Asymptotic analysis of partially and locally dissipated hyperbolic systems

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Partially dissipative systems

Introduction

We look at $n$-component linear hyperbolic systems of the form:

\[
\frac{\partial V}{\partial t} + \sum_{j=1}^{d} A_j \frac{\partial V}{\partial x_j} = -\frac{LV}{\varepsilon}.
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Such that:

- the matrices $A^j$ are symmetric $\rightarrow$ Hyperbolicity of the system
- $L = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$ with $D > 0$ $\rightarrow$ Partial dissipation
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These systems models physical phenomena with finite speed of propagation or equilibrium laws, such as the compressible Euler equation with damping:

$$\begin{cases} 
\partial_t \rho + \text{div}(\rho u) = 0, \\
\partial_t u + u \cdot \nabla u + \nabla P(\rho) + \frac{u}{\varepsilon} = 0.
\end{cases}$$

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We are interested in the following questions:

- Limit as \( \varepsilon \to 0 \)?
- Behavior as \( t \to \infty \)?
Global existence of solutions

Q: Since the dissipation is only present in some equations of the system, how can one ensure the global existence of solutions?

As a toy-model, let us look at the damped p-system

\[
\begin{align*}
\partial_t u + \partial_x v &= 0, \\
\partial_t v + \partial_x u + v &= 0.
\end{align*}
\]

For this simple system, performing standard energy estimates leads to:

\[
dt \| (u, v) \|^2_{L^2} + \| v \|^2_{L^2} \leq 0 \rightarrow \text{no time-decay information on } u.
\]

Idea: consider the following perturbed functional

\[
L_2 = \| (u, v, \partial_x u, \partial_x v) \|^2_{L^2} + \int_R v \partial_x u,
\]

which allows to recover dissipation properties on all the components. Indeed, after basic computations, we obtain

\[
dt L_2 + \| v \|^2_{L^2} + \| (\partial_x u, \partial_x v) \|^2_{L^2} \leq 0.
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And since

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\[\mathcal{L}^2 = \|(u, v, \partial_x u, \partial_x v)\|_{L^2}^2 + \int_{\mathbb{R}} v \partial_x u,\]

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Indeed, after basic computations, we obtain

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\frac{d}{dt} \mathcal{L}^2 + \|v\|_{L^2}^2 + \|\partial_x u, \partial_x v\|_{L^2}^2 \leq 0.
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And since \( \mathcal{L}^2 \sim \|(u, v, \partial_x u, \partial_x v)\|_{L^2}^2 \), we can obtain time-decay estimates.
For the general system, the idea is the same if one assume the (SK) condition:

\[ \forall \xi \in \mathbb{R}^d, \quad \ker L \cap \{\text{eigenvectors of } \sum_j A_j \xi_j \} = \{0\}. \quad (SK) \]
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Such condition is actually equivalent the Kalman rank condition and inspired by hypercoercivity theory, Beauchard and Zuazua defined

\[
L^2 \triangleq \|V\|_{L^2}^2 + \int_{\mathbb{R}^d} \min(\rho, \rho^{-1}) I \quad \text{where} \quad I \triangleq \sum_{k=1}^{n-1} \mathbb{E}_k (L \omega_{k-1} V \cdot L \omega_k \tilde{V})
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as a Lyapunov function to recover the decay estimates.
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\]

as a Lyapunov function to recover the decay estimates.

Again, we obtain

\[
\frac{d}{dt} \mathcal{L} + \kappa \min(1, |\xi|^2) \mathcal{L} \leq 0
\]
With this estimates at hand, one deduces the global existence of small $H^s$ solutions and

$$
\| V^h(t) \|_{L^2(\mathbb{R}^d, \mathbb{R}^n)} \leq C e^{-\lambda t} \| V_0 \|_{L^2(\mathbb{R}^d, \mathbb{R}^n)},
$$

$$
\| V^\ell(t) \|_{L^\infty(\mathbb{R}^d, \mathbb{R}^n)} \leq C t^{-d/2} \| V_0 \|_{L^1(\mathbb{R}^d, \mathbb{R}^n)} \tag{3}
$$

where $V^h$ and $V^\ell$ correspond, respectively, to the high and low frequencies of the solution.
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Moreover, this technique also allow to treat situation when the (SK) condition is not satisfied.

However, these decay estimates do not depict the full story in the low frequencies regime and do not allow to consider the limit as $\varepsilon \to 0$. 
"New" observations

- Back to the damped \( p \)-system:

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\begin{align*}
\partial_t u + \partial_x v &= 0, \\
\partial_t v + \partial_x u + \frac{v}{\varepsilon} &= 0.
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(4)

A spectral analysis of the matrix

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\begin{pmatrix}
0 & i\xi \\
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- In low frequencies ($|\xi| \ll \varepsilon^{-1}$), there are two real eigenvalues $\frac{1}{\varepsilon}$ and $\varepsilon\xi^2$. 
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- In high frequencies ($|\xi| \gg \varepsilon^{-1}$), two complex conjugate eigenvalues coexist, whose real parts are asymptotically equal to $\frac{1}{2\varepsilon}$. 
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- In high frequencies ($|\xi| \gg \varepsilon^{-1}$), two complex conjugate eigenvalues coexist, whose real parts are asymptotically equal to $\frac{1}{2\varepsilon}$.
- the threshold between low and high frequencies is at $\frac{1}{\varepsilon}$. 
Insights from the spectral analysis

- There exists a damped mode in the low frequencies regime associated to the eigenvalue $\frac{1}{\varepsilon} \rightarrow \text{uniform estimates.}$
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  - However, the so-called *overdamping* effect occurs: the decay rate behaves like $(\varepsilon, 1/\varepsilon)$.

![Decay rates graph]

- This is related to the fact that as $\varepsilon \to 0$, the low frequencies "invade" the whole space of frequency.
Low frequencies in a simple case

Our idea: reproduce exactly what the spectral analysis tells us using:

\[ \| f \|_{B^s_{2,1}}^h \triangleq \sum_{j \geq \frac{1}{\varepsilon}} 2^j \| \hat{\Delta}_j f \|_{L^2} \quad \text{and} \quad \| f \|_{B^s_{p,1}}^\ell \triangleq \sum_{j \leq \frac{1}{\varepsilon}} 2^{j'} \| \hat{\Delta}_j f \|_{L^p}. \]
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Back (again) to the damped p-system:

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\begin{cases}
\partial_{t} u + \partial_{x} v = 0 \\
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Defining the damped mode \( w = v + \varepsilon \partial_x u \), the system can be rewritten

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\begin{cases}
\partial_t u - \varepsilon \partial_{xx} u = -\partial_x w \\
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→ We directly get the behaviour observed in the spectral analysis, not just heat effect.
Partially dissipative systems

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→ We directly get the behaviour observed in the spectral analysis, not just heat effect.
→ It is possible to study the two equations in a decoupled way as the source terms can be absorbed in the low-frequency regime:

\[ \| \partial_x f \|_{B^s_{p,1}}^\ell \leq \| f \|_{B^s_{p,1}}^\ell \]
To Sum-up

- The hypocoercivity approach does not give the full story of the low-frequency behavior.
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- From the low-frequency analysis presented here and the high frequencies computation à la Beauchard et Zuazua, we are able to get a uniform global existence result.
To Sum-up

- The hypocoercivity approach does not give the full story of the low-frequency behavior.
- From the low-frequency analysis presented here and the high frequencies computation à la Beauchard et Zuazua, we are able to get a uniform global existence result.
- And from these uniform estimates we can justify, almost directly, the relaxation limit when $\varepsilon \to 0$ in the ill-prepared case.
**Theorem (Danchin, C-B ’21- ill-prepared relaxation limit)**

Let $d \geq 1$, $p \in [2, 4]$ and $\varepsilon > 0$. Let $\overline{\rho}$ be a strictly positive constant and $(\rho - \overline{\rho}, v)$ be the solution obtained with the previous theorem.

Let the positive function $N_0$ such that $N_0 - \overline{\rho}$ is small enough in $\dot{B}^{d}_{p,1}$, and let $N \in C_b(\mathbb{R}^+; \dot{B}^{d}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{d+2}_{p,1})$ be the unique solution associated to the Cauchy problem:

\[
\begin{cases}
\partial_t N - \Delta P(N) = 0 \\
N(0, x) = N_0
\end{cases}
\]

If we assume that

\[
\|\overline{\rho}_0 - N_0\|_{\dot{B}^{d-1}_{p,1}} \leq C\varepsilon,
\]

then

\[
\|\overline{\rho}^\varepsilon - N\|_{L^\infty(\mathbb{R}^+; \dot{B}^{d-1}_{p,1})} + \|\overline{\rho}^\varepsilon - N\|_{L^1(\mathbb{R}^+; \dot{B}^{d+1}_{p,1})} + \left\| \frac{\nabla P(\overline{\rho}^\varepsilon)}{\overline{\rho}^\varepsilon} + \overline{v} \right\|_{L^1(\mathbb{R}^+; \dot{B}^{d}_{p,1})} \leq C\varepsilon.
\]
Localized damping
Damping active outside of a ball

We consider the one-dimensional linear hyperbolic system

\[
\begin{aligned}
\partial_t U + A\partial_x U &= -BU1_\omega, \\
U(0, x) &= U_0(x),
\end{aligned}
\]

where \( U = (u_1, u_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \) and 

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\omega := \mathbb{R} \setminus B_R(0) = \{ x \in \mathbb{R} : \|x\| \geq R \} \quad \text{for a fixed } R > 0.
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We assume:

- \( B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \) with \( D > 0 \)
- The matrix \( A \) is a \textit{strictly hyperbolic matrix}, i.e. \( A \) has \( n \) real distinct eigenvalues

\[
\lambda_1 < \lambda_p < 0 < \lambda_{p+1} < \lambda_n.
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- The couple \((A, B)\) satisfies the (SK) condition.
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In other words: we are in the same situation as before but the damping is only effective in \( \omega \) (the complementary of a ball).
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- The approach depicted previously is bound to fail as it relies on the Fourier transform.
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Idea:

- The characteristic lines of the system spend only a finite time in the undamped region.
- When a characteristic is outside the undamped region, the solution decays as in the classical analysis.
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Idea:
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- When a characteristic is outside the undamped region, the solution decays as in the classical analysis.

→ This motivates us to develop a method involving only the consideration of the characteristics curves and a semigroup-wise decomposition.
Propagation of characteristics and their location with respect to the region \( \omega = \mathbb{R} \setminus B_R \) where the damping is active.

(a) **Case 1:** The initial support is in the damped region and the characteristics are going away from the un-damped region.

(b) **Case 2:** The initial support is in the damped region and the characteristics cross the un-damped region.

(c) **Case 3:** The initial support is in the un-damped region.

(d) **Case 4:** There is one zero eigenvalue. \( \rightarrow \) Standing wave
Reformulation of the system

As $A$ is symmetric with $n$ real distinct eigenvalues, there exists a matrix $P \in O(n, \mathbb{R})$ such that

$$P^{-1}AP = \Lambda \quad \text{where} \quad \Lambda = \text{diag}(\lambda_1, ..., \lambda_n).$$

Setting $V = P^{-1}U$, the system can be reformulated into

$$\begin{cases}
\partial_t V + \Lambda \partial_x V = P^{-1}BPV 1_\omega(x), & (t, x) \in (0, \infty) \times \mathbb{R}, \\
V(0, x) = V_0(x), & x \in \mathbb{R},
\end{cases}$$

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Decomposing $V = (v_1, \ldots, v_n)$, (5) is equivalent to the following system of coupled transport equations:

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\vdots \\
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For all $1 \leq i \leq n$, the characteristic lines $X_i$ of each equations passing through the point $(x_0, t_0) \in \mathbb{R} \times [0, T]$ are given by

$$X_i(t, x_0, t_0) := \lambda_i(t - t_0) + x_0, \quad t \in [0, T].$$
Figure: Characteristics passing through a point \((x, t) \in \mathbb{R} \times \mathbb{R}_+\).
The choice of $\omega$ as an exterior domain is motivated by a *geometric control condition*: the ray of geometric optics may escape the damping effect if the inclusion $\{\|x\| \geq r\} \subset \omega$ is not satisfied for some $r > 0$. Indeed, once a characteristic has crossed and exited the undamped region $\omega^c$ it will never cross it again. The time spent by each characteristics $\tau_i$ in $\omega^c$ satisfies:

$$\tau_i \leq 2R\lambda_i.$$ 

1st Principle: as we have only a finite number of components, the total time spend by all the characteristics in the undamped region is finite. Since our system has a partially dissipative nature, the dissipation of each variable arises from the coupling between each equations.

$\rightarrow$ 2nd principle: Whenever one of the characteristic is in the undamped region, then the solution does not, in general, undergo any decay. These considerations led us to the following Theorem.
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Partially dissipative systems

**General presentation**

**Damping active outside of a ball**

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These considerations led us to the following Theorem.
Main Theorem

**Theorem (De Nitti-Zuazua-CB ’22)**

Assume that the matrix $A$ is symmetric, strictly hyperbolic and does not admit the eigenvalue $0$ and that the couple $(A, B)$ satisfies the (SK) condition. Let $U_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Then, there exists a constant $C > 0$ and a finite time $\bar{\tau} > 0$ such that for $t \geq \bar{\tau}$, the solution satisfies

$$\| U^h(\cdot, t) \|_{L^2(\mathbb{R})} \leq C e^{-\gamma (t - \bar{\tau})} \| U_0 \|_{L^2(\mathbb{R})},$$

$$\| U^\ell(\cdot, t) \|_{L^\infty(\mathbb{R})} \leq C (t - \bar{\tau})^{-1/2} \| U_0 \|_{L^1(\mathbb{R})}$$

where

$$\bar{\tau} = \max \left( \sum_{i=1}^{p} \frac{2R}{|\lambda_i|}, \sum_{i=p+1}^{n} \frac{2R}{|\lambda_i|} \right).$$
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The decay estimates are delayed by the time each characteristic spend in the undamped region.
Idea of proof

We define $S_d$ the dissipative semigroup associated to the equation without localization. This semigroup is active when all the characteristics are outside the undamped region. Recall that we have

$$\|W_h(\cdot, t)\|_{L^2(\mathbb{R}^d)} \leq \|S_{h d}(t, 0)W_h0\|_{L^2(\mathbb{R}^d)} \leq C e^{-\gamma t} \|W_h0\|_{L^2(\mathbb{R}^d)},$$

$$\|W_\ell(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \leq \|S_{\ell d}(t, 0)W_\ell0\|_{L^\infty(\mathbb{R}^d)} \leq Ct^{-d/2} \|W_\ell0\|_{L^1(\mathbb{R}^d)}.$$
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We define $S_c$ the conservative semigroup associated to the equation without dissipation at all. Essentially, this semigroup will be active whenever one of the characteristic is inside the undamped region. We have

$$
\|Z(t, \cdot)\|_{L^p(\mathbb{R})} = \|S_c(t, 0)Z_0\|_{L^p(\mathbb{R})} = \|Z_0\|_{L^p(\mathbb{R})}.
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\]

Then, for every $(x, t) \in \mathbb{R}^2$, we can always find suitable times $t_1, t_2$ such that each components of the solution can be rewritten:

\[
\nu_i(x, t) = S_{d,i}(t)S_{c,i}(t_1)S_{d,i}(t_2)\nu_{i,0}(x). \tag{6}
\]

where $S_{d,i}$ and $S_{c,i}$ are the semigroup associated to each components.
Partially dissipative systems

General presentation

Damping active outside of a ball

Figure: Illustration on the semigroups and the quantities $t_1$ and $t_2$. 

$S_{d,1}$, $S_{c,1}$, $X_1$, $S_{d,p+1}$, $X_{p+1}$
Figure: Illustration on the semigroups and the quantities $t_1$ and $t_2$.

Difficulty: The time-quantities $t_1$ and $t_2$ depend on the point $(x, t)$. 
Partially dissipative systems

General presentation
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Figure: Illustration on the semigroups and the quantities $t_1$ and $t_2$.

Difficulty: The time-quantities $t_1$ and $t_2$ depend on the point $(x, t)$.

But the difference $t_1 - t_2$ is always uniformly bounded!
Difficulties due to partial dissipation:

It is only possible to obtain dissipation for the solution $V$ if all the semigroups $S_{d, i}$ are active on a same time-interval i.e. the “full” semigroup $S_{d} = (S_{d, 1}, \ldots, S_{d, p})$ needs to be active.

For instance, the action of $S_{d, 1}$ on the first component does not, in general, imply any time-decay properties for the component $v_1$.

This means that if one of the conservative semigroups $S_{c, i}$ is active on a time-interval then the whole solution does not experience any decay on this time-interval; Of course, it is possible that some components decay “on their own”. But in general the whole solution does not decay.

Still, when one semigroup $S_{d, i}$ is active, the $L^p$ norms of the solutions stay bounded thanks to the positive semidefiniteness of $B$. 

Crin-Barat Timothée
Partially and locally dissipated hyperbolic systems
Partially dissipative systems

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Other remarks/difficulties

Difficulties due to partial dissipation:

- It is only possible to obtain dissipation for the solution \( V \) if all the semigroups \( S_{d,i} \) are active on a same time-interval i.e. the "full" semigroup \( S_d = (S_{d,1}, \cdots, S_{d,p}) \) needs to be active.
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- Of course, it is possible that some components decay "on their own". But in general the whole solution does not decay.

- Still, when one semigroups $S_{d,i}$ is active, the $L^p$ norms of the solutions stay bounded thanks to the positive semidefiniteness of $B$. 
Recalling that $\forall i \in [1, p]$ 

$$v_i(x, t) = S_{d, i}(t)S_{c, i}(t_1)S_{d, i}(t_2)v_{i, 0}(x).$$
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With the previous considerations one ends up studying:

$$\mathcal{I}(x, t) = \bigcup_{i=1}^{p}[t_{1,i}(x, t), t_{2,i}(x, t)]$$

which corresponds to the union of time-interval where the dissipation is not active. $\rightarrow |\mathcal{I}|$ quantifies the delay.
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And, essentially, our theorem derives from

\[ \sup_{x \geq R, t > 0} |\mathcal{I}(x, t)| \leq \sum_{i=1}^{p} \frac{2R}{|\lambda_i|} = \bar{T}. \] (8)
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And, essentially, our theorem derives from

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(8)

And computations of the following type:

$$\|v_1(., t)\|_{L^2(\mathbb{R})} = \|S_{d,1}(t)S_{c,1}(t_1)S_{d,1}(t_2)v_{1,0}\|_{L^2(\mathbb{R})}$$

$$\leq e^{-c(t-t_1)}\|S_{c,1}(t_1)S_{d,1}(t_2)v_{1,0}\|_{L^2}$$

$$\leq e^{-c(t-t_1)}\|S_{d,1}(t_2)v_{1,0}\|_{L^2}$$

$$\leq e^{-c(t-t_1)}e^{-c(t_2-0)}\|v_{1,0}\|_{L^2}$$

$$\leq e^{-c(t-(t_1-t_2))}\|v_{1,0}\|_{L^2}$$
Optimality for shorter times

The decay estimates we obtain are optimal for times $t$ large enough but they are not totally sharp for small times. The length of $\bar{\tau}$ can be smaller than $\bar{\tau}$.

Indeed the characteristics may overlap in the undamped region for short time and therefore "reduce the delay".

The result from our theorem is optimal for times $t \leq \bar{\tau}$ if $| \lambda_i | = | \lambda_i + 1 | = | \lambda_i + 2 | \quad \forall i \in [1, p-2]$ or $\forall i \in [p+1, n-2]$, (9)
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\frac{|\lambda_i|}{|\lambda_{i+1}|} = \frac{|\lambda_{i+1}|}{|\lambda_{i+2}|} \quad \forall i \in [1, p-2] \text{ or } \forall i \in [p+1, n-2],
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What happens when this proportionality condition is not satisfied is nontrivial and depend on the length of the finite union of finite intervals $|\mathcal{I}|$. 
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We are able to provide a precise result in the case of three negative eigenvalues.
Asymptotic for 3 components with negative eigenvalues

Figure: The magenta curve is the exact upper bound of the energy.