Optimization and stability problems for eigenvalues of linear and non linear operators

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Isoperimetric Problems

Classical Isoperimetric Inequality

Let $n \geq 2$. Balls have maximal measure among Borel sets of $\mathbb{R}^n$ with finite Lebesgue measure of given perimeter, that is

$$V(\Omega) \leq V(\Omega^*),$$

where Ω^* is the ball such that $P(\Omega) = P(\Omega^*)$. We denote by $V(\cdot)$ the volume and by $P(\cdot)$ the perimeter of a set. Moreover, equality holds if and only if Ω is a ball.

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- A (not exhaustive) list of References: De Giorgi (*Atti Accad. Naz. Lincei*, 1958), Osserman (*Bull. Amer. Math. Soc.*, 1979), Talenti (*Handbook of convex geom.*, 1993), Chavel (*Cambridge Tract. in Math.*, 2001), Fusco (*Bull. Math. Sci.*, 2015).

Dirichlet boundary condition

Dirichlet boundary condition

Let $\Omega \subseteq \mathbb{R}^n$, with $n \geq 2$, be an open set with finite Lebesgue measure. The first Dirichlet eigenvalue of Ω is the least positive λ such that

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits non-trivial solution in $H_0^1(\Omega)$. Let us denote by $\lambda_1(\Omega)$ the first Dirichlet eigenvalue.

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Faber-Krahn inequality [Faber, 1923; Krahn, 1925; Pólya and Szegö, 1951]

Let $\Omega \subseteq \mathbb{R}^n$ be an open set with finite Lebesgue measure, then

$$\lambda_1(\Omega)V(\Omega)^{2/n} \geq \lambda_1(B)V(B)^{2/n},$$

and there is equality if and only if Ω is equivalent to a ball.

Neumann boundary condition

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Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open and Lipschitz domain; the first non-zero Neumann eigenvalue of Ω is the least strictly positive μ such that

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

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Szegö-Weinberger inequality [Szegö, 1954, Weinberger, 1956]

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open and Lipschitz domain. Then

$$\mu_2(\Omega) V(\Omega)^{2/n} \leq \mu_2(B) V(B)^{2/n}$$

and there is equality if and only if Ω is equivalent to a ball.

The Stability Issue

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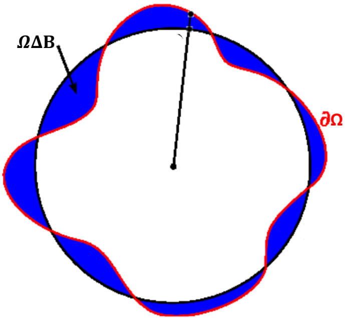
that is:

If the differences $P(\Omega) - P(\Omega^\sharp)$, $\lambda_1(\Omega) - \lambda_1(\Omega^\sharp)$ or $\mu_2(\Omega^\sharp) - \mu_2(\Omega)$ are small, can we say that Ω is "close" to a ball? And in what sense?

Quantitative Spectral Inequalities

Definition of Fraenkel Asymmetry

$$\mathcal{A}_F(\Omega) := \inf_{x\in\mathbb{R}^n} \left\{ \frac{V(\Omega\Delta B_R(x))}{V(B_R(x))}\ ,\ V(B_R(x)) = V(\Omega) \right\}.$$



Quantitative spectral inequalities

Quantitative Isoperimetric Inequality [Fusco- Maggi-Pratelli, *Ann. of Math.*, 2008]

Let $\Omega \subseteq \mathbb{R}^n$ set of finite measure

$$V(\Omega)^{(1-n)/n}P(\Omega) - V(B)^{(1-n)/n}P(B) \geq \alpha_n \mathcal{A}_F(\Omega)^2,$$

The exponent 2 is sharp

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- History: Bernstein, 1905; Bonnensen 1924; Hadwiger, 1948; Fuglede, 1989; Hall, 1992...
- New proofs: Fusco-Maggi-Figalli, 2010; Cicalese-Leonardi, 2013

Quantitative spectral inequalities

Quantitative Faber-Krahn [Brasco-De Phillippis-Velichkov, *Duke Math. J.*, 2015]

Let $\Omega \subseteq \mathbb{R}^n$ set of finite measure

$$V(\Omega)^{2/n}\lambda_1(\Omega) - V(B)^{2/n}\lambda_1(B) \geq \beta_n \mathcal{A}_F(\Omega)^2.$$

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- Melas, 1992;
- Hansen-Nadirashvili, 1994;
- Bhattacharya, 2001;
- Fusco-Maggi-Pratelli, 2009.

Quantitative spectral inequalities

Quantitative Szegö-Weinberger [Brasco-Pratelli, *Geometric and Functional Anal.*, 2012]

Let $\Omega \subseteq \mathbb{R}^n$ open set with Lipschitz boundary

$$V(B)^{2/n}\mu_2(B) - V(\Omega)^{2/n}\mu_2(\Omega) \geq \gamma_n \mathcal{A}_F(\Omega)^2,$$

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- Nadirashvili, 1997.

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First non-zero Steklov eigenvalue

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded, connected, open set with Lipschitz boundary.

The first non-zero Steklov eigenvalue of Ω is defined by

$$\sigma(\Omega) := \min \left\{ \frac{\displaystyle\int_\Omega |\nabla v|^2 dx}{\displaystyle\int_{\partial\Omega} v^2 d\sigma_x} : v \in H^1(\Omega) \setminus \{0\}, \int_{\partial\Omega} v \, d\sigma_x = 0 \right\}.$$

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Any minimizer satisfies

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\Omega. \end{cases}$$

First non-zero Steklov eigenvalue

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The sequence of Steklov eigenvalues

$$0 = \sigma_1(\Omega) < \sigma_2(\Omega)(= \sigma(\Omega)) \le \sigma_3(\Omega) \le \sigma_3(\Omega) \cdots \nearrow +\infty$$

as in the Neumann case, starts with zero.

First non-zero Steklov eigenvalue

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \\ \dfrac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\Omega \end{cases}$$

- $\sigma(\Omega)$ is invariant under translations;
- $\sigma(t\Omega) = t^{-1}\sigma(\Omega)$.

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Weinstock inequality in dimension 2

Theorem [Weinstock, *J. Rational Mech. Anal.*, 1954]

If $\Omega \subseteq \mathbb{R}^2$ is a bounded, Lipschitz simply connected open set, then

$$\sigma(\Omega)P(\Omega) \leq \sigma(B)P(B), \tag{1}$$

where $P(\Omega)$ stands for the perimeter of Ω and $B \subseteq \mathbb{R}^2$ is a ball. Equality holds if and only if Ω is a ball.

In other words: "among all simply connected sets of $\mathbb{R}^2$ with prescribed perimeter, the disc maximises the first non-zero Steklov eigenvalue".

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Remark (Girouard-Polterovich, *J. Spectral Theory*, 2017)

Weinstock inequality fails for planar domains which are not simply connected. Namely, one can find an annulus $\Omega_\varepsilon = B_1 \setminus \overline{B}_\varepsilon$, $\varepsilon \approx 0$, such that

$$\sigma(\Omega_\varepsilon)P(\Omega_\varepsilon) > \sigma(B)P(B).$$

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The isoperimetric inequality in (1) gives

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What about the n-dimensional case, $n \geq 3$?

Brock-Weinstock inequality in $\mathbb{R}^n$

Theorem [Brock, *ZAMM*, 2001]

For every Lipschitz bounded open set $\Omega \subseteq \mathbb{R}^n$, it holds true

$$\sigma(\Omega)V(\Omega)^{\frac{1}{n}} \leq \sigma(B)V(B)^{\frac{1}{n}}.$$

The equality holds iff Ω is a ball.

In other words: "Among all Lipschitz sets of $\mathbb{R}^n$ with prescribed volume, balls maximise the first non-zero Steklov eigenvalue".

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The equality holds iff Ω is a ball.

In other words: "Among all Lipschitz sets of $\mathbb{R}^n$ with prescribed volume, balls maximise the first non-zero Steklov eigenvalue".

Theorem [Brasco-De Philippis-Ruffini, *J. Funct. Anal.*, 2012]

For every $\Omega \subset \mathbb{R}^n$, bounded, Lipschitz open set, there exists a positive constant $C = C(n)$ such that it holds

$$V(B)^{\frac{1}{n}}\sigma(B) - V(\Omega)^{\frac{1}{n}}\sigma(\Omega) \geq C(n)\mathcal{A}_F(\Omega)^2.$$

The exponent 2 is sharp.

Weinstock inequality in $\mathbb{R}^n$

Weinstock inequality in $\mathbb{R}^n$

Theorem [Bucur-Ferone-Nitsch-Trombetti, *J. Differential Geom.*, 2018]

Let Ω be a bounded, open and convex set of $\mathbb{R}^n$. Then

$$\sigma(\Omega)P(\Omega)^{\frac{1}{n-1}} \leq \sigma(B)P(B)^{\frac{1}{n-1}}.$$

Equality holds only if Ω is a ball.

The above inequality cannot hold for simply connected sets in $\mathbb{R}^n$. Namely, one can find a spherical shell $\Omega_\varepsilon = B_1 \setminus \overline{B}_\varepsilon$, $\varepsilon \approx 0$, (B_r denotes the ball of radius r centered at the origin) such that

$$\sigma(\Omega_\varepsilon)P(\Omega_\varepsilon)^{\frac{1}{n-1}} > \sigma(B)P(B)^{\frac{1}{n-1}}.$$

Stability of the Weinstock Inequaliy

Theorem [Gavitone-La Manna - P. - Trani, *Calc. Var.*, 2019]

Among open, bounded and convex sets Ω, we have

- for $n = 2$
$$P(B)\sigma(B) - P(\Omega)\sigma(\Omega) \geq C\mathcal{A}^{5/2}(\Omega);$$
- for $n = 3$
$$P(B)^{1/2}\sigma(B) - P(\Omega)^{1/2}\sigma(\Omega) \geq C\, g\left(\mathcal{A}^2(\Omega)\right),$$
where g is the inverse function of $f(t) = t\log\left(\frac{1}{t}\right)$, for $0 < t < e^{-1}$;
- for $n \geq 4$
$$P(B)^{1/(n-1)}\sigma(B) - P(\Omega)^{1/(n-1)}\sigma(\Omega) \geq C\mathcal{A}(\Omega)^{(n+1)/2}.$$

Moreover, all the exponents are sharp.

Hausdorff Distance

- Definition of Hausdorff distance between two convex sets of $\mathbb{R}^n$:

$$d_{\mathcal{H}}(C, K) := \inf \{\varepsilon > 0 \ : \ C \subset K + B_\varepsilon, \ K \subset C + B_\varepsilon\},$$

where B_ϵ a ball of radius ϵ and $+$ the Minkowski sum between sets, i.e.

$$A + B = \{x + y \mid x \in A, \ \ y \in B\}.$$

Definition of Spherical Asymmetry:

$$\mathcal{A}(\Omega) := \min_{x \in \mathbb{R}^n} \left\{ \left(\frac{d_{\mathcal{H}}\left(\Omega, B_R(x)\right)}{R} \right), \, P(B_R(x)) = P(\Omega) \right\}.$$

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The Torsion Problem with Dirichlet boundary conditions

Let $\Omega \subset \mathbb{R}^n$ an open, bounded and convex set. The torsional rigidity $T(\Omega)$ is defined as

$$T(\Omega) = \int_\Omega u(x)\, dx,$$

where u is the unique solution of the PDE problem

$$\begin{cases} -\Delta u(x) = 1 & \text{in } \Omega \\ u \in H_0^1(\Omega). \end{cases}$$

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Variational characterization of Torsional Rigidity

$$T(\Omega) = \max_{\substack{\varphi \in H_0^1(\Omega) \\ \varphi \not\equiv 0}} \frac{\left(\int_\Omega \varphi(x)\, dx\right)^2}{\int_\Omega \nabla\varphi(x)^2\, dx}$$

Lower estimate for the Torsion in terms of area and perimeter

We recall the following scaling propertiesfor every $t > 0$:

$$V(t\Omega) = t^n V(\Omega), \qquad P(t\Omega) = t^{n-1} P(\Omega)$$

and

$$T(t\Omega) = t^{n+2} T(\Omega).$$

Theorem [Pólya, *J. Indian Math Soc*, (1960)]

Let Ω be an open, bounded and convex set of $\mathbb{R}^n$. It holds:

$$\frac{T(\Omega)P^2(\Omega)}{V(\Omega)^3} \geq \frac{1}{3}$$

and the equality sign is attained by a sequence of thinning cylinders.

- Pólya, *J. Indian Math Soc*, (1960);
- Fragalá-Gazzola-Lamboley, *Geom. for parabolic and elliptic PDE's*, (2013);
- Della Pietra-Gavitone, *Math. Nachr.*, 2014;

Some Definitions

Let us denote by w_Ω the minimal width and by $\mathrm{diam}(\Omega)$ the diameter of Ω.

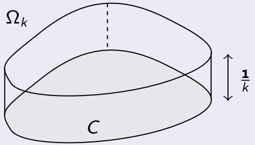
Definition

- Let Ω_k be a sequence of open, bounded and convex sets of $\mathbb{R}^n$. We say that Ω_k is a sequence of thinning domains if

$$\frac{w_{\Omega_k}}{\mathrm{diam}(\Omega_k)} \xrightarrow{k\to 0} 0.$$

- In particular, if $k > 0$ and C is an open, bounded and convex set of $\mathbb{R}^{n-1}$, then, if $k \to 0$, the sequence

$$\Omega_k = C \times \left[-\frac{1}{2k}, \frac{1}{2k}\right]$$



is called a sequence of thinning cylinders. Moreover, in the case $n = 2$ the above sequence is called sequence of thinning rectangles.

A first quantitative result

Theorem 1 [Amato, Masiello, P., Sannipoli, preprint on arxiv (2021)]

Let Ω be an open, bounded and convex set of $\mathbb{R}^2$ and let $f \equiv 1$. Then,

$$\frac{T(\Omega)P^2(\Omega)}{V(\Omega)^3} - \frac{1}{3} \geq K(2)\frac{w_\Omega}{\mathrm{diam}(\Omega)},$$

where $K(2)$ is a positive constant that can be computed explicitly.

Moreover, the exponent of the quantity $\dfrac{w_\Omega}{\mathrm{diam}(\Omega)}$ is sharp.

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- Generalization to the case of the p-Laplacian, with

$$K(p) = \frac{(p-1)p}{2^{\frac{p}{p-1}}3(3p-2)(2p-1)}.$$

- Generalization in dimension $n > 2$

A second quantitative result in the planar case

Theorem 2 [Amato, Masiello, P., Sannipoli, preprint on arxiv (2021)]

Let Ω be an open, bounded and convex set of $\mathbb{R}^2$. Then, there exists a positive constant M such that

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} - \frac{1}{3} \geq M\left(\frac{|\Omega \triangle Q|}{|\Omega|}\right)^3,$$

where $\Omega \triangle Q$ denotes the symmetric difference between Ω and a rectangle Q with sides $P(\Omega)/2$ and w_Ω, containing Ω.

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- Sharpness of the exponent of the asymmetry?
- Extend the second quantitative results contained in Theorem 2 in dimension $n > 2$?

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A quantitative isoperimetric type inequality for the Dirichlet Laplacian in terms of the perimeter

The starting point is the following conjecture.

Conjecture [Fthoui-Lamboley, 2020, SIAM]

Let $\Omega \subseteq \mathbb{R}^2$ an open and convex sets such that $V(\Omega) = 1$, then

$$\lambda_1(\Omega) - \lambda_1(B) \geq \beta \left(P(\Omega) - P(B)\right)^{3/2},$$

where $B \subseteq \mathbb{R}^2$ is a ball of area 1, $\beta := \frac{4 \cdot 3^{3/2}\, \zeta(3)}{\pi^{11/4}}$ and $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ is the Riemann zeta function.

- Analytic support: [Grinfeld- Strang, *Journal of Math. Anal. and Appl.*, 2012; Molinari, *Journal of Physics* , 1997] Let P_k^* be the regular polygon with k edges and area equal to 1. Then, as k goes to $+\infty$,

$$\lambda_1(P_k^*) - \lambda_1(B) \sim \beta\,(P(P_k^*) - P(B))^{3/2}\,.$$

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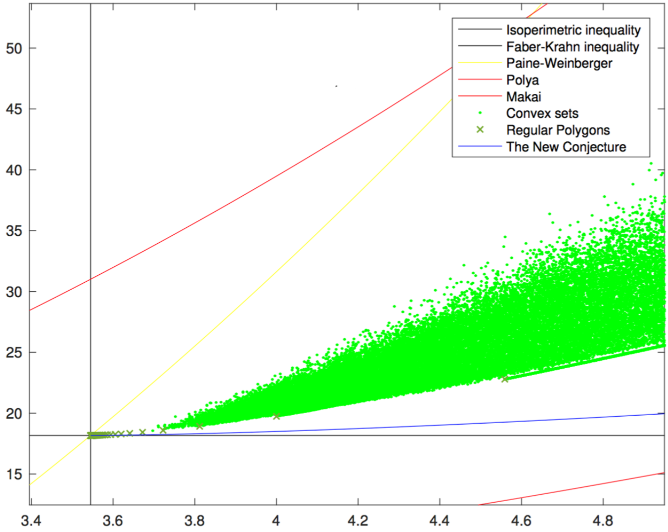
$$\lambda_1(P_k^*) - \lambda_1(B) \sim \beta\,(P(P_k^*) - P(B))^{3/2}\,.$$

- Numerical support: [Fthoui-Lamboley, 2020, preprint] the Blaschke-Santaló diagram for the triplet $(P(\cdot),\,\lambda_1(\cdot),\,V(\cdot))$, that is the sets of points

$$\{(P(\Omega),\lambda_1(\Omega))\ |\ V(\Omega)=1,\ \Omega\subset\mathbb{R}^2,\ \ \Omega\,\text{convex}\}.$$

Blaschke-Santaló Diagram (Fthoui-Lamboley, SIAM)



- We define the following class of admissible sets, with $n \geq 2$:

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Theorem [P., *Rend. Lincei*, 2021]

Let $n \geq 2$; there exists a constant $c > 0$, depending only on n, such that, for every $\Omega \in \mathcal{C}_n$, it holds

$$\lambda_1(\Omega) - \lambda_1(B) \geq c \left(P(\Omega) - P(B)\right)^2.$$

Intermediate Step

Main Ingredients:

- Quantitative Faber-Krahn [Brasco-De Philippis-Velichkov, 2015] For every open set Ω with $V(\Omega) = 1$, it holds

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The target power would be $\delta = 3/4$, but unfortunately the best power is $\delta = 1$ and the "bad" sets are the polygons:

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Thank you for your attention!