The fractional Brezis–Nirenberg problem in low dimensions

(joint work with Nicola De Nitti, arXiv:2111.13417, submitted)

Miniworkshop FAU Erlangen

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May 25, 2022

The Brezis–Nirenberg problem

For $N \geq 3$, $\Omega \subset \mathbb{R}^N$ open bounded, $a \in C(\overline{\Omega})$ [Brezis–Nirenberg 1983] consider

$$\begin{aligned} -\Delta u + au &= N(N-2)u^{\frac{N+2}{N-2}} && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1}$$

[BN] observe: When $a \equiv 0$ and Ω is strictly starshaped, (1) has no solution.

Proof.

By **Pohozaev's identity** (integrate (1) against $\nabla u(x) \cdot x$), when $a \equiv 0$

$$-\int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 \nu(x) \cdot x \, \mathrm{d}x = \frac{N-2}{2} \int_\Omega |\nabla u|^2 - \frac{N(N-2)^2}{2} \int_\Omega u^{\frac{2N}{N-2}} = 0,$$

because $N(N-2) \int_\Omega u^{\frac{2N}{N-2}} = \int_\Omega |\nabla u|^2$ (integrate (1) against u). If Ω is strictly starshaped wrt 0, this implies $\frac{\partial u}{\partial \nu} \equiv 0$ on $\partial\Omega$. Hence $\int_\Omega u^{\frac{N+2}{N-2}} = \int_\Omega -\Delta u = -\int_{\partial\Omega} \frac{\partial u}{\partial \nu} = 0$. But this is a contradiction to $u > 0$. □

So [BN] ask: **Under what conditions on a does (1) have a solution?**

Existence of solutions in high and low dimensions

Solutions are given by positive minimizers to the variational problem

$$S(a) := \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \mathcal{S}_a[u], \qquad \mathcal{S}_a[u] := \frac{\int_\Omega (|\nabla u|^2 + au^2)\, dx}{\left(\int_\Omega |u|^{\frac{2N}{N-2}}\, dx\right)^{\frac{N-2}{N}}} \tag{2}$$

Theorem ([Brezis–Nirenberg 1983])

If $S(a) < S(0)$, then a minimizer exists (due to E. H. Lieb).

- *Let $N \geq 4$. Then $\quad S(a) < S(0) \quad \Leftrightarrow \quad \{a < 0\} \neq \emptyset$.*
- *Let $N = 3$. Then $S(a) = S(0)$ whenever $\|a\|_\infty$ is small enough !*

Where does the different behavior in low dimension $N = 3$ come from ?

Let $U_{x,\lambda}(y) = \left(\frac{\lambda}{1+\lambda^2|x-y|^2}\right)^{\frac{N-2}{2}}$ with $x \in \mathbb{R}^N$ and $\lambda > 0$. We notice

$$\|\nabla U_{x,\lambda}\|^2_{L^2(\mathbb{R}^N)} = S(0) \| U_{x,\lambda}\|^2_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \quad \text{and} \quad -\Delta U_{x,\lambda} = N(N-2)\, U_{x,\lambda}^{\frac{N+2}{N-2}}.$$

Idea: Fix $x \in \Omega$ and $\varphi \in C_0^\infty(\Omega)$. Then test $\mathcal{S}_a$ with $\varphi U_{x,\lambda} \in H_0^1(\Omega)$ and let $\lambda \to \infty$!

The Brezis–Nirenberg dimensional effect

High dimension: If $N \geq 4$, [BN] compute that, as $\lambda \to \infty$,

$$\|\nabla(\varphi U_{x,\lambda})\|_2^2 = K_1 + \mathcal{O}(\lambda^{-N+2}),$$

$$\|\varphi U_{x,\lambda}\|_{\frac{2N}{N-2}}^2 = K_2 + \mathcal{O}(\lambda^{-N}),$$

$$\int_\Omega a(\varphi U_{x,\lambda})^2 = K_3 a(x)\lambda^{-2} + \mathcal{O}(\lambda^{-N+2}) \qquad (K_3 a(x)\lambda^{-2}\log\lambda \text{ if } N = 4).$$

$\implies \mathcal{S}_a[\varphi U_{x,\lambda}] = S(0) + a(x)\lambda^{-2} + \mathcal{O}(\lambda^{-N+2}) < S(0)$ as soon as $a(x) < 0$!

Low dimension: Let $N = 3$, $\Omega = B$ and a, φ radial. Then, as $\lambda \to \infty$,

$$\|\nabla(\varphi U_{x,\lambda})\|_2^2 = K_1 + \lambda^{-1}|\mathbb{S}^2| \int_0^1 |\varphi'(r)|^2 \, \mathrm{d}r + \mathcal{O}(\lambda^{-2}),$$

$$\|\varphi U_{x,\lambda}\|_6^2 = K_2 + \mathcal{O}(\lambda^{-2}),$$

$$\int_\Omega a(\varphi U_{x,\lambda})^2 = \lambda^{-1}|\mathbb{S}^2| \int_0^1 a(r)|\varphi(r)|^2 \, \mathrm{d}r + \mathcal{O}(\lambda^{-2}).$$

and hence

$$\mathcal{S}_a[\varphi U_{x,\lambda}] = S(0) + |\mathbb{S}^2| K_2^{-1} \left(\int_0^1 |\varphi'(r)|^2 \, \mathrm{d}r + \int_0^1 a(r)|\varphi(r)|^2 \, \mathrm{d}r \right) \lambda^{-1} + \mathcal{O}(\lambda^{-2}).$$

Competing lower-order terms!

Critical functions and their Green's function

Question [Brezis 1986]: Is $S(a) < S(0)$ also necessary for $S(a)$ to be achieved?

Theorem ([Druet 2002])

Let $N = 3$. Then $S(a)$ is achieved $\Leftrightarrow \quad S(a) < S(0) \quad \Leftrightarrow \quad \{\phi_a < 0\} \neq \emptyset$.
As a consequence, a is critical if and only if $\min_\Omega \phi_a = 0$.

Here, we define the **Robin function**

$$\phi_a(x) = H_a(x,x), \qquad \text{where} \qquad G_a(x,y) = \frac{1}{|x-y|^{N-2}} - H_a(x,y)$$

is the Green's function of $-\Delta + a$.

Following [Hebey–Vaugon 2001], we call a function $a \in C(\overline{\Omega})$ **critical** if $S(a) = S(0)$ and $S(\tilde{a}) < S(a)$ for all $\tilde{a} \leq a$ with $\tilde{a} \not\equiv a$.

Corollary

- *If $N \geq 4$, the only critical function is $a \equiv 0$.*
- *If $N = 3$, a is critical if and only if $\min_\Omega \phi_a = 0$. In particular, there are critical functions of all shapes.*

Concentration of solutions

Let a be critical and let $(u_\epsilon) \subset H_0^1(\Omega)$ be a sequence of positive solutions to

$$-\Delta u_\epsilon + (a + \epsilon V) u_\epsilon = N(N-2) u_\epsilon^{\frac{N+2}{N-2}} \quad \text{on } \Omega, \qquad u_\epsilon = 0 \quad \text{on } \partial\Omega. \tag{3}$$

which blows up at a single point. To leading order, $u_\epsilon \sim U_{x_\epsilon, \lambda_\epsilon}$ with $x_\epsilon \to x_0 \in \overline{\Omega}$ and $\lambda_\epsilon \to \infty$.

In these terms, x_0 is the concentration point and λ_ϵ is the concentration speed. Notice that $\|u_\epsilon\|_\infty \sim \lambda_\epsilon^{\frac{N-2}{2}}$.

Theorem

- *If* $N \geq 4$ *(hence* $a \equiv 0$*),* $V \equiv -1$*, then* $\nabla \phi_0(x_0) = 0$*. Moreover,* $\lim_{\epsilon \to 0} \epsilon \|u_\epsilon\|_\infty^{2\frac{N-4}{N-2}} = d_N \phi_0(x_0)$*. [Han 1991, Rey 1989]*
- *If* $N = 3$*, then* $\phi_a(x_0) = 0$*. If* x_0 *is non-degenerate as a minimum of* ϕ_a*, with* $a(x_0) < 0$*, then* $\lim_{\epsilon \to 0} \epsilon \|u_\epsilon\|_\infty^2 = 4\pi^2 \frac{|a(x_0)|}{|\int_\Omega V(y) G_a^2(x_0, y)\, dy|}$*. [Frank, K., Kovarik CVPDE + arXiv 2021]*

Concentration asymptotics – Proof ideas

Key idea: Refine the [BN] test function choice $\varphi U_{x,\lambda}$ and expand energy quantities to sufficient precision!

We optimize the H_0^1 cutoff procedure: Let $PU_{x,\lambda}$ be the unique function s.t.

$$-\Delta PU_{x,\lambda} = -\Delta U_{x,\lambda} \text{ in } \Omega, \qquad PU_{x,\lambda} = 0 \text{ on } \partial\Omega.$$

High dimensions: [Rey 1989]
Let $N \geq 4$ and $V \equiv -1$. As $\epsilon \to 0$, write $u_\epsilon = PU_{x_\epsilon,\lambda_\epsilon} + w_\epsilon$ with $\|\nabla w_\epsilon\|_2 = o(1)$ and $w \perp \{PU_{x,\lambda}, \partial_\lambda PU_{x,\lambda}, \partial_{x_i} PU_{x,\lambda}\}$. Expand Pohozaev

$$\epsilon \int_\Omega u_\epsilon^2 = \int_{\partial\Omega} (x - x_0) \cdot \nu(x) \left| \frac{\partial u_\epsilon}{\partial \nu} \right|^2$$

using the quantitative bound $\|w_\epsilon\|_2 = \mathcal{O}(\lambda_\epsilon^{-\frac{N+2}{2}})$, we get

$$a_N \epsilon \lambda_\epsilon^2 + o(\epsilon\lambda_\epsilon^2) = b_N \phi_0(x_0) \lambda_\epsilon^{-N+2} + o(\lambda_\epsilon^{-N+2}).$$

Low dimension: [Frank, K., Kovarik 2021]
When $N = 3$, terms in energy expansion are again arranged differently. To expand to the desired precision, we need the refined development

$$u_\epsilon = PU_{x_\epsilon,\lambda_\epsilon} + \lambda_\epsilon^{-1/2}(H_0(x_\epsilon,\cdot) - H_a(x_\epsilon,\cdot)) + q_\epsilon, \qquad \|\nabla q_\epsilon\|_2^2 = \lambda_\epsilon^{-1} + \epsilon\lambda^{-1/2}.$$

Going fractional

The appropriate generalization of the BN problem to fractional orders of derivatives $2s < N$ with $s \in (0,1)$ is

$$\begin{aligned} (-\Delta)^s u + (a + \epsilon V) u &= c_{N,s} u^{\frac{N+2s}{N-2s}} && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \mathbb{R}^N \setminus \Omega. \end{aligned} \tag{4}$$

with the **fractional Laplacian** $(-\Delta)^s$ given as

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F} u) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, \mathrm{d}y.$$

The variational problem associated to (4) is

$$S_{N,s}(a) := \inf \frac{\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + \int_\Omega a u^2}{\left(\int_\Omega u^{\frac{2N}{N-2s}} \right)^{\frac{N-2s}{N}}}$$

where the inf is taken over $\widetilde{H}^s(\Omega) := \left\{ u \in H^s(\mathbb{R}^N) \,:\, u \equiv 0 \quad \text{on } \mathbb{R}^N \setminus \Omega \right\}$.

The notions of Green's function $G_a(x,y) = \frac{1}{|x-y|^{N-2s}} - H_a(x,y)$, and critical function a carry over without problem.

Some known results for the fractional BN problem

The analysis of the fractional BN problem with $s \in (0,1)$ presents some additional difficulties which mostly stem from the fact that the operator $(-\Delta)^s$ is **non-local**.

Still: $U_{x,\lambda}(y) = \left(\frac{\lambda}{1+\lambda^2|x-y|^2}\right)^{\frac{N-2s}{2}}$ satisfies $(-\Delta)^s U_{x,\lambda} = c_{N,s} U_{x,\lambda}^{\frac{N+2s}{N-2s}}$.

Theorem (Servadei–Valdinoci 2013, 2015)

- *If $N \geq 4s$, then $S_{N,s}(a) < S_{N,s}$ whenever $a(x) < 0$ for some $x \in \Omega$.*
- *If $2s < N < 4s$, then $S_{N,s}(a) < S_{N,s}$ if $a(x) < -\mu_s < 0$.*

(Compare [Brezis–Nirenberg 1983].)

Theorem (Choi–Kim–Lee 2014)

If $N > 4s$, and solutions u_ϵ to (4) with $a \equiv 0$, $V = -1$ blow up at exactly one point $x_0 \in \Omega$, then $\lim_{\epsilon \to 0} \epsilon \|u_\epsilon\|_\infty^{2\frac{n-4s}{n-2s}} = \delta_{N,s}\phi_0(x_0)$. (Compare [Han 1991, Rey 1989].)

$\Longrightarrow$ **What about low dimensions $2s < N < 4s$??**

Main results

Theorem 1 (De Nitti, K., 2021)

Let $s \in (0,1)$ *and* $2s < N < 4s$ *and let* $a \in C(\overline{\Omega})$*. Then*

$$(i)\, S_{N,s}(a) \text{ is achieved } \Leftrightarrow \quad (ii)\, S_{N,s}(a) < S_{N,s} \quad \Leftrightarrow \quad (iii)\, \{\phi_a < 0\} \neq \emptyset.$$

As a consequence, a *is critical if and only if* $\min_\Omega \phi_a = 0$.

- This is the fractional version of [Druet 2002].
- $(i) \Leftarrow (ii)$ follows similarly to [BN]. $(ii) \Leftarrow (iii)$ follows from

$$\mathcal{S}_a[\psi_{x,\lambda}] = S(0) + c\phi_a(x)\lambda^{-N+2s} + o(\lambda^{-N+2s}) \qquad (5)$$

as $\lambda \to \infty$, where $\psi_{x,\lambda} = PU_{x,\lambda} + \lambda^{-\frac{N-2s}{2}}(H_0(x,\cdot) - H_a(x,\cdot))$.

- The forward implications are the hard part. E.g. $(iii) \Rightarrow (ii)$ requires to show: For a critical and u_ϵ minimizers of $S(a-\epsilon)$, one has

$$S(0) > S(a-\epsilon) = S(0) + c(\phi_a(x_\epsilon) + o(1))\lambda^{-N+2s}.$$

Thus $\phi_{a-\epsilon}(x_0) < \phi_a(x_0) \leq 0$ and (iii) follows.

Main results

Write $Q_V(x) := \int_\Omega V(y) G_a(x,y)^2 \, dy$.

Theorem 2 (De Nitti, K., 2021)

Let $s \in (0,1)$ and $\frac{8}{3}s < N < 4s$ and let $a \in C(\overline{\Omega})$ be critical. Suppose that $(u_\epsilon) \subset \widetilde{H}^s(\Omega)$ is a sequence of energy-minimizing solutions to (4). Then the u_ϵ blow up in exactly one point $x_0 \in \Omega$ satisfying $\phi_a(x_0) = 0$. Moreover,

$$u_\epsilon = PU_{x_\epsilon, \lambda_\epsilon} + \lambda_\epsilon^{-\frac{N-2s}{2}} (H_0(x_\epsilon, \cdot) - H_a(x_\epsilon, \cdot)) + q_\epsilon$$

with $\|((-\Delta)^{s/2}) q_\epsilon\|_2^2 = o(\lambda_\epsilon^{-2s} + \lambda_\epsilon^{-2N+4s})$ and $x_\epsilon \to x_0$ and

$$\lim_{\epsilon \to 0} \epsilon \lambda_\epsilon^{4s-N} = d_{N,s} \frac{|a(x_0)|}{|Q_V(x_0)|}. \tag{6}$$

The point x_0 maximizes $\frac{|Q_V(x_0)|^{\frac{2s}{4s-N}}}{|a(x_0)|^{\frac{N-2s}{4s-N}}}$ among all x with $\phi_a(x) = 0$ and $V(x) < 0$.

This is the fractional version of [Frank, K., Kovarik CVPDE 2021]

Some comments

- Our proof strategy (for both Theorems 1 and 2) is **variational**. In particular Theorem 2 also holds for almost-minimizers u_ϵ of $S_{N,s}(a+\epsilon V)$, which need not satisfy any PDE.
- If we knew that $\|q_\epsilon\|_\infty = o(\lambda_\epsilon^{\frac{N-2s}{2}})$, Theorem 2 would yield the value of $\lim_{\epsilon\to 0} \epsilon \|u_\epsilon\|_\infty^{2\frac{4s-N}{N-2s}}$.
- For $s=1$, the recent preprint [Frank, K., Kovarik arXiv 2021] removes the energy-minimizing assumption, thus proving a conjecture from [Brezis–Peletier 1989]. The analogous question for $s \in (0,1)$ is open.
- For $N > 4s$ we prove (work in preparation)

$$\lim_{\epsilon\to 0} \epsilon \lambda_\epsilon^{N-4s} = d_{N,s} \frac{\phi_0(x_0)}{|V(x_0)|}.$$

  and x_0 achieves $\max_{\{x:\, V(x)<0\}} \phi_0(x)^{-\frac{2s}{N-4s}} |V(x)|^{\frac{N-2s}{N-4s}}$
- If $N < 4s$, the 'renormalized energy' $|Q_V(x)|^{\frac{2s}{4s-N}} |a(x)|^{-\frac{N-2s}{4s-N}}$ is non-local in V !

The proof for fractional $s \in (0,1)$

Basic results:

- Concentration compactness [Palatucci–Pisante 2015] (Compare [Struwe 1984].)
- $PU_{x,\lambda}$ and orthogonality conditions [Abdelhedi–Chtioui–Hajajiej 2017] (Compare [Bahri–Coron 1988].)

Thus we can write $u_\epsilon = \alpha_\epsilon(PU_{x_\epsilon,\lambda_\epsilon} + w_\epsilon)$ with $\alpha_\epsilon \to 1$, $\|(-\Delta)^s w_\epsilon\|_{L^2(\mathbb{R}^N)} = o(1)$ and $w_\epsilon \perp \{PU_{x,\lambda}, \partial_\lambda PU_{x,\lambda}, \partial_{x_i} PU_{x,\lambda}\}$.

New ingredients:

- Precise analysis of the functions $PU_{x,\lambda}$ and $H_a(x,y)$
- Spectral **coercivity inequality**: For $p = \frac{2N}{N-2s}$,

$$\|(-\Delta)^{s/2} w_\epsilon\|_2^2 - c_{N,s}(p-1)\int_\Omega U_{x_\epsilon,\lambda_\epsilon}^{p-2} w_\epsilon^2 \geq \frac{4s}{N+2s+2}\|(-\Delta)^{s/2} w_\epsilon\|_2^2.$$

  by stereographic projection to $\mathbb{S}^N$. (Compare [Rey 1990, Bianchi–Egnell 1991]. For $s \in (0,1)$, see also [Chen–Frank–Weth 2012].)
- Non-existence of a minimizer for $S(a)$ when a critical. (Compare [Druet 2002].)

The proof for fractional $s \in (0,1)$ - cont'd

- The assumption $8s/3 < N$ should be technical. The phenomenon that makes it necessary to be imposed has no analogue when $s = 1$ and $N = 3$: To absorb error terms in

$$S(a+\epsilon V) = S_{N,s} + (\phi_a(x) + \epsilon \int_\Omega V G_a(x_\epsilon, \cdot))\lambda^{-N+2s} + a(x)\lambda^{-2s} + o(\lambda^{-2s}) + (\epsilon + \phi_a(x))\lambda^{-N+2s}).$$

  we need $\lambda^{-k(N-2s)} \overset{!}{=} o(\lambda^{-2s})$ for all $k \geq 3$.

  It would be very interesting to understand more precisely the impact of such 'lower-order BN dimensional effects'!

- Simplifications / differences with respect to previous works
  - Avoid the intermediate spectral cutoff argument from [Frank–K.–Kovarik CVPDE 2021].
  - Avoid the formulation of $(-\Delta)^s$ as a Dirichlet-to-Neumann problem for a degenerate-elliptic local PDE in $N+1$ dimension as in [Choi–Kim–Lee 2014] and others

Summary

- **Brezis–Nirenberg problem:** On $\Omega \subset \mathbb{R}^N$, positive solutions to $(-\Delta)^s u + au = u^{\frac{N+2s}{N-2s}}$ exist sometimes, but not always.
  - The picture is more intricate in low dimensions $2s < N < 4s$!
- **Main Result:** In $2s < N < 4s$:
  - a is critical $\Leftrightarrow \min_\Omega \phi_a = 0$.
  - Let (u_ϵ) almost minimizers of the perturbed critical Brezis–Nirenberg energy $S(a + \epsilon V)$ at a critical. We precisely characterize
    - the concentration point x_0 and the concentration speed λ_ϵ in terms of the data a, V, Ω
    - the blow-up profile: $u_\epsilon \sim PU_{x_\epsilon,\lambda_\epsilon} + \lambda_\epsilon^{-\frac{N-2s}{2}}(H_0(x_\epsilon,\cdot) - H_a(x_\epsilon,\cdot))$.
- **Method of proof:** Purely variational $\longrightarrow$ Refine iteratively the energy expansion and improve error bounds by exploiting coercivity + energy minimality.
- **Perspectives:**
  - Treat very low dimensions $2s < N \leq \frac{8s}{3}$ by understanding better the related cancellation phenomenon.
  - Remove energy bounds: simple blow-up for non-minimizers, multibubble blow-up.

Vielen Dank für Eure Aufmerksamkeit! :-)