Understanding the Role of Network Structure in Controlling Complex Systems

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Fabio Pasqualetti

My Research

Distributed control of network systems with applications to science and technology

Basic goals

- Understand mechanisms that make complex networks function
- Use knowledge to enable reliable engineering design w/ predictable behavior

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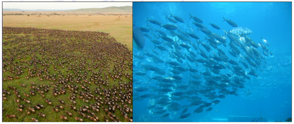

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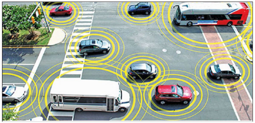

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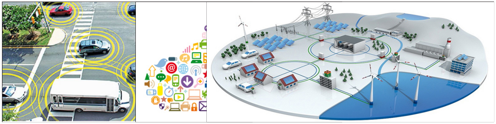

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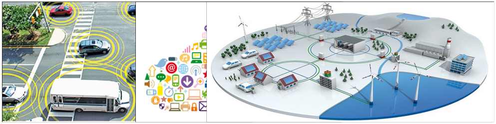


Sample questions

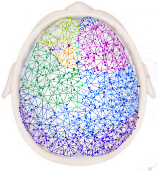
- Architectures for coordination and decision making
- Interplay between network structure, dynamics, and function
- Dynamically-sound understanding of network mechanisms
- Understanding what is doable with available resources
- Information: access, value, uncertainty
- Theoretical guarantees on correctness and robustness

Today: Role of Structure in Network Function

How does network structure shape network behavior?

What mechanisms explain the way networks function?

How to “fix” a network or enhance its function?



Two sample problems:

1. Benefits of time-invariant versus time-varying actuation in complex systems

2. Goal-driven selective attention in the brain

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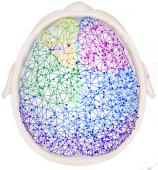


Two sample problems:

1. Benefits of time-invariant versus time-varying actuation in complex systems

   E. Nozari, F. Pasqualetti, and J. Cortés. Heterogeneity of central nodes explains the benefits of time-varying control scheduling in complex dynamical networks. *Journal of Complex Networks*, 7(5):659–701, 2019

2. Goal-driven selective attention in the brain

   E. Nozari and J. Cortés. Hierarchical selective recruitment in linear-threshold brain networks. Part I: Intra-layer dynamics and selective inhibition. *IEEE Transactions on Automatic Control*, 66(3):949–964, 2021

   E. Nozari and J. Cortés. Hierarchical selective recruitment in linear-threshold brain networks. Part II: Inter-layer dynamics and top-down recruitment. *IEEE Transactions on Automatic Control*, 66(3):965–980, 2021

Outline

1. Motivation

2. Time-Invariant versus Time-Varying Actuation
   - controllability of linear dynamical networks
   - $2k$-communicability as node centrality
   - networks that benefit from time-varying actuation

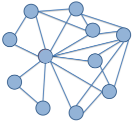
3. Goal-driven Selective Attention
   - hierarchical linear-threshold networks
   - selective inhibition
   - top-down recruitment

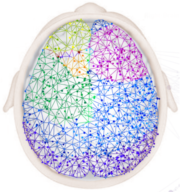
Time-Invariant versus Time-Varying Actuation

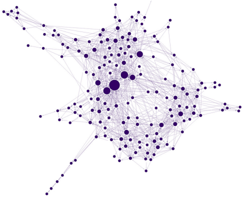
A Case in Point: Tips on Moving Heavy Furniture by Yourself

https://youtu.be/PhA9sv-Iu4w

How About for Large-Scale Networks?



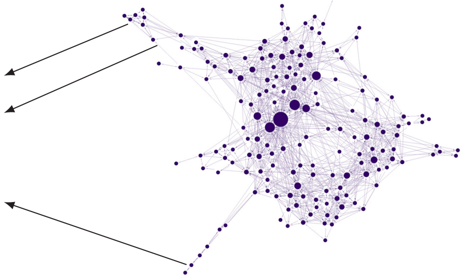




- ✓ Closed-form solutions
- ✓ Numerical tests
- ✓ Combinatorial methods
- ✓ Accurate parameters

Linear Complex Dynamical Networks

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$



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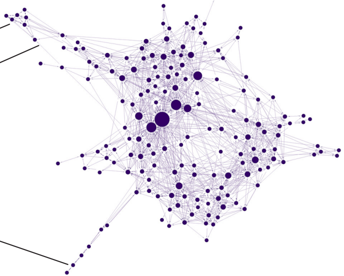

$$x(k+1) = Ax(k) + B(k)u(k), \qquad k \in \mathbb{Z}_{\geq 0}$$

(static) interconnection

each column vector $\in \{e_1, \ldots, e_n\}$

control

Network Controllability

Controllability [Kalman et. al., 1963]

Network is **controllable at time** K *if it is possible to steer its state* x *from any* $x(0) = x_0$ *to any* $x(K) = x_f$ *using* $u(0), \ldots, u(K-1) \in \mathbb{R}^m$

Deciding controllability: network is controllable at time K iff Gramian is nonsingular

$$\mathcal{W}_K \triangleq \sum_{k=0}^{K-1} A^k B(K-1-k) B(K-1-k)^T (A^T)^k$$

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Deciding controllability: How does network topology impact controllability? Dynamics-based framework for

- structural controllability (N. J. Cowan et al, PLoS ONE 12)
- degree distribution (Y. Liu, J. Slotine, and A. Barabási, Nature 2011)
- how many input nodes (A. Olshevsky, IEEE TAC 14)
- special structure: e.g., consensus-type linear networks (A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt, SIAM JCO, 09, C. Aguilar and B. Gharesifard, IEEE TAC 15)

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Minimum energy control: if network is controllable at time K,

$$u^*(k) = B(k)^T (A^T)^{K-1-k} \mathcal{W}_K^{-1} x_f, \quad k \in \{0, \ldots, K-1\}$$

with energy $E_{\min} = x_f^T \mathcal{W}_K^{-1} x_f$

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Controllability metrics: dynamics-based framework for

- how much energy required to steer the network (G. Yan et al, PRL 12)
- regularity properties, asymptotic behavior, impact of network topology and input location (F. Pasqualetti, S. Zampieri, and F. Bullo, IEEE TCNS 14, T. H. Summers, F. L. Cortesi, and J. Lygeros, IEEE TCNS 16)
- explaining structural controllability of brain networks (Gu et al, Nature Comm 15):

average, modal, boundary controllabilities

Optimal Actuator Selection/Placement

How to choose $B(0), B(1), \ldots, B(K-1)$ such that network controllability is maximized?

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More controllability $\equiv$ smaller $x_f^T \mathcal{W}_K^{-1} x_f$ $\equiv$ larger $\mathcal{W}_K$

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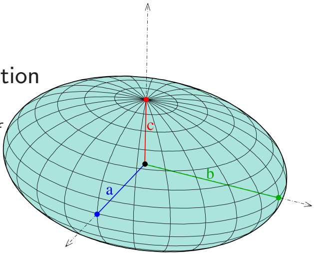
How to choose $B(0), B(1), \ldots, B(K-1)$ such that network controllability is maximized?

More controllability $\equiv$ smaller $x_f^T \mathcal{W}_K^{-1} x_f$ $\equiv$ larger $\mathcal{W}_K$

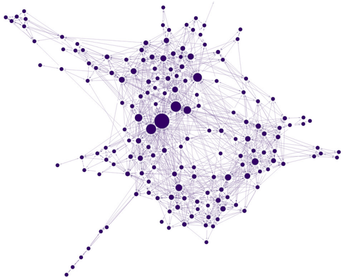
- $\lambda_{\min}(\mathcal{W}_K)$: controllability in the most difficult direction
- $\operatorname{tr}(\mathcal{W}_K^{-1})^{-1}$: average control energy over random x_f
- $\det(\mathcal{W}_K)$: volume of reachability ellipsoid
- $\operatorname{tr}(\mathcal{W}_K)$: related to average control energy over random x_f



[Image attribution: Ag2gaeh]

Literature focuses on **time-invariant** control schedules **(TICS)**:

$$B(0) = B(1) = \cdots = B(K-1)$$



Problem Statement

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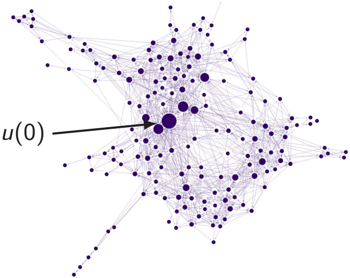

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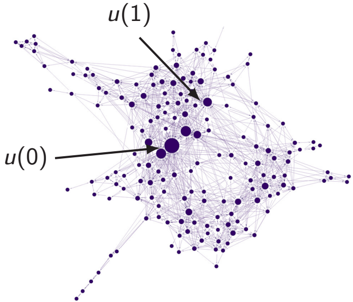

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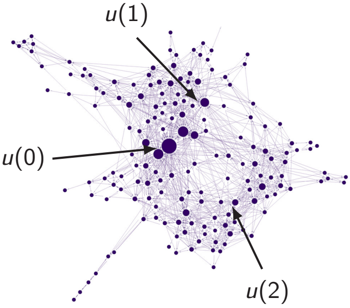

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Time-Invariant vs Time-Varying Actuation

Literature focuses on **time-invariant** control schedules **(TICS)**:
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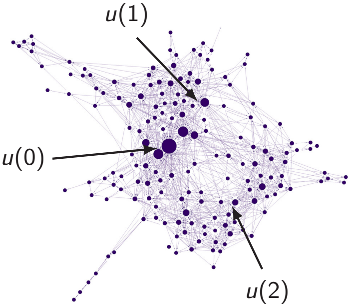
Time-varying schedules **(TVCS)** useful when

- ✓ sizable activation energy due to dead-zone
- ✓ only small number simultaneously operated b/c bandwidth limitations
- ✓ actuator interference due to proximity
- ✓ difficult precise coordination of geographically disperse actuators



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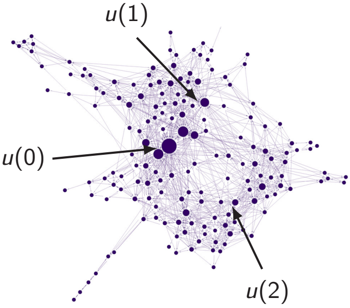

But more costly in terms of hardware&complexity

When is TV scheduling worth the cost?

What is Relative Benefit of TVCS over TICS?

$$f_{\max}^{\mathrm{TI}} = \max_{TI} f(\mathcal{W}_K) \qquad f_{\max}^{\mathrm{TV}} = \max_{TV} f(\mathcal{W}_K)$$

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Small-scale example with 5 nodes, 1 actuator, time horizon 10

f	$f^{\mathrm{TI}}_{\max}$	$f^{\mathrm{TV}}_{\max}$	Rel. Improv.
tr	5	5	0
$\mathrm{tr}(^{-1})^{-1}$	0.2	0.2	0
det	1	1	0
$\lambda_{\min}$	1	1	0

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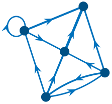
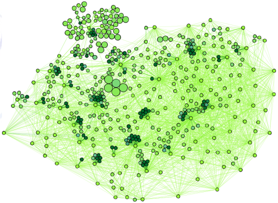


f	$f_{\max}^{\text{TI}}$	$f_{\max}^{\text{TV}}$	Rel. Improv.
tr	2.00	2.70	0.35
$\text{tr}(^{-1})^{-1}$	1.26×10^{-7}	8.22×10^{-4}	6.5×10^{3}
det	9.90×10^{-11}	7.42×10^{-10}	6.49
$\lambda_{\min}$	1.27×10^{-7}	1.10×10^{-4}	8.7×10^{2}

How About Large-Scale Networks?



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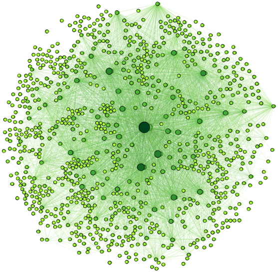

Social network of students at UC Irvine

Which networks do/which networks do not benefit from TVCS?
How can one distinguish between them?

2k-Communicability: Node Centrality Tied to Dynamics

For simplicity, **single input** ($m = 1$) case from here on

$$\operatorname{tr}(\mathcal{W}_K) = \operatorname{tr}\Big(\sum_{k=0}^{K-1} A^k b(K-1-k) b(K-1-k)^T (A^T)^k\Big)$$

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Maximization is decoupled across time steps

$$\max_{i_0,\dots,i_{K-1}} \mathrm{tr}(\mathcal{W}_K) = \sum_{k=0}^{K-1} \max_{i_{K-1-k}} ((A^k)^T A^k)_{i_k i_k}$$

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$2k$-communicability of node i: $R_i(k) \triangleq ((A^k)^T A^k)_{ii}$

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2k-Communicability: Interpretation

At time $K - k - 1$, actuate node with largest 2k-communicability

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$R_i(k)$ is sum of squares of total number of paths of length k from i to i:

- 2-communicability $\sim$ **local**
  (out-degree centrality if weights homogeneous)
- ∞-communicability $\sim$ **global**
  (same ordering as eigenvector centrality)
- #switches in $\text{argmax}_i R_i(k) = O(n^3) < \infty$

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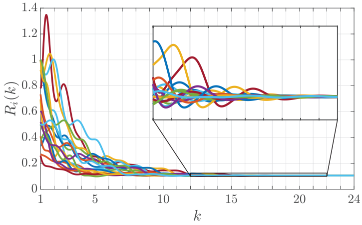
20-node network w/ heterogeneous weights



Main Insight

At time $K - k - 1$, actuate node with largest $2k$-communicability

Optimal scheduling involves

- application of $u(0)$ to node $r(K-1)$ w/ highest **global** centrality
- gradually moving control node, until $u(K-2)$ applied to node $r(1)$ w/ highest **local** centrality

Main Insight

At time $K-k-1$, actuate node with largest $2k$-communicability

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- application of $u(0)$ to node $r(K-1)$ w/ highest **global** centrality
- gradually moving control node, until $u(K-2)$ applied to node $r(1)$ w/ highest **local** centrality

Intuition is simple

- at $k=0$, control input has enough time to propagate through the network (hence highest globally-central node)
- as we approach control horizon K, control input has only a few time steps to disseminate through the network (hence locally-central nodes)

Optimal TI Schedules < Optimal TV Schedules

Theorem (2k-Communicability as Litmus Test)

Assume adjacency matrix A is irreducible, aperiodic, and diagonalizable. If

$$\operatorname{argmax}_{i\in\{1,\dots,n\}} R_i(1) \cap \operatorname{argmax}_{i\in\{1,\dots,n\}} R_i(\infty) = \emptyset,$$

then optimal actuator schedule is TV for sufficiently large K

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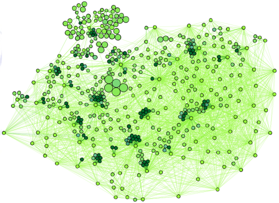
irreducible ⇔ strongly connected network

aperiodic automatically satisfied w/ self-loops

set of non-diagonalizable matrices has measure zero

Not the same node with maximum 2-communicability (local) and ∞-communicability (global)

What is Relative Benefit for Large-Scale Networks?



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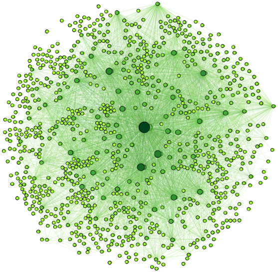

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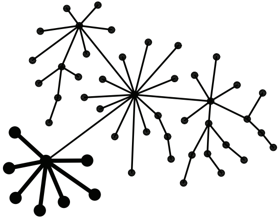
Legend ($m = 1$ actuator, time horizon $K = 10$)

- **Color intensity** of nodes represent value of $R_i(1)$
- **Size** of nodes represent value of $R_i(K-1)$

Optimal TI Schedules < Optimal TV Schedules: Examples

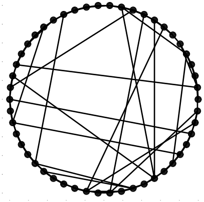
- Deterministic:

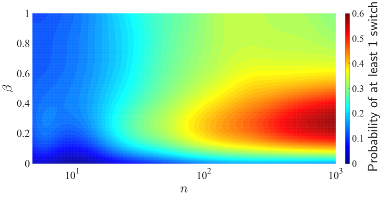
  ➢ Networks with strong subnetworks



- Stochastic:

  ➢ Watts-Strogatz small world





Optimal TI Schedules = Optimal TV Schedules

Theorem (Undirected&Normal Networks)

If **any** of the following holds:

1. $\frac{1-v_{11}^2}{v_{11}^2} \leq \frac{|\lambda_1|-|\lambda_2|}{|\lambda_1|-|\lambda_n|}$,
2. $v_{11}^2 + v_{12}^2 = 1$,
3. ≤ 3 nonzero eigenvalues with different absolute values and $1 \in \text{argmax}_i R_i(1)$,

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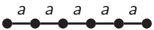
Optimal TV schedule = Optimal TI schedule

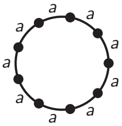
Networks where there is a **sufficiently distinct authority** and network dynamics are dominated by λ_1

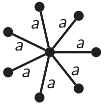
Networks where the centrality of all nodes is determined by the weight of the link to the **most central node**

Complete bipartite networks and connected strongly regular networks with **a distinct authority**

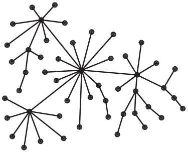
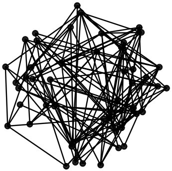
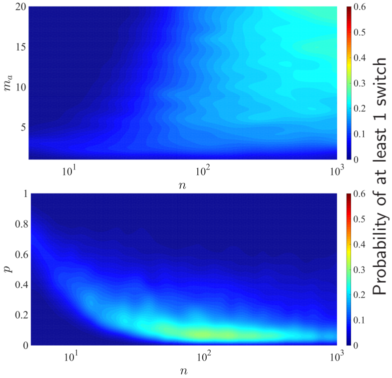
- Deterministic:







- Stochastic:
  - ➢ Barabási-Albert scale-free
  - ➢ Erdös-Rényi random network

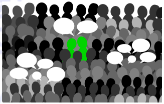
Outline

1. Motivation
2. Time-Invariant versus Time-Varying Actuation
   - controllability of linear dynamical networks
   - $2k$-communicability as node centrality
   - networks that benefit from time-varying actuation
3. Goal-driven Selective Attention
   - hierarchical linear-threshold networks
   - selective inhibition
   - top-down recruitment

Goal-Driven Selective Attention (GDSA): An Experiment

https://youtu.be/PhA9sv-Iu4w

How Does Brain Achieve Control of Attention?

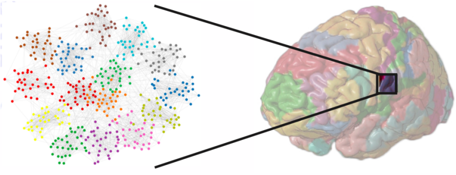


selective listening

selective taste/smell

selective memory recall

How Does Brain Achieve Control of Attention?



Highly complex dynamical system with selective population activity

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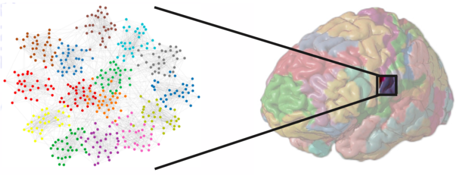

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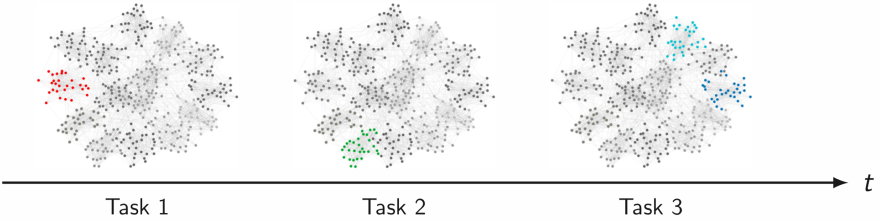

[Image attribution: Levie et. al., 2017]

Experimental Observations: General Brain Organization

(First-order approximations to full complexity of brain organization)

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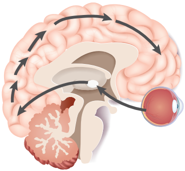
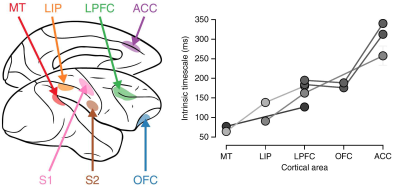
1. Sensory information processing pathways

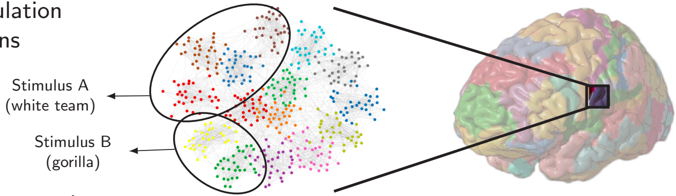


2. Timescale hierarchy



[Murray et al, Nat Neurosci, 2014]

3. Disjoint population representations

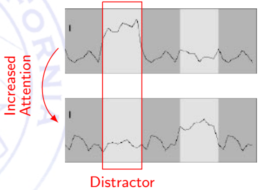


⋮

[Levie et al, 2017]

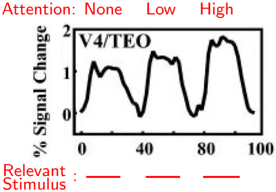
Experimental Observations: Specific to Selective Attention

4. Inhibitory Control of $\mathbf{x}_i^0$



Adapted from [Rees et al, Science, 1997]

5. Recruitment Control of $\mathbf{x}_i^1$



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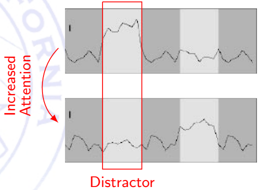

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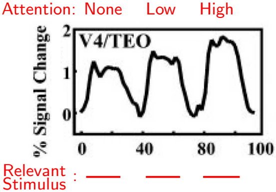

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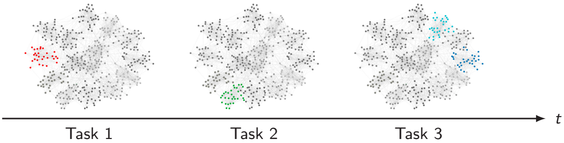
⇒ Population activity is selective (sparse)



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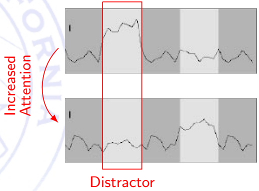

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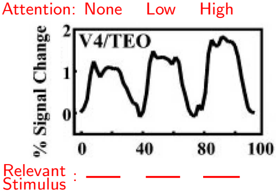

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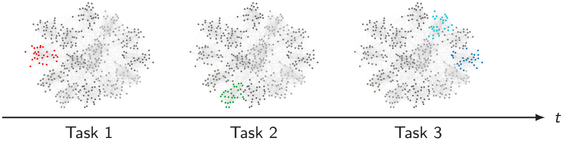

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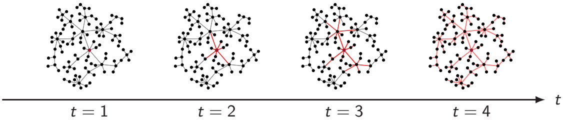

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selecti…

What happens at the level of network dynamics?

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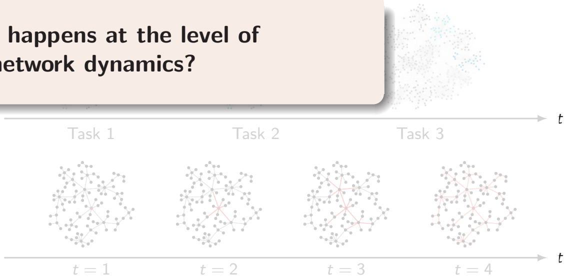
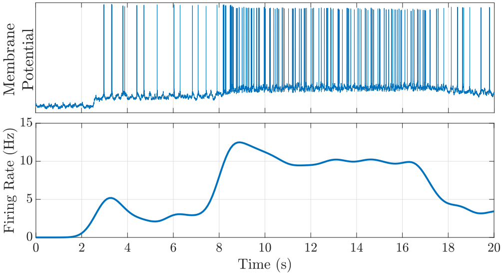


Modeling Brain Activity

Mesoscale Models: Rate Dynamics

- Information mostly encoded in **firing rate** (#spikes/s)
- $x_i(t) =$ **firing rate** of neuron i



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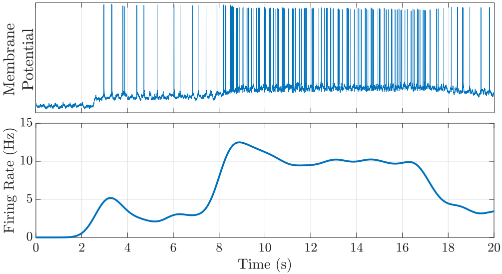

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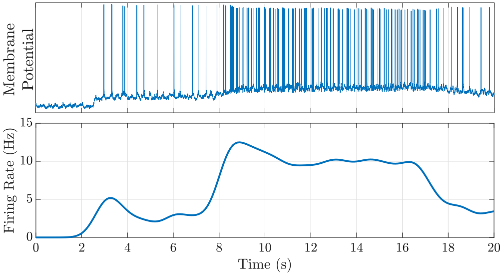
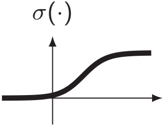


$$x_i = \sigma(I_{\text{in},i})$$



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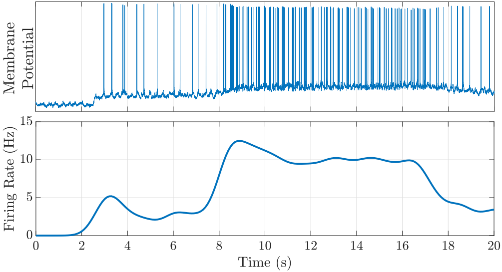

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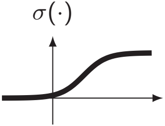


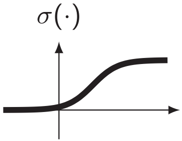
2b. For time-varying input $I_{\mathrm{in},i}(t)$

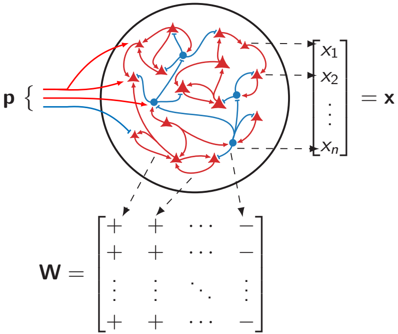
$$\tau \dot{x}_i(t) = -x_i(t) + \sigma(I_{\mathrm{in},i}(t))$$

Brain Network Dynamics

- Node = population of neurons
- State = average firing rate
- Network **dynamics** (mean-field approximation):

$$\tau \dot{\mathbf{x}}(t) = -\mathbf{x}(t) + \sigma\big(\mathbf{W}\mathbf{x}(t) + \mathbf{p}(t)\big)$$





Approximating the Sigmoidal Nonlinearity

✓ **Kuramoto:** Cubic approximation in $\mathbf{x}_i$, linearization in $\{W_{ij}\}$, change to polar coordinates

$$\dot{\theta}_i = \omega_i + \sum_j K_{ij} \sin(\theta_j - \theta_i)$$

→ For weakly-coupled oscillators, explicit phase dynamics, $\frac{n}{2}$ states, smooth

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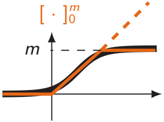

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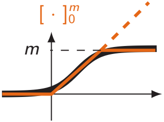

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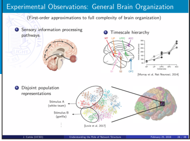
Encoding General Experimental Observations



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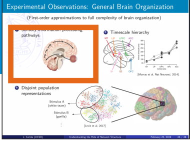

Each layer:

$$\tau_i \dot{\mathbf{x}}_i(t) = -\mathbf{x}_i(t) + [\mathbf{W}_{i,i}\mathbf{x}_i(t) + \mathbf{p}_i(t)]_{\mathbf{0}}^{\mathbf{m}}$$

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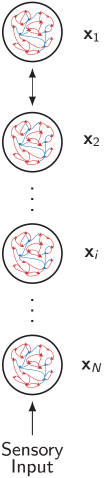
$$\mathbf{p}_i(t) = \mathbf{B}_i \mathbf{u}_i(t) + \mathbf{W}_{i,i-1}\mathbf{x}_{i-1}(t) + \mathbf{W}_{i,i+1}\mathbf{x}_{i+1}(t) + \mathbf{c}_i$$



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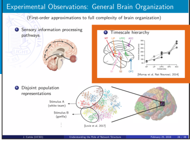

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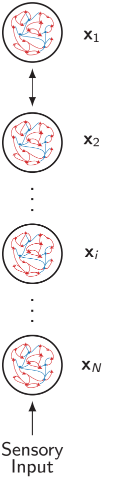
2. Timescale hierarchy:

$$\tau_1 \gg \tau_2 \gg \cdots \gg \tau_i \gg \cdots \gg \tau_N$$



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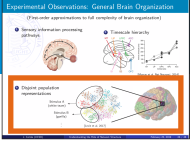

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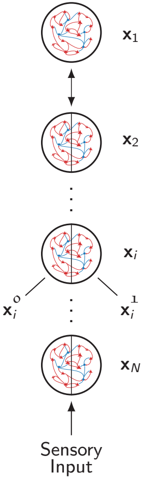
3. Disjoint population representation:

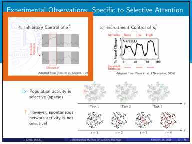
$$\mathbf{x}_i = \begin{bmatrix} \mathbf{x}_i^0 \\ \mathbf{x}_i^1 \end{bmatrix} \quad , \quad \mathbf{W}_{i,j} = \begin{bmatrix} \mathbf{W}_{i,j}^{00} & \mathbf{W}_{i,j}^{01} \\ \mathbf{W}_{i,j}^{10} & \mathbf{W}_{i,j}^{11} \end{bmatrix}$$

$\mathbf{x}_i^0$ → task-irrelevant

$\mathbf{x}_i^1$ → task-relevant



Inhibitory Control of Task-Irrelevant Nodes



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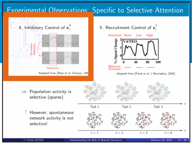

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- Start from the simplest layer: layer N

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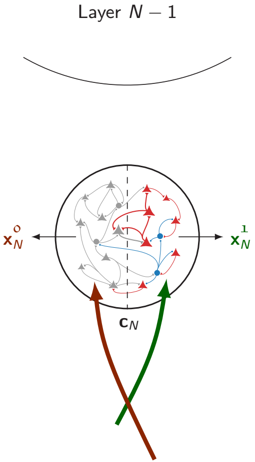

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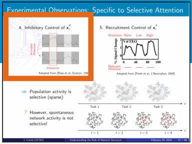

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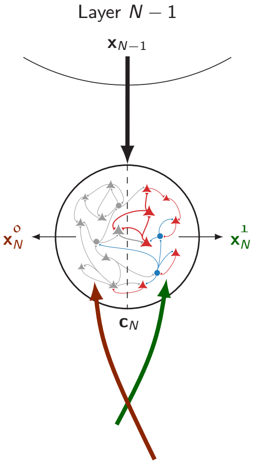
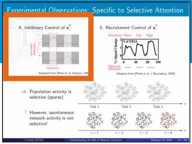
- Separate inhibitory/recruitment control inputs

$$\mathbf{B}_N = \begin{bmatrix}\mathbf{B}_N^0 \\ \mathbf{0}\end{bmatrix}, \quad \mathbf{B}_N^0 \leq \mathbf{0}$$
$$\mathbf{W}_{N,N-1} = 0$$



Inhibitory Control of Task-Irrelevant Nodes



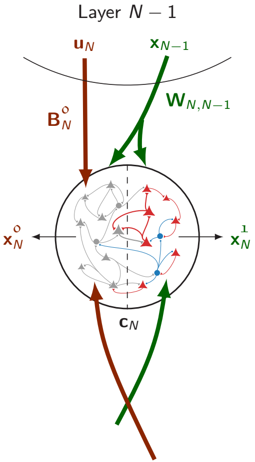
- **Hypothesis:** stabilization of $\mathbf{x}_i^{o}$ to origin by $\mathbf{x}_{i-1}$
- Start from the simplest layer: layer N

$$\tau_N \dot{\mathbf{x}}_N = -\mathbf{x}_N + [\mathbf{W}_{N,N}\mathbf{x}_N + \mathbf{p}_N(t)]_0^{m_N}$$
$$\mathbf{p}_N(t) = \mathbf{B}_N\mathbf{u}_N(t) + \mathbf{W}_{N,N-1}\mathbf{x}_{N-1}(t) + \mathbf{c}_N$$

- Separate inhibitory/recruitment control inputs

$$\mathbf{B}_N = \begin{bmatrix} \mathbf{B}_N^{o} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_N^{o} \leq \mathbf{0}$$
$$\mathbf{W}_{N,N-1} = 0$$

? **How does $\mathbf{u}_N(t)$ stabilize $\mathbf{x}_N(t)$ to $\mathbf{x}_N^* = \begin{bmatrix} \mathbf{0} \\ \mathbf{x}_N^{*1}(\mathbf{c}_N) \end{bmatrix}$?**



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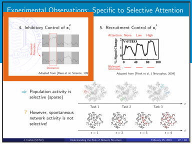

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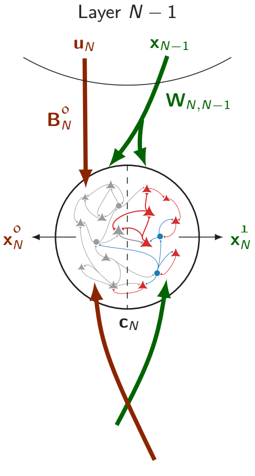

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? **Conditions for stability?**

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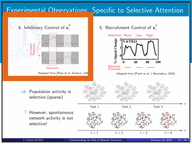

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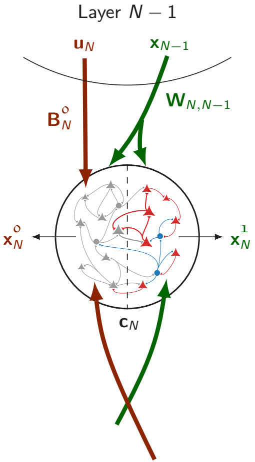

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? **How many equilibria?**

Network Dynamics as Switched Affine System

$$\tau_i \dot{x}_i = -x_i + \Big[\underbrace{\sum_j W_{ij} x_j + p_i}_{I_{\text{in},i}} \Big]_0^{m_i}$$

✓ Dynamics of each node i can be in 3 modes $\Rightarrow 3^n$ switching regions

$$\begin{aligned}
\tau_i \dot{x}_i &= -x_i && \text{if} && I_{\text{in},i} \leq 0 \\
\tau_i \dot{x}_i &= -x_i + I_{\text{in},i} && \text{if} && 0 \leq I_{\text{in},i} \leq m_i \\
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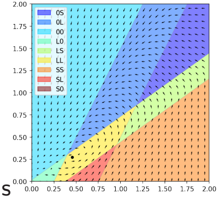

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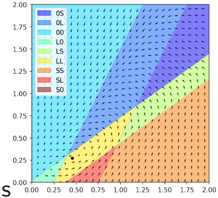

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$\tau \dot{\mathbf{x}} = (-\mathbf{I} + \boldsymbol{\Sigma}^{\ell}_{\boldsymbol{\sigma}(\mathbf{x})} \mathbf{W})\mathbf{x} + \boldsymbol{\Sigma}^{\ell}_{\boldsymbol{\sigma}(\mathbf{x})} \mathbf{p} + \boldsymbol{\Sigma}^{\mathrm{s}}_{\boldsymbol{\sigma}(\mathbf{x})} \mathbf{m}, \ \boldsymbol{\sigma}(\mathbf{x}) \in \{0, \ell, \mathrm{s}\}^n$

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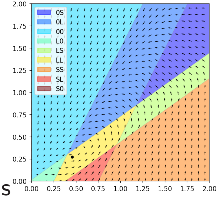

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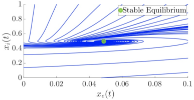
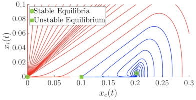
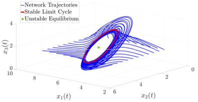
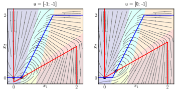
✓ **Rich nonlinearity:** mono- and multi-stability, limit cycles, bifurcations, chaos

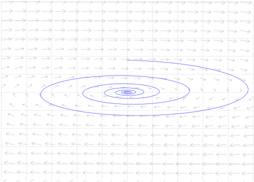




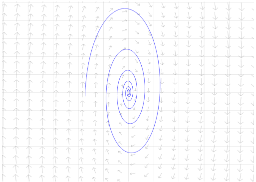




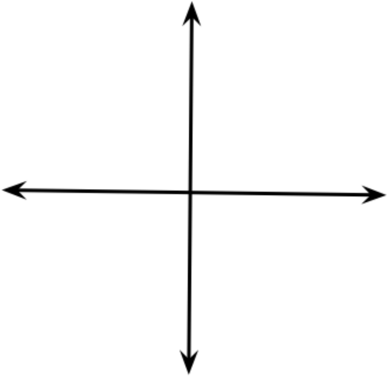
Stability of State-Dependent Switched Systems is Tricky...



Mode 1



Mode 2



Relevant Network Structures

A matrix $\mathbf{W} \in \mathbb{R}^{n \times n}$ is

- **absolutely Schur stable** if $\rho(|\mathbf{W}|) < 1$
- **totally $\mathcal{L}$-stable** ($\mathbf{W} \in \mathcal{L}$) if $\exists \mathbf{P} = \mathbf{P}^T > \mathbf{0}$ such that $\forall \boldsymbol{\sigma} \in \{0,1\}^n$

$$(-\mathbf{I} + \mathbf{W}^T \operatorname{diag}(\boldsymbol{\sigma}))\mathbf{P} + \mathbf{P}(-\mathbf{I} + \operatorname{diag}(\boldsymbol{\sigma})\mathbf{W}) < \mathbf{0}$$

- **totally Hurwitz** ($\mathbf{W} \in \mathcal{H}$) if all its principal submatrices are Hurwitz
- **a P-matrix** ($\mathbf{W} \in \mathcal{P}$) if all its principal minors are positive

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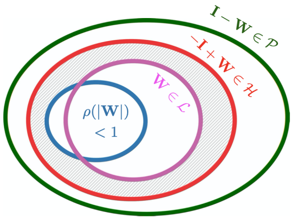

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Stability as a Function of Structure

- Characterized equilibria and stability of linear-threshold networks, e.g.

$$\mathbf{I} - \mathbf{W} \in \mathcal{P} \Leftrightarrow \text{EUE} \qquad \mathbf{W} \in \mathcal{L} \text{ or } \rho(|\mathbf{W}|) < 1 \Rightarrow \text{GES} \qquad -\mathbf{I} + \mathbf{W} \in \mathcal{H} \Leftrightarrow \text{LAS}$$

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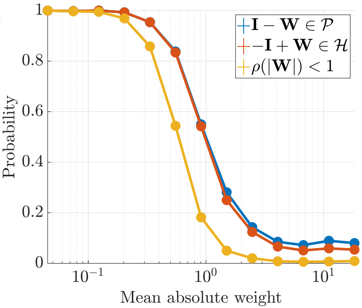
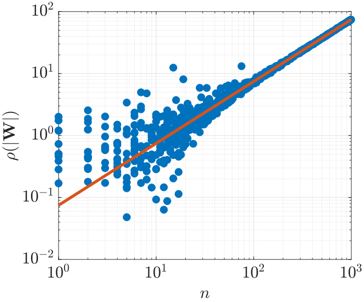
- The **stronger** or **larger** a network, the **more unstable** it becomes





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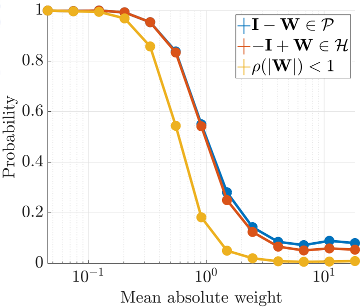

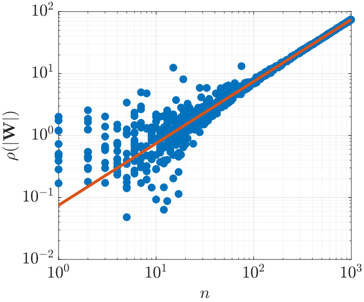

- How can brain networks
  - be *very* large but not deeply unstable?
  - become stronger during learning without losing stability?

Inhibitory Control of Task-Irrelevant Nodes – revisited

? Recall: How $\mathbf{u}_N(t)$ stabilizes $\mathbf{x}_N(t)$ to $\mathbf{x}_N^* = (\mathbf{0}, \mathbf{x}_N^{*^1}(\mathbf{c}_N))$?

- Assume $\dim(\mathbf{u}_N) \geq \dim(\mathbf{x}_N^{\text{o}})$

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Feedforward inhibition

$\exists \mathbf{u}_N(t) \equiv \bar{\mathbf{u}}_N$ such that:

$\mathbf{x}_N$ dynamics **becomes** GES towards $\mathbf{x}_N^* = (\mathbf{0}, \mathbf{x}_N^{*1})$

$\Updownarrow$

$\mathbf{x}_N^{1}$ sub-dynamics is intrinsically GES

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$\exists \mathbf{u}_N(t) = \mathbf{K}\mathbf{x}_N(t)$ such that:

1. $\mathbf{I} - (\mathbf{W}_{N,N} + \mathbf{B}_N\mathbf{K}) \in \mathcal{P} \Leftrightarrow \mathbf{I} - \mathbf{W}_{N,N}^{\mathtt{11}} \in \mathcal{P}$
2. $-\mathbf{I} + (\mathbf{W}_{N,N} + \mathbf{B}_N\mathbf{K}) \in \mathcal{H} \Leftrightarrow -\mathbf{I} + \mathbf{W}_{N,N}^{\mathtt{11}} \in \mathcal{H}$
3. $\mathbf{W}_{N,N} + \mathbf{B}_N\mathbf{K} \in \mathcal{L} \Leftrightarrow \mathbf{W}_{N,N}^{\mathtt{11}} \in \mathcal{L}$
4. $\rho(|\mathbf{W}_{N,N} + \mathbf{B}_N\mathbf{K}|) < 1 \Leftrightarrow \rho(|\mathbf{W}_{N,N}^{\mathtt{11}}|) < 1$

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4. $\rho(|\mathbf{W}_{N,N} + \mathbf{B}_N\mathbf{K}|) < 1 \Leftrightarrow \rho(|\mathbf{W}_{N,N}^{\mathtt{11}}|) < 1$

$\Rightarrow$ **Intrinsic properties of $\mathbf{W}^{\mathtt{11}}$ are sole determiner** of dynamical properties achievable via inhibitory selective stabilization

Implications for the Brain – revisited

The **stronger** or **larger** a network, the more unstable it becomes

Hypothesis: *at any given time, the active subnetwork is sufficiently* **small** *&* **weak** *to prevent (strong) instability*

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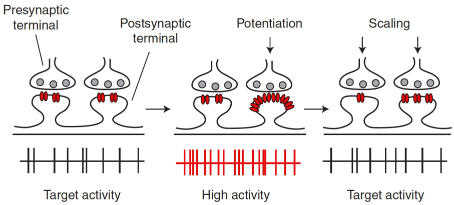
✓ Supported by 2 observations in neuroscience:

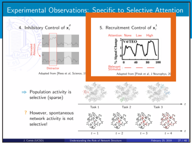
1. **Homeostasis:** [Turrigiano, 2012]



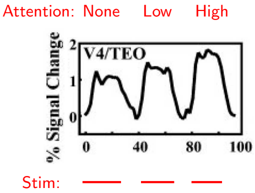
2. **Attention capacity:** [Lavie & de Fockert, 2003]

> "... distractor processing depends on the extent to which high perceptual load **exhausts attention** in relevant processing ..."

Recruitment Control of Task-Relevant Nodes

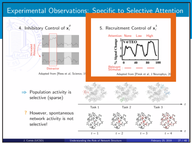


- Layer N, now $\mathbf{W}_{N,N-1} \neq \mathbf{0}$, no task-irrelevant nodes ($\mathbf{x}_N = \mathbf{x}_N^1$)



Adapted from [Pinsk et. al., J. Neurophys., 2004]

Recruitment Control of Task-Relevant Nodes



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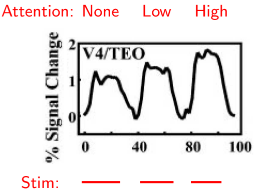

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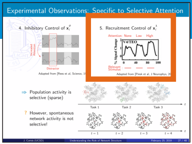

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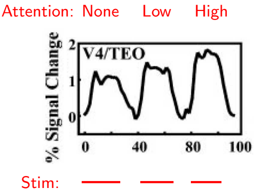
  - ? $f(\cdot)$
  - ? Conditions for $\mathbf{x}_N(t) \to f(\mathbf{x}_{N-1}(t))$



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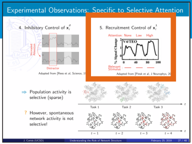
Recruitment Control of Task-Relevant Nodes



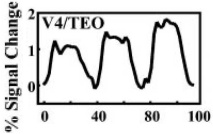
- Layer N, now $\mathbf{W}_{N,N-1} \neq \mathbf{0}$, no task-irrelevant nodes ($\mathbf{x}_N = \mathbf{x}_N^1$)
- **Hypothesis:** $\mathbf{x}_N(t) \to f(\mathbf{x}_{N-1}(t))$ (tracking)
  - ? $f(\cdot)$
  - ? Conditions for $\mathbf{x}_N(t) \to f(\mathbf{x}_{N-1}(t))$

1. Equilibria: solutions of $\mathbf{x}_N = [\mathbf{W}_{N,N}\mathbf{x}_N + \mathbf{c}_N]^+$ are

$$h_{\mathbf{W}_{N,N}}(\mathbf{c}_N) = \{(\mathbf{I} - \mathbf{\Sigma}\mathbf{W}_{N,N})^{-1}\mathbf{\Sigma}\mathbf{c}_N \mid (2\mathbf{\Sigma} - \mathbf{I})(\mathbf{I} - \mathbf{W}_{N,N}\mathbf{\Sigma})\mathbf{c}_N \geq 0\}$$

2. h is globally Lipschitz

Attention: None Low High



Stim:

Adapted from [Pinsk et. al., J. Neurophys., 2004]

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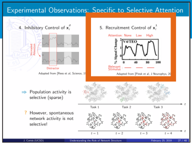

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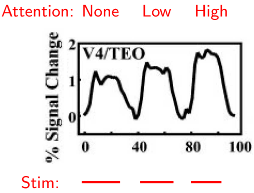

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2. h is globally Lipschitz

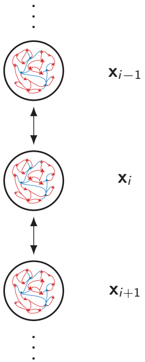
If (i) $\mathbf{x}_{N-1}(t)$ is bounded

(ii) $\dot{\mathbf{x}}_N = -\mathbf{x}_N + [\mathbf{W}_{N,N}\mathbf{x}_N + \mathbf{c}_N]^+$ is GES

then $\mathbf{x}_N(t) \to h_{\mathbf{W}_{N,N}}(\mathbf{W}_{N,N-1}\mathbf{x}_{N-1}(t) + \mathbf{c}_N)$ as $\frac{\tau_N}{\tau_{N-1}} \to 0$ (practically $\sim \frac{1}{2}$)

Recruitment Control: Multiple Layers

✓ Still no task-irrelevant nodes at any layers ($\mathbf{x}_i = \mathbf{x}_i^1$)

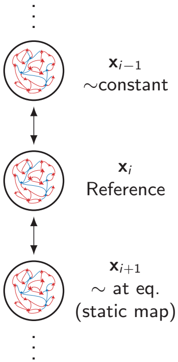


Recruitment Control: Multiple Layers

✓ Still no task-irrelevant nodes at any layers ($\mathbf{x}_i = \mathbf{x}_i^1$)

✓ Due to time-scale separation, relative to $\mathbf{x}_i$

- $\mathbf{x}_{i-1}$ is almost constant
- $\mathbf{x}_{i+1}$ is almost at equilibrium $f(\mathbf{x}_i)$



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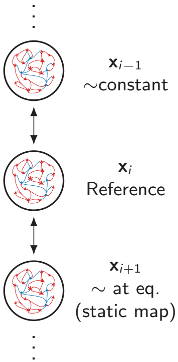
$$\mathbf{x}_i = [\mathbf{W}_{i,i+1} \underbrace{h_{i+1}(\mathbf{W}_{i+1,i}\mathbf{x}_i + \mathbf{c}_{i+1})}_{\mathbf{x}_{i+1}} + \mathbf{W}_{i,i}\mathbf{x}_i + \mathbf{c}_i]^+$$



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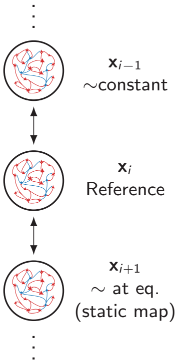
1. $\forall i$, h_i **has piecewise-affine form**

$$h_i(\mathbf{x}_{i-1}) = \mathbf{F}_{i,\lambda}\mathbf{x}_{i-1} + \mathbf{f}_{i,\lambda}, \ \forall \mathbf{x}_{i-1} \in \Psi_{i,\lambda}, \lambda \in \Lambda_i$$

   where $\Lambda_i, \mathbf{F}_{i,\lambda}, \mathbf{f}_{i,\lambda}, \Psi_{i,\lambda}$ have recursive expressions

2. $\forall i$, h_i is globally Lipschitz



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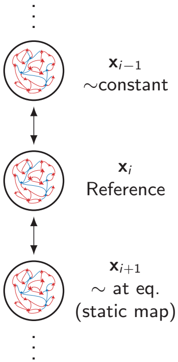

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where $\Lambda_i, \mathbf{F}_{i,\lambda}, \mathbf{f}_{i,\lambda}, \Psi_{i,\lambda}$ have recursive expressions

2. $\forall i, h_i$ is globally Lipschitz

Let $\bar{\mathbf{F}}_i \triangleq \max_{\lambda \in \Lambda_i} |\mathbf{F}_{i,\lambda}|$. $\forall i, \mathbf{x}_i$ is GES if

$$\rho(|\mathbf{W}_{i,i}| + |\mathbf{W}_{i,i+1}|\bar{\mathbf{F}}_{i+1}|\mathbf{W}_{i+1,i}|) < 1$$

Putting Everything Together

Hierarchical Selective Recruitment

Inhibitory and Recruitment Control

If GES conditions across hierarchy $\left(\rho\left(|\mathbf{W}_{i,i}^{11}| + |\mathbf{W}_{i,i+1}^{11}|\bar{\mathbf{F}}_{i+1}|\mathbf{W}_{i+1,i}^{11}|\right) < 1,\ \forall i\right)$, there exists $\mathbf{u}_i(t) = \mathbf{K}_i \mathbf{x}_i(t) + \bar{\mathbf{u}}_i(t)$ such that, as $\tau_i/\tau_{i-1} \to 0$,

$$\mathbf{x}_i^0(t) \to \mathbf{0} \qquad \text{(inhibition)}$$

$$\mathbf{x}_i^1(t) \to h_i(\mathbf{W}_{i,i-1}^{11}\mathbf{x}_{i-1}^1(t) + \mathbf{c}_i^1) \qquad \text{(recruitment)}$$

Technical approach

- Piecewise affine set-valued equilibria maps
- Layer interconnection ensuring GES of coupled systems
- Converse Lyapunov theorem for GES state-dependent switched affine systems
- Recursive application of singular perturbation theory

Summary and Outlook

Interplay between network **structure** and **dynamics**

- **time-invariant versus time-varying actuation**
  2K-communicability as measure of node centrality
  scale-heterogeneity of central nodes explains benefits of TVCS
  synthetic and real-world networks
- **hierarchical selective recruitment**
  multilayer linear-threshold networks
  selective inhibition and top-down recruitment of subnetworks
  analytical support for selective attention in brain dynamics

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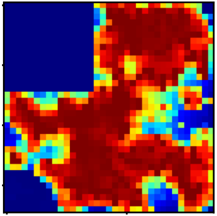
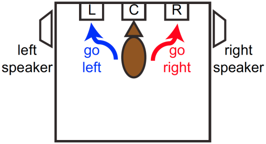


Challenging, exciting problems

- thalamocortical networks
  - role of thalamus, general interconnection topologies, impact on control magnitude, speed of convergence, robustness
- oscillations, bifurcations, and spatio-temporal patterns
  - interconnections of Wilson-Cowan oscillators, 1 inhibitory+arbitrary excitatory, purely inhibitory networks
- interventions to control oscillation containtment & spreading
  - modify interconnection structure so that desired set of nodes is/is not oscillatory

Case Study: Selective Listening in Rats

- **Task:**



[Rodgers & DeWeese, "Neural correlates of task switching in prefrontal cortex and primary auditory cortex in a novel stimulus selection task for rodents." Neuron 82 (5): 1157-1170, 2014]

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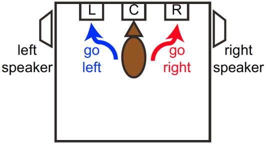
- **Task:**
  - 2 simultaneous stimuli (S) (possibly conflicting)

	S1	S2



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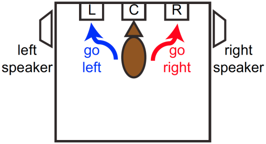
R1		
R2		



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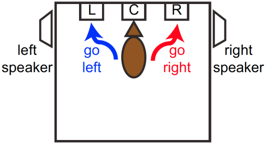
R1	Attend	Ignore
R2	Ignore	Attend



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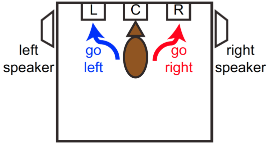
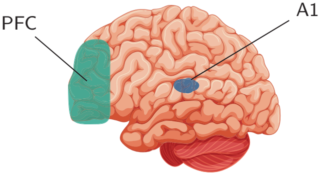


- **Data:**
  - Spike times of individual neurons
  - 2 regions: PFC (control) & A1 (auditory)
  - $\sim$ 100 neurons in each region



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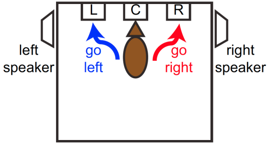

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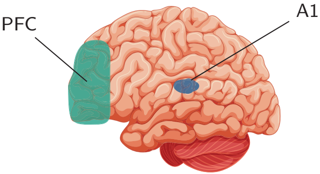
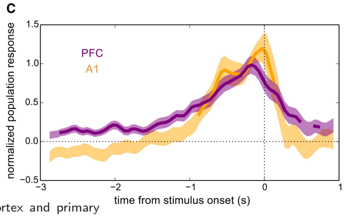


- **Original Findings:**
  - PFC neurons $\sim$ R, A1 neurons $\sim$ S
  - Disrupting PFC $\Rightarrow$ error $\uparrow$
  - PFC activity **leads (causes?)** A1 activity



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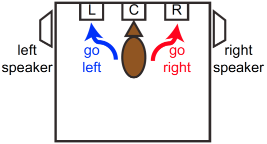

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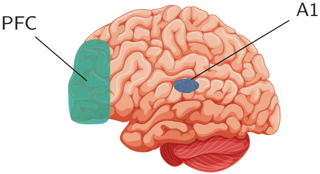

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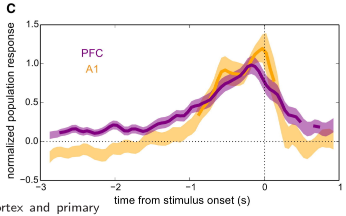
$\Rightarrow$ **PFC (seems to be) controling A1**



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Case Study: Selective Listening in Rats – cont'd

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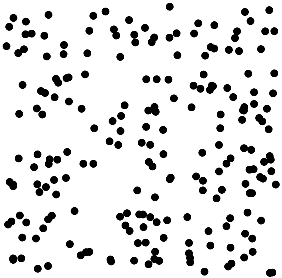
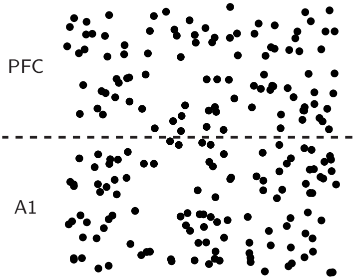


Case Study: Selective Listening in Rats – cont'd

1. Defining nodes:
   - Clustering neurons



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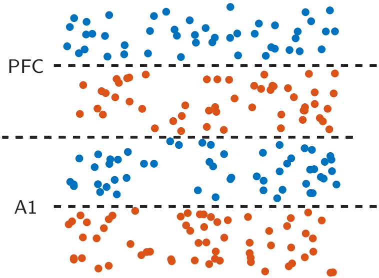

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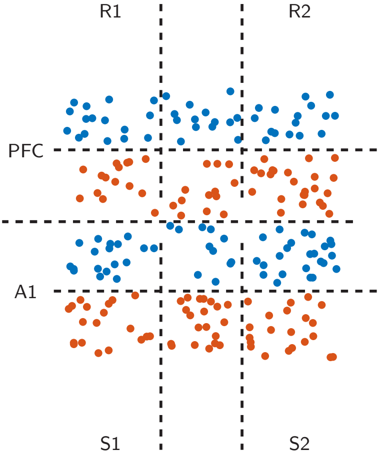

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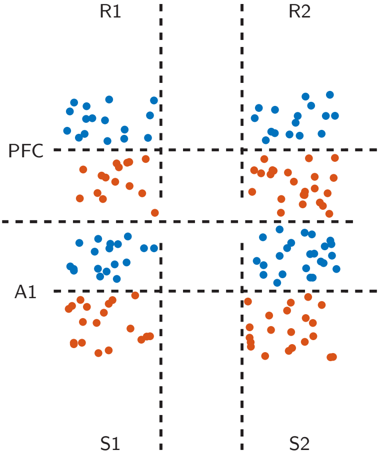

Case Study: Selective Listening in Rats – cont'd

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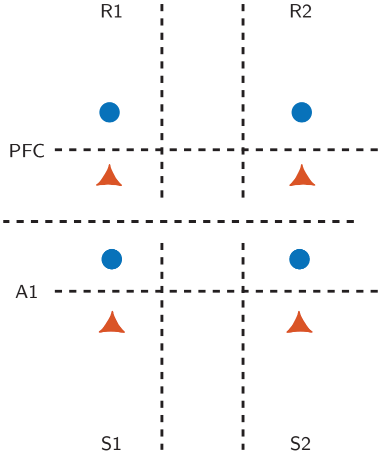
   - Population = node



Case Study: Selective Listening in Rats – cont'd

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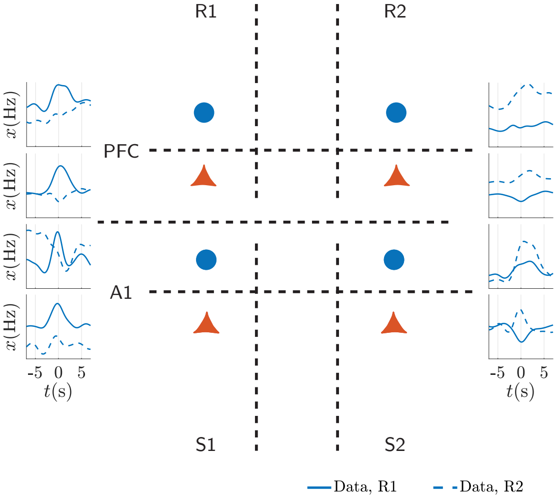
   - Population = node
   - Computing state



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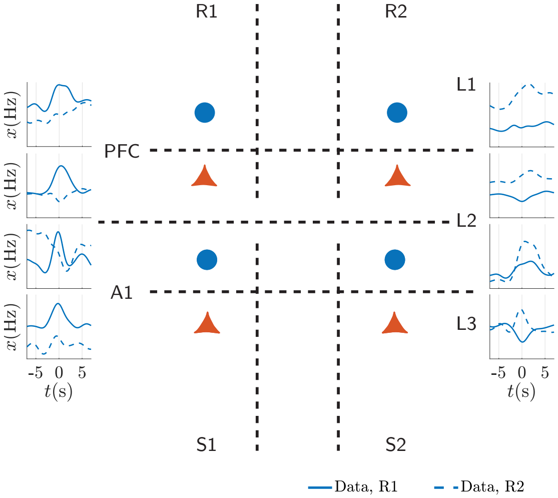
2. Defining Layers:
   - Timescale estimation



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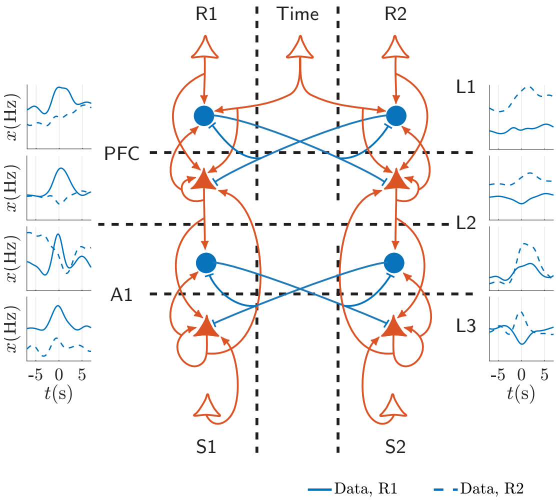
3. Defining edges:
   - Physiology



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3. Defining edges:
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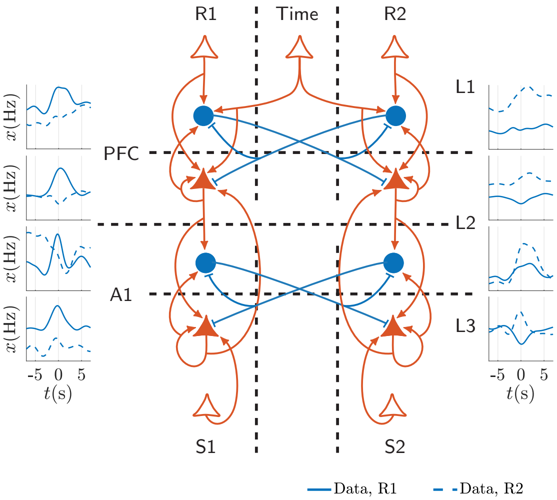
4. Finding edge weights:

$$\min_{\theta} \quad d(\mathbf{x}_{\text{data}}, \mathbf{x}_{\text{model}})$$

$$\theta = [w_{i,j}, b_{i,j}, c_i, \tau_i, x_i(0)]_{i,j}$$



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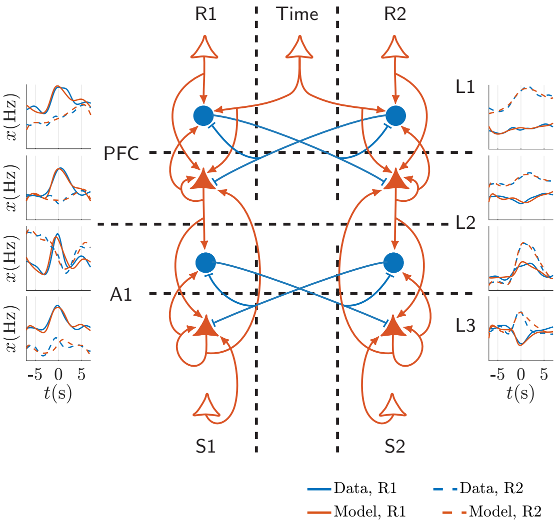

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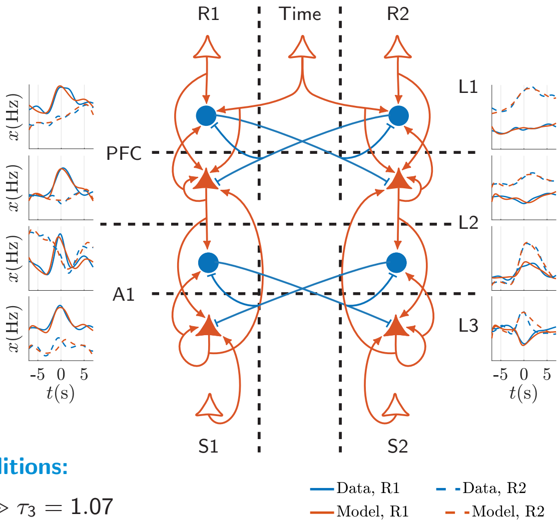

5. **Checking theoretical conditions:**

✓ $\tau_1 = 4.70 \gg \tau_2 = 2.33 \gg \tau_3 = 1.07$

✓ Under R1: $\rho\big(|\mathbf{W}^{11}_{2,2}| + |\mathbf{W}^{11}_{2,3}|\bar{F}^{1}_{3}|\mathbf{W}^{11}_{3,2}|\big) = 0.42 < 1$

✓ Under R2: $\rho\big(|\mathbf{W}^{11}_{2,2}| + |\mathbf{W}^{11}_{2,3}|\bar{F}^{1}_{3}|\mathbf{W}^{11}_{3,2}|\big) = 0.13 < 1$

References

Time-invariant vs time-varying actuation

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