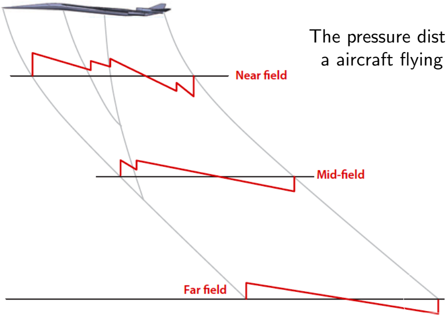
Inverse design of one-dimensional Burgers equation [1]

Enrique Zuazua

1. Based on joint work with Thibault Liard : Liard, T., & Zuazua, E. (2021). Initial data identification for the one-dimensional Burgers equation. IEEE Transactions on Automatic Control, 67(6), 3098-3104.

1. Introduction
    - Sonic boom minimization
    - Presentation of the control optimal problem under consideration
2. Preliminaries and notations
    - Wave-front tracking algorithm
    - The backward operator S_t^-
3. Main result : full characterization of minimizers
4. Find randomly all possible minimizers using
    - a backward-forward method
    - a wave-front tracking algorithm
5. Conclusion and open problems

Sonic boom and supersonic airplanes



The pressure disturbance $P(\sigma_0, \cdot)$ created by a aircraft flying above the speed of sound

"Augmented Burger equations equation"

$$\frac{\partial P}{\partial_\sigma} = P\frac{\partial P}{\partial_\tau} + \frac{1}{\gamma}\frac{\partial^2 P}{\partial_\sigma^2} + \frac{1}{2A}\frac{\partial A}{\partial_\sigma}P + \sum_\nu \frac{C_\nu}{\theta_\nu}\int_{-\infty}^{\tau} e^{\frac{(\xi-\tau)}{\theta_\nu}}\frac{\partial^2 P(\xi)}{\partial\tau^2}\, d\xi$$

σ = distance of the perturbation

τ = time of the perturbation

$P(\xi, \cdot)$ $\rightarrow$ Creation of **boom noises**

Objective : Tailoring the shape of the aircraft to minimize the ground sonic boom effects

The optimal control problem is $\min_{P_0 \in \mathcal{A}} d(P(\xi, \cdot), P^*(\cdot))$

The admissible set $\mathcal{A}$ is chosen to ensure feasible aircraft design (for instance aerodynamic lift).
$d(\cdot, \cdot)$ is chosen to be a robust and realistic metric for boom noises (Perceived loudness (PLdB)
P^* a desired ground signature and ξ the distance of the propagation

References : [Whitham, 1952 ; Cleveland, 1995 ; Alonso-Colonno, 2012 ; Rallabhandi, 2011 ; Adimurthi-Ghoshal-Gowda, 2014 ; Allahverdi-Pozo-Zuazua, 2016]

The one-dimensional Burgers equation

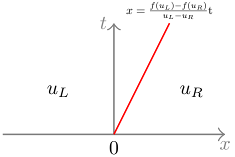
The one-dimensional Burgers equation

$$\begin{cases} u_t + f(u)_x = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u_0(x). & x \in \mathbb{R}, \end{cases} \qquad \text{(PDE)}$$

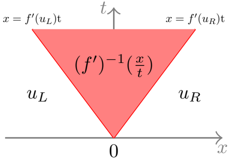
- The flux $f : u \to \frac{u^2}{2}$
- $u_0 \in BV(\mathbb{R})$

⟶ The function u is a **weak solution** to (PDE), for $(t,x) \in (0,+\infty) \times \mathbb{R}$, i.e for all $\varphi \in C_c^1(\mathbb{R}^2, \mathbb{R})$,

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} (u\partial_t \varphi + f(u)\partial_x \varphi) dxdt + \int_{\mathbb{R}} u_0(x)\varphi(0,x)dx = 0.$$



$u_L < u_R$. A weak solution of (PDE)



$u_L < u_R$. A weak-entropy solution of (PDE)

⟶ The function u is **an entropy solution** to (PDE) For every $k \in \mathbb{R}$, for all $\varphi \in C_c^1(\mathbb{R}^2, \mathbb{R}_+)$, it holds

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} (|u-k|\partial_t \varphi + \text{sgn}(u-k)(f(u)-f(k))\partial_x \varphi) dxdt + \int_{\mathbb{R}} |u_0 - k|\varphi(0,x)dx \geq 0.$$

Optimal control problem

For any initial datum $u_0 \in BV(\mathbb{R})$ there exists a unique weak-entropy solution $S_t^+(u_0) \in L^\infty([0,T] \times \mathbb{R}) \cap C^0([0,T], L^1_{loc}(\mathbb{R}))$ of (PDE)

Our aim is to solve the following optimal control problem

$$\min_{u_0 \in \mathcal{U}_{ad}^0} J_0(u_0) := \int_{\mathbb{R}} \left(u^T(x) - S_T^+(u_0) \right)^2 dx, \qquad \text{(Opt-Pb)}$$

Above $u^T \in BV(\mathbb{R})$ and the class of admissible initial data is defined by

$$\mathcal{U}_{ad}^0 = \{ u_0 \in BV(\mathbb{R}) / \|u_0\|_{BV(\mathbb{R})} < C \text{ and } \mathrm{Supp}(u_0) \subset K_0 \}.$$

Objectives :

⟶ Construction of a minimizer of (Opt-Pb) via a **backward-forward method**.

⟶ **Implementation of an algorithm** to find (randomly) all possible minimizers of (Opt-Pb)

References

Definition : u^T is **reachable** at time T if there exists $u_0 \in BV(\mathbb{R})$ such that $S_T^+(u_0) = u^T$.

If u^T is **reachable** at time T :

⟶ Characterization of reachable u^T : *[Colombo-Perrollaz, 2019],[Gosse-Zuazua, 2017]*

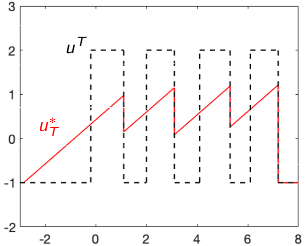
⟶ Fully characterization of initial data u_0 leading to u^T : *[Colombo-Perrollaz, 2019]*

If u^T is **unreachable** at time T :

⟶ Notion of weak-differentiability of the cost function J_0 in (Opt-Pb) :
[Majda, 1983 ; Bardos-Pironneau, 2005 ; Bouchut-James, 1999 ; Bressan-Marson, 1995]

⟶ Implementation of Gradient descent method to solve (Opt-Pb) :
[Castro-Palacios-Zuazua, 2008-2010 ; Allahverdi-Pozo-Zuazua, 2016 ; Gosse-Zuazua, 2017]

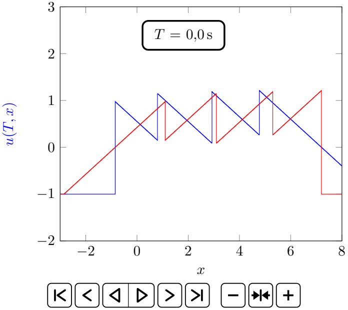
An amuse-bouche

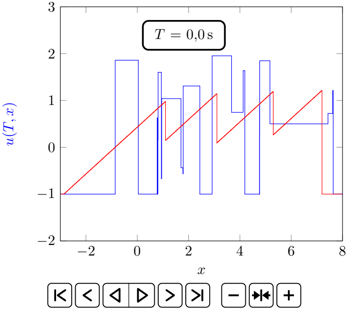


A target $u^T \in \{-1, 2\}$.

Plotting of two minimizers u_0 and u_1 of (Opt-Pb) such that

$$S_T^+(u_0) = S_T^+(u_1) = u_T^*$$



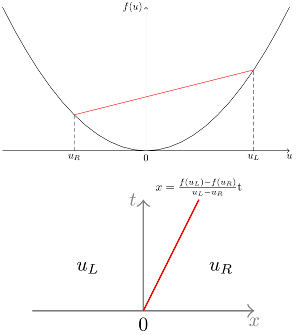


Wave-front tracking algorithm

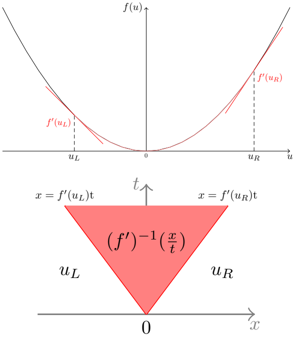
Conservation laws and Riemann solutions

The Burgers equation with Riemann type initial data

$$\partial_t \rho + \partial_x(f(\rho)) = 0, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R},$$
$$u(0,x) = \begin{cases} u_L & \text{if} \quad x < 0 \\ u_R & \text{if} \quad x > 0 \end{cases}, \qquad x \in \mathbb{R}.$$



Riemann solution when
$u_L > u_R$: a shock wave

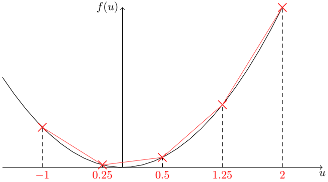


Riemann solution when
$u_L < u_R$: a rarefaction wave

A Wave-front tracking method

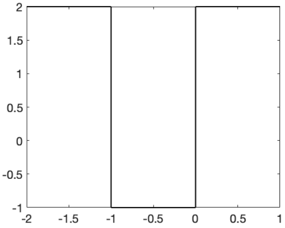
Assuming that there exists $\underline{u}, \bar{u}$ such that $\underline{u} \leq u_0 \leq \bar{u}$.

- Construction of a state mesh $\mathcal{M}_n = \underline{u} + (\bar{u} - \underline{u})2^{-n}\mathbb{N} \cap [0, 1]$
- We approximate $u_0 \in BV(\mathbb{R})$ by a piecewise constant function $u_0^n \in \mathcal{M}_n$.

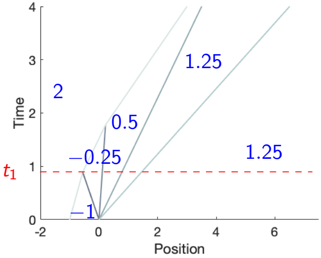


$M_n = [-1, -0.25, 0.5, 1.25, 2]$ with $n = 2$.

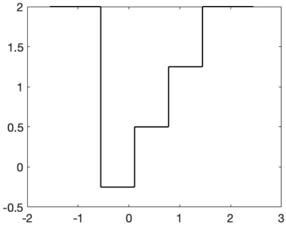
- We solve approximately the Riemann problem at each point of discontinuity $(x_i)_{i \in \{1, \cdots, N\}}$ of u_0^n.
  - if $u_0^n(x_i-) > u_0^n(x_i+))$, a shock wave is generated with speed given by the Rankine-Hugoniot condition.
  - if $u_0^n(x_i-) < u_0^n(x_i+))$, we decompose the rarefaction wave into a fan of rarefaction shocks traveling with speed given by Rankine-Hugoniot condition.



$$u_0 = 2\mathbb{1}_{(-\infty,-1)} - \mathbb{1}_{(-1,0)} + 2\mathbb{1}_{(0,\infty)}$$

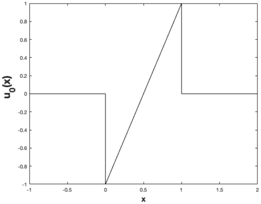


- We construct an approximate solution $u^n(t,x)$ until a time t_1, where at least two wave fronts interact together.
- At $t = t_1^+$ a new Riemann problem arises and we repeat the previous strategy replacing $t = 0$ and u_0^n by $t = t_1$ and $u^n(t_1,\cdot)$ respectively.

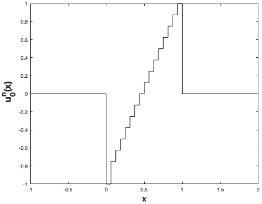


$u^n(t_1,\cdot)$

Illustration of a WFT method



Initial datum u_0



Construction of an approximate initial datum $u_0^n : x \to \mathcal{M}_n$ of u_0 with $n = 5$

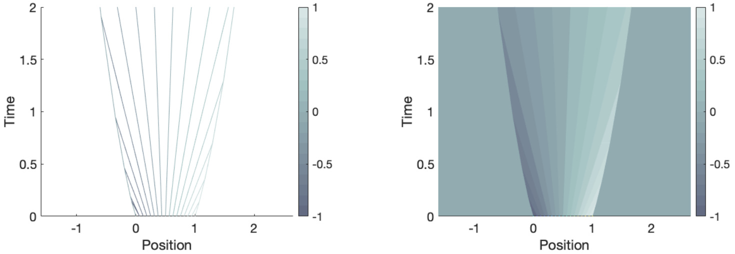
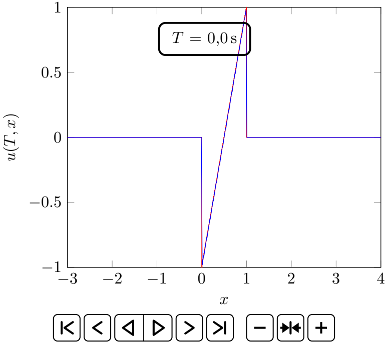


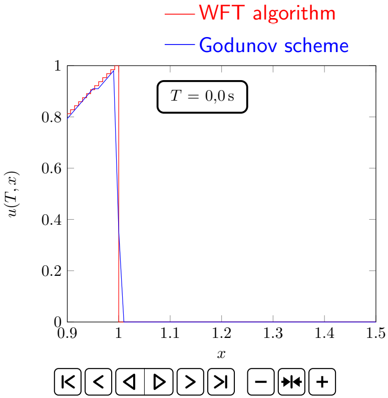
Illustration of the "wave-front" objects

Godunov scheme is a conservative three-point numerical scheme having the following form

$$u_{j+1}^n = u_j^n - \frac{\Delta t}{\Delta x}\left(g(u_j^n, u_{j+1}^n) - g(u_{j-1}^n, u_j^n)\right),$$

with g a numerical flux and $u(n\Delta t, j\Delta x) \approx u_j^n$, $n \in \mathbb{N}$, $j \in \mathbb{Z}$.





Wave-front tracking methods VS Godunov scheme

Godunov scheme :

- Discretization in space Δx and time Δt,
- "Backward uniqueness" because of diffusion effects,
- Easy to implement,
- A CFL condition has to be satisfied $\left(\frac{\Delta t}{\Delta x} \max_{u \in [\underline{u}, \bar{u}]} |f'(u)| \leq \frac{1}{2}\right) \rightarrow$ The final time T is small.

Wave-front tracking method :

- Discretization in state Δu,
- No Backward uniqueness because shocks may be created,
- Hard to implement (creation of objects and find interaction points between objects),
- No CFL condition is imposed $\rightarrow$ The final time T may be large.

The backward operator S_t^-

The backward operator S_t^-

The backward operator S_t^- associated to the Burgers dynamic is defined by

$$S_t^-(u^T)(x) = S_t^+(x \to u^T(-x))(-x),$$

for every $t \in [0, T]$ and for a.e $x \in \mathbb{R}$.

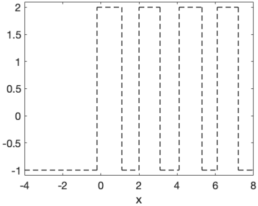
Remark : The solution $S_t^-(u^T)$ may be regarded as the zero viscosity limit of $S_T^{-,\epsilon}(u^T)$ solution of the following backward equation

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u(t,x)) = -\epsilon\partial_{xx}^2 u(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(T,\cdot) = u^T(x), & x \in \mathbb{R}. \end{cases}$$

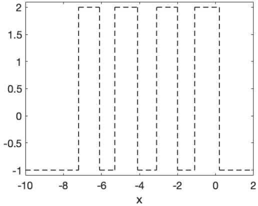
Using the change of variable $(t,x) \quad \to \quad (T - t, -x)$, we notice that the backward equation above is well-defined.

Thus, $S_T^-(u^T)$ is also called the backward entropy solution with final target u^T.

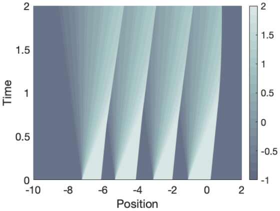
$$u^T(x) = \begin{cases} 2 & \text{if } x \in (-0.2, 1.1) \bigcup (2, 3.1) \bigcup (4.1, 5.3) \bigcup (6.1, 7.2), \\ -1 & \text{otherwise.} \end{cases}$$



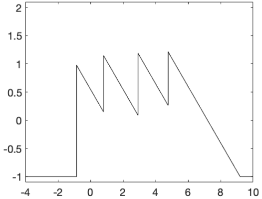
u^T



$x \to u^T(-x)$



$(t, x) \to S_t^+(x \to u^T(-x))$



$S_t^-(u^T) : (t, x) \to S_t^+(x \to u^T(-x))(-x)$

Main result

Main result

Our aim is to solve the following optimal control problem

$$\min_{u_0 \in \mathcal{U}_{ad}^0} J_0(u_0) := \int_{\mathbb{R}} \left(u^T(x) - S_T^+(u_0) \right)^2 dx, \qquad \text{(Opt-Pb)}$$

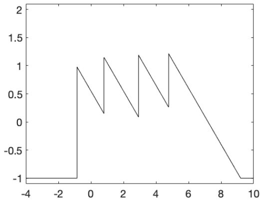
with $u^T \in BV(\mathbb{R})$ and $\mathcal{U}_{ad}^0 = \{u_0 \in BV(\mathbb{R}) / \|u_0\|_{BV(\mathbb{R})} < C \text{ and } \mathrm{Supp}(u_0) \subset K_0\}$.

Theorem

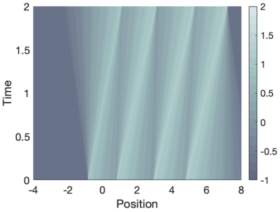
For a.e $T > 0$, the optimal control problem (Opt-Pb) *admits multiple optimal solutions. Moreover, the initial datum $u_0 \in BV(\mathbb{R})$ is an optimal solution of* (Opt-Pb) *if and only if $u_0 \in BV(\mathbb{R})$ verifies $S_T^+(u_0) = S_T^+(S_T^-(u^T))$.*

- A full characterization of the set of initial data $u_0 \in BV(\mathbb{R})$ such that $S_T^+(u_0) = S_T^+(S_T^-(u^T))$ is given in [**Colombo-Perrolaz, 2019**].
- If there exists an initial datum $u_0 \in BV(\mathbb{R})$ such that $S_T^+(u_0) = u^T$ then $S_T^+(S_T^-(u^T)) = u^T$.

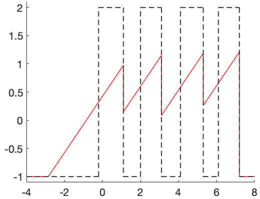
$$u^T(x) = \begin{cases} 2 & \text{if } x \in (-0.2, 1.1) \bigcup (2, 3.1) \bigcup (4.1, 5.3) \bigcup (6.1, 7.2), \\ -1 & \text{otherwise.} \end{cases}$$



$x \rightarrow S_T^-(u^T)(x)$



$(t, x) \rightarrow S_t^+(S_T^-(u^T))(x)$



u^T and $x \rightarrow S_T^+(S_T^-(u^T))(x)$

A target u^T with finite number of shocks

The two following results are given in [**Colombo-Perrolaz, 2019**].

- There exists an initial datum $u_0 \in BV(\mathbb{R})$ such that $S_T^+(u_0) = u^T$ iff u^T satisfies the Oleinik condition, means that $\partial_x u^T \leq \frac{1}{T}$ in the sense of distributions.

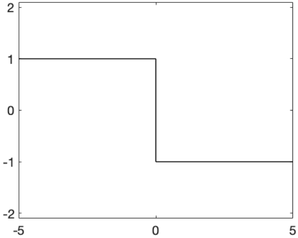
- A map $u_0 \in BV(\mathbb{R})$ verifies $S_T^+(u_0) = u^T$ if and only if the two following statements hold :

    - For every $x \in \mathbb{R} \setminus \cup_{i=1}^N [a_i, b_i]$, $u_0(x-) = S_T^-(u^T)(x-)$.
    - For every $x \in \cup_{i=1}^N [a_i, b_i]$

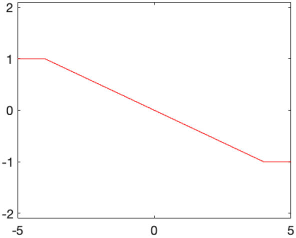
$$\int_{a_i}^{x} u_0(s)\, ds \geq \int_{a_i}^{x} S_T^-(u^T)(s)\, ds,$$

$$\int_{a_i}^{b_i} u_0(s)\, ds = \int_{a_i}^{b_i} S_T^-(u^T)(s)\, ds.$$

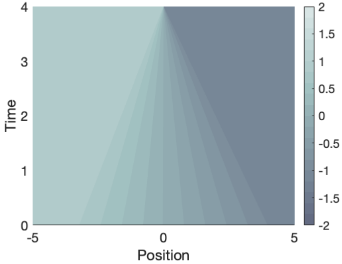
with $a_i := x_i^T - Tf'(u^T(x_i^T-))$ and $b_i := x_i^T - Tf'(u^T(x_i^T+))$ and $(x_i^T)_{i\in\{0,\cdots,N\}}$ the $N \in \mathbb{N} \cup \{\infty\}$ discontinuous points of u^T such that $u^T(x_i^T+) < u^T(x_i^T-)$.



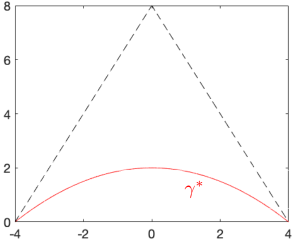
$u^T = \mathbb{1}_{(-\infty,0)} - \mathbb{1}_{(0,\infty)}$



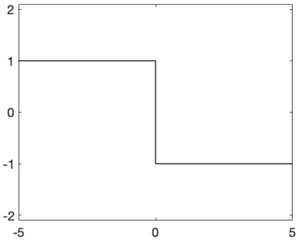
$S_T^-(u^T)$ such that $S_T^+(S_T^-(u^T)) = u^T$.



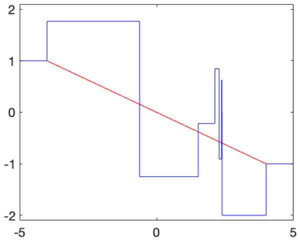
$(t, x) \to S_t^+(S_T^-(u^T))(x)$



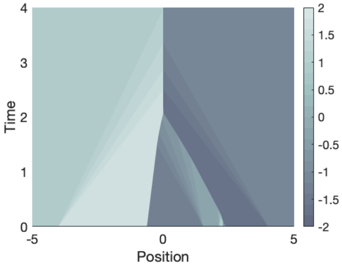
$x \to \gamma^*(x) := \int_{-4}^{x} S_T^-(u^T)(s)\, ds.$



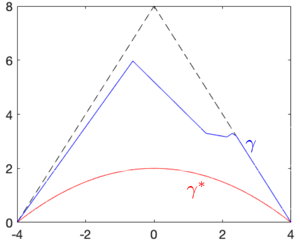
$$u^T = \mathbb{1}_{(-\infty,0)} - \mathbb{1}_{(0,\infty)}$$



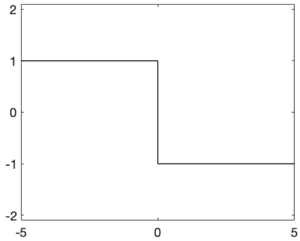
$S_T^-(u^T)$ and u_0 such that
$$S_T^+(S_T^-(u^T)) = S_T^+(u_0) = u^T.$$



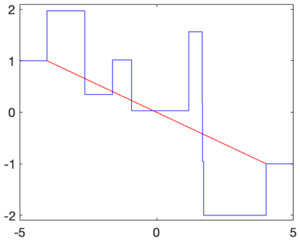
$$(t,x) \rightarrow S_t^+(u_0)(x)$$



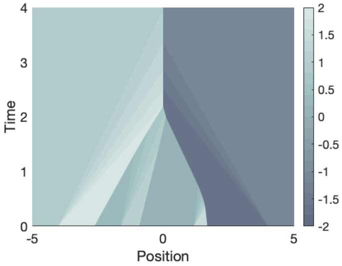
$$\gamma(x) := \int_{-4}^{x} u_0(s)\, ds \geq \gamma^*(x), \ \forall x \in [-4,4]$$



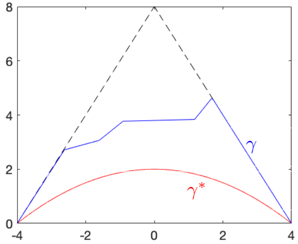
$u^T = \mathbb{1}_{(-\infty,0)} - \mathbb{1}_{(0,\infty)}$



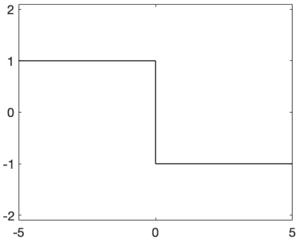
$S_T^-(u^T)$ and u_0 such that
$S_T^+(S_T^-(u^T)) = S_T^+(u_0) = u^T.$



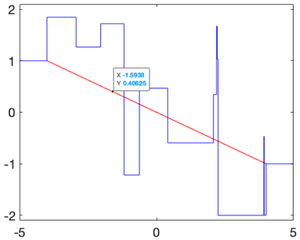
$(t, x) \rightarrow S_t^+(u_0)(x)$



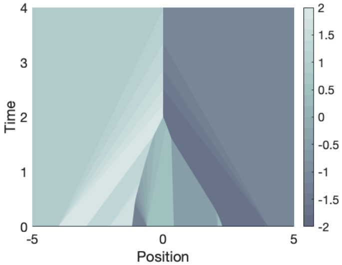
$\gamma(x) := \int_{-4}^{x} u_0(s)\, ds \geq \gamma^*(x), \ \forall x \in [-4, 4]$



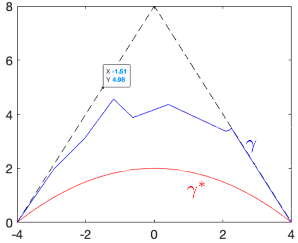
$$u^T = \mathbb{1}_{(-\infty,0)} - \mathbb{1}_{(0,\infty)}$$



$S_T^-(u^T)$ and u_0 such that
$$S_T^+(S_T^-(u^T)) = S_T^+(u_0) = u^T.$$



$$(t,x) \to S_t^+(u_0)(x)$$



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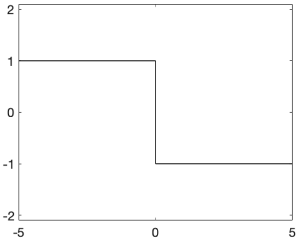

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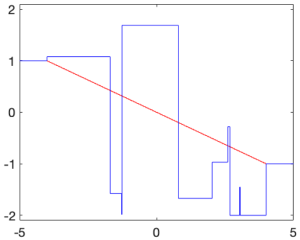

$S_T^-(u^T)$ and u_0 such that
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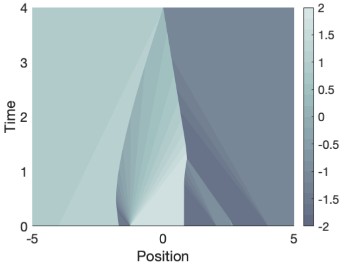
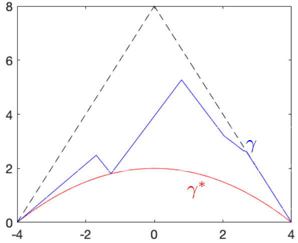
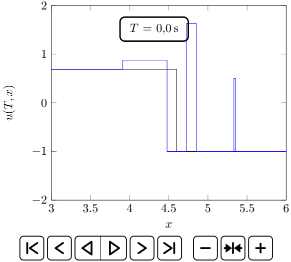


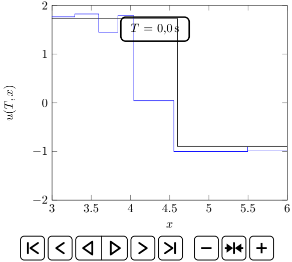
$$(t,x) \to S_t^+(u_0)(x)$$



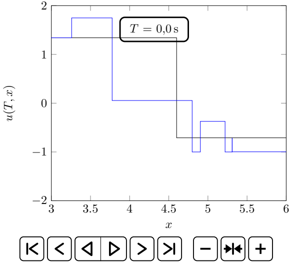
$$\gamma(x) := \int_{-4}^{x} u_0(s)\,ds \geq \gamma^*(x),\ \forall x \in [-4,4]$$

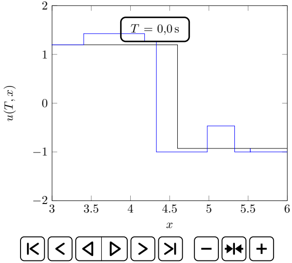
Multiple initial data leading to a shock u^T





$$S_T^+(u_0) = u^T$$





Construction of multiple initial data leading to a shock

Assuming that $u^T = \begin{cases} u_L & \text{if} \quad x < \bar{x}, \\ u_R & \text{if} \quad x > \bar{x}. \end{cases}$

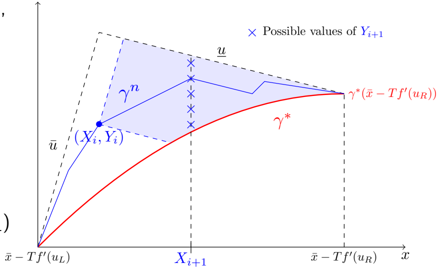
We construct a state mesh $\mathcal{M}_n = \underline{u} + (\bar{u} - \underline{u})2^{-n}\mathbb{N} \cap [0,1]$ such that $\underline{u} \leq u_0 \leq \bar{u}$ and $u_L, u_R \in \mathcal{M}_n$

Construction of a path γ^n such that

- $\gamma^n(x) \geq \gamma^*(x)$, $\forall x \in [\bar{x} - Tf'(u_L), \bar{x} - Tf'(u_R)]$,
- $\gamma^n(\bar{x} - Tf'(u_L)) = 0$,
- $\gamma^n(\bar{x} - Tf'(u_R)) = \gamma^*(\bar{x} - Tf'(u_R))$,
- $\dot{\gamma}^n \in \mathcal{M}_n$.

Construction of u_0 such that $S_T^+(u_0) = u^T$:

$$u_0 = \begin{cases} u_L & \text{for} \quad x < \bar{x} - Tf'(u_L) \\ \dot{\gamma}^n & \text{for a.e} \quad \bar{x} - Tf'(u_L) \leq x \leq \bar{x} - Tf'(u_L) \\ u_R & \text{for} \quad \bar{x} - Tf'(u_R) < x \end{cases}$$



Ideas of the proof

We consider the following optimal control problem

$$\min_{u_0 \in \mathcal{U}_{ad}^0} J_0(u_0) := \int_{\mathbb{R}} \left(u^T(x) - S_T^+(u_0)\right)^2 dx, \qquad \text{(Opt-Pb-1)}$$

$$\mathcal{U}_{ad}^0 = \{u_0 \in BV(\mathbb{R}) / \|u_0\|_{BV(\mathbb{R})} < C \text{ and } \mathrm{Supp}(u_0) \subset K_0\}.$$

$\{\exists u_0 \in BV(\mathbb{R}) / S_T^+(u_0) = q\}$ *iff* $\{q \in BV(\mathbb{R}) /\ \partial_x q \leq \frac{1}{T}\}$

$$\min_{q \in \mathcal{U}_{ad}^1} J_1(q) := \|u^T - q\|_{L^2(\mathbb{R})}, \qquad \text{(Opt-Pb-2)}$$

$$\mathcal{U}_{ad}^1 = \{q \in BV(\mathbb{R}) /\ \partial_x q \leq \frac{1}{T} \text{ and } \|q\|_{BV(\mathbb{R})} \leq C \text{ and } \mathrm{Supp}(q) \subset K_1\}.$$

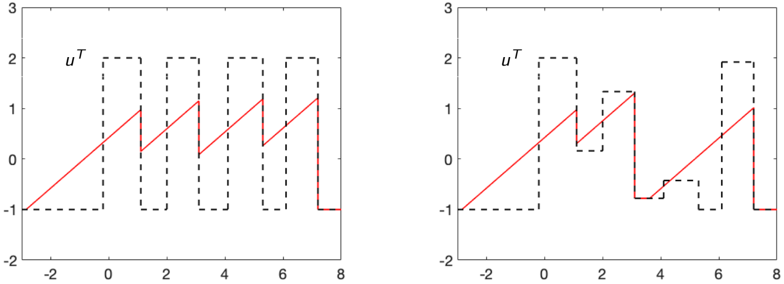
Using $S_T^-(S_T^+(S_T^-(u^T)) = S_T^-(u^T)$ *and a full characterization of* u_0 *such that* $S_T^-(u_0) = S_T^-(u^T)$

$S_T^+(S_T^-(u^T))$ is the unique critical point of (Opt-Pb-2).

Construction of an optimal solution

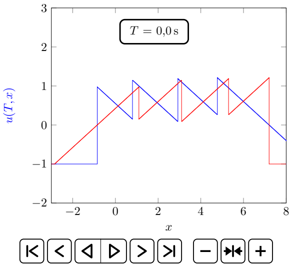
We consider the following optimal control problem

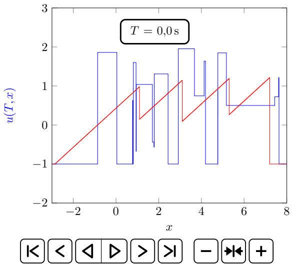
$$\min_{u_0} \int_{\mathbb{R}} \left(u^T(x) - S_T^+(u_0)\right)^2 dx,$$



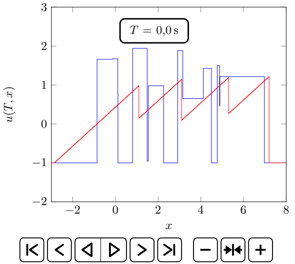
Plotting of the target u^T and $x \to S_T^+(S_T^-(u^T))(x)$
with $S_T^-(u^T)$ an optimal solution.

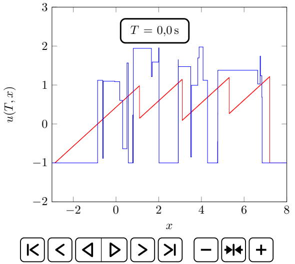
Plotting of multiple optimal solutions





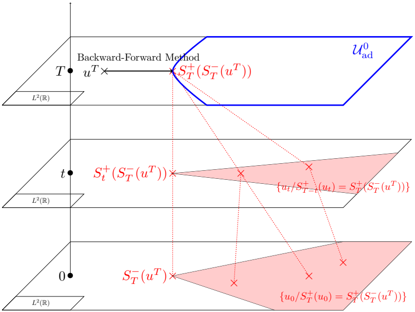
$$S_T^+(u_0) = S_T^+(S_T^-(u^T))$$





Conclusion

$$\text{Optimal problem : } \min_{u_0 \in \mathcal{U}^0_{ad}} \int_{\mathbb{R}} \left(u^T(x) - S^+_T(u_0)\right)^2 dx \qquad \text{(Opt-Pb)}$$



→ Fully characterization of minimizers for (Opt-Pb)

- Construction of the minimizer $S^+_T(S^-_T(u^T))$ of (Opt-Pb) via a **backward-forward method**
- u_0 is a minimizer of (Opt-Pb) iff $S^+_T(u_0) = S^+_T(S^-_T(u^T))$

→ **Implementation of a WFT algorithm** to pick up ramdomly one of the minimizer of (Opt-Pb)

Open problems

1. It would be interesting to extend this work to an "augmented Burgers equation" in order to minimize the sonic boom effects caused by supersonic aircrafts.

2. We may also consider a convex-concave function as a flux function in (PDE) which is for instance a more realistic choice to describe the flow of pedestrian.

3. We can also investigate systems of conservation laws in one dimension (Euler equations, Shallow water equations).

4. To finish, it would be interesting to study numerically the inverse design of multidimensional Burgers equation.