On the stability of critical points of the Sobolev inequality

"Calculus of Variations and Functional Inequalities" - FAU Erlangen-Nürnberg

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Sobolev inequality

The Sobolev inequality states that, for any $u \in \dot{H}^1(\mathbb{R}^n)$, we have

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Equality is achieved by Talenti bubbles (Aubin 1976, Talenti 1976) Given $z \in \mathbb{R}^n$ and $\lambda > 0$, the Talenti bubble with center z and concentration λ is given by

$$U[z, \lambda] := \big(n(n-2)\big)^{\frac{n-2}{4}} \lambda^{\frac{n-2}{2}} \frac{1}{(1 + \lambda^2 |x - z|^2)^{\frac{n-2}{2}}}.$$

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Remark: The scalar coefficient in the definition of U guarantees that $\|\nabla U[z,\lambda]\|_{L^2}$ $(= S^{n/2})$ and $\|U[z,\lambda]\|_{L^{2^*}}$ do not depend on z and λ.

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For any $u \in \dot{H}^1(\mathbb{R}^n)$, it holds

$$\int_{\mathbb{R}^n} |\nabla u|^2 - S^2\Big(\int_{\mathbb{R}^n} u^{2^*}\Big)^{\frac{2}{2^*}} \gtrsim \operatorname{dist}_{\dot{H}^1}(u, \mathcal{M}_{TB})^2,$$

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where $\mathcal{M}_{TB}$ denotes the manifold of Talenti bubbles, i.e.,

$$\operatorname{dist}_{\dot{H}^1}(u, \mathcal{M}_{TB}) \coloneqq \inf_{c\in\mathbb{R}, z\in\mathbb{R}^n, \lambda>0} \Big\|\nabla\Big(u - cU[z,\lambda]\Big)\Big\|_{L^2}.$$

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Remark: This result is sharp both with respect to the exponent and with respect to the choice of the distance.

Critical points

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Hence $\mathrm{d}F(u)[\varphi] = 0$ if and only if $\int_{\mathbb{R}^n} \nabla u \cdot \nabla\varphi = \lambda \int_{\mathbb{R}^n} |u|^{2^*-2}u\varphi$ (where λ depends on u but not on φ),

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$$\mathrm{d}F(u) = 0 \iff -\Delta u = \lambda |u|^{2^*-2} u.$$

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If $u \in \dot{H}^1(\mathbb{R}^n)$ solves (EL) and $u \geq 0$ in $\mathbb{R}^n$, then u is a Talenti bubble.

Thus, the positive critical points of the Sobolev inequality are exactly the Talenti bubbles, so critical points and minimizers coincide.

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Yes, as was shown in Struwe 1984.

Qualitative stability of critical points

Theorem

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$$\inf_{(z_i,\lambda_i)_{1\leq i\leq k}} \left\| \nabla \Big(u - \sum_{i=1}^{k} U[z_i, \lambda_i] \Big) \right\|_{L^2} = \omega\big(\|\Delta u + u^{2^*-1}\|_{H^{-1}} \big),$$

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for a suitable modulus of continuity ω (which does not depend on u). Moreover the Talenti bubbles $U[z_1, \lambda_1], U[z_2, \lambda_2], \ldots, U[z_k, \lambda_k]$ are weakly interacting.

Quantitative stability – Single bubble case

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Remark: While this result paved the way for the multibubble case, the latter is significantly more delicate and features unexpectedly different behaviors depending on the dimension.

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for some $k \in \mathbb{N}$ (which represents the number of bubbles). Then, there exist k Talenti bubbles $U_1, U_2, \dots, U_k$ such that

$$\left\| \nabla \Big(u - \sum_{i=1}^{k} U_i \Big) \right\|_{L^2} \lesssim \| \Delta u + u^{2^*-1} \|_{H^{-1}}.$$

Moreover, the interaction between the bubbles can be estimated by

$$\int_{\mathbb{R}^n} U_i^{2^*-1} U_j \lesssim \| \Delta u + u^{2^*-1} \|_{H^{-1}}.$$

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$$\left\| \nabla \Big(u - \sum_{i=1}^{k} U_i \Big) \right\|_{L^2} \lesssim \Phi(\|\Delta u + u^{2^*-1}\|_{H^{-1}}),$$

where $\Phi(t) := t|\log(t)|^{\frac{1}{2}}$ if $n = 6$ and $\Phi(t) = t^{(n+2)/(2(n-2))}$ for $n \geq 7$.

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By computing the first variation, we deduce that $\rho \perp_{\dot{H}^1} U_i, \partial_z U_i, \partial_\lambda U_i$.

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2. The second term will be controlled through a *nondegeneracy condition* (using that ρ is orthogonal to the first eigenfunctions of a spectral operator).

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$$+ \|\nabla\rho\|_{L^2}\sum_{i\neq j}\int U_i^{2^*-1}U_j \leq \|\nabla\rho\|_{L^2}\cdot\Big(\varepsilon\|\nabla\rho\|_{L^2} + C\|\Delta u + u^{2^*-1}\|_{H^{-1}}\Big)$$

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2. The second term will be controlled through a *nondegeneracy condition* (using that ρ is orthogonal to the first eigenfunctions of a spectral operator).
3. The third term is expected to be small because it involves interactions between different bubbles, which should be weakly interacting. It is the hard term to control.

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Bianchi-Egnell 1991 show that 1 and $2^* - 1$ are the first two eigenvalues of the operator and the corresponding eigenfunctions are $U, \partial_\lambda U, \partial_z U$. Since ρ is orthogonal to all of them, (ND) follows.

Quantitative stability – Hard term

It remains to prove, for $i \neq j$,

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This is the core of the proof and is done by induction over i, starting from the most concentrated bubble U_i and through a delicate localization procedure.

Ongoing work and future developments

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- The analogous result for the **isoperimetric inequality** is an interesting open problem. Currently it is known that if a domain has *almost constant mean curvature* then it is close to a *necklace of spheres*. The sharp quantitative version of this result with the natural norms is not known.

Thank you for your attention

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