission Condition



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$$u_\varepsilon(\sigma) = -\frac{1+\beta h}{\beta}\frac{\partial u_\varepsilon^-}{\partial \nu}$$

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If we are interested in studying the problem

$$\inf \left\{ \, \mathcal{F}_0(v,h) \,\middle|\, v \in W^{1,2}(\Omega_\varepsilon), h \in \mathcal{H} \, \right\},$$

we may try to solve

$$\inf_{v\in W^{1,2}(\Omega)} \inf_{h\in\mathcal{H}} \mathcal{F}_0.$$

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Let $\beta, m > 0$, and let

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$$h(\sigma) = \begin{cases} \dfrac{1}{\beta}\left(\dfrac{u_0(\sigma)}{c_u} - 1\right) & \text{if } u_0(\sigma) \geq c_u, \\ 0 & \text{otherwise,} \end{cases}$$

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where c_u is the unique positive constant such that $\int_{\partial\Omega} h \, d\mathcal{H}^{n-1} = m$. Finally, if Ω is connected, then (u_0, h) is the unique solution minimizing $\mathcal{F}_0$.

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The previous approximation result is a quite powerful tool to gather information about the non-approximated functional $\mathcal{F}_\varepsilon$. Nevertheless, it fails in other directions.

For instance, let h be constant, and define

$$I_\beta(\Omega, h) = \inf_{\substack{v \in W^{1,2}(\Omega_\varepsilon) \\ v \geq 1 \text{ in } \Omega}} \left\{ \int_{\Omega_\varepsilon} |\nabla v|^2 \, dx + \frac{\beta}{\varepsilon} \int_{\partial\Omega_\varepsilon} v^2 \, d\mathcal{H}^{n-1} \right\},$$

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then we have

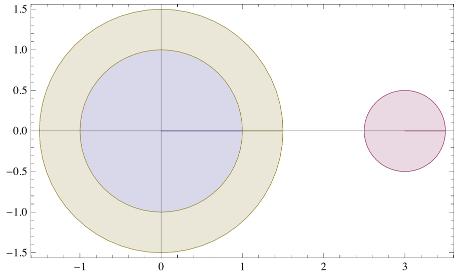
Theorem (Della Pietra, Nitsch, Trombetti, *Mathematische Annalen* - 2022)

Let $\Omega = B_R(0) \subset \mathbb{R}^n$. If

$$\frac{\beta}{\varepsilon} < \frac{n-1}{R},$$

and $0 < m \leq m_0(\varepsilon, R)$, then

$$I_\beta(B_R, h) > I_\beta(B_R, 0).$$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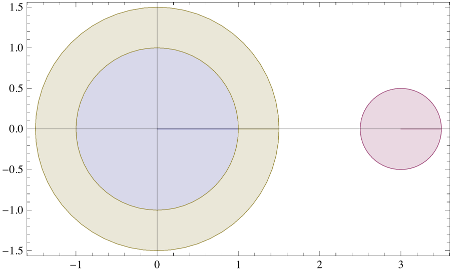
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Thermal insulation: first order approximation

In this last section we let

$$\Sigma_\varepsilon = \left\{ \sigma + t\nu(\sigma) \mid \sigma \in \partial\Omega,\, 0 < t < \varepsilon h(\sigma) \right\},$$
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and we study the functional

$$\mathcal{F}_\varepsilon(v,h) = \begin{cases} \varepsilon \displaystyle\int_{\Sigma_\varepsilon} |\nabla v|^2\,dx + \beta \int_{\partial\Omega_\varepsilon} v^2\,d\mathcal{H}^{n-1} & \text{if } v \in K_\varepsilon, \\ +\infty & \text{if } v \in L^2(\mathbb{R}^n) \setminus K_\varepsilon. \end{cases}$$

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Following the arguments in F. Della Pietra, C. Nitsch, R. Scala, and C. Trombetti (2021) ["An optimization problem in thermal insulation with Robin boundary conditions", *Communications in Partial Differential Equations*.] we have

Theorem 1 (A., Cristoforoni, Nitsch, Trombetti, *In preparation* - 2023)

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive Lispchitz function $h \colon \partial\Omega \to \mathbb{R}$. Then $\mathcal{F}_\varepsilon(\cdot, h)$ Γ-converges, as $\varepsilon \to 0^+$, in the strong $L^2(\mathbb{R}^n)$ topology, to the functional $\mathcal{F}_0(\cdot, h)$.

We now state the main result in A., E. Cristoforoni, C. Nitsch, and C. Trombetti (2023) ["On the optimal shape of a thin insulating layer", *In preparation.*] let

$$\delta\mathcal{F}_\varepsilon(v,h) = \frac{\mathcal{F}_\varepsilon(v,h) - \mathcal{F}_0(h)}{\varepsilon},$$

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$$\delta\mathcal{F}_\varepsilon(v,h) = \frac{\mathcal{F}_\varepsilon(v,h) - \mathcal{F}_0(h)}{\varepsilon},$$

and let

$$\mathcal{F}^{(1)}(v,h) = \begin{cases} \beta \displaystyle\int_{\partial\Omega} \frac{Hh(2+\beta h)}{2(1+\beta h)^2}\, d\mathcal{H}^{n-1} & \text{if } v \in K_0, \\ +\infty & \text{if } v \in L^2(\mathbb{R}^n) \setminus K_0. \end{cases}$$

Theorem 2 (A., Cristoforoni, Nitsch, Trombetti, *In preparation* - 2023)

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with C^3 boundary, and fix a positive C^2 function $h\colon \partial\Omega \to \mathbb{R}$. Then $\delta\mathcal{F}_\varepsilon(\cdot, h)$ Γ-converges, as $\varepsilon \to 0^+$, in the strong $L^2(\mathbb{R}^n)$ topology, to the functional $\mathcal{F}^{(1)}(\cdot, h)$.

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The proof of the theorem relies on finding suitable supersolution and subsolution to the Euler-Lagrange equation, in order to approximate the linear function (with respect to the distance) satisfying the Robin condition. This theorem allows us to approximate

$$\min_{v \in K_\varepsilon} \mathcal{F}_\varepsilon(v, h) \sim \mathcal{F}_0(h) + \varepsilon \mathcal{F}^{(1)}(h)$$

We now focus on the functional

$$\mathcal{G}_\varepsilon(\Omega, h) = \beta \int_{\partial\Omega} \left(\frac{1}{1+\beta h} + \varepsilon H \frac{h(2+\beta h)}{2(1+\beta h)^2} \right) \, d\mathcal{H}^{n-1}.$$

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In particular, we are interested in the problem

$$\min \left\{ \mathcal{G}_\varepsilon(h) \;\middle|\; h \in L^1(\partial\Omega),\ h \geq 0,\ \int_{\partial\Omega} h \, d\mathcal{H}^{n-1} \leq m \right\}. \tag{1}$$

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$$\mu_k(\sigma) = \begin{cases} \dfrac{1}{\beta}(y_k(\sigma) - 1) & \text{if } H(\sigma) < 1 - k. \\[2ex] 0 & \text{if } H(\sigma) \geq 1 - k, \end{cases}$$

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where $y_k(\sigma)$ is the root of a suitable polynomial (depending on σ).

Theorem 3 (A., Cristoforoni, Nitsch, Trombetti - *In preparation* (2023))

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we can prove that the best configuration is given by $\mu \equiv 0$;
- the approximation gives a better solution for annuli and disjoint balls;

- the functional $\mathcal{G}_\varepsilon$ is coherent with the results in F. Della Pietra, C. Nitsch, and C. Trombetti (2022) ["An optimal insulation problem", *Math. Ann.*.] if $\beta/\varepsilon < (n-1)/R$, then

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$$\mathcal{G}_\varepsilon(B_R, h) > \mathcal{G}_\varepsilon(B_R, 0);$$

- the minimizer may be helpful to the study of the shape optimization problems of the kind

$$\inf \{ \mathcal{G}_\varepsilon(\Omega, h) \mid (\Omega, h) \in \mathcal{K} \times \mathcal{H} \},$$

where $\mathcal{K}$ and $\mathcal{H}$ are suitable classes of sets and functions respectively.

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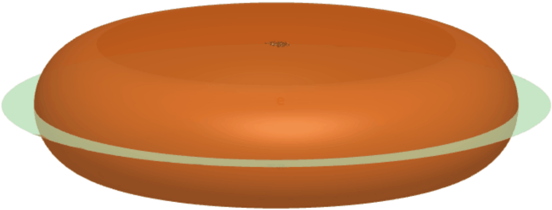
Theorem 4 (A., Cristoforoni, Nitsch, Trombetti - *In preparation* (2023))

There exists a minimizing sequence $(\Omega_k, \mu_{\Omega_k})$ to (2) such that Ω_k converges in the Hausdorff sense to the $(n-1)$-dimensional disc D for which $2\mathcal{H}^{n-1}(D) = P$.



Coarea formula

Let $d\colon \mathbb{R}^n \to \mathbb{R}$ be the distance function from $\partial\Omega$, let $g\colon \mathbb{R}^n \to \mathbb{R}$ be an $L^1(\mathbb{R}^n)$ function, and let $U \subset \mathbb{R}^n$ be an open set, then

$$\int_U g(x)\,dx = \int_{\mathbb{R}} \int_{U\cap\{\,d=t\,\}} g(y)\,d\mathcal{H}^{n-1}(y)\,dt.$$

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Let Ω be an open bounded set with C^1 boundary, and for every $\sigma \in \partial\Omega$ let $\{\tau_1(\sigma), \dots, \tau_{n-1}(\sigma)\}$ be an orthonormal basis of the tangent plane at $\partial\Omega$ in σ. Let U be an open neighbourhood of $\partial\Omega$, and $\phi : U \to \mathbb{R}^n$ a C^1 vector field. We define, for $i = 1, \dots, n$ and $j = 1, \dots, n-1$,

$$(D_\tau \phi)_{ij} = D\phi_i \cdot \tau_j, \qquad J_\tau \phi = \sqrt{\det\left((D_\tau\phi)^T D_\tau \phi\right)},$$

$$\operatorname{div}_\tau \phi = \sum_{j=1}^{n-1} D\phi\tau_j \cdot \tau_j.$$

If ν is the normal unit vector at $\partial\Omega$, we define

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Area formula on surfaces

Let $U \subseteq \mathbb{R}^n$ be a neighbourhood of $\partial\Omega$, let $\phi : U \to \mathbb{R}^n$ be a C^1 function, and let $g : \mathbb{R}^n \to \mathbb{R}$ be a positive Borel function. We have that

$$\int_{\partial\Omega} g(\phi(\sigma))\, J_\tau \phi \, d\mathcal{H}^{n-1} = \int_{\phi(\partial\Omega)} g(\sigma)\, d\mathcal{H}^{n-1}.$$

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If we choose $\phi(x) = x + tX(x)$, with X extension of ν, we can appproximate $J_\tau \phi$ as follows:

$$J_\tau \phi(\sigma) = 1 + tH(\sigma) + t^2 R(t, \sigma).$$

Combining coarea formula, area formula, and the approximation of the jacobian, we get the following fundamental equalities: for every $g : \Omega_\varepsilon \to \mathbb{R}$ positive Borel function,

$$\int_{\Sigma_\varepsilon} g(x)\, dx = \int_{\partial\Omega} \int_0^{\varepsilon h(\sigma)} g(\sigma + t\nu) \left(1 + tH(\sigma) + \varepsilon^2 R_1(\sigma, t, \varepsilon)\right)\, dt\, d\mathcal{H}^{n-1}, \tag{3}$$

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and

$$\int_{\partial\Omega_\varepsilon} g(\sigma)\,d\mathcal{H}^{n-1} = \int_{\partial\Omega} g(\sigma + \varepsilon h\nu)\left(1 + \varepsilon h(\sigma)H(\sigma) + \varepsilon^2 R_2(\sigma, \varepsilon)\right)\,d\mathcal{H}^{n-1}, \tag{4}$$

where R_1 and R_2 are bounded.

In general, we can not expect u_ε to be linear, but we can approach the problem as follows: we let $\alpha \in (-\alpha_0, \alpha_0) \setminus \{0\}$, and we define functions $v_{\varepsilon,\alpha}$ as follows

$$v_{\varepsilon,\gamma}(\sigma + th\nu) = 1 - \left(\frac{t}{\varepsilon}\right)^{1+\alpha} \frac{\beta h}{(1+\alpha)(1+\beta h)}.$$

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It can be proved that, for positive α, $v_{\varepsilon,\alpha}$ and $v_{\varepsilon,-\alpha}$ are respectively a supersolution and a subsolution for our PDE, and that, if we choose wisely α, these functions behave like a linear approximation in in the limit for $\varepsilon \to 0^+$.

Further Developments

- Adapting these techniques to the case with the heat source $-\Delta u = f$ in Ω;

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$$\frac{\varepsilon}{\beta}\sup_{\partial\Omega} H > \frac{2}{3};$$

- Studying the problem of minimizing or maximizing $\mathcal{G}_\varepsilon$ with respect both h and Ω when we replace the constraint on the perimeter (for instance fixed volume, equi-bounded curvature, quermassintegrals constriant, ...);

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Thank you for your attention!