On the Fractional Schrödinger Equation

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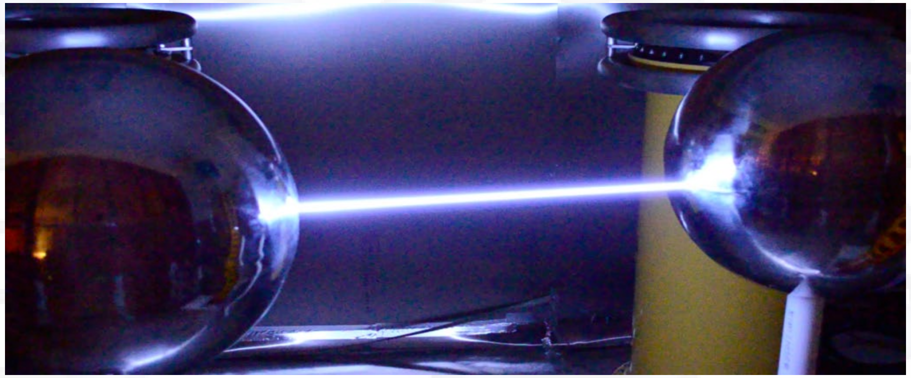
Some general thoughts

$$i\frac{\partial U(z,x)}{\partial z} - \Delta_{\underline{x}} U - |U|^{2\sigma} U = 0$$

Continuum systems

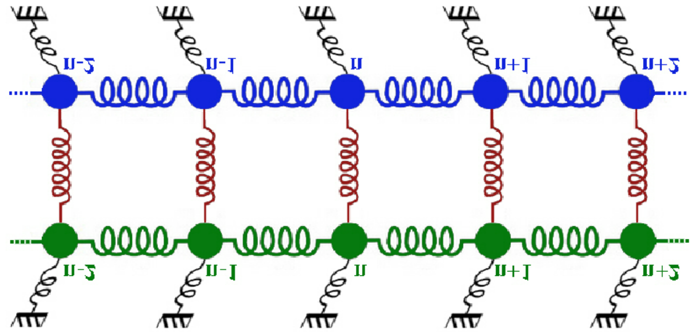


Rogue waves

Light filaments

Kuramoto

Fermi-Pasta-Ulam-Tsingou



Discrete systems

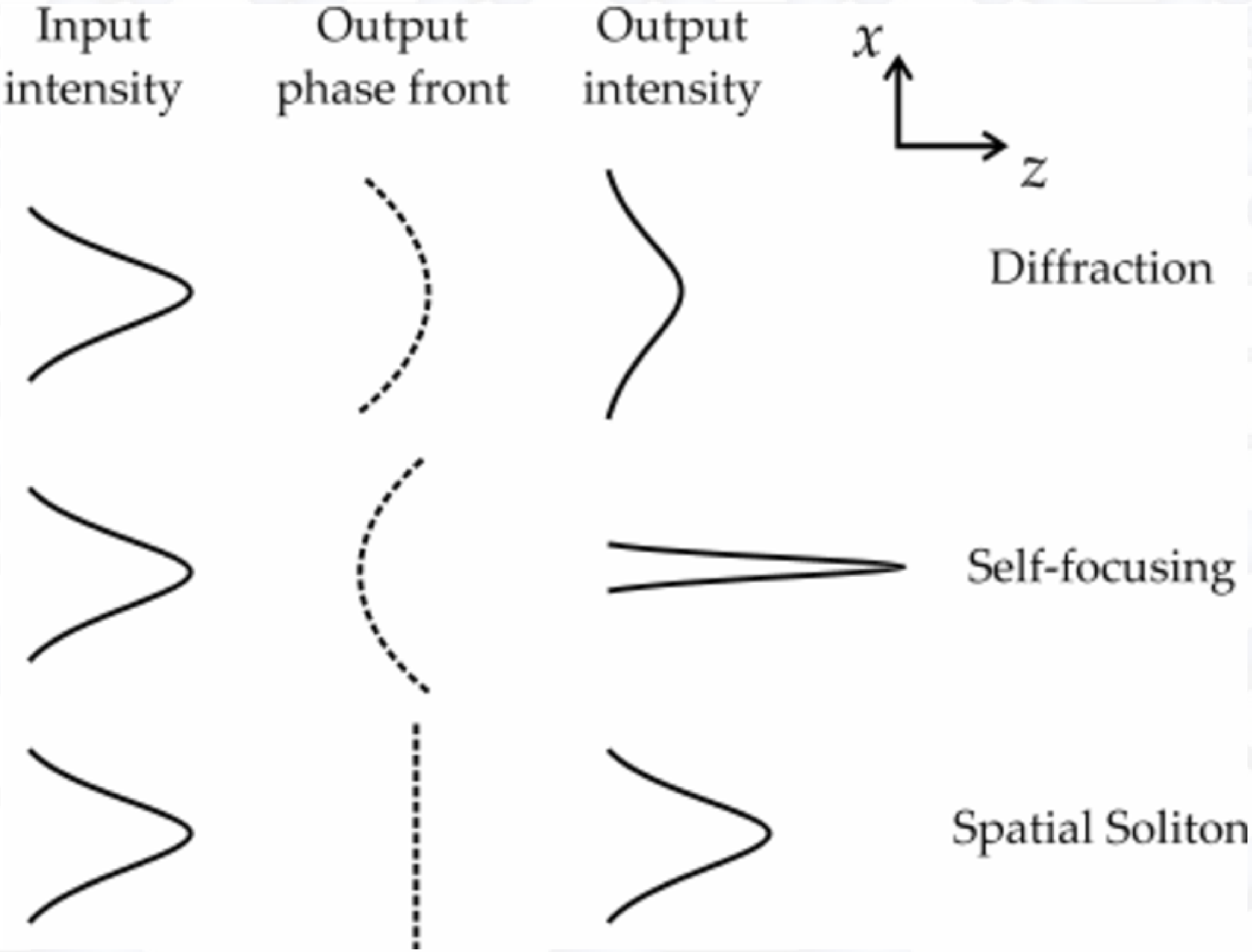
α-model $\quad \ddot{x}_n = (x_{n+1} - 2x_n + x_{n-1}) + \alpha\left[(x_{n+1} - x_n)^2 - (x_n - x_{n-1})^2\right]$

β-model $\quad \ddot{x}_n = (x_{n+1} - 2x_n + x_{n-1}) + \beta\left[(x_{n+1} - x_n)^3 - (x_n - x_{n-1})^3\right]$

Local vs. non-local (global) coupling

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

Diffraction + Nonlinearity



Δ in (x,y) $|k|^2$ In Fourier space

Fractional Laplacian

	Fractional Equation	Domain
Diffusion-Reaction [2]	$\partial_t u + (-\Delta)^{\alpha/2} u + c(t,x)u = 0$	$(0,+\infty) \times \mathbb{R}^d$
Quasi-geostrophic [3]	$\partial_t \theta + u \cdot \nabla\theta + \kappa(-\Delta)^{\alpha/2}\theta = f$	$[0,T] \times \mathbb{R}^2$
Cahn-Hilliard [4, 5]	$\partial_t u + (-\Delta)^{\alpha/2}(-\varepsilon^2 \Delta u + f(u)) = 0$	$(0,T] \times (0,2\pi)^2$
Porous Medium [4, 6]	$\partial_t u + (-\Delta)^{\alpha/2}(\lvert u\rvert^{m-1} \mathrm{sign} u) = 0$	$(0,+\infty) \times \mathbb{R}^d$
Schrödinger [7]	$i\hbar\partial_t \psi = D_\alpha(-\hbar^2\Delta)^{\alpha/2}\psi + V(r,t)\psi$	$(r,t) \in \mathbb{R}^3 \times (0,+\infty)$
Ultrasound [8, 9]	$\frac{1}{c_0^2}\partial_t^2 p = \nabla^2 p - \left\{\tau\partial_t(-\Delta)^{\alpha/2} + \eta(-\Delta)^{(\alpha+1)/2}\right\} p$	$(-\infty,+\infty) \times \mathbb{R}^d$

Table 1: Important equations involving the fractional Laplacian.

Fractional Schrödinger Equation in QM

In quantum mechanics, Laskin* proposed a different generalization of the **Feynman path integral representation** based on Brownian trajectories. **If instead path integrals are replaced by Levy fights, one arrives to what is known as space-fractional quantum mechanics** (SFQM). SFQM provides an interesting fractional physical model in quantum physics, but as with Anderson localization or PT-symmetry, there is no QM-example to date where this principle has been observed. The similarity of Optics and QM at the level of modeling is most evident in the Laplacian spatial operator, which in QM represents the squared momentum operator whereas in optics it models diffraction in free space. In both instances it is difficult To envision the fractional property.

*N. Laskin, Fractional quantum mechanics, Phys. Rev. E 62, 31353145 (2000)

A Lévy Flight for Light

(P. Barthelemy et al, Nature 2008)

- New optical material in which light performs a Lévy flight
- Ideal experimental system to study Lévy flights in a controlled way
- Precisely chosen distribution of glass microspheres of different diameters d

$P(d) \sim d^{-(2+\alpha)}$

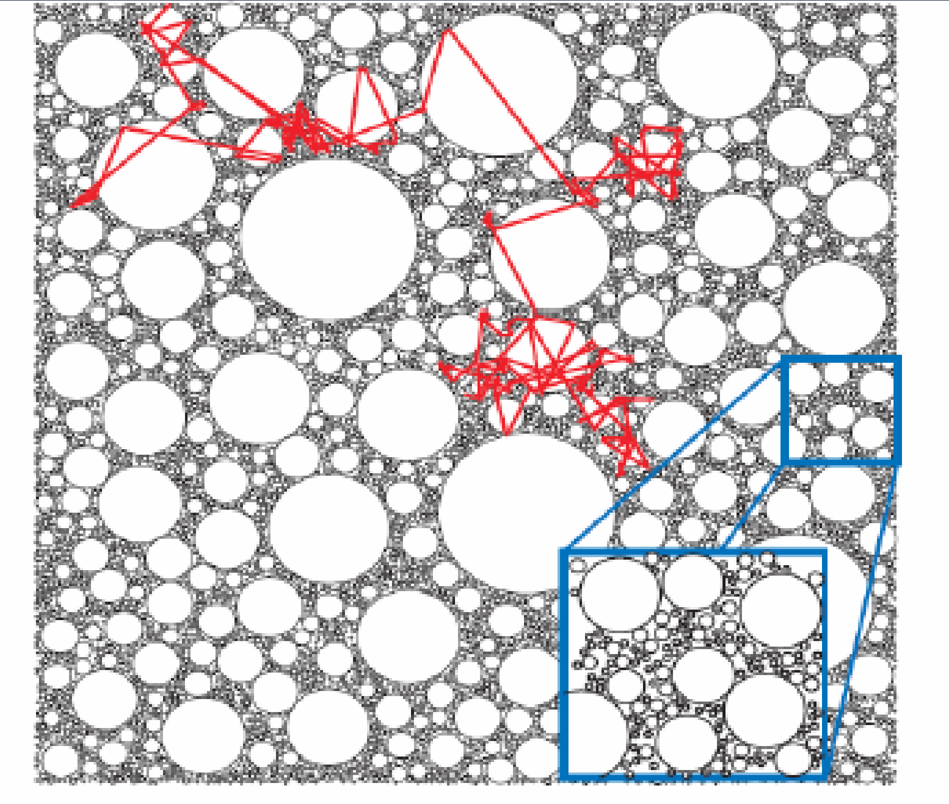
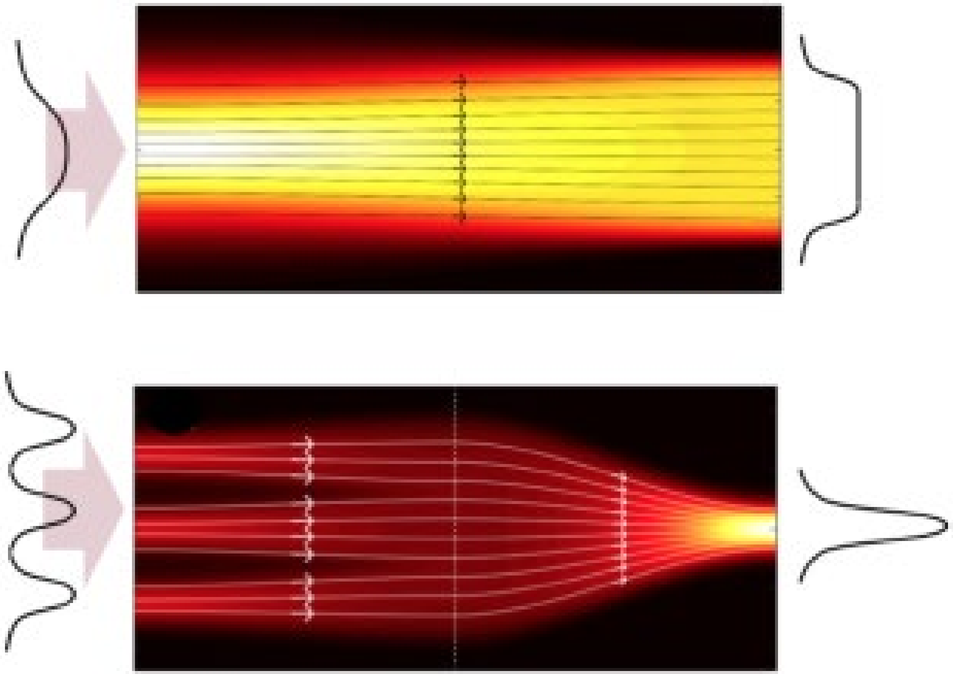


Figure: Lévy walker trajectory in a scale – invarian Levy glass

Work by Dr. Jimmie Adriazola, NJIT USA

Problem 2: Laser Beam Reshaping

Classical Diffraction



Studied by Kunkel and Leger, (2018).

Optimal Control Framework

The optimization problem is to maximize the fidelity:

$$\min_{u \in \mathcal{U}} \left\{ \frac{1}{2} \left(\|\varphi_d\|^4_{L^2(\mathbb{R}^n)} - |\langle \varphi_d, \psi(x,l) \rangle|^2_{L^2(\mathbb{R}^n)} \right) + \frac{\gamma}{2} \int_0^l |\partial_z u|^2 \, dz \right\}$$

subject to Schrödinger's equation

$$i\partial_z \psi + \frac{1}{2}\partial_x^2 \psi - V(x,z)\psi = 0,$$

$$\psi(x,0) = \varphi_0(x),$$

where $\mathcal{U} = \{u \in H^1([0,l]) : u(0) = u_0, u(l) = u_l\}$.

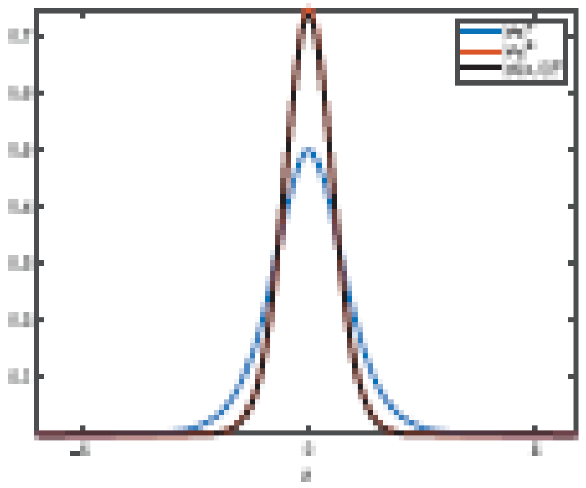
Initial and Desired States

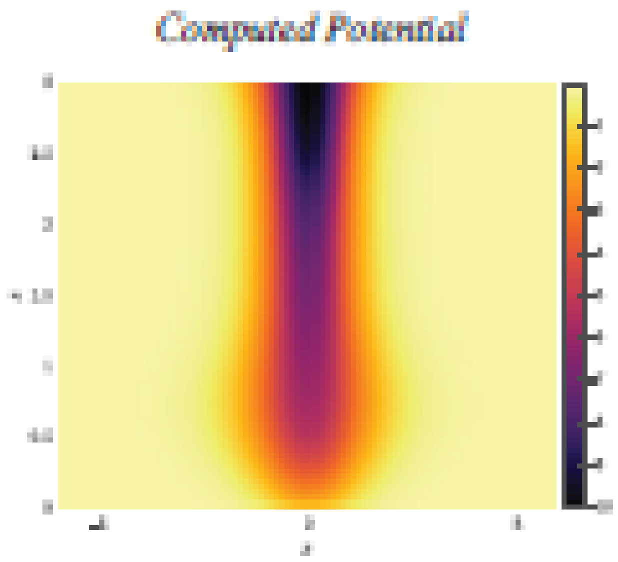
They're eigenfunctions of the form

$$\left(\frac{1}{2}\partial_x^2 - V(x,0)\right)\varphi_0(x) = \lambda_0\varphi_0(x),$$

$$\left(\frac{1}{2}\partial_x^2 - V(x,l)\right)\varphi_d(x) = \lambda_d\varphi_d(x).$$

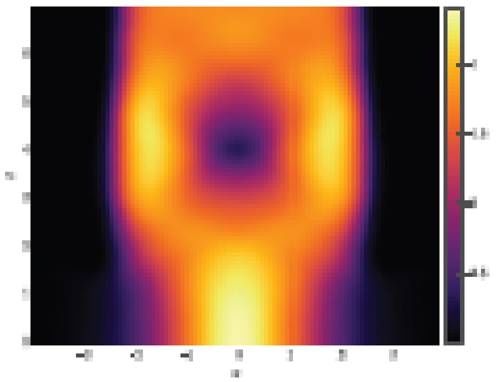
Result 1: Squeezing a Wave



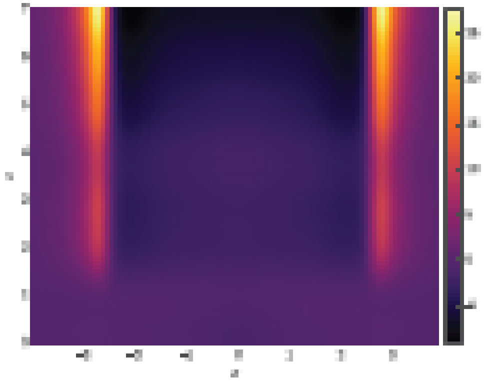


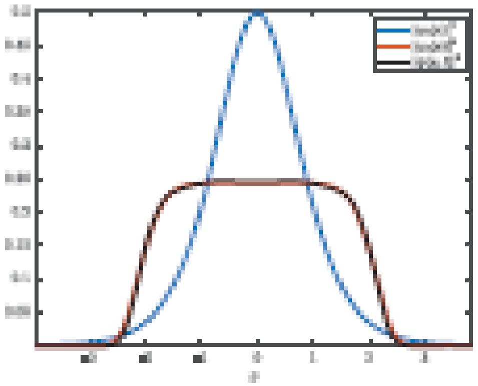
Result 2: Mapping to a Square-like Wave

Solution of Schrodinger's Equation



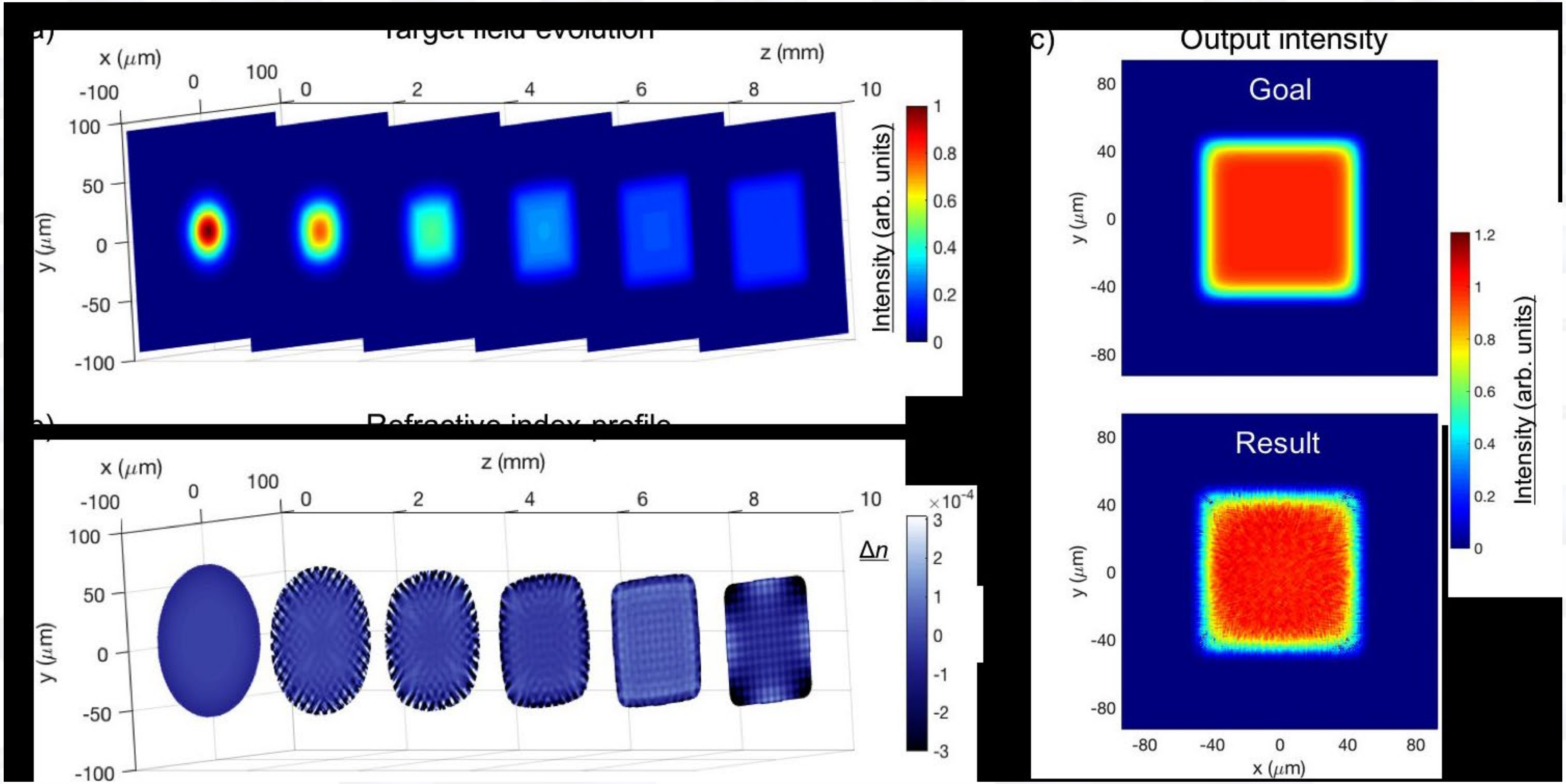
Computed Potential



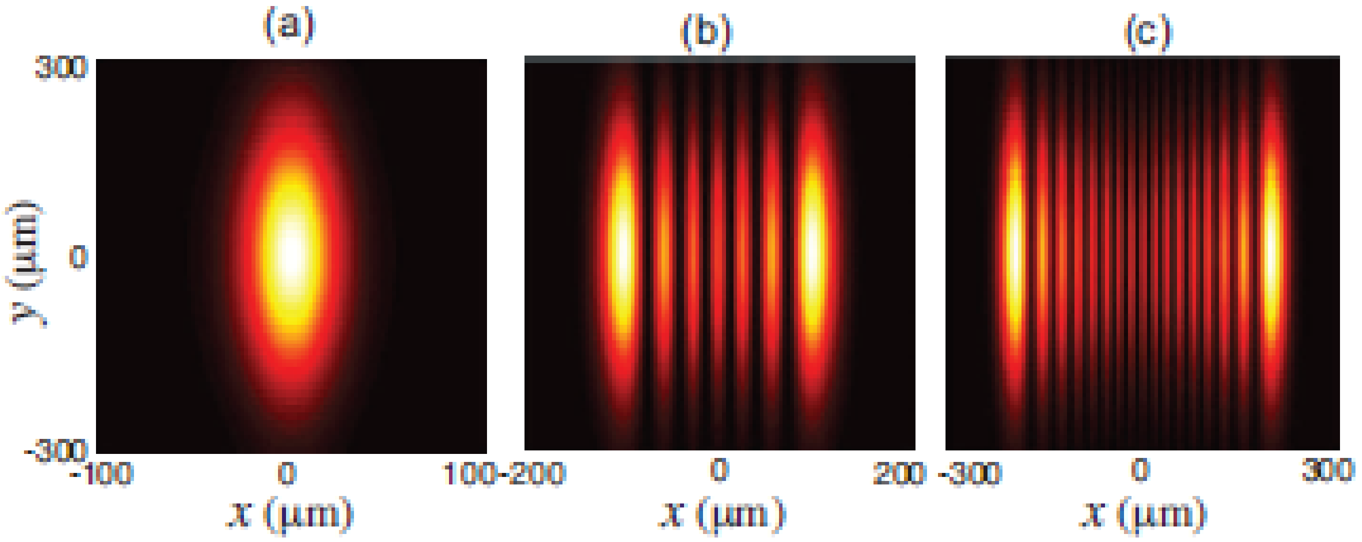


Experiment

W. Minster Kunkel, Ali Ghoreyshi, Glen Douglass, Simon Gross, Michael J. Withford, James R. Leger, "Gradient-index beam shapers: fabricated devices and 3D design method," Proc. SPIE 10518, Laser Resonators, Microresonators, and Beam Control XX, 105181S (16 February 2018); doi:10.1117/12.2290909



What if it is desirable to have a more complex profile



Perhaps introduce fractional diffraction……

…..Perhaps no ! (Neural networks). Fresh from the press

Research Article | Vol. 30, No. 13 / 20 Jun 2022 / *Optics Express* 22798

Optics EXPRESS

Advanced beam shaping for laser materials processing based on diffractive neural networks

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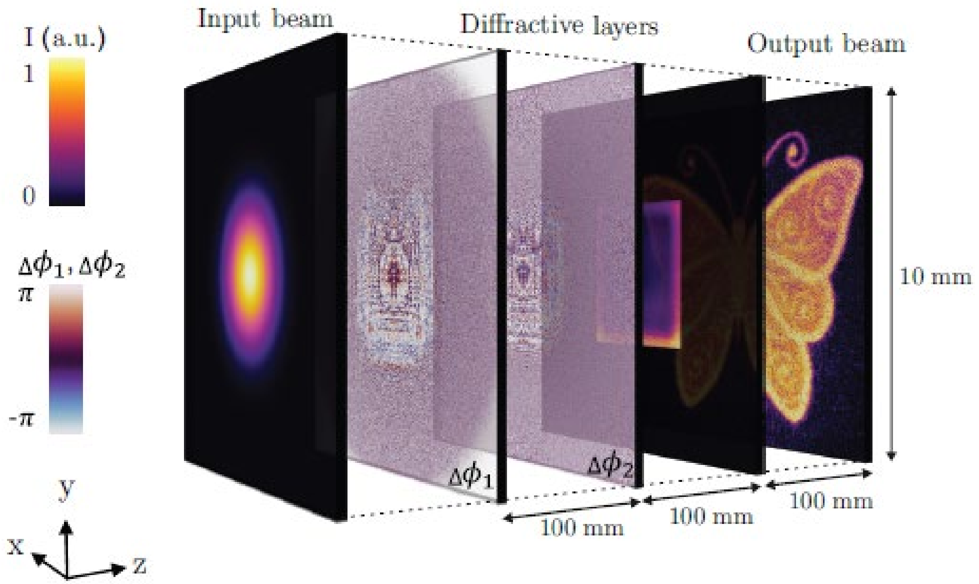
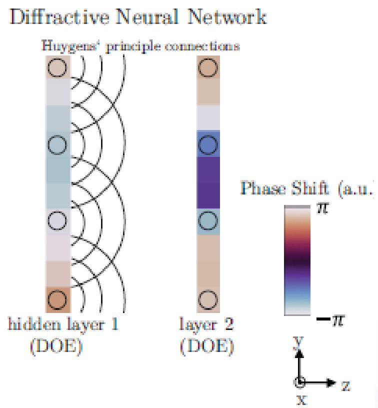


Fig. 1. Example setup with two diffractive layers ($\Delta\Phi_1$ and $\Delta\Phi_2$) and two output layers



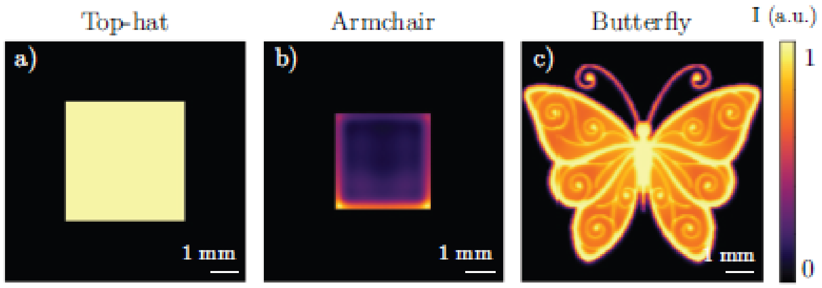
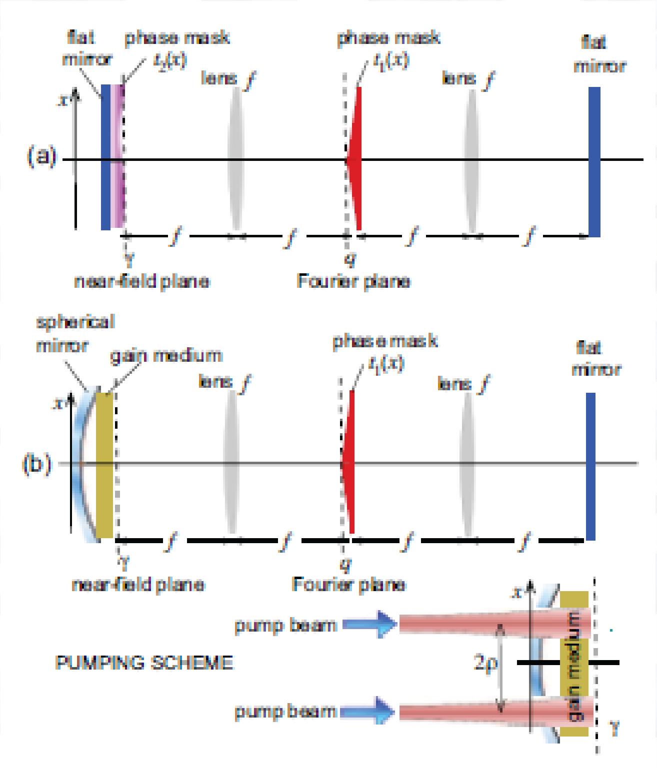
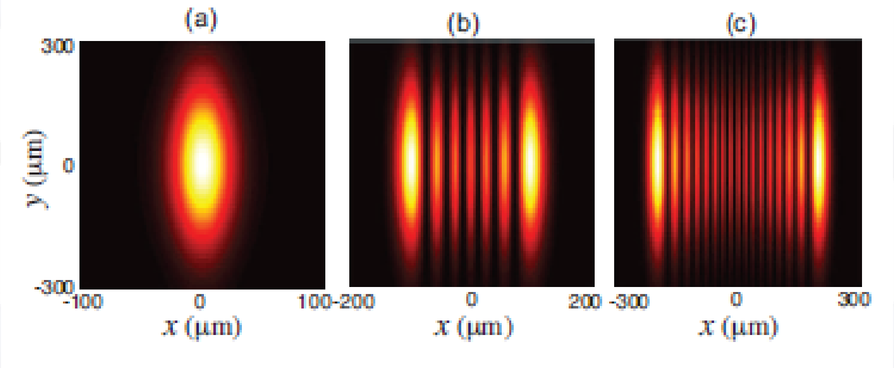


Fig. 3. Target intensity distributions.

Longhi's resonator model:

From Longhi: Fractional Schrödinger equation in optics (OL 2015)





Map

$$\psi^{(n+1)}(x) = \int d\theta K(x,\theta)\,\psi^{(n)}(\theta)d\theta \approx \int \left[(1 - iV(x))\delta(x-\theta) - \hat{f}(x-\theta)\right]\psi^{(n)}(\theta)d\theta$$

$$= (1 - iV(x))\psi^{(n)}(x) - \int \hat{f}(x-\theta)\,\psi^{(n)}(\theta)d\theta$$

$$\frac{\partial\psi(x,T)}{\partial T} = -iV(x)\psi(x,T) - i\int \hat{f}(x-\theta)\,\psi^{(n)}(\theta)d\theta, \quad f(\theta) \propto |\theta|^{\alpha}$$

Mean field PDE

$$i\frac{\partial\psi(x,T)}{\partial T} = \left[D_\alpha\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2} + V(x)\right]\psi(x,T) - i[\gamma(x) + g(x)]\psi$$

Some brief observations for the nonlinear FSE

$$\mathrm{i}\frac{\partial \Psi}{\partial z} - (-\Delta)^{\frac{\alpha}{2}}\Psi \pm \alpha|\Psi|^2\Psi = 0$$

$$i\frac{\partial U}{\partial z} = C_s P.V.\int \frac{U(x,z) - U(y,z)}{|x-y|^{1+2s}}dy \quad 0 < s < \infty$$

Orbital stability, rigorous results

"Stability and instability of standing waves for the fractional nonlinear Schrödinger equations, B Feng, S Zhu, Journal of Differential Equations 292 (2021) 287–324

Abstract

In this paper, we make a comprehensive study for the orbital stability of standing waves for the fractional Schrödinger equation with combined power-type nonlinearities

$$i\partial_t\psi - (-\Delta)^s\psi + a|\psi|^{p_1}\psi + |\psi|^{p_2}\psi = 0. \qquad (FNLS)$$

We prove that when $p_2 = \frac{4s}{N}$ and $a(p_1 - \frac{4s}{N}) < 0$, there exist the standing waves of (FNLS), which are orbitally stable. When $a = 0$ and $\frac{4s}{N} < p_2 < \frac{4s}{N-2s}$, we present a new, simpler method to study the strong instability of standing waves. When $a = -1$, $0 < p_1 < p_2$ and $\frac{4s}{N} \leq p_2 < \frac{4s}{N-2s}$, or $a = 1$ and $\frac{4s}{N} \leq p_1 < p_2 < \frac{4s}{N-2s}$, or $a = 1$, $0 < p_1 < \frac{4s}{N} < p_2 < \frac{4s}{N-2s}$ and $\partial_\lambda^2 S_\omega(u_\omega^\lambda)|_{\lambda=1} \leq 0$, we deduce that the ground state standing waves of (FNLS) are strongly unstable by blow-up.

If $u(\vec{x},t)$ is a solution to the 2d NLS

$$i\partial_t u = -\Delta u + \mu|u|^{p-1}u, \tag{1}$$

then so is $u_\lambda(\vec{x},t) = \lambda^{-\frac{2}{p-1}} u(\frac{\vec{x}}{\lambda}, \frac{t}{\lambda^2})$. For $s_c = 1 - \frac{2}{p-1}$, since

$$\|\lambda^{-\frac{2}{p-1}} f(\frac{\cdot}{\lambda})\|_{\dot{H}^s(\mathbb{R}^2)} = \lambda^{s_c - s}\|f\|_{\dot{H}^s(\mathbb{R}^2)},$$

eq. (1) is well-posed for any initial data in $H^s(\mathbb{R}^2)$ for $s \geq s_c$.
Do we have an analogous scaling symmetry for the mixed FNLS? For

$$i\partial_t u = (D_1^2 + D_2^\alpha)u + \mu|u|^{p-1}u, \tag{2}$$

$u_\lambda(x,y,y) = \lambda^{-\frac{2}{p-1}} u(\frac{x}{\lambda}, \frac{y}{\lambda^{2/\alpha}}, \frac{t}{\lambda^2})$ **defines a one-parameter family of solutions. To account for the mixed dispersion, define the** α**-adapted anisotropic Sobolev space** $H^s_\alpha(\mathbb{R}^2)$ **with the symbol** $(1+\xi^2+|\eta|^\alpha)^{\frac{s}{2}}$ **as the multiplier. Then for** $s_c(\alpha) = \frac{1}{2} + \frac{1}{\alpha} - \frac{2}{p-1}$**,**

$$\|\lambda^{-\frac{2}{p-1}} f(\frac{x}{\lambda}, \frac{y}{\lambda^{2/\alpha}})\|_{\dot{H}^s_\alpha(\mathbb{R}^2)} = \lambda^{s_c(\alpha)-s}\|f\|_{\dot{H}^s_\alpha(\mathbb{R}^2)}.$$

Adapting the proofs of [Hong-Sire '15] and [Dinh '16],

Theorem (Subcritical Well-posedness)

For $\alpha \in (1,2)$, suppose $s \in (\frac{1}{\alpha} - \frac{1}{2}, \lfloor p \rfloor]$ for $p \in (1,3)$, $s \in (s_c, \lfloor p \rfloor]$ for $p \geq 3$ not an odd integer, and $s \in (s_c, \infty)$ for $p \geq 3$, an odd integer. Then, eq. (2) is locally well-posed in $H^s_\alpha(\mathbb{R}^2)$.

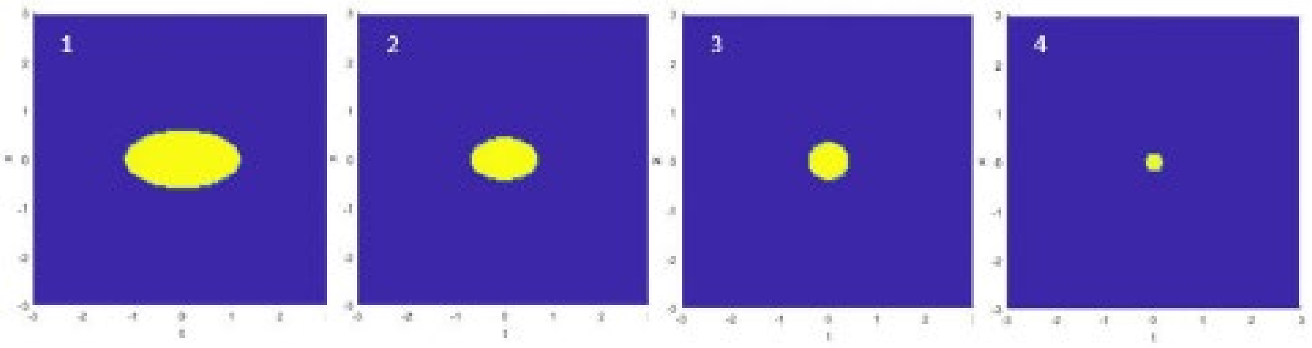
Main differences and difficulties:

- For $\alpha = 2$, the Strichartz estimates (spacetime estimate of the linear evolution) have no derivative loss.
- For $\alpha < 2$, the Strichartz estimates suffer derivative loss (except in L^2), portraying the nonlocal effect.
- For the mixed dispersion and $\alpha < 2$, the competition between the local and nonlocal effects needs to be investigated. Furthermore, the standard results in Fourier analysis may or may not hold due to the non-smooth symbol $(1 + \xi^2 + |\eta|^\alpha)^{1/2}$.

Mixed Nonlinear FSE:

$$i\frac{\partial\Psi}{\partial z} - \left(-\frac{\partial^2}{\partial x^2}\right)^{\frac{\alpha_1}{2}}\Psi - \left(-\frac{\partial^2}{\partial y^2}\right)^{\frac{\alpha_2}{2}}\Psi \pm |\Psi|^2\Psi = 0$$

$\alpha_1 = 0, \quad \alpha_2 = 2$: Integrable NLSE
$\alpha_1 = \alpha_2 = 2$: 2d-NLSE



Possible finite time blow-up event (Austin Copeland, 2020) $\alpha_1 = 1, \quad \alpha_2 = 2$.
Q: Is there a finite time singular blow-up. Is it radially symmetric ?

Q2: What if we have nonlocal coupling in discrete systems ? Continuum limit ?

$$i\frac{d}{dz}u_m = h\sum_{n\neq m}\frac{u_m-u_n}{h^{1+2s}|m-n|^{1+2s}} + \gamma|u_m|^2u_m \qquad 0 < s < \infty,\ u_m = u(z,mh)$$

Long range interactions

Q: Under which conditions limit when h approaches 0 is valid ?

$$i\frac{\partial U(z,x)}{\partial z} = c(-\Delta)^{\alpha}U + \gamma|U|^2U$$

where $(-\Delta)^{\alpha}$ Is the continuous fractional Laplacian.
And what is the relation between α and s ?

- For *s* below 1 , the long-range interactions in the discrete NLS-type equation remain long-range in the continuum limit, producing a fractional NLS with a nonlocal character coming from the Laplacian of order *a* = *s*.

- For *s* above 1, the interaction strength decays quickly enough that only local effects survive in the continuum limit, which is exactly the “classical” NLS, *a* = 1.

- For *s* = 1 , we get the classical NLS in the continuum limit, with a logarithmic factor appearing in the scaling constants

“On the Continuum Limit for Discrete NLS with Long-Range Lattice Interactions”, Kirkpatrick et.al., Commun. Math. Phys. 317, 563–591 (2013)

Our recent work

- Biccari, Umberto and AA, "*WKB expansion for a fractional Schrödinger equation with applications to controllability*", 2018 https://doi.org/10.48550/arxiv.1809.08099.

- A Copeland, A Aceves, "*Spatiotemporal Dynamics in the Fractional Nonlinear Schrödinger Equation*", Proceedings Nonlinear Photonics, 2020.

- Brian Choi, A A, "*Well-posedness of the mixed-fractional nonlinear Schrödinger equation on R2*", Partial Differential Equations in Applied Mathematics, (2022), 100406. https://www.sciencedirect.com/science/article/pii/S2666818122000754

- B Choi, A A, "*On Properties of the mixed-Fractional Nonlinear Schrödinger Equation*", preprint arXiv:2109.03921, 2021

Conclusions

- Presented a brief mathematical introduction of the concept of fractional Laplacian
- Highlighted some applications in photonics
- Part of the next generation of photonic devices !

Thanks for your attention