Analysis, control, and singular limits for hyperbolic conservation laws

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Analysis, Steuerung und singuläre Limites für hyperbolische Erhaltungsgleichungen

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vorgelegt von
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Prof. Dr. Mauro Garavello
Prof. Dr. Paola Goatin

"To whom it may concern"

"*Peer's law*: The solution to a problem changes the nature of the problem."

A. Bloch, *Murphy's Law: Complete.* Arrow Books, 2002, p. 54.

"When things get too complicated, it sometimes makes sense to stop and wonder: have I asked the right question?"

E. Bombieri, *Prime Territory. Exploring the Infinite Landscape at the Base of the Number System*, *The Sciences*, vol. 32, no. 5, pp. 32, Sep.–Oct. 1992.

"Lejos un trino.
El ruiseñor no sabe
que te consuela."

J. L. Borges, *Diecisiete haiku (n. 16)*, in *La cifra*, Emecé, 1981.

Contents

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Abstract

The main focus of this thesis is on the study of singular limits related to scalar conservation laws. These are first-order partial differential equations that describe how the amount of a physical quantity in a given region of space changes over time, solely determined by the flux of that quantity across the boundary of the region.

The first part of this manuscript deals with nonlocal regularizations of scalar conservation laws, where the flux function depends on the solution through the convolution with a given kernel. These models are widely used to describe vehicular traffic, where each car adjusts its velocity based on a weighted average of the traffic density ahead. First, we establish the existence, uniqueness, and maximum principle for solutions of the nonlocal problem under mild assumptions on the kernel and flux function. We then investigate the convergence of the solution to that of the corresponding local conservation law when the nonlocality is shrunk to a local evaluation (i.e., when the kernel tends to a Dirac delta distribution). For kernels of exponential type, we analyze this singular limit for initial data of bounded variation as well as for merely bounded ones, using Oleĭnik-type estimates. We also demonstrate how the techniques developed in this analysis can be used to study the long-time behavior of a nonlocal regularization of the Burgers equation and to show that the asymptotic profile is given by the N-wave entropy admissible solution. We also investigate the role played by artificial viscosity in the nonlocal–to–local singular limit process. Finally, we study the boundary controllability problem for nonlocal traffic models.

In the second part of this thesis, we address the controllability of scalar conservation laws on networks and its relationship to the vanishing viscosity singular limit. Our main analysis is carried out in the linear case: for a linear advection-diffusion equation, we show that the cost of controllability blows up exponentially as the viscosity parameter vanishes for small times and decays exponentially for a sufficiently long time-horizon. Finally, for nonlinear conservation laws, we prove a controllability result for entropy solutions using a Lyapunov approach and highlight the stability of this result when a small viscosity is added.

Zusammenfassung

Der Schwerpunkt dieser Dissertation liegt auf der Untersuchung singulärer Grenzwerte im Kontext von skalaren Erhaltungsgleichungen. Dies sind partielle Differentialgleichungen erster Ordnung, die beschreiben, wie sich die Menge einer physikalischen Größe in einer bestimmten Raumregion im Laufe der Zeit ändert, allein bestimmt durch den Fluss dieser Größe über die Grenze der Region.

Der erste Teil dieses Manuskripts befasst sich mit nichtlokalen Regularisierungen von skalaren Erhaltungsgleichungen, bei denen die Flussfunktion von der Lösung abhängt, indem sie mit einem gegebenen Kernel konvolutiert wird. Diese Modelle werden häufig verwendet, um den Fahrzeugverkehr zu beschreiben, bei dem jedes Auto seine Geschwindigkeit anhand eines gewichteten Durchschnitts der Verkehrsdichte voraus anpasst. Zunächst stellen wir die Existenz, Eindeutigkeit und das Maximumprinzip für Lösungen des nichtlokalen Problems unter milden Annahmen über den Kernel und die Flussfunktion fest. Anschließend untersuchen wir die Konvergenz der Lösung zu der des entsprechenden lokalen Erhaltungsgesetzes, wenn die Nichtlokalität zu einer lokalen Auswertung schrumpft (d.h. wenn der Kern gegen eine Dirac-Delta-Verteilung tendiert). Für Kerne exponentiellen Typs analysieren wir diesen singulären Grenzwert für Anfangsdaten mit beschränkter Variation sowie für lediglich beschränkte Anfangsdaten unter Verwendung von Oleĭnik-Typ-Schätzungen. Techniken verwendet werden können, um das Langzeitverhalten einer nichtlokalen Regularisierung der Burgers-Gleichung zu untersuchen und nachzuweisen, dass das asymptotische Profil durch die N-Wellen-Entropie zulässige Lösung gegeben ist. Wir untersuchen auch die Rolle, die künstliche Viskosität im nichtlokalen-zu-lokalen singulären Grenzprozess spielt. Schließlich untersuchen wir das Randsteuerungsproblem für nichtlokale Verkehrsmodelle.

Im zweiten Teil dieser Arbeit beschäftigen wir uns mit der Steuerbarkeit skalärer Erhaltungsgleichungen auf Netzwerken und ihrem Zusammenhang mit dem Grenzwert des verschwindenden Viskositätskoeffizienten. Unsere Hauptanalyse wird im linearen Fall durchgeführt: Für eine lineare Advektions-Diffusions-Gleichung zeigen wir, dass die Kosten der Steuerbarkeit exponentiell ansteigen, wenn der Viskositätskoeffizient für kurze Zeiten gegen Null geht, und für einen ausreichend langen Zeitraum exponentiell abfallen. Schließlich beweisen wir für nichtlineare Erhaltungsgleichungen ein Steuerbarkeitsresultat für Entropielösungen unter Verwendung eines Lyapunov-Ansatzes und heben die Stabilität dieses Ergebnisses hervor, wenn eine geringe Viskosität hinzugefügt wird.

CHAPTER 1

Introduction

The main object of study of this thesis is the first-order partial differential equation

$$\partial_t \rho(t,x) + \partial_x f(\rho(t,x)) = 0, \qquad (t,x) \in (0,+\infty) \times \mathbb{R}, \tag{1.0.1}$$

where $\rho : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is called *conserved quantity* and $f : \mathbb{R} \to \mathbb{R}$ *flux (of the conserved quantity)*. This type of equation is called *scalar conservation law* because it expresses the fact that the variation of ρ between two points is equal to the difference of the flux at these two points. Indeed, if we (formally) integrate (1.0.1) between two points $a, b \in \mathbb{R}$ (with $a < b$), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^b \rho(t,x)\,\mathrm{d}x &= \int_a^b \partial_t \rho(t,x)\,\mathrm{d}x = -\int_a^b \partial_x f(\rho(t,x))\,\mathrm{d}x = f(\rho(t,a)) - f(\rho(t,b)) \\ &= [\text{inflow at } x = a \text{ and time } t] - [\text{outflow at } x = b \text{ and time } t]\,. \end{aligned} \tag{1.0.2}$$

Under integrability assumptions, this leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} \rho(t,x)\,\mathrm{d}x = 0.$$

In other words, the physical entity ρ is neither created nor destroyed. Thus, (1.0.1) arises when modeling the evolution of a quantity that is conserved: mass, momentum, energy, etc.

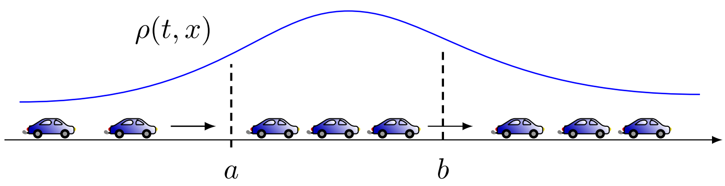

FIGURE 1.1. Illustration of (1.0.2) in case ρ represents a density of cars. Cf. [**44**, Figure 2].

Another way to view the dynamics of (1.0.1) is to rewrite the equation in advective form:

$$\partial_t \rho(t,x) + f'(\rho(t,x)) \partial_x \rho(t,x) = 0, \qquad (t,x) \in (0,+\infty) \times \mathbb{R}.$$

When the flux f is not linear, the quantity ρ is transported at a speed of $f'(\rho)$ that depends on the solution itself.

Let us now focus on solving (1.0.1) and, more specifically, the associated Cauchy problem

$$\begin{cases} \partial_t \rho(t,x) + \partial_x (f(\rho(t,x))) = 0, & (t,x) \in (0,+\infty) \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.0.3}$$

where $\rho_0 : \mathbb{R} \to \mathbb{R}$ is a given initial condition. Among the key features of scalar conservation laws, there is the emergence of singularities in finite time (even starting from smooth initial data; see Figure 1.2). Supposing that ρ_0 and f are smooth, we can build classical solutions of (1.0.3) by the *method of characteristics*, at least on a sufficiently small time-horizon. The idea is as follows: a classical solution of (1.0.3) is constant along the curves satisfying $x'(t) = f'(\rho(t,x(t)))$. Indeed, we compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\rho(t,x(t))) &= \partial_t \rho(t,x(t)) + x'(t) \partial_x \rho(t,x(t)) \\ &= \partial_t \rho(t,x(t)) + f'(\rho(t,x(t))) \partial_x \rho(t,x(t)) = 0, \qquad t > 0. \end{aligned}$$

This allows us to define the solution ρ by fixing a point $(t,x) \in (0,+\infty) \times \mathbb{R}$, solving the equation

$$x = y + t f'(\rho_0(y)), \qquad y \in \mathbb{R}, \tag{1.0.4}$$

and setting $\rho(t,x) = \rho_0(y)$. In the linear case, the characteristic curves are a priori known straight lines and the graph of the solution at a given time is obtained by translating the graph of the initial data. In the nonlinear case, the speed depends on the solution itself, which thus deforms over time (as illustrated in Figure 1.2) and it may happen that equation (1.0.4) has multiple solutions, thus preventing the construction of a classical solution. More precisely, by [**222**, Proposition 2.1.1], for $\rho_0 \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, the Cauchy problem (1.0.3) has one and only one C^1 solution defined on $[0,T^*) \times \mathbb{R}$, where

$$T^* := \begin{cases} +\infty & \text{if } f' \circ \rho_0 \text{ is increasing,} \\ -\frac{1}{\inf(f' \circ \rho_0)'} & \text{otherwise,} \end{cases}$$

and does not have one in any larger strip $[0,T] \times \mathbb{R}$.

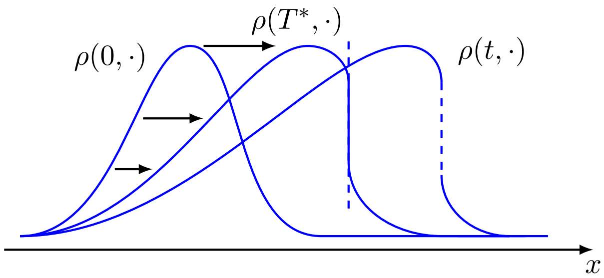

FIGURE 1.2. Finite-time shock formation for scalar conservation laws. Cf. [**44**, Figure 5].

We have thus to resort to extending the notion of solutions to possibly discontinuous functions, such as essentially bounded ones.

Another fundamental feature of conservation laws is the need to prescribe an *entropy condition* in order to single out a unique "physically meaningful" solution among the many possible weak ones. Let us recall the notion of entropy solutions for the Cauchy problem (1.0.3), which is inspired by the second principle of thermodynamics (see [**135, 115**]).

DEFINITION 1.0.1 (Entropy solutions). *A function $\rho : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is an* entropy solution *of the Cauchy problem* (1.0.3) *with initial datum $\rho_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ if $\rho \in L^\infty_{\mathrm{loc}}((0,+\infty) \times \mathbb{R})$ and, for every non-negative test function $\varphi \in C_c^\infty([0,+\infty) \times \mathbb{R}; \mathbb{R}_+)$, we have*

$$\int_0^\infty \int_{\mathbb{R}} \Big(\eta(\rho(t,x)) \partial_t \varphi(t,x) + q(\rho(t,x)) \partial_x \varphi(t,x) \Big) \, \mathrm{d}t \, \mathrm{d}x + \int_{\mathbb{R}} \eta(\rho_0(x)) \varphi(0,x) \, \mathrm{d}x \geq 0 \tag{1.0.5}$$

for every convex entropy η *with* entropy-flux q, *i.e.* $\eta, q \in C^2(\mathbb{R})$ *with* $\eta'' \geq 0$ *and* $\eta' f' = q'$.

One of the main contributions in the classical theory of conservation laws, due to Kružkov (see [**174**]), is the fact that the solution operator of the scalar conservation law (1.0.1) is a L^1-contraction: using the entropy–entropy-flux pair

$$\eta(\rho,k) := |\rho - k| \quad \text{and} \quad q(\rho,k) := \operatorname{sign}(\rho - k)|f(\rho) - f(k)|, \qquad \text{for every } k \in \mathbb{R},$$

and a "doubling of variables" argument, he proved that two entropy solutions satisfy

$$\int_{\mathbb{R}} |\rho_1(t,x) - \rho_2(t,x)| \, \mathrm{d}x \leq \int_{\mathbb{R}} |\rho_1(0,x) - \rho_2(0,x)| \, \mathrm{d}x, \qquad t > 0.$$

From this, uniqueness and BV bounds follow. For a modern presentation of the proof of the well-posedness of entropy solutions for scalar conservation laws, we refer the reader to the monographs [**116, 161, 203, 145, 222, 43**].

In this thesis, we shall focus on two aspects of scalar conservation laws:

1. their nonlocal regularization;
2. their study on a metric graph.

1.1. Nonlocal regularizations of scalar conservation laws

1.1.1. Motivation and formulation of the Cauchy problem. Several mathematical models are formulated in terms of conservation laws.

For instance, the velocity u of a field of particles that do not interact with each other (an isolated one-dimensional medium) is modeled by the *(inviscid) Burgers equation*[1]:

$$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0, \qquad (t,x) \in (0,+\infty) \times \mathbb{R}.$$

In most of what follows, we will be motivated by the study of macroscopic traffic flow (as introduced by Lighthill–Whitham–Richards, [**218, 187**]): ρ represents the car density and, typically, we assume that the flux is a concave function given by

$$f(\rho) \coloneqq V(\rho)\rho,$$

for a suitable velocity function $V : \mathbb{R} \to \mathbb{R}$ that is monotonically decreasing (i.e., the higher the density of cars on the road, the lower their speed)—e.g., the celebrated *LWR–Greenshields model* $V(\xi) \coloneqq v_{\max}(1 - \xi/\rho_{\max})$, with $v_{\max} > 0$ and $\rho_{\max} > 0$ (see, e.g., [**147**] and [**138**, Section 3.1.2]).

The dynamics given by the Cauchy problem

$$\begin{cases} \partial_t \rho(t,x) + \partial_x\big(V(\rho(t,x))\rho(t,x)\big) = 0, & (t,x) \in (0,+\infty) \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1.1}$$

where $\rho_0 : \mathbb{R} \to \mathbb{R}$ represents the (non-negative) initial traffic density, does not allow for a car to change its velocity looking at the traffic ahead, but only based on the density at the given space-time point. This is one reason for introducing a *nonlocal* variant of this model, which can be written as

$$\begin{cases} \partial_t \rho(t,x) + \partial_x\big(V(W[\rho](t,x))\rho(t,x)\big) = 0, & (t,x) \in (0,+\infty) \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1.2}$$

with

$$W[\rho](t,x) \coloneqq (\gamma * \rho)(t,x) \coloneqq \int_{\mathbb{R}} \gamma(x-y)\rho(t,y)\,\mathrm{d}y. \tag{1.1.3}$$

We shall call $W[\rho] : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$ the *nonlocal impact* affecting the velocity function $V : \mathbb{R} \to \mathbb{R}$ of the nonlocal conservation law in (1.1.2). The *nonlocal weight* γ influences how the density is averaged; for traffic flow modeling, it is reasonable to assume that γ is anisotropic and, in particular, supported on the negative axis $\mathbb{R}_-$ and monotonically non-decreasing. This means that the drivers adjust their speed based only on the "downstream" traffic density (i.e., only looking forward and not backward) and give it more consideration the closer it is to their position.

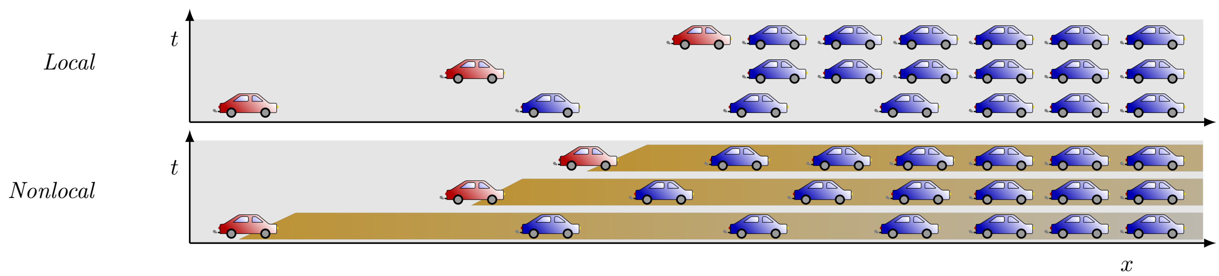

FIGURE 1.3. Comparison of local and nonlocal traffic flow models. In the nonlocal case, the red car looks ahead within the golden region and adjusts its velocity in response to the high downstream density (which illustrates the effect of the nonlocal impact on the dynamics). Cf. [**170**, Figure 1].

In recent decades, nonlocal balance laws have been used to describe various physical phenomena: from the aforementioned traffic flow [**144, 35, 133, 65, 64, 63**], to supply chains [**166, 223, 146**], crowd dynamics [**4, 96, 94, 95**], opinion formation [**6, 215**], chemical engineering processes [**214,**

[1] The origins of the *viscous* Burgers equation can be traced back to [**49, 85, 29**].

226], sedimentation [**33**], slow erosion of granular matter [**7, 83, 81**], materials with fading memory effects [**60**], various biological and industrial models [**97**], and conveyor belt dynamics [**220**]. We refer to [**170**] for a survey. Moreover, a nonlocal regularization of the Burgers equation (the α-*Burgers equation*; see [**225, 32, 149**]) has long been used to describe the averaged motion of an ideal incompressible fluid, filtering over spatial scales smaller than some a priori fixed $\alpha > 0$:

$$\begin{cases} \partial_t u_\alpha(t,x) + v_\alpha(t,x)\partial_x u_\alpha(t,x) = 0, & (t,x) \in (0,+\infty)\times\mathbb{R}, \\ -\alpha\partial^2_{xx} v_\alpha(t,x) + v_\alpha(t,x) = u_\alpha(t,x), & (t,x) \in (0,+\infty)\times\mathbb{R}. \end{cases}$$

For the Cauchy problem (1.1.2), various well-posedness results have been presented in the literature under different sets of assumptions regarding initial data, velocity function, and nonlocal weight. For example, in [**35, 33**], existence was established via numerical schemes, while, in [**83**], the vanishing viscosity technique was employed; both approaches required an entropy condition to be imposed to prove uniqueness. More recently, the existence and uniqueness of weak solutions were established using fixed-point methods, without necessitating an entropy condition (see, e.g., [**109, 167, 172, 171**]). For measure-valued solutions, a similar approach was adopted in [**112**]. A key point in this analysis is that, for the Cauchy problem (1.1.2), the regularity of the initial data is essentially preserved (under suitable assumptions) because of the nonlocal term, which means that no shocks may arise from a smooth initial density.

The first contribution of this thesis, presented in CHAPTER 2[2], is to extend the previously known well-posedness results to a more general situation: essentially bounded initial data, locally Lipschitz continuous velocity, and assuming that the nonlocal weight is non-negative and belongs to BV. This generalizes the standard condition $\gamma \in W^{1,\infty}$ commonly found in the literature (see, for example, [**167**]).

Following [**8**], we recall that the *total variation* of a function $u \in L^1(\mathbb{R})$ is given by

$$|u|_{\mathrm{TV}(\mathbb{R})} \coloneqq \sup\left\{ \int_{\mathbb{R}} u\,\psi'\,\mathrm{d}x : \ \psi \in C^\infty_{\mathrm{c}}(\mathbb{R}),\ \|\psi\|_{C^0(\mathbb{R})} \le 1 \right\}.$$

The norm $\|u\|_{\mathrm{BV}(\mathbb{R})} \coloneqq \|u\|_{L^1(\mathbb{R})} + |u|_{\mathrm{TV}(\mathbb{R})}$ endows the space of L^1 functions with bounded variation, $\mathrm{BV}(\mathbb{R})$, with a Banach space structure. We also use the notation $\mathrm{TV}(\mathbb{R})$ to denote the space of measurable functions with bounded variation.

The key idea to establish the well-posedness result is to interpret the nonlocal conservation law in (1.1.2) as a fixed-point problem. To this end, we may rewrite it as a coupled system

$$\begin{cases} \partial_t \rho(t,x) + \partial_x(V(w(t,x))\rho(t,x)) = 0, & (t,x) \in (0,+\infty)\times\mathbb{R}, \\ w(t,x) \coloneqq W[\rho](t,x), & (t,x) \in (0,+\infty)\times\mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.1.4}$$

As long as w is Lipschitz continuous, we can view the first equation in (1.1.4) as a linear conservation law with a Lipschitz continuous space-time dependent velocity field. Using the method of characteristics, we can then express the solution as

$$\rho(t,x) = \rho_0(\xi[w](t,x;0))\partial_x\xi[w](t,x;0), \quad (t,x) \in (0,T)\times\mathbb{R},$$

for $T > 0$, where $\xi[w]$ is the solution of the characteristic equation (written here as a Volterra-type integral equation):

$$\xi(t,x;\tau) = x + \int_t^\tau V(w(s,\xi(t,x;s)))\,\mathrm{d}s, \quad (t,x,\tau) \in (0,T)\times\mathbb{R}\times(0,T).$$

Plugging this back into the second equation of (1.1.4) yields the following fixed-point problem in w:

$$w(t,x) = \int_{\mathbb{R}} \gamma(x-y)\rho(t,y)\,\mathrm{d}y = \int_{\mathbb{R}} \gamma(x-\xi[w](0,y;t))\rho_0(y)\,\mathrm{d}y, \quad (t,x) \in (0,T)\times\mathbb{R}. \tag{1.1.5}$$

Banach's fixed-point theorem provides the existence of a unique solution for problem (1.1.5) in $L^\infty((0,T^*);W^{1,\infty}(\mathbb{R}))$ for a suitably small time-horizon $T^* > 0$. Under the physically reasonable

[2] G. M. Coclite, **N. De Nitti**, A. Keimer, and L. Pflug. On existence and uniqueness of weak solutions to nonlocal conservation laws with BV kernels. *Z. Angew. Math. Phys.*, 73(6):Paper No. 241, 10, 2022.

additional monotonicity assumptions on the velocity and the weight mentioned above, we obtain the existence result on an arbitrary time-horizon due to a comparison principle.

1.1.2. Nonlocal–to–local singular limit problem. Having established a well-posedness result for the nonlocal problem, a natural question arises: whether we can recover the entropy-admissible solution of the local equation as the nonlocality is shrunk to a local evaluation (i.e., when the weight γ approaches a Dirac delta distribution). In other words, given a *nonlocal average parameter* $\alpha > 0$, we consider the rescaled problem

$$\begin{cases} \partial_t \rho_\alpha(t,x) + \partial_x\big(V(W_\alpha[\rho_\alpha](t,x))\rho_\alpha(t,x)\big) = 0, & (t,x) \in (0,+\infty)\times\mathbb{R}, \\ \rho_\alpha(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1.6}$$

with

$$W_\alpha[\rho_\alpha](t,x) \coloneqq (\gamma_\alpha * \rho_\alpha)(t,x) \coloneqq \frac{1}{\alpha}\int_{\mathbb{R}} \gamma\left(\frac{x-y}{\alpha}\right)\rho_\alpha(t,y)\,\mathrm{d}y; \tag{1.1.7}$$

the aim is then to study the limit of the family $\{\rho_\alpha\}_{\alpha>0}$ as $\alpha \to 0^+$.

The interest in a nonlocal–to–local convergence result lies in bridging the gap between nonlocal and local modeling of phenomena described by conservation laws. In particular, this type of singular limit presents an alternative to the classical vanishing viscosity approach (for which we refer, e.g., to [**161**, Appendix B]): it allows us to define entropy-admissible solutions of local conservation laws by viewing them as limits of weak solutions to nonlocal conservation laws. In contrast to the case of a parabolic viscous regularization, the nature of the approximating nonlocal equation retains a somewhat “hyperbolic” character (namely, with finite propagation of mass—albeit with infinite propagation of information).

First, in [**13**], it was observed that, at least numerically, the solution of the nonlocal conservation law appears to converge to the entropy solution of the corresponding local problem as $\alpha \to 0^+$. However, in [**90**], several counterexamples highlighted that this is not generally true. Positive results on the nonlocal–to–local convergence were obtained in [**234**] provided that the limit entropy solution is smooth and the convolution kernel is even; in [**168**], for a large class of nonlocal conservation laws with monotone initial data, exploiting the fact that monotonicity is preserved along the evolution; and, in [**88**], under the assumption that the initial datum has bounded total variation, is bounded away from zero, and satisfies a one-sided Lipschitz condition. More recently, in [**45, 46**], Bressan and Shen proved a convergence result for an exponential nonlocal weight—i.e., for $\gamma(\cdot) \coloneqq \mathbb{1}_{(-\infty,0]}(\cdot)\exp(\cdot)$—provided that the initial datum is bounded away from zero and has bounded total variation. The core of their argument is the observation that, under suitable changes of variables, the nonlocal problem can be rewritten as a hyperbolic system with relaxation terms: indeed, letting $s \coloneqq t - x/\lambda$, $y \coloneqq x$ (with $\lambda > 0$ to be later chosen sufficiently large), the PDE in (1.1.6) can be rewritten as

$$\begin{cases} \partial_s\left(\lambda\rho_\alpha - \rho_\alpha V(W_\alpha)\right) + \partial_y(\lambda\rho_\alpha V(W_\alpha)) = 0, & (s,y) \in (0,+\infty)\times\mathbb{R}, \\ \partial_s W_\alpha - \lambda\partial_y W_\alpha = \frac{\lambda}{\alpha}(\rho_\alpha - W_\alpha), & (s,y) \in (0,+\infty)\times\mathbb{R}. \end{cases}$$

The assumption on the initial data being bounded away from zero played a key role in showing a uniform total variation bound for the solution of the nonlocal problem—specifically, for the quantities $\log(\rho_\alpha)$ and $\log(\lambda - V(W_\alpha))$. On the other hand, in [**88**], a counterexample was provided to demonstrate that the total variation of the solution can blow up if the initial datum is not bounded away from zero.

The limitations illustrated by the aforementioned results lead us to focus, in CHAPTER 3[3], on the particular case of the exponential weight $\gamma(\cdot) \coloneqq \mathbb{1}_{(-\infty,0]}(\cdot)\exp(\cdot)$ and study the nonlocal impact W_α instead of the solution ρ_α itself. By analyzing W_α, it is possible to remove the additional assumption that the initial datum is bounded away from zero. This is because W_α enjoys extra

[3]G. M. Coclite, J.-M. Coron, **N. De Nitti**, A. Keimer, and L. Pflug. A general result on the approximation of local conservation laws by nonlocal conservation laws: The singular limit problem for exponential kernels. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 2022.

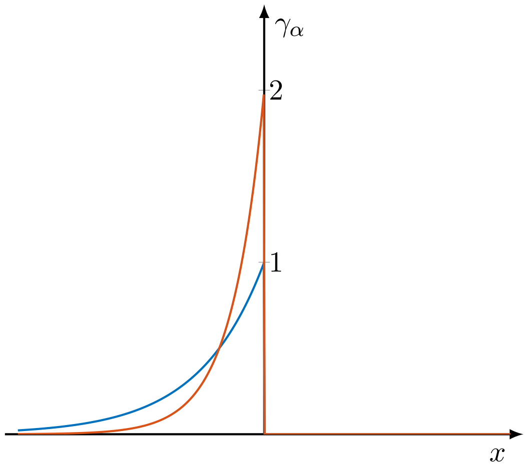

FIGURE 1.4. Plot of the one-sided exponential weight $\gamma_\alpha \coloneqq \alpha^{-1}\mathbb{1}_{(-\infty,0]}(\cdot)\exp(\cdot/\alpha)$ for $\alpha = 1$ (blue) and $\alpha = 0.5$ (red).

regularity; in particular, its total variation remains uniformly bounded by the one of the initial datum. The key idea is to use the ODE given by

$$\begin{aligned}\alpha\partial_x W_\alpha[\rho_\alpha](t,x) &= \partial_x \int_x^\infty \exp\left(\frac{x-y}{\alpha}\right)\rho_\alpha(t,y)\,\mathrm{d}y \\ &= W_\alpha[\rho_\alpha](t,x) - \rho_\alpha(t,x), \quad (t,x)\in(0,+\infty)\times\mathbb{R},\end{aligned}$$

to deduce a conservation law in W_α with a nonlocal source (which acts as a regularization term):

$$\partial_t W_\alpha + \partial_x(V(W_\alpha)W_\alpha) = g_\alpha - g_\alpha * \gamma_\alpha, \qquad \text{where } g_\alpha \coloneqq V'(W_\alpha)W_\alpha\partial_x W_\alpha. \tag{1.1.8}$$

Using (1.1.8), we establish a uniform total variation bound on W_α without assuming that the initial datum is bounded away from zero. Furthermore, due to the specific structure of the exponential weight, this directly implies that the solution of the conservation law converges strongly in L^1 to the entropy-admissible solution of the corresponding local conservation law.

It is worth noting that, in traffic models, the use of an exponential weight is not common; instead, one with compact support is typically chosen to reflect the fact that cars only look ahead within a finite space-horizon. However, the behavior of a compactly supported weight can be effectively approximated by an exponentially-decaying one.

Our approach of using W_α to study the convergence of the nonlocal conservation law was later extended in [**89**] to establish convergence results, with rates, for more general classes of weights.

The key assumption in all the contributions cited above is that the initial datum is of bounded variation. However, it is of interest to consider the case of an initial datum that is merely bounded. To this end, in CHAPTER 4[4], we establish an Oleĭnik-type inequality. For the scalar local conservation law (1.0.1), Oleĭnik's result [**206**] (see also Lax [**181**], Ladyženskaya [**175**], and Hopf [**162**]) states that, if f is uniformly strictly convex (i.e. $f'' \geq \kappa > 0$ on $\mathbb{R}$), then any entropy solution of (1.0.1) satisfies the following one-sided Lipschitz bound:

$$\rho(t,y) - \rho(t,x) \leq \frac{y-x}{\kappa t}, \qquad t>0,\ x,y\in\mathbb{R},\ x\leq y.$$

This inequality can be written in a sharp form (see [**113, 159**]) assuming $f'' \geq 0$ and that there are no non-trivial intervals where f is affine (*Tartar's condition*; see [**227**]):

$$f'(\rho(t,y)) - f'(\rho(t,x)) \leq \frac{y-x}{t}, \qquad t>0,\ x,y\in\mathbb{R},\ x\leq y.$$

The Oleĭnik estimate provides an equivalent characterization of entropy solutions and exemplifies how the nonlinearity in a PDE can produce a regularizing effect on the solution: initial data in L^∞ are instantaneously regularized to functions of locally bounded variation ($\mathrm{BV}_{\mathrm{loc}}$).

[4]G. M. Coclite, M. Colombo, G. Crippa, **N. De Nitti**, A. Keimer, E. Marconi, L. Pflug, and L. V. Spinolo. Oleĭnik-type estimates for nonlocal conservation laws and applications to the nonlocal-to-local limit. *Submitted*, 2023.

For nonlocal conservation laws, the only previous results in this direction were obtained under rather restrictive assumptions: an Oleĭnik-type estimate was established in [**88**, Theorem 3] for initial data that satisfy a one-sided Lipschitz condition and are bounded away from zero and, in [**108**, Theorem 3.10] (for a slightly different class of nonlocal conservation laws in advective form), for initial data that are quasi-concave and have a one-sided bound on the derivative.

Arguing again with (1.1.8), we establish Oleĭnik-type estimates for the nonlocal term W_α and a one-sided bound for the nonlocal source $V'(W_\alpha)W_\alpha\partial_x W_\alpha$. As a corollary, we deduce the nonlocal–to–local convergence results from Chapter 3 without requiring the initial data to have bounded total variation. However, the trade-off is that we need stronger assumptions on the velocity function V (which are still satisfied by several relevant models).

1.1.3. Long-time asymptotics for a nonlocal Burgers equation and N-waves. The Oleĭnik inequalities established in Chapter 4 serve as a basis for exploring a distinct, albeit somewhat related, problem: the long-time behavior of the solution to a nonlocal Burgers equation. In Chapter 5[5], we consider

$$\begin{cases} \partial_t\rho(t,x) + \partial_x\big(W[\rho](t,x)\rho(t,x)\big) = 0, & (t,x) \in (0,+\infty)\times\mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x\in\mathbb{R}, \end{cases} \tag{1.1.9}$$

with initial datum $\rho_0 \in L^\infty(\mathbb{R};\mathbb{R}_{\geq 0}) \cap L^1(\mathbb{R};\mathbb{R}_{\geq 0})$ and nonlocal impact

$$W[\rho](t,x) \coloneqq \int_{-\infty}^{x} \exp(y-x)\rho(t,y)\,\mathrm{d}y, \qquad (t,x)\in(0,+\infty)\times\mathbb{R}, \tag{1.1.10}$$

which also satisfies the identity

$$\partial_x W[\rho](t,x) = \rho(t,x) - W[\rho](t,x), \qquad (t,x)\in(0,+\infty)\times\mathbb{R}. \tag{1.1.11}$$

We then show that, as $t\to+\infty$, $\rho(t,\cdot)$ converges to the (unique) *N-wave solution* (or *source-type solution*) w of the local Burgers equation (see [**192**, Eq. (2.1)]), i.e., the solution of the Burgers equation with initial data given by a Dirac delta,

$$\begin{cases} \partial_t w(t,x) + \partial_x(w^2(t,x)) = 0, & (t,x)\in(0,+\infty)\times\mathbb{R}, \\ w(0,x) = M\delta_{\{x=0\}}, & x\in\mathbb{R}, \end{cases} \tag{1.1.12}$$

which is given explicitly by

$$w(t,x) = \begin{cases} \dfrac{x}{2t} & \text{if } x\in(0,\sqrt{4Mt}), \\ 0 & \text{otherwise.} \end{cases} \tag{1.1.13}$$

Here, M denotes the mass of the initial datum ρ_0.

Proving this result can be reduced to studying a nonlocal–to–local singular limit problem that is very similar to the ones of Chapters 3 and 4. Indeed, following [**192**], for a given $\lambda>0$, we consider the rescaled function

$$\rho_\lambda(t,x) \coloneqq \lambda\rho(\lambda^2 t, \lambda x), \tag{1.1.14}$$

which solves

$$\begin{cases} \partial_t\rho_\lambda(t,x) + \partial_x\big(W_\lambda[\rho_\lambda](t,x)\rho_\lambda(t,x)\big) = 0, & (t,x)\in(0,+\infty)\times\mathbb{R}, \\ \rho_\lambda(0,x) = \rho_{0,\lambda}(x) \coloneqq \lambda\rho_0(\lambda x), & x\in\mathbb{R}, \end{cases} \tag{1.1.15}$$

with

$$W_\lambda[\rho_\lambda](t,x) \coloneqq \lambda\int_{-\infty}^{x} \exp(\lambda(y-x))\rho_\lambda(t,y)\,\mathrm{d}y, \qquad (t,x)\in(0,+\infty)\times\mathbb{R}. \tag{1.1.16}$$

We show that, for a fixed $t>0$, $\rho_\lambda(t,\cdot)\to w(t,\cdot)$ in $L^1(\mathbb{R})$ as $\lambda\to\infty$, which, in turn, implies

$$\|\rho(t,\cdot) - w(t,\cdot)\|_{L^1(\mathbb{R})} \to 0 \quad \text{as } t\to+\infty,$$

with w defined in (1.1.13).

[5] G. M. Coclite, **N. De Nitti**, A. Keimer, L. Pflug, and E. Zuazua. Long-time convergence of a nonlocal Burgers' equation towards the local N-wave. *Nonlinearity* (to appear), 2023.

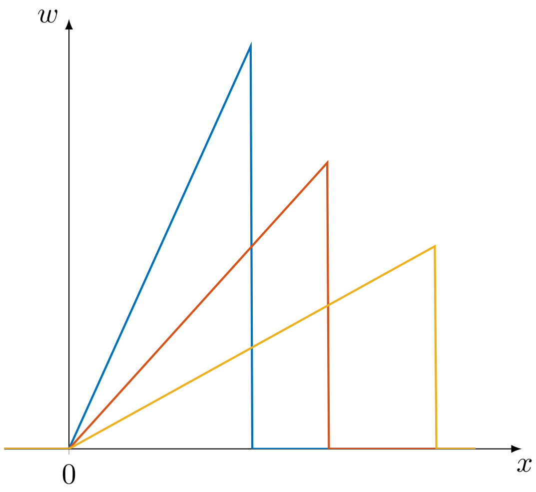

FIGURE 1.5. Plot of an N-wave solution (1.1.13) (with $M = 1$) for $t = 0.5$ (blue), $t = 1$ (red), and $t = 2$ (yellow). Cf. [**72**, Figure 1].

A significant difference and added level of complexity, compared to CHAPTER 3, is the fact that the initial data converge to a Dirac delta, i.e.,

$$\rho_\lambda(0,\cdot) \to M\delta_0 \quad \text{and} \quad W_\lambda(0,\cdot) \to M\delta_0 \tag{1.1.17}$$

in the sense of distributions as $\lambda \to +\infty$. In other words, the sole uniform bound on the initial data ρ_0, with respect to λ, is given by its L^1-norm.

To overcome this difficulty, we leverage the Oleĭnik-type inequality satisfied by W_λ, i.e.

$$\frac{W_\lambda(t,x) - W_\lambda(t,y)}{x-y} \le \frac{1}{t}, \qquad t > 0,\ x,y \in \mathbb{R},\ x \neq y.$$

Combining it with the uniform L^1-bound

$$\int_{\mathbb{R}} W_\lambda(t,x)\,\mathrm{d}x = M, \qquad t > 0,$$

we deduce an L^∞-bound for $t > 0$:

$$0 \le W_\lambda(t,x) \le \sqrt{\frac{2M}{t}}, \qquad (t,x) \in (0,+\infty)\times\mathbb{R}.$$

With these ingredients, we can establish the convergence of $\{W_\lambda\}_{\lambda>0}$ and $\{\rho_\lambda\}_{\lambda>0}$ to the N-wave solution of the (local) Burgers equation following the general framework proposed in [**127**].

1.1.4. Nonlocal–to–local singular limit with an artificial viscosity. Investigating the effects of artificial viscosity during the nonlocal–to–local approximation process is also relevant. Indeed, most numerical tests used to conjecture the convergence results employed a (dissipative) Lax–Friedrichs scheme; a detailed analysis of the effect of numerical viscosity on the study of the nonlocal–to–local limit can be found in [**87**].

The incorporation of viscous perturbations was extensively explored in the context of smooth, non-negative, compactly supported weights given by standard mollifiers (see [**90, 87, 86**] and also [**51**] in the case of more regular initial data and linear velocity).

In the particular case of an exponential nonlocal weight, in CHAPTER 6[6], we improve the convergence result in [**90**]. For $\alpha, \nu > 0$, we consider

$$\begin{cases} \partial_t \rho_{\alpha,\nu}(t,x) + \partial_x(V(W_\alpha[\rho_{\alpha,\nu}](t,x))\rho_{\alpha,\nu}(t,x)) = \nu\partial^2_{xx}\rho_{\alpha,\nu}(t,x), & (t,x) \in (0,+\infty)\times\mathbb{R}, \\ \rho_{\alpha,\nu}(0,x) = \rho_{0,\nu}(x), & x \in \mathbb{R}, \end{cases} \tag{1.1.18}$$

[6]G. M. Coclite, **N. De Nitti**, A. Keimer, and L. Pflug. Singular limits with vanishing viscosity for nonlocal conservation laws. *Nonlinear Anal.*, 211:Paper No. 112370, 12, 2021.

with

$$W_\alpha[\rho_{\alpha,\nu}](t,x) \coloneqq \frac{1}{\alpha}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right)\rho_{\alpha,\nu}(t,y)\,\mathrm{d}y, \qquad (t,x)\in(0,+\infty)\times\mathbb{R}, \tag{1.1.19}$$

and smoothed initial data $\rho_{0,\nu}$.

As $\alpha,\nu\to 0^+$, we show that the family of solutions $\{\rho_{\alpha,\nu}\}_{\alpha,\nu>0}$ converges to the entropy solution of the corresponding local conservation law (1.1.1) under the assumption $\alpha/\nu\to 0$ (which is weaker than the condition $\alpha\le e^{-C\nu^{-\beta}}$, with C, $\beta>0$, used in [**90**]).

A critical observation for our proof is that (1.1.18)–(1.1.19) can be reformulated as a higher-order equation with competing diffusion and dispersion effects:

$$\begin{cases} \partial_t W_{\alpha,\nu} + \partial_x(V(W_{\alpha,\nu})W_{\alpha,\nu}) \\ = \alpha\partial_{tx}^2 W_{\alpha,\nu} + \nu\partial_{xx}^2 W_{\alpha,\nu} + \alpha\partial_x(V(W_{\alpha,\nu})\partial_x W_{\alpha,\nu}) - \alpha\nu\partial_{xxx}^3 W_{\alpha,\nu}, & (t,x)\in(0,+\infty)\times\mathbb{R}, \\ W_{\alpha,\nu}(0,x) = \frac{1}{\alpha}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right)\rho_{0,\nu}(y)\,\mathrm{d}y, & x\in\mathbb{R}. \end{cases} \tag{1.1.20}$$

While the proof presented in [**90**] (which considers more general kernels) relies heavily on energy estimates for the heat kernel and Duhamel's principle, our approach differs. Specifically, we establish an energy estimate for the nonlocal impact $W_{\alpha,\nu}$ by leveraging the structure of (1.1.20); next, we apply Tatar's compensated compactness technique (see [**197, 227**]) to deduce the L^p compactness of the family $\{\rho_{\alpha,\nu}\}_{\alpha,\nu>0}$.

This strategy draws some inspiration from the study of the singular limit problem for the Camassa–Holm equation (see [**84, 74**]) and the Ostrovski–Hunter equation (see [**76, 73**]): in these cases, the approximating equations are higher-order ones that can be equivalently rewritten as parabolic–elliptic systems or as conservation laws with nonlocal perturbations. These works were, in turn, influenced by the seminal paper [**221**] by Schonbek on the zero diffusion-dispersion limits for the Korteweg–de Vries–Burgers and Benjamin–Bona–Mahony–Burgers equations. However, (1.1.20) presents several peculiarities compared to the equations appearing in the references mentioned above: it involves a mixed derivative $\partial_{tx}^2 W_{\alpha,\nu}$; (1.1.6) represents a PDE–ODE coupling rather than a parabolic–elliptic system; and we do not need to rely on the L^p compensated compactness developed by Schonbek—instead, we are able to deduce an L^∞-estimate from (1.1.18), which allows us to apply the standard L^∞ compensated compactness theorem by Tartar.

1.1.5. Controllability of nonlocal conservation laws. Recent research has also focused on the initial-boundary value problem (IBVP) for nonlocal conservation laws. In [**172**], a fixed-point approach based on characteristics (similar to the approach also used in CHAPTER 2 for the Cauchy problem) was utilized to show the existence and uniqueness of a weak solution. This naturally leads to the question of whether it is possible to use the boundary data to "control" the nonlocal dynamics.

The literature addressing the boundary controllability and stabilization of solutions of nonlocal conservation laws is limited, despite the significant interest in this problem. This interest stems from the question of whether it is possible to direct the traffic on a road segment toward a target end-state or to achieve a given out-flux. In [**107, 223, 110, 109, 111, 62**], some results on end-state and out-flux controllability, as well as asymptotic exponential stabilization, were obtained for a nonlocal model introduced in [**24**] to describe semiconductor manufacturing systems:

$$\begin{cases} \partial_t\rho(t,x) + \partial_x(\rho(t,x)V(W[\rho](t))) = 0, & (t,x)\in(0,+\infty)\times(0,1), \\ \rho(0,x) = \rho_0(x), & x\in(0,1), \\ V(W(t))\rho(t,0) = u(t), & t\in(0,+\infty), \end{cases} \tag{1.1.21}$$

with a strictly positive velocity function $V\in C^1(\mathbb{R};\mathbb{R}_{>0})$ and a nonlocal impact that is independent of the spatial domain—namely, $V(\xi)\coloneqq 1/(1+\xi)$, for $\xi\ge 0$, and $W[\rho](t)\coloneqq\int_0^1\rho(t,x)\,\mathrm{d}x$. These results were generalized in [**66**], where (local) end-state controllability and out-flux controllability results were established for a space-dependent velocity (but space-independent nonlocal impact), $V(x,W[\rho](t))$.

For a related system of scalar nonlocal conservation laws on networks (modeling a highly re-entrant multi-commodity manufacturing system), the optimal control problem was analyzed in [**155**].

A more general conservation law featuring an explicitly space-dependent nonlocal flux was considered in [**91**] in the context of modeling supply chains and pedestrian flow.

In Chapter 7[7], we extend the previous results by investigating the controllability of

$$\begin{cases} \partial_t \rho_\alpha(t,x) + \partial_x\big(V\big(\mathcal{W}_\alpha[\rho_\alpha](t,x)\big)\rho_\alpha(t,x)\big) = 0, & (t,x) \in (0,+\infty)\times(0,1),\\ \rho_\alpha(0,x) = \rho_0(x), & x \in (0,1),\\ V(\mathcal{W}[\rho_\alpha](t,0))\rho_\alpha(t,0) = V(\mathcal{W}[\rho_\alpha](t,0))u_\ell(t), & t \in (0,+\infty), \end{cases}$$

with

$$\mathcal{W}[\rho_\alpha](t,x) \coloneqq \frac{1}{\alpha}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right)\Big(\mathbb{1}_{(-\infty,1)}(y)\rho_\alpha(t,y) + \mathbb{1}_{[1,+\infty)}(y)u_r(t)\Big)\,\mathrm{d}y,$$

for $(t,x) \in (0,+\infty)\times(0,1)$. At the entry point of the road, $x=0$, an in-flux boundary condition with density u_ℓ is prescribed, which can be interpreted as "on-ramp metering". The function u_r in the nonlocal operator can be interpreted as a parameter that influences the velocity with which the density (normalized between 0 and 1 for this discussion) leaves the domain at $x=1$; for instance, it can be used to model traffic lights: if $u_r \equiv 1$, no density leaves (red light); if $u_r \equiv 0$, the adjacent road is fully evacuated and the density can leave as fast as possible (green light). We stress that it is necessary to define u_r not only at $x=1$, but also on $(1,+\infty)$ due to the structure of the nonlocal term.

In the context of control, both u_ℓ and u_r are relevant, and we demonstrate that they can be employed to steer the system toward a desired end-state or out-flux. We establish that exact controllability is equivalent to the existence of weak solutions to the backward-in-time problem. In particular, we prove controllability to constant states. We also analyze the long-time behavior of the solutions under suitable hypotheses on the boundary data.

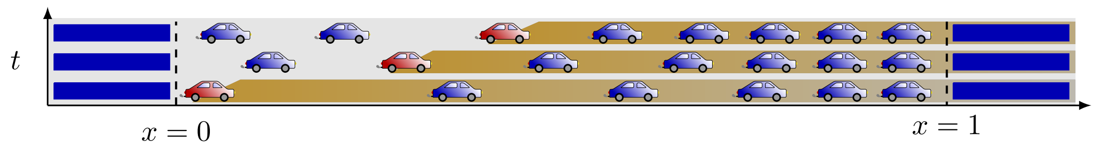

Figure 1.6. As in Figure 1.1, the red car looks ahead within the golden region and adjusts its velocity accordingly. The blue areas on either side of the road segment indicate the boundary data, u_ℓ and u_r: they represent the in-flux at $x=0$ and the "speed control" of the cars leaving the domain at $x=1$, respectively. Cf. [**30**, Figure 1.1].

1.2. Conservation laws models on networks

The study of conservation laws on *networks* has a rich history dating back to [**160, 80**]. Over the past decades, this field has received significant attention due to its relevance in a variety of applied problems in diverse domains, such as blood circulation [**217, 130**], gas pipelines [**92, 93**], vehicular traffic [**138, 137**], irrigation channels [**126**], and supply chains [**118**], among others. Surveys and additional references are available in [**138, 47**].

In particular, the well-posedness of (suitable notions of) entropy solutions for conservation laws on networks was extensively investigated. In the second part of this thesis, we concentrate on the (unique) entropy-admissible solution that arises from a specific vanishing viscosity approximation process (see [**79, 19, 77, 202**]; cf. also [**52**] for Hamilton–Jacobi equations). Our main aim is to study several aspects of the controllability of scalar conservation laws and, in particular, the interplay between controllability and vanishing viscosity on *tree-shaped* networks (i.e., networks without loops).

[7]A. Bayen, J.-M. Coron, **N. De Nitti**, A. Keimer, and L. Pflug. Boundary controllability and asymptotic stabilization of a nonlocal traffic flow model. *Vietnam J. Math.*, 49(3):957–985, 2021.

1.2.1. Controllability of linear advection-diffusion equations and vanishing viscosity limit. In Chapter 8[8], we begin by considering the case of a linear flux function and incorporate a small viscosity effect. We study a model that has been used to describe the flow of a fluid containing a dissolved contaminant through a network of one-dimensional cracks (see [**207, 125**]).

Following the notation in [**124**], we represent the network by a finite, directed, and connected graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ with vertices $\mathcal{V} := \{v_1, \dots, v_n\}$ and edges $\mathcal{E} := \{e_1, \dots, e_m\} \subset \mathcal{V} \times \mathcal{V}$. We use the notation $\mathcal{E}(v) := \{e \in \mathcal{E} : e = (v, \cdot) \text{ or } e = (\cdot, v)\}$ for the set of edges incident to a vertex $v \in \mathcal{V}$; $\mathcal{V}_\partial := \{v \in \mathcal{V} : |\mathcal{E}(v)| = 1\}$ for the set of boundary vertices; and $\mathcal{V}_0 := \mathcal{V} \backslash \mathcal{V}_\partial = \{v \in \mathcal{V} : |\mathcal{E}(v)| \geq 2\}$ for the sets of internal vertices or *junctions* (here, $|S|$ is the cardinality of a finite set S). For every edge $e = \left(v^{\text{in}}, v^{\text{out}}\right)$, we define $n^e\left(v^{\text{in}}\right) := -1$ and $n^e\left(v^{\text{out}}\right) := 1$ (to indicate its starting and end point) and $n^e(v) := 0$ for $v \in \mathcal{V} \setminus \left\{v^{\text{in}}, v^{\text{out}}\right\}$. We write $\mathcal{E}^{\text{in}}(v) := \{e \in \mathcal{E} : n^e(v) > 0\}$ and $\mathcal{E}^{\text{out}}(v) := \{e \in \mathcal{E} : n^e(v) < 0\}$ for the sets of edges pointing into and out of the vertex $v \in \mathcal{V}$, respectively. Furthermore, we split $\mathcal{V}_\partial$ into a set of boundary vertices $\mathcal{V}_\partial^{\text{in}} := \{v \in \mathcal{V}_\partial : n^e(v) < 0 \text{ for } e \in \mathcal{E}(v)\}$, from which edges go into the network, and $\mathcal{V}_\partial^{\text{out}} := \{v \in \mathcal{V}_\partial : n^e(v) > 0 \text{ for } e \in \mathcal{E}(v)\}$, in which edges end. We identify each edge with an interval of length ℓ^e: e.g., $e \simeq (0, \ell^e)$.

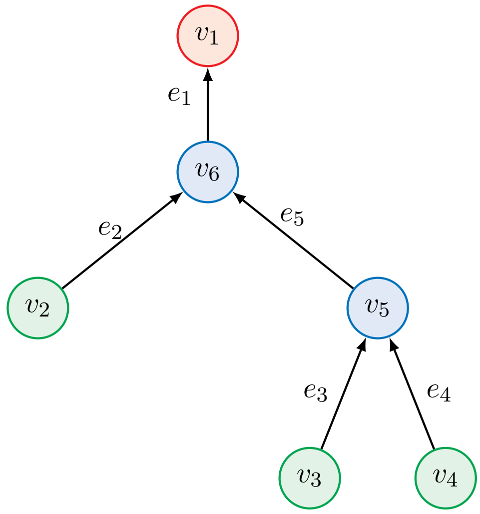

Figure 1.7. Tree-shaped network with edges $e_1 = (v_6, v_1)$, $e_2 = (v_2, v_6)$, $e_3 = (v_3, v_5)$, $e_4 = (v_4, v_5)$, $e_5 = (v_3, v_5)$; inner vertices $\mathcal{V}_0 = \{v_5, v_6\}$ (blue), and boundary vertices $\mathcal{V}_\partial = \{v_1, v_2, v_3, v_4\}$. We split the set of boundary vertices into $\mathcal{V}_\partial^{\text{in}} = \{v_2, v_3, v_4\}$ (green) and $\mathcal{V}_\partial^{\text{out}} = \{v_1\}$ (red). The set $\mathcal{E}(v_6) = \{e_1, e_2, e_5\}$ denotes the edges adjacent to the junction v_6, which can be divided into $\mathcal{E}^{\text{in}}(v_6) = \{e_2, e_5\}$ and $\mathcal{E}^{\text{out}}(v_6) = \{e_1\}$. The arrows illustrate the direction. Cf. [**50**, Figure 1].

With these notations, we write the system under consideration as follows:

$$
\begin{cases}
a^e \partial_t y_\varepsilon^e(t,x) + b^e \partial_x y_\varepsilon^e(t,x) - \varepsilon \partial_{xx}^2 y_\varepsilon^e(t,x) = 0, & t \in (0,T),\ x \in e,\ \forall e \in \mathcal{E}, \\
y_\varepsilon^e(t,v) = u_\varepsilon^v(t), & t \in (0,T),\ v \in \mathcal{V}_\partial, \\
y_\varepsilon^{e_1}(t,v) = y_\varepsilon^{e_2}(t,v), & t \in (0,T),\ v \in \mathcal{V}_0,\ \forall e_1, e_2 \in \mathcal{E}(v), \\
\sum_{e \in \mathcal{E}(v)} b^e y_\varepsilon^e(t,v) n^e(v) - \varepsilon \partial_{n^e(v)} y_\varepsilon^e(t,v) = 0, & t \in (0,T),\ v \in \mathcal{V}_0, \\
y_\varepsilon^e(0,x) = y_0^e(x), & x \in e,\ \forall e \in \mathcal{E},
\end{cases}
\tag{1.2.1}
$$

where $(a^e)_{e \in \mathcal{E}}$ and $(b^e)_{e \in \mathcal{E}}$ are *strictly positive* constants depending on the edge, $\varepsilon \in (0,1]$ is a viscosity parameter, $T > 0$ is a fixed time-horizon, and $\partial_{n^e(v)} y_\varepsilon^e(t,v) := n^e(v) \partial_x y_\varepsilon^e(t,v)$. Here, y_ε^e represents the contaminant concentration, b^e is the flow rate on each edge of the graph, and u_ε^v is the datum at each boundary vertex. We also use the notation $y_\varepsilon := (y_\varepsilon^{e_1}, \dots, y_\varepsilon^{e_m})$ and $u_\varepsilon := (u_\varepsilon^{v_1}, \dots, u_\varepsilon^{v_k})$, where $k = |\mathcal{V}_\partial|$ and $m = |\mathcal{E}|$. We impose *Kirchhoff-type junction conditions*: $(1.2.1)_3$ is a continuity condition in the internal nodes and $(1.2.1)_4$ implies that the flux of the mass is null.

[8]J. A. Bárcena-Petisco, M. Cavalcante, G. M. Coclite, **N. De Nitti**, and E. Zuazua. Control of hyperbolic and parabolic equations on networks and singular limits. *Submitted*, 2023.

Our main focus is steering the solution of (1.2.1) to rest, by using boundary controls, and analyzing the asymptotic behavior of the associated cost of null-controllability as $\varepsilon \to 0^+$, i.e. as the solution of (1.2.1) converges to that of the hyperbolic problem

$$
\begin{cases}
a^e \partial_t y^e(t,x) + b^e \partial_x y^e(t,x) = 0, & t \in (0,T),\ x \in e,\ \forall e \in \mathcal{E}, \\
y^e(t,v) = u^v(t), & t \in (0,T),\ v \in \mathcal{V}_\partial^{\mathrm{in}}, \\
y^{e_1}(t,v) = y^{e_2}(t,v), & t \in (0,T),\ v \in \mathcal{V}_0,\ \forall e_1, e_2 \in \mathcal{E}^{\mathrm{out}}(v), \\
\sum_{e \in \mathcal{E}^{\mathrm{in}}(v)} b^e y^e(t,v) = y^{e_1}(t,v) \sum_{e \in \mathcal{E}^{\mathrm{out}}(v)} b^e, & t \in (0,T),\ v \in \mathcal{V}_0,\ e_1 \in \mathcal{E}^{\mathrm{out}}(v), \\
y^e(0,x) = y_0^e(x), & x \in e,\ \forall e \in \mathcal{E},
\end{cases}
\tag{1.2.2}
$$

where the contaminant does not undergo diffusion and is only driven by the velocity of the liquid flow. In the hyperbolic problem, boundary conditions can only be prescribed at $v \in \mathcal{V}_\partial^{\mathrm{in}}$; in addition, we point out that each $v \in \mathcal{V}_0$ has $|\mathcal{E}^{\mathrm{out}}(v)|$ coupling conditions, which is only sufficient to ensure mass conservation at the junctions and to specify the concentrations flowing into the outgoing edges. On the other hand, in the parabolic problem, each $v \in \mathcal{V}_0$ has $|\mathcal{E}(v)|$ coupling conditions, guaranteeing the conservation of mass and also the continuity of the solution at the junctions.

For the asymptotic result, we additionally assume

$$
\sum_{e \in \mathcal{E}(v)} b^e n^e(v) = 0, \qquad v \in \mathcal{V}_0,
\tag{1.2.3}
$$

which is a balance relation for the flow, ensuring that the energy does not increase at the junctions. This condition is also important for proving the vanishing viscosity approximation result in [**125**].

We first show that the system (1.2.2) is null-controllable (using zero boundary data) for sufficiently large times (without needing assumption (1.2.3)) and not controllable for small times.

By the results in [**164**], it is known that the parabolic problem (1.2.1) is null-controllable when the boundary control acts on all the external vertices (except at most one; however, it may not be null-controllable if it only acts on a smaller subset of the boundary vertices). As established in the classical literature (see, e.g., [**104**, Chapter 2.3]), proving the null-controllability of (1.2.1) is equivalent to establishing an *observability inequality* for the adjoint system:

$$
\begin{cases}
-a^e \partial_t \varphi_\varepsilon^e(t,x) - b^e \partial_x \varphi_\varepsilon^e(t,x) - \varepsilon \partial_{xx}^2 \varphi_\varepsilon^e(t,x) = 0, & t \in (0,T),\ x \in e,\ \forall e \in \mathcal{E}, \\
\varphi_\varepsilon^e(t,v) = 0, & t \in (0,T),\ v \in \mathcal{V}_\partial, \\
\varphi_\varepsilon^{e_1}(t,v) = \varphi_\varepsilon^{e_2}(t,v), & t \in (0,T),\ v \in \mathcal{V}_0,\ \forall e_1, e_2 \in \mathcal{E}(v), \\
\sum_{e \in \mathcal{E}(v)} \varepsilon \partial_{n^e(v)} \varphi_\varepsilon^e(t,v) = 0, & t \in (0,T),\ v \in \mathcal{V}_0, \\
\varphi_\varepsilon^e(T,x) = \varphi_T^e(x), & x \in e,\ \forall e \in \mathcal{E},
\end{cases}
\tag{1.2.4}
$$

where we have used (1.2.3) to obtain $(1.2.4)_4$. The aforementioned *cost of controllability* is given by

$$
K(\varepsilon, T, a^e, b^e, \mathcal{G}) := \sup_{y_0 \in L^2(\mathcal{E}) \setminus \{0\}} \inf_{u_\varepsilon \in \mathcal{U}} \frac{\left(\int_0^T \sum_{v \in \mathcal{V}_\partial} |u_\varepsilon^v(t)|^2 \, \mathrm{d}t \right)^{1/2}}{\|y_0\|_{L^2(\mathcal{E})}},
$$

where $\mathcal{U}$ denotes the subset of controls in $L^2((0,T); \ell^2(\mathcal{V}_\partial))$ such that the solution of (1.2.1) satisfies $y(T,\cdot) \equiv 0$, and the equivalent *cost of observability* of the adjoint variable by

$$
\widetilde{K}(\varepsilon, T, a^e, b^e, \mathcal{G}) := \frac{1}{\sqrt{\varepsilon}} \sup_{\varphi_T \in L^2(\mathcal{E}) \setminus \{0\}} \frac{\|a\, \varphi_\varepsilon(0,\cdot)\|_{L^2(\mathcal{E})}}{\left(\int_0^T \sum_{v \in V_\partial} |\partial_{n^e(v)} \varphi_\varepsilon^e(t,v)|^2 \, \mathrm{d}t \right)^{1/2}}.
$$

From the controllability result for the hyperbolic problem, we expect the following behavior: for small times, $K \to +\infty$ as $\varepsilon \to 0^+$; on the other hand, for T sufficiently large, $K \to 0$ as $\varepsilon \to 0^+$.

The main results of this Chapter are quantitative versions of these claims:

- we show that the cost explodes exponentially for small times by considering a datum for the adjoint problem supported far away from the observation domains and checking that, while its mass remains within the network, the portion that reaches the observation vertices is of order $\exp(C\varepsilon^{-1})$;

– we prove that the cost decays exponentially for large times by first using the decay of the free solutions and then, when the mass of the state is almost null, by observing it exactly.

Our strategy is based on the *Hilbert Uniqueness Method* (H.U.M., introduced by J.-L. Lions; see [**191**]) and, in particular, on the ideas of [**106, 150**]. To prove the blow-up, we rely on a "non-degeneracy" estimate and an Agmon-type inequality. For the proof of the decay, we need to establish a decay property for the L^2-mass of the adjoint system and a Carleman-type inequality (keeping track of the viscosity parameter). The proof of the Carleman inequality is particularly demanding due to the presence of boundary terms at the junctions. To address this issue, we need to introduce a general construction of *Fursikov–Imanuvilov weights* (see [**136**]) using a piecewise-C^2 auxiliary function.

Several previous contributions on the controllability of various classes of PDEs on networks can be found in the literature. For example, results on wave, Schrödinger, heat, beam, and other equations were collected in [**117**]. In particular, [**117**, Chapter 8.1] demonstrated that the heat equation with Kirchhoff-type junction conditions can be controlled to zero under suitable assumptions. We also refer to [**178, 177, 176, 179**] for the well-posedness, controllability, and stabilization of several models of thermoelastic beams, linked plates, and plate–beam systems and to [**28, 152, 157, 233, 121, 151, 156**] for models arising in water flow, gas transport, etc.

No results were available on uniform controllability in the context of singular limits on networks, despite the problem's long history on Euclidean domains. The study of uniform controllability problems for singular perturbations of PDEs began with the pioneering works [**191, 189, 190, 195, 194**]. In the context of linear advection-diffusion equations in the vanishing viscosity limit, the first result was obtained by Coron and Guerrero in [**106**], where they also made a conjecture on the minimal time needed to achieve uniform controllability. Subsequently, in [**140**], Glass refined the available estimates. The result of [**106**] was generalized to several space dimensions and (non-constant) Lipschitz continuous transport speed in [**150**]. Nonlinear transport terms were later taken into account by Glass and Guerrero, who studied the Burgers equation in [**141**], and by Léautaud, who extended their results to more general flux functions in [**183**].

Related results have recently been obtained for other systems as well: namely, the Stokes system (see [**25**]), an artificial advection-diffusion problem (see [**100, 101**]), and fourth-order parabolic equations (see [**55, 196, 165**]).

1.2.2. Controllability of scalar conservation laws on networks. In Chapter 9[9], we return to the nonlinear problem and examine the controllability of entropy solutions.

Previously, there were no known controllability results for scalar conservation laws on networks in the context of entropy solutions. Some studies focused on optimization problems (see [**15, 14**]) and stabilization issues (as in [**123**]). Further results were obtained on the related topic of conservation laws on the real line with space-discontinuous flux in [**2, 16**]. On the contrary, the controllability and stabilization of (systems of) conservation laws on networks were extensively studied in the context of smooth solutions (see, e.g., [**153, 154**] and references therein).

The study of controllability in the case of the IBVP associated with (systems of) conservation laws on a segment has a longer history. In the framework of classical solutions, controls drive the state to the desired target and prevent the formation of singularities (see [**185, 186, 104, 28**]). In the context of entropy solutions, the set of admissible target states has been investigated in [**17, 18, 20, 21, 99**]. Several controllability results were obtained relying on the Lax–Oleĭnik representation formula, which applies when the flux function is strictly convex or concave (see [**3, 20**]); on the method of generalized characteristics introduced by Dafermos in [**114**], which allows for one inflection point (see [**18, 21, 99, 163, 211**]); or on the return method proposed by Coron in [**102**], which also covers the case of a finite number of inflection points (see [**163, 141, 183**]).

In the recent development [**122**], Donadello and Perrollaz introduced a novel approach to the controllability problem for multi-dimensional conservation laws building on the classic concept of *Lyapunov functionals*, which originated in the study of asymptotic stabilization (see [**212, 36, 28,**

[9] **N. De Nitti** and E. Zuazua. On the controllability of entropy solutions of scalar conservation laws at a junction via Lyapunov methods. *Vietnam J. Math.*, 51(1):71–88, 2023.

103, 105]). They demonstrated a null-controllability result without needing convexity/concavity assumptions, instead relying on appropriate geometric monotonicity-type conditions.

To simplify the notation, we focus on networks composed of a single junction with n incoming and m outgoing edges (i.e., *star-shaped* graphs; see the illustration in Figure 1.8). Following the notation of [**138**], the incoming edges are labeled by $i \in \mathcal{I}_{\text{in}} := \{1, \dots, n\}$ and parameterized by the segments $I_i := (-L_i, 0)$; the outgoing edges are labeled by $j \in \mathcal{I}_{\text{out}} := \{n+1, \dots, n+m\}$ and parameterized by the segments $I_j := (0, L_j)$, with $L_i, L_j > 0$; the junction is at $x = 0$. As in CHAPTER 8, we shall also use the notation $\mathcal{G} := (0, \mathcal{E})$, where $\mathcal{E} = \{I_\ell\}_{\ell \in \{1,\dots,n+m\}}$.

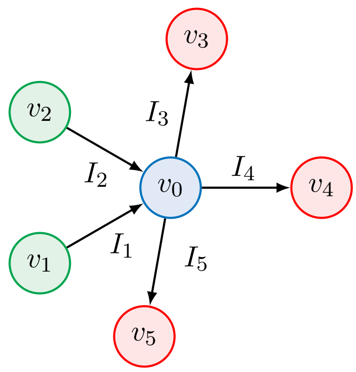

FIGURE 1.8. Junction with $n = 2$ incoming and $m = 3$ outgoing edges. Cf. [**50**].

For each edge of the graph, we consider the dynamics given by a scalar hyperbolic conservation law with flux f_ℓ (with $\ell \in \{1, \dots, n+m\}$) satisfying the following assumptions:

(F1) $f_\ell \in \mathrm{Lip}(\mathbb{R}; \mathbb{R}_+)$;
(F2) f_ℓ is *non-degenerate*: for all $(\xi, \zeta) \in \mathbb{R} \times \mathbb{R} \setminus \{(0,0)\}$, we have $\mathcal{L}\left(\{z \in \mathbb{R} : \xi + \zeta f'_\ell(z) = 0\}\right) = 0$, where $\mathcal{L}$ denotes the Lebesgue measure;
(F3) $\inf f'_\ell \geq c_\ell > 0$.

With these assumptions, we consider the system

$$\begin{cases} \partial_t u_i + \partial_x f_i\left(u_i\right) = 0, & t > 0,\ x \in I_i, \\ \partial_t u_j + \partial_x f_j\left(u_j\right) = 0, & t > 0,\ x \in I_j, \\ u_i(0,x) = u_{0,i}(x), & x \in I_i, \\ u_j(0,x) = u_{0,j}(x), & x \in I_j, \\ u_i(t, -L_i) = u_{b,i}(t), & t > 0, \\ \sum_{i=1}^n f_i(u_i(t, 0-)) = \sum_{j=n+1}^{n+m} f_j(u_j(t, 0+)), & t > 0, \end{cases} \tag{1.2.5}$$

for all $i \in \{1, \dots, n\}$ and $j \in \{n+1, \dots, n+m\}$. Here, for every $\ell \in \{1, \dots, n+m\}$, the initial data satisfy $u_{0,\ell} \in L^\infty(I_\ell; \mathbb{R}_+)$ and, at the entry points of the network, we prescribed *in-flux boundary conditions* with data $u_{b,i} \in L^\infty((0, +\infty); \mathbb{R}_+)$ for $i \in \{1, \dots, n\}$. As in CHAPTER 8, we impose a junction condition that ensures mass conservation; moreover, as in [**19**], we will need to introduce additional entropy-admissibility conditions (which are motivated by the vanishing viscosity approximation) to guarantee uniqueness.

The main contribution of this Chapter is adapting the result established by Donadello and Perrollaz in [**122**, Proposition 4] to (1.2.5). For the proof, we also rely on a Lyapunov functional consisting of an exponentially-weighted L^1-norm; however, we have to face additional difficulties: we need to take into account an adapted entropy-admissibility condition to propagate information across the junction. This is similar in spirit to the considerations in CHAPTER 8 regarding the controllability of (1.2.2).

Assumptions (F1)–(F3) are key in our study, as they were in [**122**]. Hypothesis (F1) is classical in Kružkov's well-posedness theory and guarantees finite speed of propagation; (F2) is a technical condition needed to ensure the existence of strong boundary traces for L^∞ entropy solutions of conservation laws (see, e.g., [**204, 209, 231**]); and (F3) encodes the requirement that all generalized characteristics emanating from $(t, x) \in \{0\} \times I_\ell$ leave the cylinder $(0, T_\ell) \times I_\ell$ by time $T_\ell := L_\ell / c_\ell$.

In other words, (F3) implies that, after a sufficiently long time, the dynamics on each edge will depend solely on the boundary data prescribed at the exterior nodes and not on the initial data; moreover, under hypothesis (F3), the effective boundary condition can be imposed only at $x = -L_i$, for $i \in \{1, \dots, n\}$.

Assumption (F3) renders (1.2.5) unsuitable for traffic flow models (which typically involve bell-shaped flux functions; see [**138**] and references therein), but it is common in supply chain modeling: for example, in the so-called *M/M/1 queuing model with capacity one* (see [**118**]), the flux function is given by $f(\xi) = \xi/(\xi + 1)$.

Finally, similarly to Chapter 8, we consider a viscous regularization of (1.2.5):

$$\begin{cases} \partial_t u_{\varepsilon,i} + \partial_x f_i(u_{\varepsilon,i}) = \varepsilon \partial^2_{xx} u_{\varepsilon,i}, & t > 0,\ x \in I_i, \\ \partial_t u_{\varepsilon,j} + \partial_x f_j(u_{\varepsilon,j}) = \varepsilon \partial^2_{xx} u_{\varepsilon,j}, & t > 0,\ x \in I_j, \\ u_{\varepsilon,i}(0,x) = u_{0,\varepsilon,i}(x), & x \in I_i, \\ u_{\varepsilon,j}(0,x) = u_{0,\varepsilon,j}(x), & x \in I_j, \\ u_{\varepsilon,i}(t,-L_i) = u_{b,i}(t), & t > 0, \\ u_{\varepsilon,j}(t,L_j) = u_{b,j}(t), & t > 0, \\ \sum_{i=1}^{n} \big(f_i(u_{\varepsilon,i}(t,0-)) - \varepsilon \partial_x u_{\varepsilon,i}(t,0-) \big) & \\ \quad = \sum_{j=n+1}^{n+m} \big(f_j(u_{\varepsilon,j}(t,0+)) - \varepsilon \partial_x u_{\varepsilon,j}(t,0+) \big), & t > 0, \\ u_{\varepsilon,i}(t,0-) = u_{\varepsilon,j}(t,0+), & t > 0, \end{cases} \tag{1.2.6}$$

for all $i \in \{1, \dots, n\}$ and $j \in \{n+1, \dots, n+m\}$. We aim to study the effect of viscosity on the inviscid control strategy. A small exponential tail remains as an error when considering the evolution of the difference of two solutions u_ε and v_ε with different initial conditions but the same boundary data, which needs to be taken into account.

Outline

The contributions contained in this thesis are organized into two parts.

Part 1 deals with nonlocal regularizations of conservation laws modeling traffic flow.

- In Chapter 2, we start by studying the well-posedness of a class of nonlocal conservation laws that includes (1.1.2). In particular, we shall focus on the case when the nonlocal impact is given by the convolution of the density with an averaging kernel γ which is only of bounded variation.
- Chapters 3–4 are dedicated to the convergence of the nonlocal model to the corresponding local one. In particular, in Chapter 3, we prove the convergence of the (unique) weak solution of (1.1.2) to the (unique) entropy solution of the local conservation law (1.1.1) under the assumption that the initial datum is non-negative, bounded, and of bounded variation. On the other hand, in Chapter 4, we establish Oleĭnik-type estimates that give (under suitable—more restrictive—assumptions on the velocity function) the convergence for initial data that are not necessarily of bounded variation.
- In Chapter 5, we use an Oleĭnik-type estimate to study the long-time asymptotics for a nonlocal regularization of the Burgers equation.
- In Chapter 6, we remark on the effect of artificial viscosity on the nonlocal–to–local limit process.
- In Chapter 7, we study the boundary controllability of nonlocal conservation laws.

Part 2 deals with conservation laws on networks.

- In Chapter 8, we focus on a linear viscous problem: we study the controllability of advection-diffusion equations on networks and the asymptotic behavior of the cost of null-controllability as the viscosity parameter vanishes.
- In Chapter 9, we consider a class of (nonlinear) conservation laws and prove a controllability result for the entropy solution.

In the final Chapter 10, we present some open problems and perspectives for future research. These include, in particular, the development of the theory of nonlocal conservation laws on networks, thereby bridging the gap between Part 1 and Part 2

Publications related to the thesis. This thesis unifies the exposition of mathematical contributions that were previously published or appeared as preprints during the course of the author's Ph.D. studies:

Chapter 2: [**71**];
Chapter 3: [**69**];
Chapter 4: [**68**];
Chapter 5: [**72**];
Chapter 6: [**70**];
Chapter 7: [**30**];
Chapter 8: [**50**];
Chapter 9: [**120**].

In line with the established practice in the mathematical community, all co-authors of these papers are listed alphabetically and have contributed.

Notably, the author of this thesis made significant contributions to the theoretical aspects of these works. These contributions include, for instance, investigating the effects of BV weights on the well-posedness of nonlocal conservation laws; exploiting the underlying structure of exponential weights to derive scalar equations instead of systems of equations with relaxation, which are crucial to the study of uniform bounds and of the nonlocal–to–local singular limit problem; examining Oleĭnik-type estimates and related compactification effects (particularly in the linear velocity case, which also led to the analysis long-time behavior for the nonlocal Burgers equation); studying backward-in-time evolutions and boundary controllability of nonlocal conservation laws in the presence of an exponential weight; analyzing the impact of junction conditions on the estimates of the cost of controllability in the vanishing viscosity limit for advection-diffusion equations on networks; and employing Lyapunov approaches for the controllability of entropy solutions of conservation laws on networks. The author of this thesis was not involved in the implementation of the numerical experiments:

- the simulations presented in the Chapters of Part 1 of the thesis have been produced by L. Pflug based on the numerical scheme and MATLAB code he had previously developed in [**213**, Chapter 3] and [**173**];
- the simulations presented in Chapter 9 have been produced by M. Munsch based on the numerical scheme studied in [**202**] and the Python code developed in [**201**].

Part 1

Nonlocal conservation laws

CHAPTER 2

Well-posedness of nonlocal conservation laws with BV weights

The main aim of this Chapter is to establish the well-posedness of weak solutions for the following class of nonlocal conservation laws:

$$\begin{cases} \partial_t \rho(t,x) + \partial_x\big(V(W[\rho](t,x))\rho(t,x)\big) = 0, & (t,x) \in (0,T)\times\mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.0.1}$$

for a fixed time horizon $T > 0$ and

$$W[\rho](t,x) := (\gamma(t,\cdot) * \rho(t,\cdot))(x), \qquad (t,x) \in (0,T)\times\mathbb{R}. \tag{2.0.2}$$

Before stating our main result, let us recall the notion of weak solution adopted in [**167**, Definition 2.13].

Definition 2.0.1 (Weak solution of the nonlocal balance law). *We say that $\rho \in C\big([0,T]; L^1_{\mathrm{loc}}(\mathbb{R})\big)$ is a* weak solution *of the nonlocal conservation law in* (2.0.1) *if, for all $\varphi \in C^1_c([0,T)\times\mathbb{R})$,*

$$\int_0^T \int_{\mathbb{R}} \Big(\partial_t\varphi(t,x)\rho(t,x) + \partial_x\varphi(t,x)V\big(W[\rho](t,x)\big)\rho(t,x)\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} \varphi(0,x)\rho_0(x)\,\mathrm{d}x = 0,$$

with $W[\rho]$ as in (2.0.2).

First, we prove the existence and uniqueness of weak solutions of (2.0.1) locally in time.

Theorem 2.0.1 (Local well-posedness of nonlocal conservation laws with rough kernels). *Let us fix $T > 0$ and suppose that the following conditions hold:*

(A1) $\gamma \in L^\infty((0,T);\mathrm{BV}(\mathbb{R}))$ *and* $\gamma \geq 0$*;*
(A2) $V \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$*;*
(A3) $\rho_0 \in L^\infty(\mathbb{R})$.

Then, there exists $T^* \in (0,T]$ such that (2.0.1) *admits a unique weak solution $\rho \in C\big([0,T^*]; L^1_{\mathrm{loc}}(\mathbb{R})\big) \cap L^\infty((0,T^*); L^\infty(\mathbb{R}))$ in the sense of Definition 2.0.1. Moreover, the weak solution can be written as*

$$\rho(t,x) = \rho_0(\xi_{w^*}(t,x;0))\partial_x\xi_{w^*}(t,x;0), \quad (t,x) \in [0,T^*]\times\mathbb{R},$$

where w^* is the unique solution on $(0,T^*)\times\mathbb{R}$ of the fixed point problem in (2.1.4) *and ξ_{w^*} the characteristics, defined in* (2.1.3).

As pointed out in Chapter 1, we emphasize that, for the nonlocal problem (3.0.1), no entropy condition is needed to select a unique solution.

Furthermore, a global well-posedness result can be achieved under additional hypotheses.

Theorem 2.0.2 (Global existence and comparison principle). *Under the assumptions of Theorem 2.0.1, let us suppose, in addition, that*

(A4) $V' \leq 0$*;*
(A5) $\mathrm{supp}(\gamma(t,\cdot)) \subseteq \mathbb{R}_{\leq 0}$ *and $\gamma(t,\cdot)$ is monotonically non-decreasing on $\mathbb{R}_{\leq 0}$ for all $t \in [0,T]$;*
(A6) $\rho_0 \in L^\infty(\mathbb{R};\mathbb{R}_{\geq 0})$.

Then, for any $T > 0$, the initial-value problem (2.0.1) *admits for every a unique solution*

$$\rho \in C\big([0,T]; L^1_{\mathrm{loc}}(\mathbb{R})\big) \cap L^\infty((0,T); L^\infty(\mathbb{R}))$$

satisfying the comparison principle

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}} \rho_0(x) \leq \rho(t,x) \leq \operatorname*{ess\,sup}_{x\in\mathbb{R}} \rho_0(x), \quad (t,x) \in (0,T)\times\mathbb{R}.$$

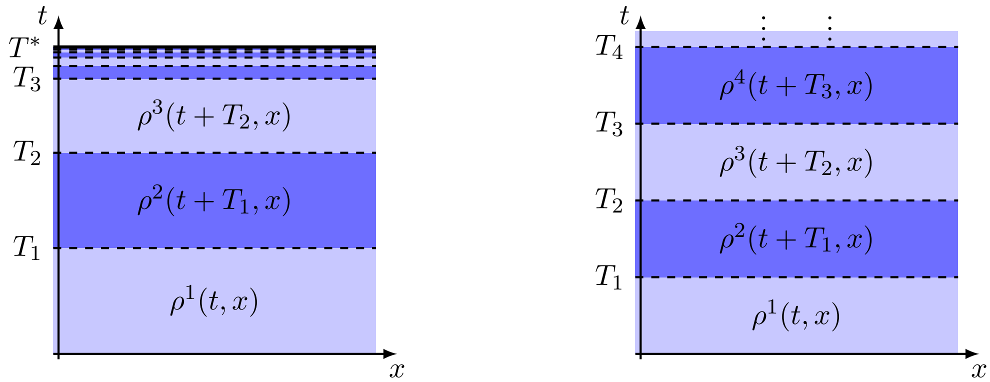

FIGURE 2.1. Illustration of the extension of the time-horizon and construction of the global solution. Cf. [**167**, Figure 3].

In Section 2.1, we prove these well-posedness results and, in Section 2.2, we present some numerical experiments to illustrate them.

2.1. Proof of the well-posedness result

The proofs of Theorem 2.0.1 and Corollary 2.0.2 follow the steps presented in [**167**]. However, we have to introduce several technical modifications to deal with the more general nonlocal weight satisfying assumption (2.0.1).

Step 1. *Formulation of the fixed-point equation in the nonlocal term w.* We assume, for the moment, that the nonlocal term

$$W[\rho](t,x) = \int_{\mathbb{R}} \gamma(t, x-y)\rho(t,y)\,\mathrm{d}y =: w(t,x), \qquad (t,x) \in (0,T)\times\mathbb{R}, \tag{2.1.1}$$

is a given Lipschitz continuous function. Then, the corresponding conservation law is linear with Lipschitz continuous velocity $V \circ w$ and we can use the method of characteristics to write the solution of the Cauchy problem as

$$\rho_w(t,x) = \rho_0(\xi_w(t,x;0))\partial_x\xi_w(t,x;0), \qquad (t,x)\in(0,T)\times\mathbb{R}, \tag{2.1.2}$$

where ξ_w solves the characteristics equation

$$\xi_w(t,x;\tau) = x + \int_t^\tau V\big(w(s,\xi_w(t,x;s)\big)\,\mathrm{d}s, \quad \tau\in[0,T]. \tag{2.1.3}$$

Plugging (2.1.3) into the nonlocal term in (2.1.1) yields, for $(t,x)\in(0,T)\times\mathbb{R}$,

$$\begin{aligned} w(t,x) &= \int_{\mathbb{R}} \gamma(t,x-y)\rho(t,y)\,\mathrm{d}y = \int_{\mathbb{R}} \gamma(t,x-y)\rho_0(\xi_w(t,y;0))\partial_y\xi_w(t,y;0)\,\mathrm{d}y \\ &= \int_{\mathbb{R}} \gamma\big(t, x-\xi_w(0;y;t)\big)\rho_0(y)\,\mathrm{d}y, \end{aligned}$$

which is a fixed-point problem in w.

Step 2. *Local existence for the nonlocal conservation law.* We use Banach's fixed-point theorem to prove the existence of a solution $w^* \in L^\infty((0,T^*);W^{1,\infty}(\mathbb{R}))$ of (2.1.3) on a sufficiently small time-horizon T^*. Then, we can build a solution of (2.0.1) in terms of characteristics as follows:

$$\rho(t,x) = \rho_0(\xi_{w^*}(t,x;0))\partial_x\xi_{w^*}(t,x;0), \qquad (t,x)\in(0,T^*)\times\mathbb{R},$$

which is presented in [**167**, Theorem 2.20] and [**169**, Theorem 3.1] in detail.

Step 3. *Uniqueness for the nonlocal conservation law.* The uniqueness of w^* can be shown to imply the uniqueness of the solution ρ. The main idea is to prove that any weak solution can be written in the same way as instantiated in (2.1.2) (see [**167**, Lemma 3.1 and Theorem 3.2]).

Step 4. *Extension of the solution for larger times.* Gluing a sequence of initial value problems with initial data equal to the terminal-time solution of the previous one, we can extend the existence result to a longer (but not necessarily arbitrary) time-horizon (as in [**167**, Theorem 4.1]).

Step 5. *Extension to arbitrary time-horizons and comparison principle.* Under the stronger assumptions (A4)–(A6), we can extend the solution to an arbitrary time horizon and show that a comparison principle holds. For the detailed argument, we refer to [**172**, Lemma 5.8]. It mainly consists of studying the time evolution of the maximum/minimum of the solution and showing that its time derivative is negative, implying that the minimum can only increase and the maximum can only decrease over time.

Extension of the proof to BV weights. The only parts of the proof outlined above that need to be adjusted from [**167**] to extend the well-posedness result to our more general setting are related to **Step 2**. More precisely, they can be summarized as follows:

1. proving that, for $t \in [0,T]$, the convolution $(x \mapsto \gamma(t,\cdot) * \rho(t,\cdot))(x)$ is in $W^{1,\infty}$ for $\gamma \in L^\infty((0,T);\mathrm{BV}(\mathbb{R}))$;
2. establishing the analog of [**167**, Proposition 2.17], where it was shown that the mapping induced in **Step 1** satisfies the assumptions of Banach's fixed-point theorem by relying on the regularity assumption $\gamma \in L^\infty\big((0,T);W^{1,\infty}(\mathbb{R})\big)$.

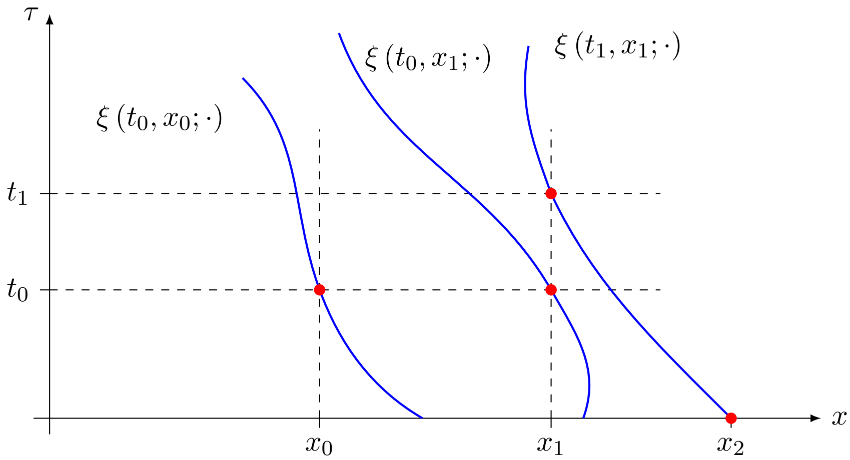

FIGURE 2.2. Characteristics for the nonlocal conservation law. Cf. [**167**, Figure 1].

We start by proving the Lipschitz-continuity (in space) of the convolution.

LEMMA 2.1.1 (Smoothing via convolution with BV functions). *Let* $\gamma \in \mathrm{BV}(\mathbb{R})$ *and* $f \in L^\infty(\mathbb{R})$. *Then* $\gamma * f \in W^{1,\infty}(\mathbb{R})$.

PROOF. From [**8**, Remark 3.5] or [**184**, Corollary 2.17], we deduce that, for $h \in \mathbb{R}$,

$$\|\tau_h\gamma - \gamma\|_{L^1(\mathbb{R})} \le |\gamma|_{\mathrm{TV}(\mathbb{R})}|h|.$$

where $\tau_h\gamma(x) := \gamma(x+h)$ for a.e. $x \in \mathbb{R}$. As a consequence, we can estimate, by using Young's convolution inequality (see [**48**, Theorem 4.33]),

$$\begin{aligned}\|\tau_h(\gamma * f) - \gamma * f\|_{L^\infty(\mathbb{R})} &= \|(\tau_h\gamma - \gamma) * f\|_{L^\infty(\mathbb{R})}\\ &\le \|f\|_{L^\infty(\mathbb{R})}\|\tau_h\gamma - \gamma\|_{L^1(\mathbb{R})} \le |\gamma|_{\mathrm{TV}(\mathbb{R})}\|f\|_{L^\infty(\mathbb{R})}|h|.\end{aligned}$$

We thus conclude that $\gamma * f \in W^{1,\infty}(\mathbb{R})$. □

We now review the proof of the fixed-point argument contained in [**167**, Proposition 2.17]. As mentioned above, this is the main step that needs to be taken to adapt the arguments of [**167**] to the case of a nonlocal impact given by the convolution of the density ρ with a rough kernel $\gamma \in L^\infty((0,T);\mathrm{BV}(\mathbb{R}))$.

PROPOSITION 2.1.1 (Properties of the fixed-point mapping). *Let*

$$F : \begin{cases} \Omega_M^{M'}(T) \to L^\infty((0,T);W^{1,\infty}(\mathbb{R})),\\ w \mapsto \Big((t,x) \mapsto \displaystyle\int_{\mathbb{R}} \gamma\big(t,x-\xi_w(0,z;t)\big)\rho_0(z)\,\mathrm{d}z\Big), \end{cases} \tag{2.1.4}$$

*be the fixed-point mapping defined in **Step 1** and let*

$$\Omega_M^{M'}(T) \coloneqq \Big\{ w \in L^\infty((0,T);W^{1,\infty}(\mathbb{R})) :\ \|w\|_{L^\infty((0,T);L^\infty(\mathbb{R}))} \leq M \\ \text{and } \|\partial_x w\|_{L^\infty((0,T);L^\infty(\mathbb{R}))} \leq M' \Big\}, \tag{2.1.5}$$

with

$$M \coloneqq \bar{\kappa}\|\gamma\|_{L^\infty((0,T);L^1(\mathbb{R}))}\|\rho_0\|_{L^\infty(\mathbb{R})}, \qquad M' \coloneqq \bar{\kappa}|\gamma|_{L^\infty((0,T);\mathrm{TV}(\mathbb{R}))}\|\rho_0\|_{L^\infty(\mathbb{R})},$$

for a fixed (sufficiently large) $\bar{\kappa} > 0$. Then, the fixed-point mapping defined in (2.1.4) *satisfies the following properties.*

(1) *There exists $T^* \in (0,T]$ such that $\|F[w]\|_{L^\infty((0,T^*);L^\infty(\mathbb{R}))} \leq M$ for all $w \in \Omega_M^{M'}(T^*)$.*
(2) *There exists $T' \in (0,T]$ such that $\|\partial_2 F[w]\|_{L^\infty((0,T');L^\infty(\mathbb{R}))} \leq M'$ for all $w \in \Omega_M^{M'}(T')$.*
(3) *F is Lipschitz continuous with respect to the uniform topology, i.e., for $w, \tilde{w} \in \Omega_M^{M'}(\bar{T})$ and $\bar{T} \coloneqq \min\{T^*, T'\}$, we have*

$$\|F[w] - F[\tilde{w}]\|_{L^\infty((0,\bar{T});L^\infty(\mathbb{R}))} \leq |\gamma|_{L^\infty((0,\bar{T});\mathrm{TV}(\mathbb{R}))}\bar{T}\|w - \tilde{w}\|_{L^\infty((0,\bar{T});L^\infty(\mathbb{R}))} \\ \times \|V'\|_{L^\infty((-M,M))} \exp\left(2\bar{T}\|V'\|_{L^\infty((-M,M))}M'\right);$$

thus, for small time $\widehat{T} \in (0,\bar{T}]$, F is a contraction on $\Omega_M^{M'}\big(\widehat{T}\big)$.

Proof. We shall prove the three claims separately.

Claim (1). For $w \in \Omega_M^{M'}(T)$ and $t \in [0,T]$, we estimate—recalling the definition of F in (2.1.4)—

$$\begin{aligned} \|F[w](t,\cdot)\|_{L^\infty(\mathbb{R})} &= \left\| \int_{\mathbb{R}} \gamma(t, \cdot - \xi_w(0,z;t))\rho_0(z)\,\mathrm{d}z \right\|_{L^\infty(\mathbb{R})} \\ &\leq \|\gamma(t,\cdot)\|_{L^1(\mathbb{R})}\|\partial_2\xi_w(t,\cdot;0)\|_{L^\infty(\mathbb{R})}\|\rho_0\|_{L^\infty(\mathbb{R})} \\ &\leq \|\gamma(t,\cdot)\|_{L^1(\mathbb{R})} \exp\left(t\|V'\|_{L^\infty((-M,M))}M'\right)\|\rho_0\|_{L^\infty(\mathbb{R})}, \end{aligned}$$

where we have used the substitution rule and the properties of the characteristics (see [**167**, Lemma 2.6(3)] and [**169**, Corollary 2.1])—in particular, the fact that

$$\|\partial_2\xi_w(t,\cdot;0)\| \leq \exp\left(t\|V'\|_{L^\infty((-M,M))}M'\right), \qquad t \in [0,T], \tag{2.1.6}$$

which is an immediate consequence of differentiating (2.1.3) with respect to $x \in \mathbb{R}$ to obtain a linear Cauchy problem in $\partial_2\xi_w$. As M and M' are fixed, we can find a time horizon $T^* \in (0,T]$ such that

$$\|\gamma\|_{L^\infty((0,T);L^1(\mathbb{R}))} \exp\left(T^*\|V'\|_{L^\infty((-M,M))}M'\right)\|\rho_0\|_{L^\infty(\mathbb{R})} \leq M \\ \iff \exp\left(T^*\|V'\|_{L^\infty((-M,M))}M'\right) \leq \bar{\kappa}.$$

Claim (2). We estimate the spatial derivative of the fixed-point mapping in (2.1.4) (which is well-defined according to (2.1.1)) for $w \in \Omega_M^{M'}(T)$ and $(t,x) \in (0,T) \times \mathbb{R}$. Some technical details are left out and can be found in [**167**, Lemma 2.6(2)]; however, the argument is essentially as follows:

$$\begin{aligned} |\partial_x F[w](t,x)| &\leq \int_{\mathbb{R}} \big|\partial_x\gamma(t, x - \xi_w(0,z;t))\rho_0(z)\big|\,\mathrm{d}z \\ &\leq \|\partial_2\xi(t,\cdot;0)\|_{L^\infty((0,T)\times\mathbb{R})}\|\rho_0\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|\partial_2\gamma(t,x-y)|\,\mathrm{d}y \\ &\leq |\gamma(t,\cdot)|_{\mathrm{TV}(\mathbb{R})} \exp\left(t\|V'\|_{L^\infty((-M,M))}M'\right)\|\rho_0\|_{L^\infty(\mathbb{R})}, \end{aligned}$$

where we applied once more the substitution rule and the estimate in (2.1.6). Making this uniform in $(t,x) \in (0,T) \times \mathbb{R}$ and since M and M' are fixed, we can find a time horizon $T' \in (0,T]$ such that

$$|\gamma|_{L^\infty((0,T);\mathrm{TV}(\mathbb{R}))} \exp\left(T'\|V'\|_{L^\infty((-M,M))}M'\right)\|\rho_0\|_{L^\infty(\mathbb{R})} \leq M' \\ \iff \exp\left(T'\|V'\|_{L^\infty((-M,M))}M'\right) \leq \bar{\kappa}.$$

We can indeed choose $T' = T^*$. Thus, combining the previous results, we conclude that

$$F\Big(\Omega_M^{M'}(T')\Big) \subseteq \Omega_M^{M'}(T'),$$

i.e., F is a self-mapping on $\Omega_M^{M'}(T')$.

Claim (3). To demonstrate the contraction property of F in $L^\infty((0,T');L^\infty(\mathbb{R}))$, we estimate, for $w,\tilde{w} \in \Omega_M^{M'}(T')$ and $(t,x) \in (0,T') \times \mathbb{R}$,

$$
\begin{aligned}
&|F[w](t,x) - F[\tilde{w}](t,x)| \\
&= \left| \int_{\mathbb{R}} \gamma(t,\xi_w(0,z;t))\rho_0(z)\,\mathrm{d}z - \int_{\mathbb{R}} \gamma(t,\xi_{\tilde{w}}(0,z;t))\rho_0(z)\,\mathrm{d}z \right| \\
&\leq \int_{\mathbb{R}} |\gamma(t,\xi_w(0,z;t)) - \gamma(t,\xi_{\tilde{w}}(0,z;t))|\,\rho_0(z)\,\mathrm{d}z \\
&\leq |\gamma|_{L^\infty((0,T);\mathrm{TV}(\mathbb{R}))} \|\xi_w - \xi_{\tilde{w}}\|_{L^\infty((0,t)\times\mathbb{R}\times(0,t))} \|\rho_0\|_{L^\infty(\mathbb{R})} \exp\left(t\|V'\|_{L^\infty((-M,M))}M'\right) \qquad (2.1.7) \\
&\leq |\gamma|_{L^\infty((0,T);\mathrm{TV}(\mathbb{R}))} t\|V'\|_{L^\infty((-M,M))} \|w - \tilde{w}\|_{L^\infty((0,t);L^\infty(\mathbb{R}))} \|\rho_0\|_{L^\infty(\mathbb{R})} \qquad (2.1.8) \\
&\qquad \times \exp\left(2t\|V'\|_{L^\infty((-M,M))}M'\right).
\end{aligned}
$$

In (2.1.7), we used the substitution rule and the uniform bound on $\partial_x \xi_w$ and $\partial_x \xi_{\tilde{w}}$ (obtained from (2.1.5)); in (2.1.8), the stability of the characteristics with respect to the nonlocal term (see [**167**, Lemma 2.6(3)] and [**169**, Theorem 2.4] for further details). Making the previous estimate uniform in (t,x) and recalling that M, M' are fixed, we conclude that there exists $\widehat{T} \in (0,T']$ such that

$$
\begin{aligned}
&|\gamma|_{L^\infty((0,T);\mathrm{TV}(\mathbb{R}))} \widehat{T} \|V'\|_{L^\infty((-M,M))} \|w - \tilde{w}\|_{L^\infty((0,\widehat{T});L^\infty(\mathbb{R}))} \|\rho_0\|_{L^\infty(\mathbb{R})} \\
&\times \exp\left(2\widehat{T}\|V'\|_{L^\infty((-M,M))}M'\right) \leq \frac{1}{2}.
\end{aligned}
$$

From this, it follows that F is also a contraction in $\Omega_M^{M'}\big(\widehat{T}\big)$ for a sufficiently small $\widehat{T} \in \big(0,\bar{T}\big)$. □

2.2. Numerical experiments

In what follows, we present some numerical simulations using a non-dissipative scheme based on the method of characteristics (see [**213**, Chapter 3] and [**173**]).

We consider the Cauchy problem (2.0.1) with initial datum

$$
\rho_0(x) = \mathbb{1}_{\left(0,\frac{1}{2}\right)}(x) + \mathbb{1}_{(1,\infty)}(x), \qquad x \in \mathbb{R} \tag{2.2.1}
$$

and focus on the LWR–Greenshields velocity (see [**161**, Eq. (1.26)]), i.e. $V(\xi) := 1 - \xi$, and on the Burgers velocity (see [**161**, Eq. (1.8)]), i.e. $V(\xi) := \xi$ (see also [**138**, Section 3.1.2] for the fundamental diagrams and for a generalized Greenshields model [**147**]). To illustrate the effect of a nonlocal weight with a discontinuity in its support, we consider

$$
\gamma_1(\cdot) = 2\,\mathbb{1}_{(-1,0)}(2\,\cdot\,), \qquad \gamma_2(\cdot) = \frac{4}{3}\mathbb{1}_{(-2,-1)}(4\,\cdot\,) + \frac{8}{3}\mathbb{1}_{(-1,0)}(4\,\cdot\,), \tag{2.2.2}
$$

which satisfy $\gamma_1, \gamma_2 \in \mathrm{BV}(\mathbb{R})$.

For the LWR–Greenshields velocity, a comparison principle is satisfied. Since the initial datum is chosen in such a way that it achieves the maximum density (and thus moves with zero velocity) in $(1,\infty)$, the initial density in $(-\infty,1)$ slows down as it gets closer to $x = 1$. The second illustration in Figure 2.3 indicates a disturbance that evolves from points where the discontinuities of γ and ρ "intersect".

For the example involving the Burgers velocity, a comparison principle does not hold (due to the chosen initial datum and the right-looking nonlocal impact—see [**167**, Example 6.1]) and the entire mass concentrates at $x = 0.5$ as time evolves. Thus, the solution ceases to exist for large time. The impact of the discontinuous weight (the fourth illustration in Figure 2.3) destroys the rather "smooth" structure of the solution that we would obtain when using a continuous kernel.

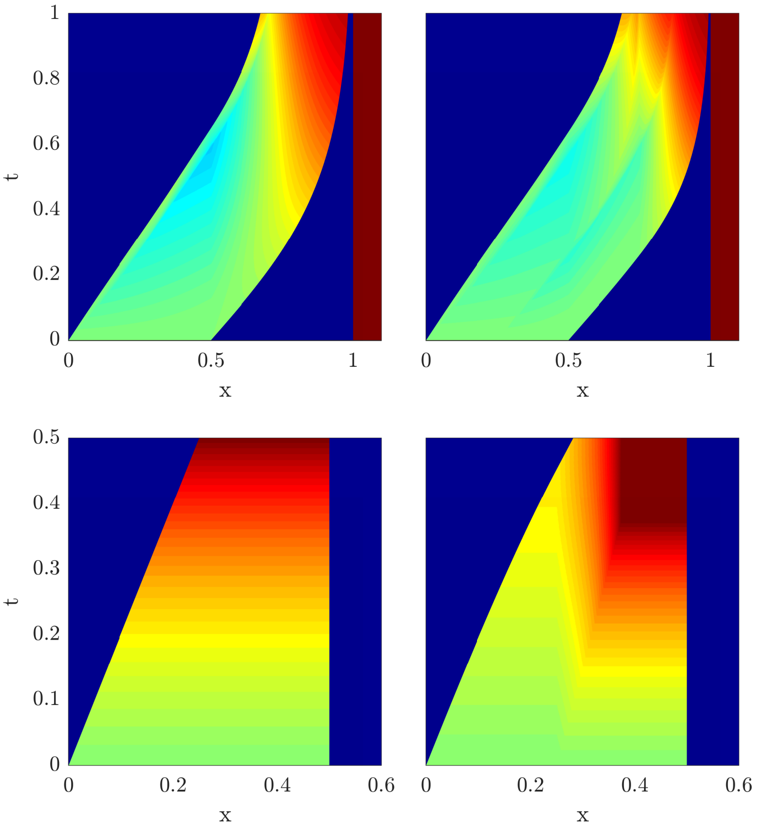

Figure 2.3. Top (left and right): Plot of the solution for the LWR–Greenshields velocity function, i.e. $V(\xi) \coloneqq 1 - \xi$, with weights γ_1 and γ_2, respectively. Bottom (left and right): Plot of the solution for the Burgers velocity, i.e. $V(\xi) \coloneqq \xi$, with weights γ_1 and γ_2, respectively. Colorbar: 0 1. *N.B.:* for the rightmost figure, the maximal density exceeds 1, but is still visualized in dark red.

CHAPTER 3

Nonlocal–to–local singular limit for BV initial data

This Chapter focuses on the nonlocal–to–local limit for a class of nonlocal conservation laws with an exponential nonlocal weight. Given $\alpha > 0$ and a time horizon $T > 0$, we consider

$$\begin{cases} \partial_t \rho_\alpha(t,x) + \partial_x\big(V(W_\alpha[\rho_\alpha](t,x))\rho_\alpha(t,x)\big) = 0, & (t,x) \in (0,T)\times\mathbb{R}, \\ \rho_\alpha(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{3.0.1}$$

with

$$W_\alpha[\rho_\alpha](t,x) \coloneqq \frac{1}{\alpha}\int_x^{+\infty} \exp\left(\frac{x-y}{\alpha}\right)\rho_\alpha(t,y)\,\mathrm{d}y, \tag{3.0.2}$$

and study the convergence of the family $\{\rho_\alpha\}_{\alpha>0}$ to the entropy solution of the corresponding local conservation law

$$\begin{cases} \partial_t \rho(t,x) + \partial_x\big(V(\rho(t,x))\rho(t,x)\big) = 0, & (t,x) \in (0,T)\times\mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{3.0.3}$$

as $\alpha \to 0^+$. As mentioned in Chapter 1, the key idea is to analyze the nonlocal impact $W_\alpha[\rho_\alpha]$. Due to the relation $\alpha\partial_x W_\alpha[\rho_\alpha] = W_\alpha[\rho_\alpha] - \rho_\alpha$, the strong convergence $\{\rho_\alpha\}_{\alpha>0}$ to a weak solution of the local conservation law follows from the strong convergence of $\{W_\alpha\}_{\alpha>0}$, for which we can prove a suitable total variation bound. Our main result is as follows.

Theorem 3.0.1 (Convergence to the entropy solution). *Let us assume that*

$$\rho_0 \in L^\infty(\mathbb{R};\mathbb{R}_{\geq 0}) \cap \mathrm{TV}(\mathbb{R}); \tag{3.0.4}$$

$$V \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}) \quad \textit{and} \quad V'(\xi) \leq 0, \qquad \xi \in [\operatorname{ess\,inf}\rho_0, \operatorname{ess\,sup}\rho_0]; \tag{3.0.5}$$

$$\xi \mapsto V(\xi)\xi \quad \textit{is concave for } \xi \in [\operatorname{ess\,inf}\rho_0, \operatorname{ess\,sup}\rho_0]. \tag{3.0.6}$$

Then the solution $\rho_\alpha \in C\big([0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$ of (3.0.1) *and the corresponding nonlocal impact $W_\alpha[\rho_\alpha]$ converge to the entropy solution of the local conservation law* (3.0.3) *in $C\big([0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$ as $\alpha \to 0^+$.*

Remark 3.0.1 (Generalizations). *Motivated by traffic flow models, we restrict ourselves to monotonically decreasing velocities and non-negative initial data. However, our results can be generalized to different setups. The assumption on V being monotonically decreasing can be changed to V being monotonically increasing as long as we also change the nonlocal impact to*

$$W_\alpha[\rho_\alpha](t,x) \coloneqq \frac{1}{\alpha}\int_{-\infty}^{x} \exp\left(\frac{y-x}{\alpha}\right)\rho_\alpha(t,y)\,\mathrm{d}y, \quad (t,x) \in (0,T)\times\mathbb{R}.$$

Analogously, the results can be extended to non-positive initial data when changing the nonlocal term accordingly. Furthermore, when assuming that $V'(\xi)\xi$ has a sign for all $\xi \in \mathbb{R}$, the initial datum can be chosen arbitrarily in $L^\infty(\mathbb{R}) \cap \mathrm{TV}(\mathbb{R})$ (no sign restrictions). However, then we do not obtain the convergence of ρ_α, but only of W_α, which remains essentially bounded and for which the total variation bound derived in Theorem 3.2.1 below still holds. Compare also Remark 3.2.2.

In Section 3.1, we recall some preliminary results on the well-posedness of the nonlocal problem (3.0.1). In Section 3.2, we deduce the key total variation bound on the nonlocal impact W_α which is needed, in Section 3.3, to prove the nonlocal–to–local convergence theorem. Finally, in Section 3.4, we illustrate the results and further conjectures with some numerical simulations.

3.1. Preliminary results

To start with, we recall some results on the existence and uniqueness of solutions and their properties. The proof is essentially covered by the general theory of CHAPTER 2 (see also [**167**, Theorem 2.20, Theorem 3.2, and Corollary 4.3]).

THEOREM 3.1.1 (Existence and uniqueness of weak solutions and maximum principle). *Let us assume that*

$$\rho_0 \in L^\infty(\mathbb{R};\mathbb{R}_{\geq 0}) \cap \mathrm{TV}(\mathbb{R}); \tag{3.1.1}$$

$$V \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}) \quad \text{and} \quad V'(\xi) \leq 0, \qquad \xi \in [\operatorname{ess\,inf} \rho_0, \operatorname{ess\,sup} \rho_0]. \tag{3.1.2}$$

Then, for every $\alpha > 0$, there exists a unique weak solution $\rho_\alpha \in C\big([0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big) \cap L^\infty((0,T);L^\infty(\mathbb{R})) \cap L^\infty((0,T);\mathrm{TV}(\mathbb{R}))$ of the nonlocal conservation law (3.0.1). *Moreover, the following maximum principle is satisfied:*

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}} \rho_0(x) \leq \rho_\alpha(t,x) \leq \operatorname*{ess\,sup}_{x\in\mathbb{R}} \rho_0(x), \quad (t,x) \in (0,T)\times\mathbb{R}. \tag{3.1.3}$$

For the arguments that follow, it is helpful to establish the following stability result, which enables us to smooth the solution.

LEMMA 3.1.1 (Stability of the nonlocal conservation law w.r.t. the initial datum). *Given $\mathcal{C}_1, \mathcal{C}_2 > 0$, let us consider the set*

$$Q(\mathcal{C}_1,\mathcal{C}_2) \coloneqq \big\{\rho \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}) : \|\rho\|_{L^\infty(\mathbb{R})} \leq \mathcal{C}_1 \text{ and } |\rho|_{\mathrm{TV}(\mathbb{R})} \leq \mathcal{C}_2\big\}.$$

and suppose that $\rho_0 \in Q(\mathcal{C}_1,\mathcal{C}_2)$. Then the solution ρ of the corresponding nonlocal conservation law (3.0.1) *satisfies the following $C([0,T];L^1(\mathbb{R}))$ stability estimate:*

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ s. t. } \ \forall \tilde\rho_0 \in Q(\mathcal{C}_1,\mathcal{C}_2): \ \|\rho_0 - \tilde\rho_0\|_{L^1(\mathbb{R})} \leq \delta \implies \|\rho - \tilde\rho\|_{C([0,T];L^1(\mathbb{R}))} \leq \varepsilon,$$

where $\tilde\rho$ is the solution to the nonlocal conservation law with initial datum $\tilde\rho_0$.

PROOF. The result we need can be essentially found in [**168**, Theorem 4.17] for kernels with compact support. Nevertheless, the modifications necessary for the argument to hold for the exponential weight are minor; we will not go into further detail. ☐

3.2. Total variation bound on the nonlocal impact

We start our analysis by showing that the nonlocal impact $W_\alpha[\rho_\alpha]$ satisfies a conservation law (in advective form) with local velocity and a nonlocal source. This will enable us to study $W_\alpha[\rho_\alpha]$ without referring to ρ_α itself.

LEMMA 3.2.1 (The transport equation with nonlocal source satisfied by the nonlocal impact). *The nonlocal impact $W_\alpha[\rho_\alpha]$ is Lipschitz continuous and solves the following Cauchy problem in the strong sense:*

$$\begin{cases} \partial_t W_\alpha(t,x) + V(W_\alpha(t,x))\partial_x W_\alpha(t,x) \\ \quad = -\dfrac{1}{\alpha}\displaystyle\int_x^\infty \exp\left(\tfrac{x-y}{\alpha}\right) V'(W_\alpha(t,y))\partial_y W_\alpha(t,y) W_\alpha(t,y)\,\mathrm{d}y, & (t,x)\in(0,T)\times\mathbb{R} \\ W_\alpha(0,x) = \dfrac{1}{\alpha}\displaystyle\int_x^{+\infty} \exp\left(\tfrac{x-y}{\alpha}\right)\rho_0(y)\,\mathrm{d}y, & x \in \mathbb{R}. \end{cases} \tag{3.2.1}$$

PROOF. We first show that $W_\alpha[\rho_\alpha]$ is Lipschitz continuous and then that it solves the Cauchy problem (3.2.1).

Step 1. *Boundedness of the space derivative.* We compute, for $(t,x)\in(0,T)\times\mathbb{R}$,

$$\partial_x W_\alpha[\rho_\alpha](t,x) = \partial_x \frac{1}{\alpha}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right)\rho_\alpha(t,y)\,\mathrm{d}y = \frac{1}{\alpha}W_\alpha[\rho_\alpha](t,x) - \frac{1}{\alpha}\rho_\alpha(t,x). \tag{3.2.2}$$

Since $\alpha > 0$, $W_\alpha[\rho_\alpha] \in L^\infty((0,T)\times\mathbb{R})$, and $\rho_\alpha \in L^\infty((0,T)\times\mathbb{R})$, owing to Theorem 3.1.1, we obtain the uniform boundedness on the spatial derivative.

Step 2. *Boundedness of the time derivative.* For the time derivative, we need to rely on the method of characteristics analyzed in [**167**, Lemma 2.6] to write down the solution ρ_α and compute, for $(t,x) \in (0,T) \times \mathbb{R}$,

$$\begin{aligned}
\partial_t W_\alpha[\rho_\alpha](t,x) &= \partial_t \frac{1}{\alpha} \int_x^\infty \exp\left(\frac{x-y}{\alpha}\right) \rho_\alpha(t,y)\,\mathrm{d}y \\
&= \partial_t \frac{1}{\alpha} \int_x^\infty \exp\left(\frac{x-y}{\alpha}\right) \rho_0(\xi(t,y;0))\partial_2\xi(t,y;0)\,\mathrm{d}y \\
&= \partial_t \frac{1}{\alpha} \int_{\xi(t,x;0)}^\infty \exp\Big(\frac{x-\xi(0,z;t)}{\alpha}\Big)\rho_0(z)\,\mathrm{d}z \\
&= -\frac{1}{\alpha^2} \int_{\xi(t,x;0)}^\infty \exp\Big(\frac{x-\xi(0,z;t)}{\alpha}\Big)\rho_0(z)\partial_3\xi(0,z;t)\,\mathrm{d}z - \frac{1}{\alpha}\rho_0(\xi(t,x;0))\partial_1\xi(t,x;0).
\end{aligned} \tag{3.2.3}$$

Recalling, from [**167**, Lemma 2.6], that

$$\begin{aligned}
\partial_3\xi(0,\xi(t,y;0);t) &= V(W_\alpha[\rho_\alpha](t,y)), && (t,y) \in (0,T) \times \mathbb{R}, \\
\partial_1\xi(t,y;0) &= -\partial_2\xi(t,y;0)V(W_\alpha[\rho_\alpha](t,y)), && (t,y) \in (0,T) \times \mathbb{R},
\end{aligned}$$

we obtain, by continuing Eq. (3.2.3),

$$\begin{aligned}
\partial_t W_\alpha[\rho_\alpha](t,x) &= -\frac{1}{\alpha^2} \int_{\xi(t,x;0)}^\infty \exp\Big(\frac{x-\xi(0,z;t)}{\alpha}\Big)\rho_0(z)\partial_3\xi(0,z;t)\,\mathrm{d}z - \frac{1}{\alpha}\rho_0(\xi(t,x;0))\partial_1\xi(t,x;0) \\
&= -\frac{1}{\alpha^2} \int_x^\infty \exp\left(\frac{x-y}{\alpha}\right) \rho_0(\xi(t,y;0))\partial_3\xi(0,\xi(t,y;0)\partial_2\xi(t,y;0)\,\mathrm{d}y \\
&\quad + \frac{1}{\alpha}\rho_0(\xi(t,x;0))\partial_2\xi(t,x;0)V(W_\alpha[\rho_\alpha](t,x)), \\
&= -\frac{1}{\alpha^2} \int_x^\infty \exp\left(\frac{x-y}{\alpha}\right) \rho_\alpha(t,y)V(W_\alpha[\rho_\alpha](t,y))\,\mathrm{d}y + \frac{1}{\alpha}\rho_\alpha(t,x)V(W_\alpha[\rho_\alpha](t,x)).
\end{aligned}$$

Since this expression is essentially bounded for $\alpha > 0$, we obtain the claimed Lipschitz continuity.

Step 3. *Evolution equation.* Next, we show that $W_\alpha[\rho_\alpha]$ solves the Cauchy problem (3.2.1). For $(t,x) \in (0,T) \times \mathbb{R}$, we compute

$$\begin{aligned}
&\partial_t W_\alpha[\rho_\alpha](t,x) + V(W_\alpha[\rho_\alpha](t,x))\partial_x W_\alpha[\rho_\alpha](t,x) \\
&= \frac{1}{\alpha}\rho_\alpha(t,x)V(W_\alpha[\rho_\alpha](t,x)) - \frac{1}{\alpha^2}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right)\rho_\alpha(t,y)V(W_\alpha[\rho_\alpha](t,y))\,\mathrm{d}y \\
&\quad + V(W_\alpha[\rho_\alpha](t,x))\left(\frac{1}{\alpha}W_\alpha[\rho_\alpha](t,x) - \frac{1}{\alpha}\rho_\alpha(t,x)\right) \\
&= V(W_\alpha[\rho_\alpha](t,x))\frac{1}{\alpha}W_\alpha[\rho_\alpha](t,x) \\
&\quad - \frac{1}{\alpha^2}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right)\Big(W_\alpha[\rho_\alpha]t,y) - \alpha\partial_y W_\alpha[\rho_\alpha](t,y))V(W_\alpha[\rho_\alpha](t,y)\Big)\,\mathrm{d}y \\
&= V(W_\alpha[\rho_\alpha](t,x))\frac{1}{\alpha}W_\alpha[\rho_\alpha](t,x) \\
&\quad - \frac{1}{\alpha^2}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right)W_\alpha[\rho_\alpha]t,y)V(W_\alpha[\rho_\alpha](t,y))\,\mathrm{d}y \\
&\quad + \frac{1}{\alpha}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right)\partial_y W_\alpha[\rho_\alpha](t,y)V(W_\alpha[\rho_\alpha](t,y))\,\mathrm{d}y \\
&= -\frac{1}{\alpha}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right)V'(W_\alpha[\rho_\alpha](t,x))\partial_y W_\alpha[\rho_\alpha](t,y)W_\alpha[\rho_\alpha](t,y)\,\mathrm{d}y,
\end{aligned}$$

where we have used the identity (3.2.2) two times and integration by parts. The last term is in fact the right-hand side of the PDE in (3.2.1). The nonlocal impact $W_\alpha[\rho_\alpha]$ also satisfies the initial condition in (3.2.1) as a direct consequence of the definition of $W_\alpha[\rho_\alpha]$ when plugging in $t = 0$, which is possible as $\rho_\alpha \in C\big([0,T]; L^1_{\text{loc}}(\mathbb{R})\big)$. □

Remark 3.2.1 (Fully local equation in W_α). *The nonlocal transport equation in* (3.2.1) *can also be transformed into a fully local equation involving higher derivatives:*

$$\begin{aligned}&\partial_t W_\alpha(t,x)+\partial_x\big(V(W_\alpha(t,x))W_\alpha(t,x)\big)\\&\quad=\alpha\partial_{tx}^2W_\alpha(t,x)+\partial_x\big(V(W_\alpha(t,x))\partial_x W_\alpha(t,x)\big), &&(t,x)\in(0,T)\times\mathbb{R}.\end{aligned}$$

This formulation will be pivotal in Chapter *6.*

The next theorem shows that the total variation of the nonlocal impact W_α cannot increase over time and thus presents the key ingredient for the proof of our main result.

Theorem 3.2.1 (Total variation bound in the spatial component of W_α, uniform in α). *The nonlocal impact W_α, which solves the Cauchy problem* (3.2.1)*, admits the following total variation bound:*

$$|W_\alpha(t,\cdot)|_{\mathrm{TV}(\mathbb{R})}\leq|W_\alpha(0,\cdot)|_{\mathrm{TV}(\mathbb{R})}\leq|\rho_0|_{\mathrm{TV}(\mathbb{R})},\quad t\in[0,T],$$

for all $\alpha>0$.

Proof. We take advantage of the stability result in Theorem 3.1.1, which tells us that, when smoothing ρ_0 by $\rho_0^\varepsilon\coloneqq\rho_0*\eta_\varepsilon$ (η_ε being a standard mollifier with smoothing parameter $\varepsilon>0$; see [**184**, Appendix C.4]), the corresponding solution ρ_α^ε is close to ρ_α in the $C([0,T];L^1(\mathbb{R}))$ topology. Furthermore, since the initial datum is smooth, so is the corresponding solution (see [**167**, Corollary 5.3]).

We now prove the total variation bound. Since ρ_α^ε is smooth, the total variation coincides with the L^1-norm of the derivative and we can estimate it as follows for $t\in[0,T]$:

$$\begin{aligned}
(3.2.4)\quad &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}|\partial_x W_\alpha^\varepsilon(t,x)|\,\mathrm{d}x\\
&=\int_{\mathbb{R}}\operatorname{sign}\left(\partial_x W_\alpha^\varepsilon(t,x)\right)\partial_{tx}^2W_\alpha^\varepsilon(t,x)\,\mathrm{d}x\\
&=-\int_{\mathbb{R}}\operatorname{sign}\left(\partial_x W_\alpha^\varepsilon(t,x)\right)V(W_\alpha^\varepsilon(t,x))\partial_{xx}^2W^\varepsilon(t,x)\,\mathrm{d}x\\
&\quad-\int_{\mathbb{R}}\operatorname{sign}\left(\partial_x W_\alpha^\varepsilon(t,x)\right)V'(W_\alpha^\varepsilon(t,x))\big(\partial_x W_\alpha^\varepsilon(t,x)\big)^2\,\mathrm{d}x\\
&\quad+\frac{1}{\alpha}\int_{\mathbb{R}}\operatorname{sign}\left(\partial_x W_\alpha^\varepsilon(t,x)\right)V'(W_\alpha^\varepsilon(t,x))W_\alpha^\varepsilon(t,x)\partial_x W_\alpha^\varepsilon(t,x)\,\mathrm{d}x\\
&\quad-\frac{1}{\alpha^2}\int_{\mathbb{R}}\operatorname{sign}\left(\partial_x W_\alpha^\varepsilon(t,x)\right)\int_x^\infty\exp\left(\frac{x-y}{\alpha}\right)V'(W_\alpha^\varepsilon(t,y))\partial_y W_\alpha^\varepsilon(t,y)W_\alpha^\varepsilon(t,y)\,\mathrm{d}y\,\mathrm{d}x\\
&=\int_{\mathbb{R}}2\delta_{\{\partial_x W_\alpha^\varepsilon(t,x)=0\}}V(W_\alpha^\varepsilon(t,x))\partial_x W_\alpha^\varepsilon(t,x)\partial_{xx}^2W_\alpha^\varepsilon(t,x)\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}}\operatorname{sign}\left(\partial_x W_\alpha^\varepsilon(t,x)\right)V'(W_\alpha^\varepsilon(t,x))\big(\partial_x W_\alpha^\varepsilon(t,x)\big)^2\,\mathrm{d}x\\
&\quad-\int_{\mathbb{R}}\operatorname{sign}\left(\partial_x W_\alpha^\varepsilon(t,x)\right)V'(W_\alpha^\varepsilon(t,x))\big(\partial_x W_\alpha^\varepsilon(t,x)\big)^2\,\mathrm{d}x\\
&\quad+\frac{1}{\alpha}\int_{\mathbb{R}}\operatorname{sign}\left(\partial_x W_\alpha^\varepsilon(t,x)\right)V'(W_\alpha^\varepsilon(t,x))W_\alpha^\varepsilon(t,x)\partial_x W_\alpha^\varepsilon(t,x)\,\mathrm{d}x\\
&\quad-\frac{1}{\alpha^2}\int_{\mathbb{R}}\operatorname{sign}\left(\partial_x W_\alpha^\varepsilon(t,x)\right)\int_x^\infty\exp\left(\frac{x-y}{\alpha}\right)V'(W_\alpha^\varepsilon(t,y))\partial_y W_\alpha^\varepsilon(t,y)W_\alpha^\varepsilon(t,y)\,\mathrm{d}y\,\mathrm{d}x\\
&\leq\frac{1}{\alpha}\int_{\mathbb{R}}|\partial_x W_\alpha^\varepsilon(t,x)|V'(W_\alpha^\varepsilon(t,x))W_\alpha^\varepsilon(t,x)\,\mathrm{d}x\\
(3.2.5)\quad&\quad-\frac{1}{\alpha^2}\int_{\mathbb{R}}V'(W_\alpha^\varepsilon(t,y))|\partial_y W_\alpha^\varepsilon(t,y)|W_\alpha^\varepsilon(t,y)\int_{-\infty}^y\exp\left(\frac{x-y}{\alpha}\right)\mathrm{d}x\,\mathrm{d}y\\
&\leq\frac{1}{\alpha}\int_{\mathbb{R}}|\partial_x W_\alpha^\varepsilon(t,x)|V'(W_\alpha^\varepsilon(t,x))W_\alpha^\varepsilon(t,x)\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{\alpha}\int_{\mathbb{R}} V'(W_\alpha^\varepsilon(t,y))|\partial_y W_\alpha^\varepsilon(t,y)|W_\alpha^\varepsilon(t,y)\exp\left(\frac{y-y}{\alpha}\right)\mathrm{d}y\\
&= 0.
\end{aligned}
$$

We thus obtain

$$
|W_\alpha^\varepsilon(t,\cdot)|_{\mathrm{TV}(\mathbb{R})} \leq |W_\alpha^\varepsilon(0,\cdot)|_{\mathrm{TV}(\mathbb{R})} \leq |\rho_0|_{\mathrm{TV}(\mathbb{R})}, \tag{3.2.6}
$$

where the last inequality follows from the following computation:

$$
\begin{aligned}
|W_\alpha^\varepsilon(0,\cdot)|_{\mathrm{TV}(\mathbb{R})} &= \sup_{\substack{\psi\in C_c^1(\mathbb{R}):\\ \|\psi\|_{L^\infty(\mathbb{R})}\leq 1}} \int_{\mathbb{R}} \psi'(x) W_\alpha^\varepsilon[\rho_0^\varepsilon](x)\,\mathrm{d}x\\
&= \sup_{\substack{\psi\in C_c^1(\mathbb{R}):\\ \|\psi\|_{L^\infty(\mathbb{R})}\leq 1}} \int_{\mathbb{R}} \psi'(x)\frac{1}{\alpha}\int_0^{+\infty} \exp\left(\frac{x-y}{\alpha}\right)\rho_0^\varepsilon(y)\,\mathrm{d}y\,\mathrm{d}x\\
&= \sup_{\substack{\psi\in C_c^1(\mathbb{R}):\\ \|\psi\|_{L^\infty(\mathbb{R})}\leq 1}} \int_{\mathbb{R}} \psi'(x)\frac{1}{\alpha}\int_{-\infty}^{0} \exp\left(\frac{z}{\alpha}\right)\rho_0^\varepsilon(x-z)\,\mathrm{d}y\,\mathrm{d}x\\
&= \sup_{\substack{\psi\in C_c^1(\mathbb{R}):\\ \|\psi\|_{L^\infty(\mathbb{R})}\leq 1}} \sup_{z\in(-\infty,0)} \int_{\mathbb{R}} \psi'(x+z)\rho_0^\varepsilon(x)\,\mathrm{d}y\\
&= \sup_{\substack{\psi\in C_c^1(\mathbb{R}):\\ \|\psi\|_{L^\infty(\mathbb{R})}\leq 1}} \int_{\mathbb{R}} \psi'(x)\int_{\mathbb{R}} \eta_\varepsilon(x-y)\rho_0(x)\,\mathrm{d}x\,\mathrm{d}y\\
&\leq \sup_{y\in\mathbb{R}}\int_{\mathbb{R}} \eta_\varepsilon(x-y) \sup_{\substack{\psi\in C_c^1(\mathbb{R}):\\ \|\psi\|_{L^\infty(\mathbb{R})}\leq 1}} \int_{\mathbb{R}} \psi'(z)\rho_0(z)\,\mathrm{d}z\,\mathrm{d}x\\
&\leq |\rho_0|_{\mathrm{TV}(\mathbb{R})}.
\end{aligned}
$$

As the bound in (3.2.6) is uniform with respect to $(\varepsilon,\alpha)\in\mathbb{R}^2_{>0}$, this concludes the proof. ☐

REMARK 3.2.2 (Total variation bound and the required assumptions on the velocity V). *The key step in the proof of the total variation bound stated in Theorem 3.2.1 can be located in the estimate around Eq.* (3.2.5). *Reconnecting to Remark 3.0.1, it is enough to assume that the velocity satisfies $V'(\xi)\xi \leq 0$ for $\xi\in\mathbb{R}$ to obtain the uniform total variation bound without any sign restriction on the initial data.*

3.3. Nonlocal–to–local convergence

Using the results in Section 3.2, we can show that the set of nonlocal terms is compact in the canonical $C\big([0,T];L^1(K)\big)$ topology for a given compact set $K\Subset\mathbb{R}$.

THEOREM 3.3.1 (Compactness of $\{W_\alpha\}_{\alpha>0}$ in $C\big([0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$). *The set $\{W_\alpha\}_{\alpha>0}\subseteq C\big([0,T];L^1(K)\big)$ of solutions to* (3.0.2) *is compactly embedded into $C\big([0,T];L^1(K)\big)$ for any compact interval $K\Subset\mathbb{R}$:*

$$
\Big\{W_\alpha\in C\big([0,T];L^1(K)\big):\ W_\alpha \textit{ satisfies } (3.0.2),\ \alpha>0\Big\} \overset{c}{\hookrightarrow} C\big([0,T];L^1(K)\big).
$$

PROOF. The proof consists of applying the Ascoli–Arzelà-type theorem in [**224**, Lemma 1]. Given a Banach space B, a set $F\subset C([0,T];B)$ is relatively compact in $C([0,T];B)$ if

1. $F(t) := \big\{f(t)\in B : f\in F\big\}$ is relatively compact in B for all $t\in[0,T]$;
2. F is uniformly equi-continuous, i.e.

$$
\forall\sigma>0\ \exists\delta>0 \text{ s. t. } \forall f\in F\ \forall(t_1,t_2)\in[0,T]^2 : |t_1-t_2|\leq\delta \implies \|f(t_1)-f(t_2)\|_B\leq\sigma.
$$

Given $K \Subset \mathbb{R}$, we start with setting $B := L^1(K)$ and $F(t) := \{W_\alpha(t,\cdot) \in L^1(K) : \alpha > 0\}$. Thanks to Theorem 3.2.1, we know that $W_\alpha(t,\cdot)$ satisfies a uniform total variation bound and, by Helly's compactness theorem (see [**184**, Theorem 13.35]),

$$F(t) \overset{\mathrm{c}}{\hookrightarrow} L^1(K), \qquad t \in [0,T].$$

To prove the uniform equi-continuity, we consider a smoothed initial datum ρ_0^ε (for $\varepsilon > 0$) as in the proof of Theorem 3.2.1 and call the corresponding smooth nonlocal impact W_α^ε for an $\alpha > 0$. Then, we can estimate, by Theorem 3.2.1,

$$\begin{aligned}
&\big\|W_\alpha^\varepsilon(t_1,\cdot) - W_\alpha^\varepsilon(t_2,\cdot)\big\|_{L^1(\mathbb{R})} = \bigg\|\int_{t_2}^{t_1} \partial_t W_\alpha^\varepsilon(t,\cdot)\,\mathrm{d}t\bigg\|_{L^1(\mathbb{R})}\\
&\leq \bigg\|\int_{t_2}^{t_1} V(W_\alpha^\varepsilon(t,\cdot))\partial_2 W_\alpha^\varepsilon(t,\cdot)\,\mathrm{d}t\bigg\|_{L^1(\mathbb{R})}\\
&\quad + \bigg\|\int_{t_2}^{t_1} \frac{1}{\alpha}\int_*^\infty \exp\Big(\frac{*-y}{\alpha}\Big) V'(W_\alpha^\varepsilon(t,y))\partial_y W_\alpha^\varepsilon(t,y) W_\alpha^\varepsilon(t,y)\,\mathrm{d}y\,\mathrm{d}t\bigg\|_{L^1(\mathbb{R})}\\
&\leq \|V\|_{L^\infty((0,\|\rho_0\|_{L^\infty(\mathbb{R})}))}|W_\alpha^\varepsilon|_{L^\infty((0,T);\mathrm{TV}(\mathbb{R}))}|t_1 - t_2|\\
&\quad + \|V'\|_{L^\infty((0,\|\rho_0\|_{L^\infty(\mathbb{R})}))}\|W_\alpha^\varepsilon\|_{L^\infty((0,T);L^\infty(\mathbb{R}))}|W_\alpha^\varepsilon|_{L^\infty((0,T);\mathrm{TV}(\mathbb{R}))}|t_1 - t_2|\\
&\leq \big(\|V\|_{L^\infty((0,\|\rho_0\|_{L^\infty(\mathbb{R})}))} + \|V'\|_{L^\infty((0,\|\rho_0\|_{L^\infty(\mathbb{R})}))}\|\rho_0\|_{L^\infty(\mathbb{R})}\big)|\rho_0|_{\mathrm{TV}(\mathbb{R})}|t_1 - t_2|.
\end{aligned}$$

Since this is a uniform bound in $\alpha > 0$ and $\varepsilon > 0$, we have the uniform equi-continuity. This concludes the proof. □

From the strong convergence of W_α, we also deduce the strong convergence of ρ_α to a weak solution of the local conservation law.

COROLLARY 3.3.1 (Limits of ρ_α and W_α are weak solutions of the local conservation law). *For every sequence $\{\alpha_k\}_{k\in\mathbb{N}_{\geq 1}} \subset \mathbb{R}_{>0}$ with $\lim_{k\to\infty}\alpha_k = 0$, there exists a subsequence (for reasons of convenience again denoted by α_k) and a function $\rho^* \in C\big([0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$ such that the solution $\rho_{\alpha_k} \in C\big([0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$ of the nonlocal conservation law* (3.0.1) *converges in $C\big([0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$ to the limit point ρ^* and so does the nonlocal weight W_{α_k}; moreover, ρ^* is a weak solution of the local conservation law* (3.0.3)*. That is,*

$$\lim_{\alpha\to 0}\|\rho_\alpha - \rho^*\|_{C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}))} = 0 \quad \textit{and} \quad \lim_{\alpha\to 0}\|W_\alpha - \rho^*\|_{C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}))} = 0,$$

and ρ^* satisfies, for all $\varphi \in C^1_c([0,T)\times\mathbb{R})$,

$$\int_0^T\int_{\mathbb{R}} (\partial_t\varphi(t,x)\rho^*(t,x) + \partial_x\varphi(t,x)V(\rho^*(t,x))\rho^*(t,x))\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}}\varphi(0,x)\rho_0(x)\,\mathrm{d}x = 0. \tag{3.3.1}$$

PROOF. Owing to Theorem 3.3.1, $\{W_{\alpha_k}\}_{k\in\mathbb{N}_{\geq 1}} \overset{\mathrm{c}}{\hookrightarrow} C\big([0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$; thus, there exists a limit point $\rho^* \in C\big([0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$ such that

$$\lim_{k\to\infty}\|W_{\alpha_k} - \rho^*\|_{C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}))} = 0.$$

The identity (3.2.2) implies

$$\|W_{\alpha_k}(t,\cdot) - \rho_{\alpha_k}(t,\cdot)\|_{L^1(\mathbb{R})} = \alpha_k|W_{\alpha_k}(t,\cdot)|_{\mathrm{TV}(\mathbb{R})} \leq \alpha_k|\rho_0|_{\mathrm{TV}(\mathbb{R})},$$

which yields

$$\lim_{k\to\infty}\|\rho_{\alpha_k} - \rho^*\|_{C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}))} = 0.$$

The fact that ρ^* is indeed a weak solution follows from Lebesgue's dominated convergence theorem, owing to the strong convergence of ρ_{α_k} to ρ^* in $C\big([0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$ and the uniform bound on ρ_α given in (3.1.3). □

To prove Theorem 3.0.1, it remains to show that every limit point ρ^* is the (unique) entropy-admissible solution of the local conservation law (3.0.3).

PROOF OF THEOREM 3.0.1. The result is a direct consequence of the convergence of W_α and ρ_α to a weak solution of the local conservation law in $C\big([0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$, as stated in Corollary 3.3.1, and of the argument in [**46**]. Therein, the minimal entropy condition of [**119, 208**] (which requires the additional assumption that the flux is strictly concave) is used to demonstrate that a solution ρ_α of the nonlocal conservation law in (3.0.1), with uniform TV-bounds, converges to the entropy solution of the local problem. When carefully examining the proof, it becomes apparent that it suffices to assume that the solution ρ_α converges strongly to a weak solution ρ^*, which is the case.

For the sake of completeness, we provide the argument for any convex entropy $\eta\in C^2(\mathbb{R})$ and associated entropy-flux $q\in C^2(\mathbb{R})$. We write the conservation law in (3.0.1) as

$$\begin{aligned}&\partial_t\rho_\alpha(t,x)+\partial_x(V(\rho_\alpha(t,x))\rho_\alpha(t,x))\\&\quad=\partial_x\Big(\rho_\alpha(t,x)\big(V(\rho_\alpha(t,x))-V(W_\alpha[\rho_\alpha](t,x))\big)\Big),\qquad (t,x)\in(0,T)\times\mathbb{R}.\end{aligned}$$

Multiplying the equation by η' and using the chain rule (here a smoothing argument can be used) yields

$$\begin{aligned}&\partial_t\eta(\rho_\alpha(t,x))+\partial_x(q(\rho_\alpha(t,x))\\&\quad=\eta'(\rho_\alpha(t,x))\partial_x\Big(\rho_\alpha(t,x)\big(V(\rho_\alpha(t,x))-V(W_\alpha[\rho_\alpha](t,x))\big)\Big),\qquad (t,x)\in(0,T)\times\mathbb{R}.\end{aligned}$$

Multiplying by a test function $\varphi\in C^1_c((0,T)\times\mathbb{R};\mathbb{R}_+)$, integrating on $(0,T)\times\mathbb{R}$, and integrating by parts yields

$$\int_0^T\int_{\mathbb{R}}(\partial_t\varphi(t,x)\rho_\alpha(t,x)+\partial_x\varphi(t,x)q(\rho_\alpha(t,x)))\,\mathrm{d}x\,\mathrm{d}t=-D,$$

where we estimate the entropy production as follows:

$$\begin{aligned}D&:=\int_0^T\int_{\mathbb{R}}\varphi\eta'(\rho_\alpha)\partial_x(\rho_\alpha(V(\rho_\alpha)-V(W_\alpha)))\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^T\int_{\mathbb{R}}\varphi\Big(\eta'(\rho_\alpha)\partial_x\rho_\alpha(V(\rho_\alpha)-V(W_\alpha))+\eta'(\rho_\alpha)\rho_\alpha\partial_x(V(\rho_\alpha)-V(W_\alpha))\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^T\int_{\mathbb{R}}\varphi\Big(\partial_x\eta(\rho_\alpha)(V(\rho_\alpha)-V(W_\alpha))+\eta'(\rho_\alpha)\rho_\alpha\partial_x(V(\rho_\alpha)-V(W_\alpha))\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&=-\int_0^T\int_{\mathbb{R}}\partial_x\varphi\big(\eta(\rho_\alpha)(V(\rho_\alpha)-V(W_\alpha))\big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\int_{\mathbb{R}}\varphi(\eta'(\rho_\alpha)\rho_\alpha-\eta(\rho_\alpha))\partial_x(V(\rho_\alpha)-V(W_\alpha))\,\mathrm{d}x\,\mathrm{d}t\\
&=\underbrace{-\int_0^T\int_{\mathbb{R}}\partial_x\varphi\big(\eta(\rho_\alpha)(V(\rho_\alpha)-V(W_\alpha))\big)\,\mathrm{d}x\,\mathrm{d}t}_{=:D_1}\\
&\quad\underbrace{-\int_0^T\int_{\mathbb{R}}\partial_x\varphi\big(P(\rho_\alpha)-P(W_\alpha)\big)\,\mathrm{d}x\,\mathrm{d}t}_{=:D_2}\\
&\quad\underbrace{+\int_0^T\int_{\mathbb{R}}\varphi\big((\eta'(W_\alpha)W_\alpha-\eta(W_\alpha))-(\eta'(\rho_\alpha)\rho_\alpha-\eta(\rho_\alpha))\big)V'(W_\alpha)\frac{W_\alpha-\rho_\alpha}{\alpha}\,\mathrm{d}x\,\mathrm{d}t}_{=:D_3},\end{aligned}$$

with $P'(\xi)=\big(\eta'(\xi)\xi-\eta(\xi)\big)V'(\xi)$. By Lagrange's mean-value theorem, we have

$$D_3=\int_{\mathbb{R}}\varphi(\eta'(\bar\xi)\bar\xi-\eta(\bar\xi))'V'(W_\alpha)\frac{(W_\alpha-\rho_\alpha)^2}{\alpha}\,\mathrm{d}x\le 0.$$

where $\bar\xi\in\big[\min\{W_\alpha(t,x),\rho_\alpha(t,x)\},\max\{W_\alpha(t,x),\rho_\alpha(t,x)\}\big]$.

Since $\rho_{\alpha_k}, W_{\alpha_k} \to \rho^*$ in $C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$, as $k \to +\infty$, we conclude that $D_1, D_2 \to 0$ and, thus,

$$\int_0^T \int_{\mathbb{R}} \left(\partial_t \varphi(t,x)\rho^*(t,x) + \partial_x \varphi(t,x) q(\rho^*(t,x))\right) \mathrm{d}x \, \mathrm{d}t \geq 0.$$

As every limit point is, by the previous argument, an entropy solution of the local conservation law (3.0.3) and the entropy solution is unique, we conclude that the whole families $\{\rho_\alpha\}_{\alpha>0}$ and $\{W_\alpha\}_{\alpha>0}$ converge to the same limit point by Urysohn's subsequence principle (see [**230**]). □

3.4. Numerical experiments

We rely on a non-dissipative solver based on characteristics (see [**173**] and [**213**, Chapter 3]). On the basis of a simple numerical example, we want to shed more light on the difference between the total variation of ρ_α and the nonlocal counterpart $W_\alpha[\rho_\alpha]$ (see Figure 3.1, top row). We consider the LWR–Greenshields velocity $V(\xi) := 1 - \xi$ (see [**138**, Chapter 3, Eq. (3.1.3)]) and the initial datum

$$\rho_0 := \frac{1}{2}\mathbb{1}_{(0,\frac{1}{3})} + \mathbb{1}_{(\frac{2}{3},+\infty)}. \tag{3.4.1}$$

The zeros of the initial datum, located in the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$, are moving along the evolution, but are kept in the nonlocal solution ρ_α for all times (see Figure 3.2). This results in an increase in the total variation. In the nonlocal impact W_α, on the other hand, there are no roots and, as proven in Theorem 3.2.1, the total variation is non-increasing.

We also perform some experiments with a piecewise-constant weight and consider

$$W_\alpha[\rho_\alpha] := \frac{1}{\alpha}\int_x^{x+\alpha} \rho_\alpha(t,y)\, \mathrm{d}y, \qquad (t,x) \in (0,T) \times \mathbb{R}. \tag{3.4.2}$$

For the initial datum mentioned above, it appears to be true that a total variation bound on the nonlocal impact is maintained even in this case and that the solution still converges to the local entropy solution (see Figure 3.1, bottom row). However, more recently, in [**89**] a particular initial datum was built for which the total variation of the nonlocal impact defined in (3.4.2) can actually increase.

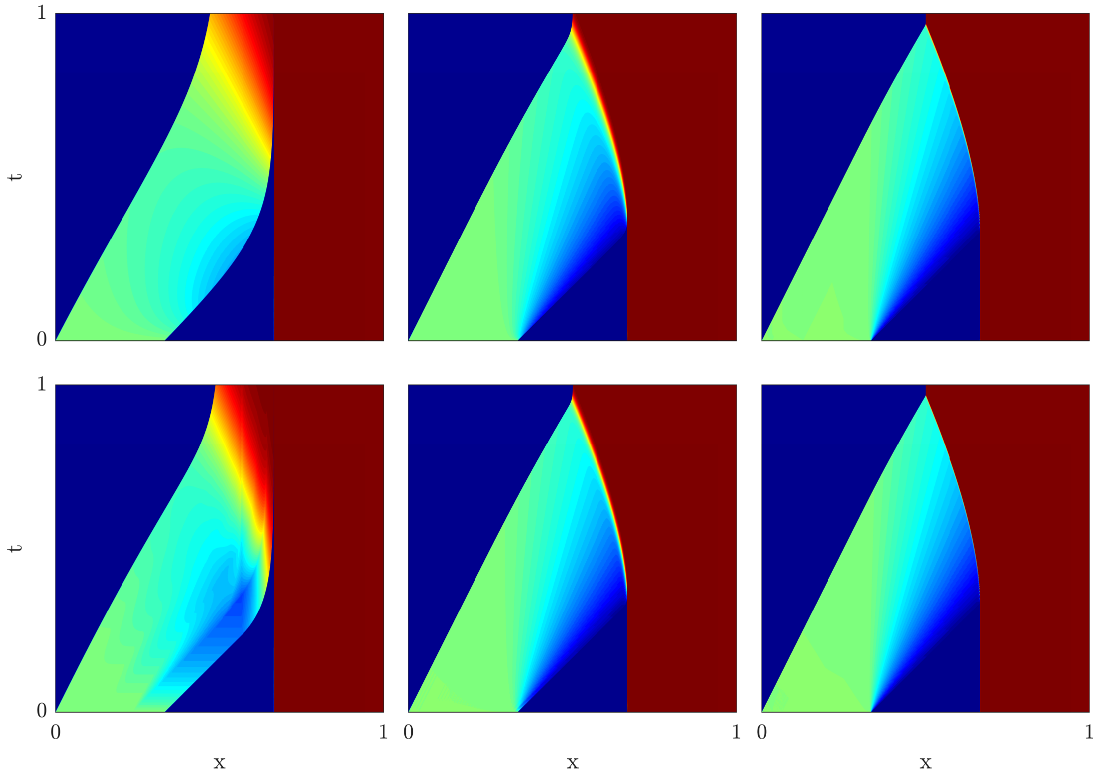

FIGURE 3.1. Solution of the nonlocal conservation law with exponential weight (TOP) or piecewise-constant weight (BOTTOM) and initial datum (3.4.1), plotted in the space-time domain. From left to right, α is decreasing: $\alpha \in \{10^{-1},\ 10^{-2}, 10^{-3}\}$.
COLORBAR: 0 1

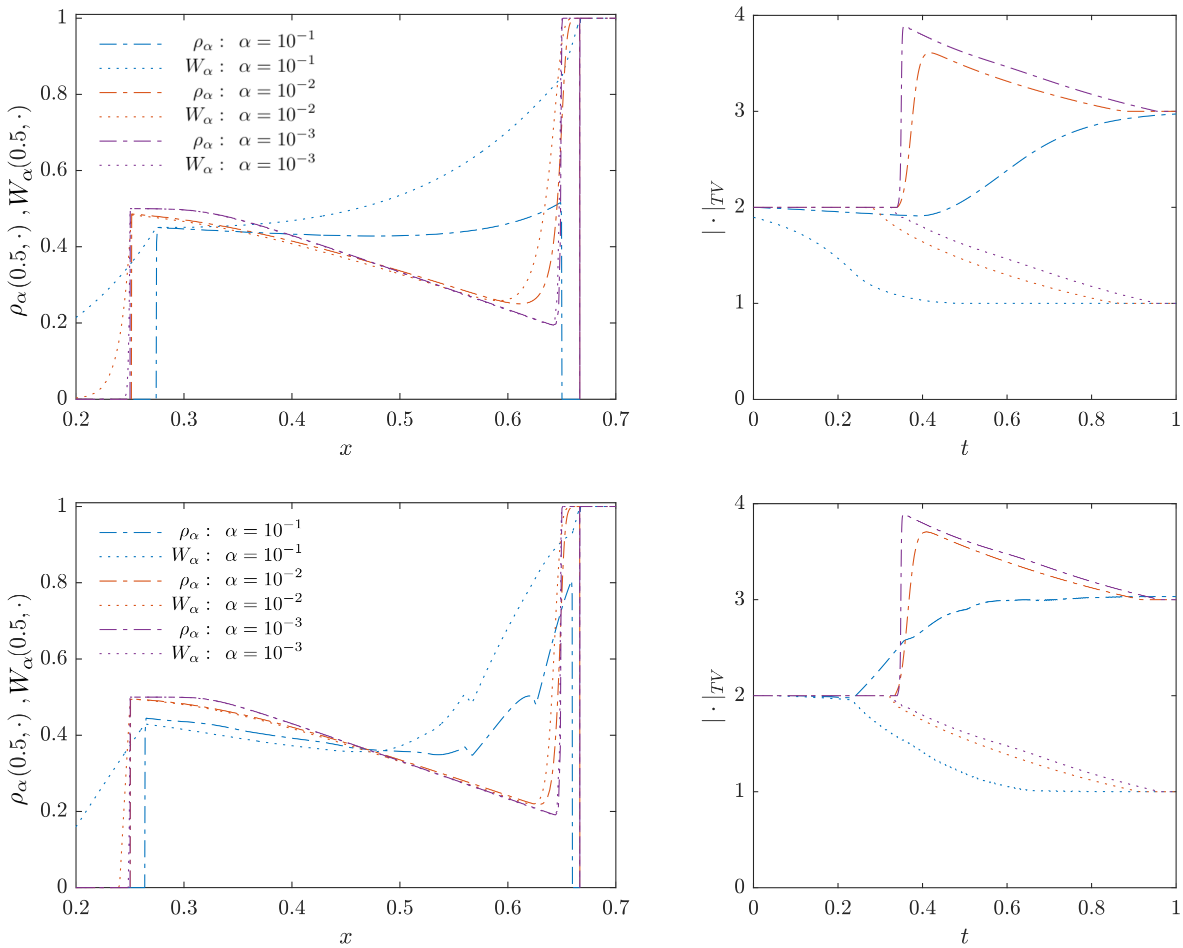

Figure 3.2. Left: Solution and nonlocal impact of the nonlocal conservation law with exponential weight (top) or piecewise-constant weight (bottom) and initial datum (3.4.1), plotted for $t = 0.5$ and $\alpha \in \{10^{-1}, 10^{-2}, 10^{-3}\}$. Right: Evolution of the corresponding total variations. The nonlocal impact (dotted lines) has non-increasing total variation; on the other hand, the total variation of the solution itself (dashed-dotted lines) is not monotone and approaches 3 for large times (while the solution of the local conservation law has total variation equal to 1 for $t \in (1, T)$).

CHAPTER 4

Oleĭnik-type estimates and nonlocal–to–local singular limit for L^∞ initial data

The main results of this Chapter are the following Oleĭnik-type estimates involving the nonlocal impact W_α. More precisely, under different sets of assumptions on the velocity function V, we show that W_α satisfies a one-sided Lipschitz condition and that $V'(W_\alpha)W_\alpha\partial_x W_\alpha$ satisfies a one-sided bound, respectively.

THEOREM 4.0.1 (Oleĭnik-type inequality for W_α). *Let $0<\kappa_1<\kappa_2$, $\rho_0\in L^\infty(\mathbb{R};\mathbb{R}_{\geq 0})$, and $V\in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ a non-increasing velocity such that at least one of the following conditions is satisfied:*

$$V'(\xi)=-\delta<0, \qquad \xi\in[\operatorname{ess\,inf}\rho_0,\operatorname{ess\,sup}\rho_0]; \tag{4.0.1}$$

$$0\leq V'(\xi)+V''(\xi)\xi\leq\kappa_1,\quad V'(\xi)\leq-\kappa_2,\quad \kappa_2-\kappa_1>0,\qquad \xi\in[\operatorname{ess\,inf}\rho_0,\operatorname{ess\,sup}\rho_0]. \tag{4.0.2}$$

Let ρ_α be the solution of the Cauchy problem associated to (3.0.1)*. Then the nonlocal impact W_α satisfies the following inequality:*

$$\frac{W_\alpha(t,x)-W_\alpha(t,y)}{x-y}\geq-\frac{1}{\kappa t},\qquad \text{for all } t>0 \text{ and } x,y\in\mathbb{R} \text{ with } x\neq y, \tag{4.0.3}$$

with $\kappa:=\delta$ (in case assumption (4.0.1) *holds) or $\kappa:=\kappa_2-\kappa_1$ (in case assumption* (4.0.2) *holds).*

REMARK 4.0.1 (Convexity/concavity assumptions). *If we assume that the flux is strictly convex (instead of strictly concave as implied by the assumptions* (4.0.1) *or* (4.0.2)*), we can establish an analogous result. We chose to prove the statement in the concave case because of its relevance to traffic models. For the computation in the case of a convex flux with linear velocity (i.e., the counterpart of the setting of* (4.0.1)*), we refer to* CHAPTER 5.

THEOREM 4.0.2 (Oleĭnik-type inequality for $V'(W_\alpha)W_\alpha\partial_x W_\alpha$). *Let $0<\kappa_1$, $\rho_0\in L^\infty(\mathbb{R};\mathbb{R}_{\geq 0})$, and $V\in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ a non-increasing velocity such that at least one of the following conditions is satisfied:*

$$0\leq(-V'(\xi)-V''(\xi)\xi)(\operatorname{ess\,sup}\rho_0-\operatorname{ess\,inf}\rho_0)\leq-V'(\xi)\xi,\qquad \xi\in[\operatorname{ess\,inf}\rho_0,\operatorname{ess\,sup}\rho_0]; \tag{4.0.4}$$

$$-V'(\xi)\leq V''(\xi)\xi\leq-(2-\kappa_1)V'(\xi),\qquad \xi\in[\operatorname{ess\,inf}\rho_0,\operatorname{ess\,sup}\rho_0]. \tag{4.0.5}$$

Let ρ_α be the solution of the Cauchy problem associated to (3.0.1)*. Then,*

$$\sup_{\mathbb{R}} V'(W_\alpha)W_\alpha\partial_x W_\alpha\leq\frac{\|\rho_0\|_{L^\infty(\mathbb{R})}}{\kappa t},\qquad \text{for all } t>0, \tag{4.0.6}$$

where $\kappa:=1$ (in case assumption (4.0.4) *holds) or $\kappa:=\kappa_1$ (in case assumption* (4.0.5) *holds).*

REMARK 4.0.2 (Independence of the constant on $|\rho_0|_{\mathrm{TV}(\mathbb{R})}$). *In Theorems 4.0.1 and 4.0.2, the initial datum is not required to be of bounded variation.*

REMARK 4.0.3 (Assumptions on the velocity function and traffic models). *The assumptions on the velocity function V in Theorems 4.0.1 and 4.0.2 may appear quite restrictive. In the proofs, we exploit such conditions when manipulating the equations satisfied by $\partial_x W_\alpha$ and $V'(W_\alpha)W_\alpha\partial_x W_\alpha$ to deduce a Riccati-type differential inequality. Despite their apparent intricacy, these assumptions are satisfied by several classes of well-known traffic models, possibly under some restrictions on the initial data.*

1. *The* LWR–Greenshields model *$V(\xi):=v_{\max}(1-\xi/\rho_{\max})$, with $v_{\max}>0$ and $\rho_{\max}>0$ (see* [**138**, Chapter 3, Eq. (3.1.3)]*), satisfies assumption* (4.0.1).

2. *The* Underwood model $V(\xi) \coloneqq v_0 \exp\left(-\frac{\xi}{\rho_{\max}}\right)$, *with* $\rho_{\max} > 0$ *and* $v_0 > 0$ *(see* [**138**, Chapter 3, Eq. (3.1.5)]*), satisfies assumption* (4.0.4) *under the constraint* $\operatorname{ess\,inf}\rho_0 \geq \frac{3-\sqrt{8}}{2}\operatorname{ess\,sup}\rho_0$.
3. *The generalized* Greenshields model $V(\xi) \coloneqq v_0\left(1 - \left(\frac{\xi}{\rho_{\max}}\right)^n\right)$, *with* $\rho_{\max} > 0$ *and* $v_0 > 0$ *(see* [**138**, Chapter 3, Eq. (3.1.6)]*), satisfies assumption* (4.0.4) *under the constraint* $\operatorname{ess\,inf}\rho_0 \geq \frac{n}{n+1}\operatorname{ess\,sup}\rho_0$.
4. *The* generalized California model $V_\mu(\xi) \coloneqq v_0\left(\frac{1}{\xi^\mu} - \frac{1}{\rho_{\max}^\mu}\right)$, *with* $\rho_{\max} > 0$, $v_0 > 0$, *and* $\mu \in (0,1)$ *(cf.* [**138**, Chapter 3, Eq. (3.1.7)]*), satisfies assumptions* (4.0.2) *and* (4.0.5). *This velocity is not locally Lipschitz continuous at* $\xi = 0$*; however, its variant* $V_\mu(\xi) \coloneqq v_{\max}\left(\frac{1}{\xi^\mu + \frac{v^\mu{}_{\max}}{v^\mu{}_{\max}+1}} - \frac{1}{\rho_{\max}^\mu}\right)$ *is and satisfies the same assumption; alternatively, we may just assume* $\rho_0 \geq c_0 > 0$.

As a consequence of Theorems 4.0.1 and 4.0.2, we deduce the following nonlocal–to–local convergence results. The key difference compared to CHAPTER 3 (and the more recent work [**89**]) is the fact that we do not require the initial datum to have bounded total variation; on the other hand, some extra assumptions on the velocity function are required.

COROLLARY 4.0.1 (Nonlocal–to–local singular limit problem). *Let us suppose that either*

– *the assumptions of Theorem 4.0.1 hold;*
– *the assumptions of Theorem 4.0.2 hold, and additionally* $V' \leq -\kappa_2 < 0$ *for some* $\kappa_2 > 0$.

Let ρ_α *be the unique weak solution of the nonlocal conservation law* (3.0.1) *and* ρ *be the unique entropy admissible solution of the local conservation law* (3.0.3). *Then, both* ρ_α *and the corresponding nonlocal impact* W_α *converge to* ρ *in* $L^1_{\text{loc}}([0,T) \times \mathbb{R})$.

In Section 4.3, these results are illustrated by several numerical simulations. Before diving into the proof of our main results, which are contained in Section 4.1 and 4.2, we recall the well-posedness result from CHAPTER 3 (in a slightly modified form) and the evolution equation satisfied by W_α.

THEOREM 4.0.3 (Existence and uniqueness of weak solutions, maximum principle, and properties of the nonlocal impact). *Let* $\rho_0 \in L^\infty(\mathbb{R};\mathbb{R}_{\geq 0})$ *and* $V \in W^{2,\infty}_{\text{loc}}(\mathbb{R})$ *be a non-increasing velocity. Then, for every* $\alpha > 0$, *there exists a unique weak solution* $\rho_\alpha \in C([0,T];L^1_{\text{loc}}(\mathbb{R})) \cap L^\infty((0,T);L^\infty(\mathbb{R}))$ *of the nonlocal conservation law* (3.0.1) *and the following maximum principle holds:*

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}} \rho_0(x) \leq \rho_\alpha(t,x) \leq \operatorname*{ess\,sup}_{x\in\mathbb{R}} \rho_0(x), \qquad \textit{for a.e. } (t,x) \in (0,T) \times \mathbb{R}. \tag{4.0.7}$$

Moreover, for the nonlocal impact W_α, *the following properties hold:*

(1) $W_\alpha \in W^{1,\infty}([0,T] \times \mathbb{R})$ *and* $\operatorname{ess\,inf}\rho_0 \leq W_\alpha \leq \operatorname{ess\,sup}\rho_0$;
(2) $W_\alpha \in C^0\left([0,T];L^1(\mathbb{R})\right)$;
(3) *if* $\rho_0 \in C^k(\mathbb{R})$, *then* $W_\alpha \in C^{k+1}([0,T] \times \mathbb{R})$ *for* $k \geq 0$.

In addition, for every $t \in [0,T]$, *the map* $t \mapsto \operatorname{Lip}^-(\rho_\alpha(t,\cdot))$ *is a locally Lipschitz continuous function from* $[0,+\infty)$ *to* $[0,+\infty)$. *Here,* $\operatorname{Lip}^-(\rho_\alpha) \coloneqq -\inf_{x<y} \frac{\rho_\alpha(y)-\rho_\alpha(x)}{y-x}$.

Furthermore, W_α *solves the following Cauchy problem in the strong sense:*

$$\begin{cases} \partial_t W_\alpha(t,x) + V(W_\alpha(t,x))\partial_x W_\alpha(t,x) & \\ = -\frac{1}{\alpha}\int_x^\infty \exp(\frac{x-y}{\alpha})V'(W_\alpha(t,y))\partial_y W_\alpha(t,y)W_\alpha(t,y)\,\mathrm{d}y, & (t,x) \in (0,T) \times \mathbb{R}, \\ W_\alpha(0,x) = \frac{1}{\alpha}\int_x^\infty \exp(\frac{x-y}{\alpha})\rho_0(y)\,\mathrm{d}y, & x \in \mathbb{R}. \end{cases} \tag{4.0.8}$$

We note that (4.0.8) *can be equivalently rewritten as*

$$\partial_t W_\alpha + \partial_x(V(W_\alpha)W_\alpha) = g_\alpha - g_\alpha * \gamma_\alpha, \qquad \textit{where } g_\alpha = V'(W_\alpha)W_\alpha\partial_x W_\alpha \tag{4.0.9}$$

and we use the notations

$$\gamma(\cdot) \coloneqq \mathbb{1}_{(-\infty,0]}(\cdot)\exp(\cdot) \quad \textit{and} \quad \gamma_\alpha = \alpha^{-1}\gamma(\cdot/\alpha). \tag{4.0.10}$$

4.1. Proof of the Oleĭnik estimates

In order to prove the Oleĭnik estimates, it is helpful to regularize the initial data of the nonlocal conservation law (3.0.1). To this end, we need the following stability result (see Chapters 3 and 5 for related results).

LEMMA 4.1.1 (Approximation). *Let us consider the Cauchy problem*

$$\begin{cases} \partial_t \rho(t,x) + \partial_x (V(W[\rho](t,x))\rho(t,x)) = 0, & (t,x) \in (0,+\infty)\times\mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{4.1.1}$$

where

$$W[\rho](t,x) := \int_x^{+\infty} \exp(x-y)\rho(t,y)\,\mathrm{d}y, \qquad (t,x) \in (0,+\infty)\times\mathbb{R}.$$

Let us also consider the family of the Cauchy problems

$$\begin{cases} \partial_t \rho_n(t,x) + \partial_x \left(V\left(W_n(t,x)\right)\rho_n(t,x)\right) = 0, & (t,x) \in (0,+\infty)\times\mathbb{R}, \\ \rho_n(0,x) = \rho_{0,n}(x), & x \in \mathbb{R}, \end{cases} \tag{4.1.2}$$

where $n \in \mathbb{N}$ and

$$W_n[\rho_n](t,x) := \int_x^{+\infty} \exp(x-y)\rho_n(t,y)\,\mathrm{d}y.$$

Let us furthermore assume that, for a suitable constant $M > 0$, it holds

$$0 \le \rho_{0,n} \le M \text{ a.e. for every } n \in \mathbb{N}, \qquad \rho_{0,n} \overset{*}{\rightharpoonup} \rho_0 \text{ weakly-}* \text{ in } L^\infty(\mathbb{R}) \text{ for } n \to +\infty. \tag{4.1.3}$$

Then,

$$W_n \to W \quad \text{strongly in } L^1_{\mathrm{loc}}\left(\mathbb{R}_+\times\mathbb{R}\right).$$

REMARK 4.1.1 (More general kernels). *The statement of Lemma 7.4.1 is still valid if we replace the exponential weight with a more general kernel*

$$\gamma \in \mathrm{Lip}\left(\mathbb{R}_-\right), \quad \int_{\mathbb{R}_-} \gamma(y)\,\mathrm{d}y = 1, \quad \gamma' \ge 0.$$

PROOF OF LEMMA 7.4.1. By the maximum principle, the first condition in (4.1.3) yields

$$0 \le \rho_n, W_n \le M \text{ a.e. and for every } n \in \mathbb{N}. \tag{4.1.4}$$

Owing to (4.1.4), we have that, up to subsequences, $\rho_n \overset{*}{\rightharpoonup} v$ in the weak-$*$ topology of $L^\infty\left(\mathbb{R}_+\times\mathbb{R}\right)$, for some bounded limit function v. By Lebesgue's dominated convergence theorem, this, in turn, implies that $W_n \to v * \mathbb{1}_{(-\infty,0]}(\cdot)\exp(\cdot)$ strongly in $L^1_{\mathrm{loc}}\left(\mathbb{R}_+\times\mathbb{R}\right)$. By passing to the limit in the distributional formulation of (4.1.2), we conclude that v coincides with the unique bounded distributional solution of (4.1.1). ☐

REMARK 4.1.2 (Continuity in time). *By using* [**116**, Lemma 1.3.3]*, we can assume—with no loss of generality—that the functions $t \mapsto \rho_\alpha(t,\cdot)$ and $t \mapsto W_\alpha(t,\cdot)$ are continuous from $\mathbb{R}_+$ to $L^\infty(\mathbb{R})$ endowed with the L^∞-weak-$*$ and the strong L^1_{loc} topology, respectively. In Section 4.2, we will use this remark to pass to the limit in the nonlocal Oleĭnik inequalities* (4.0.3) *or* (4.0.6) *for every $t > 0$.*

4.1.1. Oleĭnik-type estimate for W_α. In this Section, we establish the Oleĭnik-type inequality in Theorem 4.0.1. The key idea is to use the transport equation with nonlocal source satisfied by W_α, i.e. (4.0.8).

PROOF OF THEOREM 4.0.1. Owing to Lemma 4.1.1, it suffices to prove the statement for initial data $\rho_0 \in \mathcal{D} \cap C^2(\mathbb{R})$ and thus for solutions $\rho_\alpha \in C^2([0,T]\times\mathbb{R})$. Here,

$$\mathcal{D} := \left\{\rho_0 \in L^\infty(\mathbb{R}) : |\rho_0|_{\mathrm{TV}(\mathbb{R})} < \infty,\ \rho_0(x) \in [0,\rho_{\max}] \text{ for a.e. } x \in \mathbb{R}\right\}. \tag{4.1.5}$$

By differentiating (4.0.8) with respect to x, we get

$$\begin{aligned} \partial^2_{tx} W_\alpha = &- V(W_\alpha)\partial^2_{xx} W_\alpha - V'(W_\alpha)(\partial_x W_\alpha)^2 + \frac{1}{\alpha} V'(W_\alpha) W_\alpha \partial_x W_\alpha \\ &- \frac{1}{\alpha^2}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right) V'(W_\alpha) W_\alpha \partial_y W_\alpha\,\mathrm{d}y. \end{aligned} \tag{4.1.6}$$

We now set $m(t) \coloneqq \min_{y\in\mathbb{R}} \partial_y W_\alpha(t,y)$ and assume without loss of generality that $m(t) \leq 0$.
Case 1: *we assume* (4.0.2). We estimate the right-hand side of (4.1.6) from below as follows:

$$\begin{aligned}
\partial^2_{tx} W_\alpha =& - V(W_\alpha)\partial^2_{xx} W_\alpha - V'(W_\alpha)(\partial_x W_\alpha)^2 + \frac{1}{\alpha} V'(W_\alpha) W_\alpha \partial_x W_\alpha \\
&- \frac{1}{\alpha^2}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right) V'(W_\alpha) W_\alpha \partial_y W_\alpha \,\mathrm{d}y \\
\geq& - V(W_\alpha)\partial^2_{xx} W_\alpha - V'(W_\alpha)(\partial_x W_\alpha)^2 + \frac{1}{\alpha} V'(W_\alpha) W_\alpha \partial_x W_\alpha \\
&- \frac{1}{\alpha^2} m \int_x^\infty \exp\left(\frac{x-y}{\alpha}\right) V'(W_\alpha) W_\alpha \,\mathrm{d}y
\end{aligned}$$

(integrating by parts in the last term)

$$\begin{aligned}
=& - V(W_\alpha)\partial^2_{xx} W_\alpha - V'(W_\alpha)(\partial_x W_\alpha)^2 + \frac{1}{\alpha} V'(W_\alpha) W_\alpha \partial_x W_\alpha \\
&- \frac{1}{\alpha} m V'(W_\alpha) W_\alpha - \frac{1}{\alpha} m \int_x^\infty \exp\left(\frac{x-y}{\alpha}\right) \big(V'(W_\alpha)\partial_y W_\alpha + V''(W_\alpha) W_\alpha \partial_y W_\alpha\big)\,\mathrm{d}y.
\end{aligned}$$

We consider $\bar{x} \coloneqq \bar{x}(t) \in \mathbb{R}$ such that $m(t) = \partial_x W_\alpha(t,\bar{x})$ and evaluate the previous expression at $x = \bar{x}$. Due to (4.0.2), we have

$$-\frac{1}{\alpha} m \int_x^\infty \exp\left(\frac{x-y}{\alpha}\right) \big(V'(W_\alpha) + V''(W_\alpha)W_\alpha\big)\partial_y W_\alpha \,\mathrm{d}y \geq -\kappa_1 m^2$$

and, then, we deduce (using [**98**, Theorem 2.1])

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t) \geq -V'(W_\alpha) m(t)^2 - \kappa_1 m^2(t) \geq (\kappa_2 - \kappa_1) m^2(t), \qquad t > 0.$$

Case 2: *we assume* (4.0.1). We estimate the right-hand side of (4.1.6) from below as follows:

$$\begin{aligned}
\partial^2_{tx} W_\alpha &= -V(W_\alpha)\partial^2_{xx} W_\alpha + \delta(\partial_x W_\alpha)^2 - \frac{\delta}{\alpha} W_\alpha \partial_x W_\alpha \\
&\quad + \frac{\delta}{\alpha^2}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right) W_\alpha \partial_y W_\alpha \,\mathrm{d}y \\
&= -V(W_\alpha)\partial^2_{xx} W_\alpha + \delta(\partial_x W_\alpha)^2 - \frac{\delta}{\alpha} W_\alpha \partial_x W_\alpha \\
&\quad + \frac{\delta}{\alpha^2}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right) \Big(\alpha \partial_y W_\alpha(t,y) + \rho_\alpha(t,y)\Big)\partial_y W_\alpha \,\mathrm{d}y \\
&= -V(W_\alpha)\partial^2_{xx} W_\alpha + \delta(\partial_x W_\alpha)^2 - \frac{\delta}{\alpha} W_\alpha \partial_x W_\alpha \\
&\quad \underbrace{+ \frac{\delta}{\alpha}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right) (\partial_y W_\alpha)^2 \,\mathrm{d}y}_{\geq 0} + \frac{\delta}{\alpha^2}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right) \rho_\alpha \partial_y W_\alpha \,\mathrm{d}y \\
&\geq -V(W_\alpha)\partial^2_{xx} W_\alpha + \delta(\partial_x W_\alpha)^2 - \frac{\delta}{\alpha} W_\alpha \partial_x W_\alpha + \frac{\delta}{\alpha^2} m \int_x^\infty \exp\left(\frac{x-y}{\alpha}\right) \rho_\alpha \,\mathrm{d}y \\
&= -V(W_\alpha)\partial^2_{xx} W_\alpha + \delta(\partial_x W_\alpha)^2 - \frac{\delta}{\alpha} W_\alpha \partial_x W_\alpha + \frac{\delta}{\alpha} m W_\alpha.
\end{aligned}$$

Using [**98**, Theorem 2.1], we fix $\bar{x} = \bar{x}(t) \in \mathbb{R}$ such that $m(t) = \partial_x W_\alpha(t,\bar{x})$, evaluate the previous expression at $x = \bar{x}$, and deduce

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t) \geq \delta m(t)^2 - \frac{\delta}{\alpha} W_\alpha(t,\bar{x}) m(t) + \frac{\delta}{\alpha} m(t) W_\alpha(t,\bar{x}) = \delta m(t)^2, \qquad t > 0.$$

Conclusion. In both cases, we arrive at the Riccati-type differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t) \geq \kappa m^2(t), \qquad t > 0$$

(with $\kappa := (\kappa_1 - \kappa_2)$ or $\kappa := \delta$, respectively), which yields

$$\frac{W_\alpha(t,x) - W_\alpha(t,y)}{x-y} = \frac{1}{x-y}\int_y^x \partial_x W_\alpha(t,\xi)\,\mathrm{d}\xi \geq -\frac{1}{\kappa t}, \qquad t>0,\ x,y\in\mathbb{R},\ x\neq y.$$

□

4.1.2. Oleĭnik-type estimate for $V'(W_\alpha)W_\alpha\partial_x W_\alpha$. The basic idea underpinning the proof of the Oleĭnik inequality for $g_\alpha = V'(W_\alpha)W_\alpha\partial_x W_\alpha$ is to observe that this quantity satisfies the equation

$$\partial_t g_\alpha = (V''(W_\alpha)W_\alpha + V'(W_\alpha))\partial_x W_\alpha \partial_t W_\alpha + V'(W_\alpha)W_\alpha \partial^2_{tx} W_\alpha.$$

Proof of Theorem 4.0.2. Owing to Lemma 4.1.1, it suffices to prove the statement for initial data $\rho_0 \in \mathcal{D}\cap C^2(\mathbb{R})$ and therefore for solutions $\rho_\alpha \in C^2([0,T]\times\mathbb{R})$. The set $\mathcal{D}$ has been defined in (4.1.5).

For the sake of brevity, we set $z_\alpha := \partial_x W_\alpha$. By differentiating (4.0.9) with respect to x, we obtain the following equation for z_α:

$$\partial_t z_\alpha = -V(W_\alpha)\partial_x z - V'(W_\alpha)z^2 - g_\alpha * \partial_x\gamma_\alpha, \quad (t,x)\in(0,T)\times\mathbb{R}. \tag{4.1.7}$$

From (4.0.9), (4.1.7), and the fact that

$$\partial_x \gamma_\alpha = \frac{1}{\alpha}\left(\gamma_\alpha - \delta_0\right), \tag{4.1.8}$$

where γ_α is the same as in (4.0.10), we get

$$\begin{aligned}\partial_t g_\alpha &= (V''(W_\alpha)W_\alpha + V'(W_\alpha))z_\alpha\partial_t W_\alpha + V'(W_\alpha)W_\alpha\partial_t z_\alpha\\ &= h_\alpha z_\alpha\big(-V(W_\alpha)z_\alpha - g_\alpha*\gamma_\alpha\big)\\ &\quad + V'(W_\alpha)W_\alpha\left(-V(W_\alpha)\partial_x z_\alpha - V'(W_\alpha)z_\alpha^2 - \frac{1}{\alpha}\left(g_\alpha*\gamma_\alpha - g_\alpha\right)\right),\end{aligned} \tag{4.1.9}$$

where

$$h_\alpha := V''(W_\alpha)W_\alpha + V'(W_\alpha) \tag{4.1.10}$$

and

$$\partial_x g_\alpha = h_\alpha z_\alpha^2 + V'(W_\alpha)W_\alpha\partial_x z_\alpha. \tag{4.1.11}$$

We now separately consider two cases:

1. for every $t\in[0,T]$, there exists $x\in\mathbb{R}$ such that $g_\alpha(t,x)>0$;
2. there exists $t\in[0,T]$ such that $g_\alpha(t,x)\leq 0$ for every $x\in\mathbb{R}$.

Case 1. Owing to Lemma 4.1.1, we can assume, with no loss of generality, that, for every $\bar t>0$, we have $\rho_\alpha(\bar t,\cdot)\in\mathcal{D}\cap C^2(\mathbb{R})$ and hence $W_\alpha(\bar t,\cdot)\in\mathcal{D}\cap C^2(\mathbb{R})$. For every $\bar t\in[0,T)$, there exists a maximum point $\bar x$ of $g_\alpha(\bar t,\cdot)$. In particular, $\partial_x g_\alpha(\bar t,\bar x)=0$; by (4.1.11), we have

$$\partial_x z_\alpha(\bar t,\bar x) = -\frac{h_\alpha}{V'(W_\alpha)W_\alpha} z_\alpha^2(\bar t,\bar x). \tag{4.1.12}$$

Evaluating (4.1.9) at $(\bar t,\bar x)$, we get

$$\begin{aligned}\partial_t g_\alpha(\bar t,\bar x) &= \left(-h_\alpha z_\alpha g_\alpha*\gamma_\alpha - (V'(W_\alpha))^2 W_\alpha z_\alpha^2 - \frac{V'(W_\alpha)W_\alpha}{\alpha}\left(g_\alpha*\gamma_\alpha - g_\alpha\right)\right)(\bar t,\bar x)\\ &=: \mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned} \tag{4.1.13}$$

We observe that $\mathrm{III}\leq 0$ since $V'\leq 0$, $W_\alpha\geq 0$, and $\bar x$ is a maximum point of $g_\alpha(\bar t,\cdot)$. Moreover, by using the definition of g_α and the maximum principle, we get

$$\mathrm{II} = -\frac{g_\alpha^2}{W_\alpha} \leq -\frac{1}{\|\rho_0\|_{L^\infty(\mathbb{R})}} g_\alpha^2. \tag{4.1.14}$$

The term I is more delicate and can be controlled using the assumptions (4.0.4) or (4.0.5).

Case 1a. Under the assumption (4.0.4), we have $h_\alpha\leq 0$. Therefore, if $g_\alpha*\gamma_\alpha(\bar t,\bar x)\geq 0$, then $\mathrm{I}\leq 0$.

Otherwise, let us assume that $g_\alpha * \gamma_\alpha(\bar t, \bar x) < 0$. Since $z_\alpha = \rho_\alpha * \partial_x \gamma_\alpha$, then, by recalling (4.1.8), we arrive at

$$|z_\alpha| = \left| \frac{1}{\alpha} (\rho_\alpha * \gamma_\alpha - \rho_\alpha) \right| \le \frac{\operatorname{Osc} \rho_\alpha}{\alpha}; \tag{4.1.15}$$

therefore,

$$|h_\alpha z_\alpha g_\alpha * \gamma_\alpha(\bar t, \bar x)| = |\mathrm{I}| \le \frac{\operatorname{Osc} \rho_\alpha}{\alpha} |h_\alpha g_\alpha * \gamma_\alpha(\bar t, \bar x)| \le \frac{|V'(W_\alpha) W_\alpha|}{\alpha} |g_\alpha * \gamma_\alpha(\bar t, \bar x)| \le |\mathrm{III}|, \tag{4.1.16}$$

where we used (4.0.4) and $h_\alpha \le 0$ in the second inequality and $g_\alpha * \gamma_\alpha(\bar t, \bar x) < 0$ in the last inequality. In particular, this shows

$$\partial_t g_\alpha(\bar t, \bar x) \le -\frac{1}{\|\rho_0\|_{L^\infty(\mathbb{R})}} g_\alpha^2(\bar t, \bar x), \tag{4.1.17}$$

which, by comparison, yields the desired claim.

Case 1b. Under the assumption (4.0.5), we have $h_\alpha \ge 0$. In case $g_\alpha * \gamma_\alpha(\bar t, \bar x) \le 0$, then $\mathrm{I} \le 0$. We then focus on the case $g_\alpha * \gamma_\alpha(\bar t, \bar x) > 0$. Since $\bar x$ is a maximum point for $g_\alpha(\bar t, \cdot)$, then $g_\alpha * \gamma_\alpha(\bar t, \bar x) \le g_\alpha(\bar t, \bar x)$; hence

$$\begin{aligned}
\mathrm{I} + \mathrm{II} \le & -\left[h_\alpha z_\alpha g_\alpha + (V'(W_\alpha))^2 W_\alpha z_\alpha^2\right](\bar t, \bar x) \\
= & -W_\alpha V'(W_\alpha) z_\alpha^2 (V''(W_\alpha) W_\alpha + 2V'(W_\alpha))(\bar t, \bar x) \\
\le & -\kappa_1 W_\alpha (V'(W_\alpha))^2 z_\alpha^2(\bar t, \bar x) \\
= & -\frac{\kappa_1}{W_\alpha} g_\alpha(\bar t, \bar x)^2 \\
\le & -\frac{\kappa_1}{\|\rho_0\|_{L^\infty(\mathbb{R})}} g_\alpha(\bar t, \bar x)^2,
\end{aligned}$$

where, in the second inequality, we used (4.0.5). This establishes (4.1.17) which, by comparison, yields (4.0.6).

Case 2. We define $\bar t \in [0, T]$ by setting

$$\bar t := \inf\{t \in [0, T] : \; g_\alpha(t, x) \le 0 \text{ for every } x \in \mathbb{R}\}. \tag{4.1.18}$$

Assuming that $\bar t > 0$, we can apply the same argument as in **Case 1** on the interval $[0, \bar t)$. Since $t \mapsto \operatorname{Lip}^- \rho_\alpha(t)$ is a continuous function, then also $t \mapsto \max g_\alpha(t, \cdot)$ is continuous and this establishes (4.0.6) on $[0, \bar t]$. Note that $g_\alpha(t, x) \le 0$ for every $x \in \mathbb{R}$ if and only if $\rho_\alpha(t, \cdot)$ is non-decreasing. Therefore, since (1.1.1) preserves the monotonicity of the initial datum (see [**35, 167**]), then, for every $t \in (\bar t, T]$, $\rho_\alpha(t, \cdot)$ is a monotone non-decreasing function, that is $g_\alpha(t) \le 0$. If $\bar t = 0$, then we can directly apply the argument for the preservation of monotonicity. This concludes the proof. □

REMARK 4.1.3 (The Greenberg model). *Let us consider the velocity function $V(\xi) := v_0 \ln(\rho_{\max}/\xi)$ with $v_0 > 0$ and $\rho_{\max} > 0$, which corresponds to a traffic model proposed by Greenberg and supported by experimental data (see* [**138**, Chapter 3, Eq. (3.1.4)]*). Formally, an Oleĭnik-type estimate still holds: indeed, going back to* (4.1.13)*, we get $h_\alpha \equiv 0$; thus $I = 0$ therefore, since $III \le 0$ and* (4.1.14)*, it follows from* (4.1.13) *that*

$$\partial_t g_\alpha(\bar t, \bar x) \le -\frac{1}{\|\rho_0\|_{L^\infty(\mathbb{R})}} g_\alpha^2(\bar t, \bar x),$$

which, by comparison, implies (4.0.6)*. Assuming that the initial density is bounded away from zero, this remark can be made rigorous.*

4.2. Proof of the nonlocal–to–local convergence

As a first step towards proving Theorem 4.0.1, we point out that the Oleĭnik inequality in Theorem 4.0.1 implies a uniform $\mathrm{BV}_{\mathrm{loc}}$-estimate for $t > 0$ and, thus, compactness for $\{W_\alpha\}_{\alpha > 0}$.

LEMMA 4.2.1 ($\mathrm{BV}_{\mathrm{loc}}$-regularization and compactness). *Let us suppose that* (4.0.3) *holds. Then, the function $W_\alpha(t,\cdot)$ belongs to $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ for every $t>0$ uniformly with respect to $\alpha>0$: for every compact interval $K \Subset \mathbb{R}$,*

$$|W_\alpha(t,\cdot)|_{\mathrm{TV}(K)} \leq 2\left(\frac{|K|}{2t} + \|W_\alpha(t,\cdot)\|_{L^\infty(K)}\right). \tag{4.2.1}$$

As a result, the set $\{W_\alpha\}_{\alpha>0}$ of solutions to (4.0.8) *is compactly embedded into $L^1_{\mathrm{loc}}((0,T)\times\mathbb{R})$.*

PROOF. The claim in (4.2.1) is contained in [**38**, Eq. (4.3)] or [**39**, Lemma 2.2 (ii) and Remark 2.3] (the proof is also presented in CHAPTER 5). The second follows an argument as in Theorem 3.3.1. □

With Lemma 4.2.1 in hand, we can directly establish Corollary 4.0.1 under assumptions (4.0.2) or (4.0.1)—i.e. using the Oleĭnik inequality from Theorem 4.0.1—by arguing similarly as in CHAPTER 3. In fact, more simply, to prove that the limit point of $\{W_\alpha\}_{\alpha>0}$ is an entropy-admissible solution of the local conservation law (3.0.3), it suffices to pass to the limit pointwise in (4.0.3).

The proof of Theorem 4.0.1 under assumptions (4.0.4) or (4.0.5)—i.e., using the Oleĭnik inequality from Theorem 4.0.2—is somewhat more delicate. Indeed, we cannot directly deduce a uniform TV bound on $\{W_\alpha\}_{\alpha>0}$. In Lemma 4.2.2 below, we rather show that $\{W_\alpha^2\}_{\alpha>0}$ is equi-bounded in $\mathrm{BV}_{\mathrm{loc}}((0,T)\times\mathbb{R})$ and, therefore, that the family $\{W_\alpha\}_{\alpha>0}$ is precompact in $L^1_{\mathrm{loc}}((0,T)\times\mathbb{R})$ and that limit points W of $\{W_\alpha\}_{\alpha>0}$ as $\alpha\to 0^+$ are weak solutions of (3.0.3). The fact that the limit point of $\{W_\alpha\}_{\alpha>0}$ so constructed is an entropy-admissible solution of the local conservation law is already known (see CHAPTER 3). In Lemma 4.2.3, we present, however, another proof. We point out that the Oleĭnik-type inequality for W_α^2 rules out the presence of non-entropic shocks in the limit W. When W does not have bounded variation it is not trivial to deduce that it is, in fact, the entropy-admissible solution: we achieve this by exploiting the recent results of [**139, 198**] on Besov regularity and on the structure of solutions of conservation laws with finite entropy production. This seems to be of independent interest.

Finally, we need to show that ρ_α converges to the same limit as W_α. If we have a total variation bound on W_α, this follows immediately from the identity (3.2.2). In case the bound holds only for W_α^2, a more subtle analysis is needed, which we perform in Lemma 4.2.4.

LEMMA 4.2.2 (Precompactness in L^1). *Let us assume that* (4.0.6) *holds. Then the sequence $\{W_\alpha\}_{\alpha>0}$ is precompact in $L^1_{\mathrm{loc}}((0,T)\times\mathbb{R})$ and every limit point of W_α is a weak solution of* (3.0.3).

PROOF. **Step 1.** *Precompactness of W_α.* Since $V' \leq -\kappa_2 < 0$, then, from $g_\alpha(t,\cdot) \leq \frac{1}{\kappa t}$, we deduce

$$\partial_x W_\alpha^2(t,\cdot) \leq \frac{2}{\kappa_2 \kappa t} \tag{4.2.2}$$

and

$$\partial_t W_\alpha^2(t,\cdot) = -V(W_\alpha)\partial_x W_\alpha^2 - 2W_\alpha g_\alpha * \gamma_\alpha \geq -\frac{2V(0)+2\max\rho_0}{\kappa\kappa_2 t}$$

for $t>0$. In particular, this yields the result that W_α^2 is equi-bounded in $\mathrm{BV}_{\mathrm{loc}}((0,T)\times\mathbb{R})$. By Helly's compactness theorem, there is a subsequence $W_{\alpha_k}^2$ which converges a.e. to some function W^2 as $k\to+\infty$. Therefore, W_{α_k} converges to W a.e. and, by Lebesgue's dominated convergence theorem, $W_{\alpha_k}\to W$ in $L^1_{\mathrm{loc}}((0,T)\times\mathbb{R})$.

Step 2. *W is a weak solution of* (3.0.3). By (4.0.9), it suffices to show that $g_\alpha - g_\alpha * \gamma_\alpha \to 0$ in $\mathcal{D}'([0,T)\times\mathbb{R})$. Let us first fix $\varphi \in C_c^\infty((0,T)\times\mathbb{R})$, then

$$\int_0^T\int_{\mathbb{R}} \varphi(g_\alpha - g_\alpha*\gamma_\alpha)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{R}} \varphi g_\alpha * (\delta_0-\gamma_\alpha)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{R}} \varphi*(\delta_0-\tilde\gamma_\alpha) g_\alpha\,\mathrm{d}x\,\mathrm{d}t,$$

where $\tilde\gamma_\alpha(x) \coloneqq \gamma_\alpha(-x)$. Since $\varphi(t,\cdot)*(\delta_0-\tilde\gamma_\alpha)$ converges uniformly to 0 and decays exponentially in space uniformly in α and

$$\int_{-L}^{L} |g_\alpha(t,x)|\,\mathrm{d}x \leq \|V'\|_{L^\infty(\mathbb{R})} |W_\alpha^2(t,\cdot)|_{\mathrm{TV}([-L,L])}$$

grows at most linearly in L owing to (4.2.2), then, for every $\varphi \in C_c^\infty((0,T)\times\mathbb{R})$, we have

$$\lim_{\alpha\to 0}\int_0^T\int_{\mathbb{R}}\varphi(g_\alpha - g_\alpha * \gamma_\alpha)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

We now fix $\varphi \in C_c^\infty([0,T)\times\mathbb{R})$; since ρ_α solves (3.0.1), then the map

$$t \mapsto \int_{\mathbb{R}}\rho_\alpha(t,x)\varphi(t,x)\,\mathrm{d}x$$

is Lipschitz continuous with respect to t uniformly with respect to α on $[0,T)$. Therefore, the same is true if we replace ρ_α by $W_\alpha \coloneqq \rho_\alpha * \gamma_\alpha$. In particular, by (4.0.9), we have that

$$t \mapsto \int_{\mathbb{R}}(g_\alpha - g_\alpha * \gamma_\alpha)\varphi(t,x)\,\mathrm{d}x$$

is Lipschitz continuous with respect to t uniformly with respect to α on $[0,T)$. Hence $g_\alpha - g_\alpha*\gamma_\alpha \to 0$ in $\mathcal{D}'([0,T)\times\mathbb{R})$. □

LEMMA 4.2.3 (Entropy admissibility of the limit point). *If W is a limit point of $\{W_\alpha\}_{\alpha>0}$ then W is the entropy solution of* (3.0.3).

PROOF. We already know from Lemma 4.2.2 that W is a weak solution of (3.0.3). Moreover, since W is a limit point of W_α, then $W^2 \in \mathrm{BV}_{\mathrm{loc}}((0,T)\times\mathbb{R})$. We check that this implies $W \in B^{1/3,3}_{\infty,\mathrm{loc}}((0,T)\times\mathbb{R})$: indeed, given Ω compactly contained in $(0,T)\times\mathbb{R}$ and $h\in\mathbb{R}^2$ sufficiently small, we have

$$\int_\Omega |D_hW_\alpha|^3\,\mathrm{d}x \le \|\rho_0\|_{L^\infty(\mathbb{R})}\int_\Omega |D_hW_\alpha|^2\,\mathrm{d}x \le \|\rho_0\|_{L^\infty(\mathbb{R})}\int_{\Omega_h}|D_hW_\alpha^2| \le \|\rho_0\|_{L^\infty(\mathbb{R})}|h||W_\alpha^2|_{\mathrm{TV}(\Omega_h)},$$

where D_h denotes the (first-order) forward-difference operator, $\Omega_h \coloneqq \{(t,x)\in(0,T)\times\mathbb{R} : \mathrm{dist}(x,\Omega)\le|h|\}$, and we used $0\le W_\alpha\le\|\rho_0\|_{L^\infty(\mathbb{R})}$. Weak solutions W to Burgers equation belonging to $B^{1/3,3}_{\infty,\mathrm{loc}}((0,T)\times\mathbb{R})$ enjoy a kinetic formulation (see [**139**, Theorem 2.6]) and for every weak solution enjoying a kinetic formulation there are countably many Lipschitz continuous curves $X_n : [0,T)\to\mathbb{R}$ such that for every entropy–entropy-flux pair (η,q) and every $\varphi\in C_c^\infty((0,T)\times\mathbb{R};\mathbb{R}_+)$ we have

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}}\left(\eta(W)\partial_t\varphi + q(W)\partial_x\varphi\right)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad=\sum_{n=1}^\infty\int_0^T\varphi\left[q(W^+)-q(W^-)-\dot{X}_n(t)(\eta(W^+)-\eta(W^-))\right](t,X_n(t))\,\mathrm{d}t,\end{aligned}\tag{4.2.3}$$

where $W^\pm$ denotes the traces of W along X_n (see [**198**]). The uniform one-sided bound on g_α proven in Proposition 4.0.2 implies that, for every $n\in\mathbb{N}$ and a.e. $t\in(0,T)$, we have $W^+(t,X_n(t)+)\ge W^-(t,X_n(t)-)$. Since $\xi\mapsto\xi V(\xi)$ is concave, then it is well-known that the shocks with $W^+\ge W^-$ are entropic: namely, for every convex entropy η and every $W^-\le W^+$, we have

$$q(W^+)-q(W^-)-\dot{X}_n(t)(\eta(W^+)-\eta(W^-))\ge 0.$$

In particular, by (4.2.3), we have that W is the entropy solution of (3.0.3). □

LEMMA 4.2.4 (Convergence of ρ_α). *The family of functions $\{\rho_\alpha\}_{\alpha>0}$ converges to W in $L^1_{\mathrm{loc}}((0,T)\times\mathbb{R})$ as $\alpha\to 0^+$.*

PROOF. Owing to the specific choice of the weight γ_α, we have the relation

$$\rho_\alpha = W_\alpha - \alpha z_\alpha.\tag{4.2.4}$$

Therefore, by (4.2.2), we deduce

$$W_\alpha^2 - W_\alpha\rho_\alpha = W_\alpha(W_\alpha-\rho_\alpha) = \alpha_k W_\alpha z_\alpha = \frac{\alpha}{2}\partial_x W_\alpha^2 \to 0 \quad \text{in } L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}),$$

so that there exists a sequence $\alpha_k\to 0$ such that ρ_{α_k} converges to W a.e. in the set $\{W\ne 0\}$.

We now discuss the convergence on the set $\{W=0\}$. Given $\bar{t}, L>0$, let us define

$$A(\bar{t},L) \coloneqq \{(t,x)\in(0,T)\times\mathbb{R} : x\in(-L-V_{\max}(\bar{t}-t), L+V_{\max}(\bar{t}-t))\},$$

where $V_{\max} := \max V = V(0)$. Up to removing a negligible set of values for $\bar t$ and L, we can assume that $\mathcal{H}^1$-a.e. point in $\partial A(\bar t, L) \cap (0,T)\times\mathbb{R}$ is a Lebesgue point of W_{α_k} and ρ_{α_k} for every $k\in\mathbb{N}$. Taking a further subsequence of α_k, which we do not rename, we can assume that W_{α_k} converges to W a.e. in $(0,T)\times\mathbb{R}$.

Given $h>0$, let us consider an increasing function $\chi_h \in C^\infty(\mathbb{R})$ such that

$$\chi_h(x) = \begin{cases} 1 & \text{if } x \geq h, \\ 0 & \text{if } x \leq 0, \end{cases}$$

and the approximation φ_h of the characteristic function of $A(\bar t, L)$ defined by

$$\varphi_h(t,x) = \chi_h(\bar t - t)\chi_h(x + L + V_{\max}(\bar t - t))\chi_h(L + V_{\max}(\bar t - t) - x).$$

Testing (3.0.1) with φ_h and letting $h\to 0$, we get

$$\int_{-L-V_{\max}\bar t}^{L+V_{\max}\bar t} \rho_0(x)\,\mathrm{d}x - \int_{-L}^{L} \rho_\alpha(\bar t, x)\,\mathrm{d}x = \int_0^{\bar t} \mathcal{F}^+(\rho_\alpha)(t)\,\mathrm{d}t + \int_0^{\bar t} \mathcal{F}^-(\rho_\alpha)(t)\,\mathrm{d}t, \tag{4.2.5}$$

where

$$\begin{aligned} \mathcal{F}^+(\rho_\alpha)(t) &:= \left(\rho_\alpha V(W_\alpha) + V_{\max}\rho_\alpha\right)(t, L + V_{\max}(\bar t - t)), \\ \mathcal{F}^-(\rho_\alpha)(t) &:= \left(-\rho_\alpha V(W_\alpha) + V_{\max}\rho_\alpha\right)(t, -L - V_{\max}(\bar t - t)) \end{aligned}$$

are the exiting fluxes of the quantity ρ_α across the lateral boundaries of $A(\bar t, L)$. Since $\rho_{\alpha_k}\to W$ in the set $\{W\neq 0\}$ and $\rho_{\alpha_k}\geq 0$, then

$$\limsup_{k\to\infty} \int_{-L-V_{\max}\bar t}^{L+V_{\max}\bar t} \rho_0(x)\,\mathrm{d}x - \int_{-L}^{L} \rho_{\alpha_k}(\bar t, x)\,\mathrm{d}x \leq \int_{-L-V_{\max}\bar t}^{L+V_{\max}\bar t} \rho_0(x)\,\mathrm{d}x - \int_{-L}^{L} W(\bar t, x)\,\mathrm{d}x. \tag{4.2.6}$$

Similarly, observing that $\xi \mapsto \mathcal{F}^\pm(\xi)$ is increasing, we have

$$\liminf_{k\to\infty} \int_0^{\bar t} \mathcal{F}^+(\rho_{\alpha_k})(t)\,\mathrm{d}t + \int_0^{\bar t} \mathcal{F}^-(\rho_{\alpha_k})\,\mathrm{d}t \geq \int_0^{\bar t} \mathcal{F}^+(W)(t)\,\mathrm{d}t + \int_0^{\bar t} \mathcal{F}^-(W)(t)\,\mathrm{d}t. \tag{4.2.7}$$

Now let us test (4.0.9) with φ_h and let $\varepsilon\to 0$: since $g_\alpha - g_\alpha * \gamma_\alpha \to 0$ in the sense of distributions on $[0,T)\times\mathbb{R}$, we get

$$\int_{(0,T)\times\mathbb{R}} \Big(W\partial_t\varphi_h + WV(W)\partial_x\varphi_h\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} \rho_0(x)\varphi_h(0,x)\,\mathrm{d}x = 0.$$

Letting $h\to 0$, we thus obtain

$$\int_{-L-V_{\max}\bar t}^{L+V_{\max}\bar t} \rho_0(x)\,\mathrm{d}x - \int_{-L}^{L} W(\bar t, x)\,\mathrm{d}x = \int_0^{\bar t} \mathcal{F}^+(W)(t)\,\mathrm{d}t + \int_0^{\bar t} \mathcal{F}^-(W)(t)\,\mathrm{d}t. \tag{4.2.8}$$

Comparing (4.2.5) and (4.2.8), we get that the two inequalities (4.2.6) and (4.2.7) are actually equalities and the liminf and limsup are actually limits. In particular, since $\rho_{\alpha_k}\geq 0$, it follows from (4.2.6) and $\rho_{\alpha_k}\to W$ in $\{W\neq 0\}$ that

$$\lim_{k\to\infty} \int_{\{W=0\}\cap[-L,L]} \rho_{\alpha_k}(\bar t, x)\,\mathrm{d}x = 0$$

and therefore $\rho_{\alpha_k}(\bar t,\cdot)\to W_\alpha(\bar t,\cdot)$ in $L^1_{\mathrm{loc}}(\mathbb{R})$. Since the limit W does not depend on the subsequence α_k we are considering, we conclude that

$$\rho_\alpha \to W \qquad \text{in } L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}).$$

□

Remark 4.2.1 (Effect of a lower-bound on the density). *The proof of the convergence result is easier and self-contained if we also assume a lower-bound on the density:*

$$\operatorname{ess\,inf} \rho_0 \geq c_0 > 0. \tag{4.2.9}$$

From (4.2.9), *we can show*

$$\operatorname{ess\,inf} \rho_\alpha \geq \operatorname{ess\,inf} \rho_0 \geq c_0 > 0. \tag{4.2.10}$$

Let us note that, in this case, the generalized California model and the Greenberg model mentioned above (which are not Lipschitz continuous at zero density) are well-posed.

From (4.0.6), (4.2.10), *and the upper-bound* $V' \leq -\kappa_2$, *we deduce that, for every* $t > 0$,

$$\sup_{\mathbb{R}} \partial_x W_\alpha(t, \cdot) \geq -\frac{1}{\kappa\kappa_2 c_0 t}. \tag{4.2.11}$$

This implies that $W_\alpha \in \mathrm{BV}_{\mathrm{loc}}((0, +\infty) \times \mathbb{R})$ *uniformly with respect to* $\alpha > 0$. *In particular, let* W *be an accumulation point of* $\{W_\alpha\}_{\alpha>0}$ *as* $\alpha \to 0^+$ *in* $L^1_{\mathrm{loc}}((0, +\infty) \times \mathbb{R})$, *then* W *solves* (3.0.3) *and, since it is one-sided Lipschitz continuous, it coincides with the entropy solution* ρ.

In order to complete the proof, we only need to show that ρ_α *also converges to* ρ. *Since* $\rho_\alpha = W_\alpha - \alpha\partial_x W_\alpha$ *and* $\partial_x W_\alpha$ *is equi-bounded in* L^1_{loc}, *the two sequences* $\{\rho_\alpha\}_{\alpha>0}$ *and* $\{W_\alpha\}_{\alpha>0}$ *converge to the same limit function* ρ.

Proof of Corollary 4.0.1. We proceed according to the following steps.

Step 1. *Proof using Theorem 4.0.1.* We assume (4.0.3) and apply Lemma 4.2.1 to deduce that $\{W_\alpha\}_{\alpha>0}$ is compactly embedded in $L^1_{\mathrm{loc}}((0, T) \times \mathbb{R})$. Then, by arguing as in Chapter 3, we obtain that $\{W_\alpha\}_{\alpha>0}$ converges to the unique entropy solution of the local conservation law (3.0.3) and so does $\{\rho_\alpha\}_{\alpha>0}$. We only need to pay extra attention to the fact that the convergence holds on every compact set contained in the open set $t > 0$. To this end, given a parameter $n \in \mathbb{N}$ and a non-negative test function $\varphi \in C^\infty_c([0, +\infty) \times \mathbb{R}; \mathbb{R}_+)$, as in Chapter 3, by the compactness of $\{\rho_\alpha\}_{\alpha>0}$ in $L^1_{\mathrm{loc}}((0, T) \times \mathbb{R})$, we can pass to the limit in the entropy inequality as $\alpha \to 0^+$ and deduce

$$\begin{aligned} 0 \leq\ & \underbrace{\int_{1/n}^{T} \int_{\mathbb{R}} \big(\eta(\rho(t, x))\partial_t\varphi(t, x) + q(\rho(t, x))\partial_x\varphi(t, x)\big) \,\mathrm{d}x \,\mathrm{d}t}_{I_{1,n}} \\ & + \underbrace{\int_{0}^{1/n} \int_{\mathbb{R}} \big(\bar\eta(t, x)\partial_t\varphi(t, x) + \bar q(t, x)\partial_x\varphi(t, x)\big) \,\mathrm{d}x \,\mathrm{d}t}_{I_{2,n}} + \int_{\mathbb{R}} \eta(\rho_0(x))\varphi(0, x) \,\mathrm{d}x, \end{aligned}$$

where $\eta(\rho_\alpha) \overset{*}{\rightharpoonup} \bar\eta$ and $q(\rho_\alpha) \overset{*}{\rightharpoonup} \bar q$ in $L^\infty(\mathbb{R})$ by the uniform L^∞-bound on $\{\rho_\alpha\}_{\alpha>0}$. By letting $n \to \infty$, we then deduce

$$0 \leq \int_{0}^{T} \int_{\mathbb{R}} \big(\eta(\rho(t, x))\partial_t\varphi(t, x) + q(\rho(t, x))\partial_x\varphi(t, x)\big) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}} \eta(\rho_0(x))\varphi(0, x) \,\mathrm{d}x,$$

where we used the fact that $I_{2,n} \to 0$ because of the L^1-bound on the integrand.

Step 2. *Proof using Theorem 4.0.2.* We assume (4.0.6), then the claim follows by combining Lemmas 4.2.2, 4.2.3, and 4.2.4, and the computation above. □

4.3. Numerical experiments

In this Section, we illustrate the results of Theorem 4.0.1 and Theorem 4.0.2 with some numerical simulations. We rely on a non-dissipative solver based on characteristics (see [**213**, Chapter 3] and [**173**]). In particular, we consider the LWR–Greenshields velocity function $V(\xi) \coloneqq 1-\xi$; in Figure 4.1 and Figure 4.2 we show the behavior of $t \mapsto \partial_x W_\alpha(t, \cdot)$ for two types of initial data, continuous (Figure 4.1) and with a jump discontinuity (Figure 4.2). We present simulations for both the exponential weight (top row of Figures 4.1 and 4.2) and for a piecewise-constant weight $\gamma \coloneqq \alpha^{-1}\mathbb{1}_{(-\alpha,0)}$ (bottom row of Figures 4.1 and 4.2) which is not covered by the results of this Chapter; the same result appears to hold in this case too. Finally, in Figure 4.3, we highlight the BV-regularization effect on W_α provided by the Oleĭnik inequality.

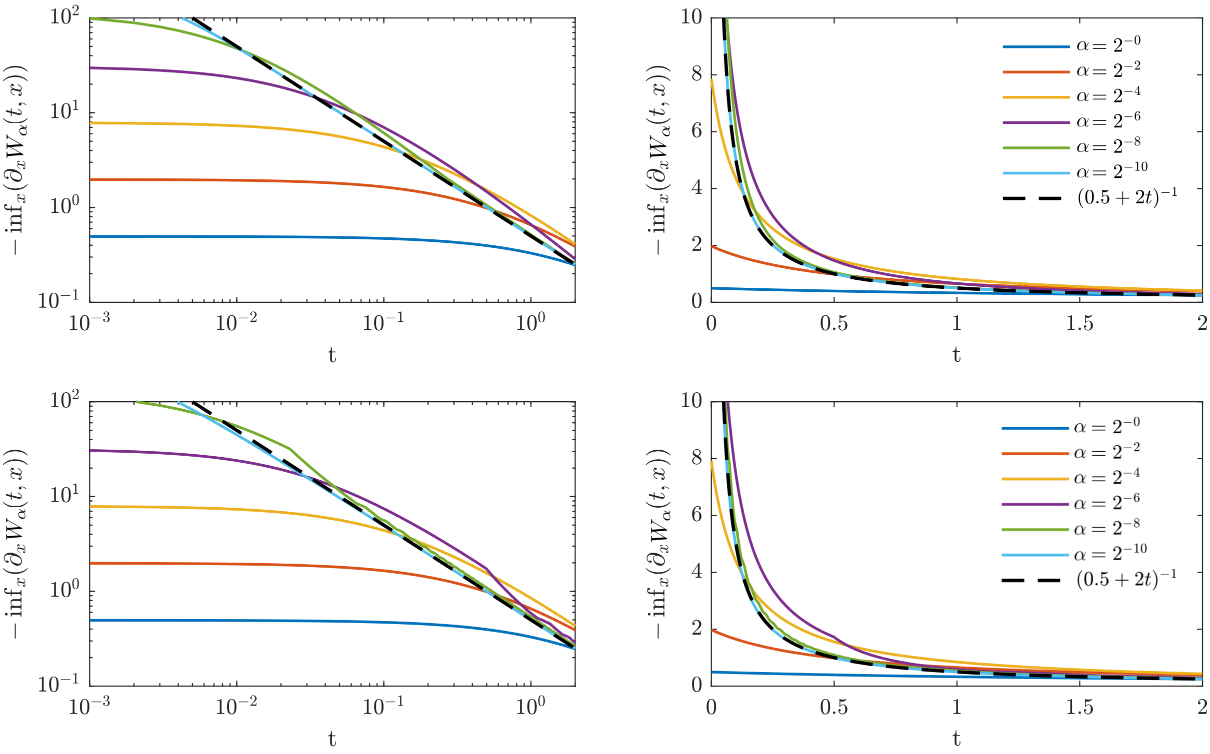

FIGURE 4.1. Illustration of $-\inf \partial_x W_\alpha$. Simulations for the initial datum $\rho_0(\cdot) := \frac{1}{2}\mathbb{1}_{(-0.5,0.5)}(\cdot)$ and the LWR–Greenshields velocity $V(\xi) := 1-\xi$. TOP ROW: weight $\gamma_\alpha(\cdot) := \alpha^{-1}\mathbb{1}_{(-\infty,0]}(\cdot)\exp(\cdot/\alpha)$. BOTTOM ROW: weight $\gamma_\alpha(\cdot) := \alpha^{-1}\mathbb{1}_{(-\alpha,0)}(\cdot)$.

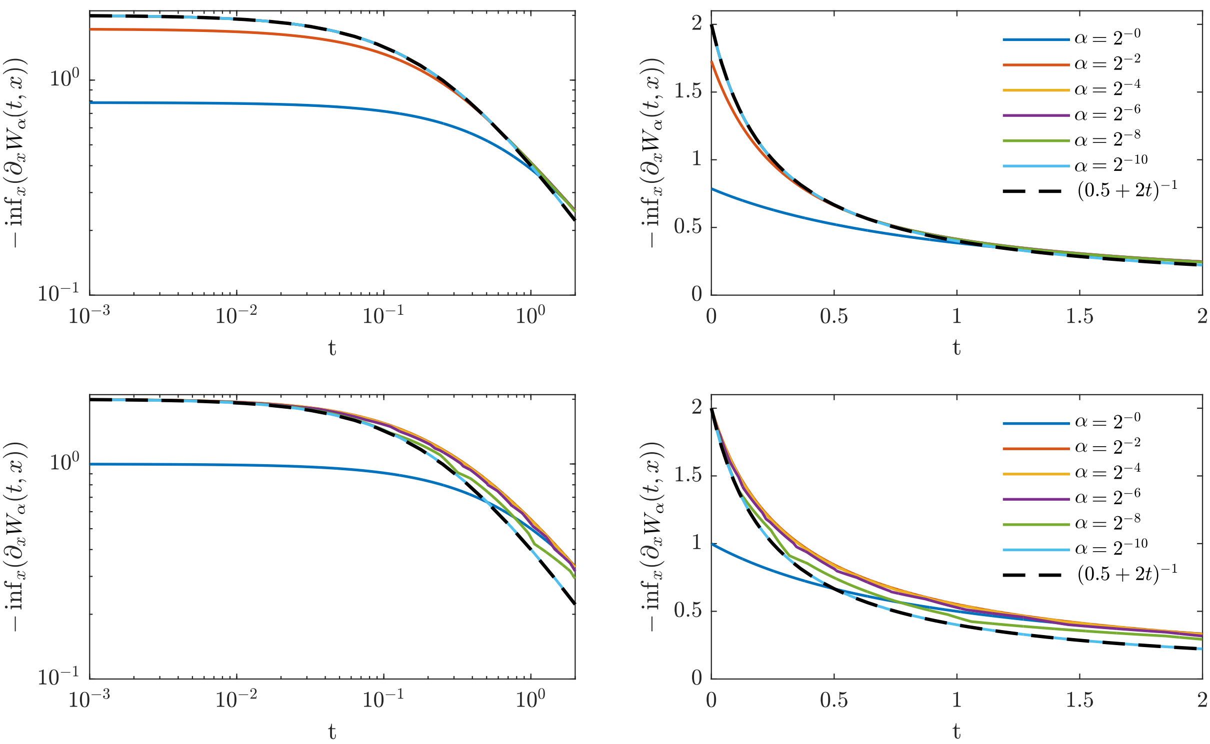

FIGURE 4.2. Illustration of $-\inf_{x\in\mathbb{R}} \partial_x W_\alpha(t,x)$. Simulations for the initial datum $\rho_0(\cdot) := (1-2|\cdot|)\mathbb{1}_{(-0.5,0.5)}(\cdot)$ and the LWR–Greenshields velocity $V(\xi) := 1-\xi$. TOP ROW: weight $\gamma_\alpha(\cdot) := \alpha^{-1}\mathbb{1}_{(-\infty,0]}(\cdot)\exp(\cdot/\alpha)$. BOTTOM ROW: weight $\gamma_\alpha(\cdot) := \alpha^{-1}\mathbb{1}_{(-\alpha,0)}(\cdot)$.

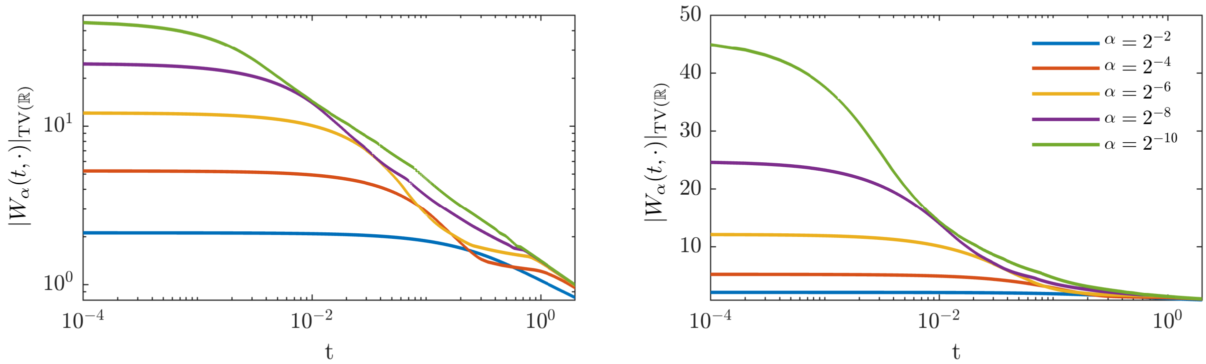

FIGURE 4.3. Illustration of $|W_\alpha(t,\cdot)|_{\mathrm{TV}(\mathbb{R})}$. Total variations of the nonlocal impact W_α for an initial datum with unbounded total variation, i.e., $\rho_0(\cdot) \coloneqq \sum_{n=1}^{\infty} \mathbb{1}_{(1/n+1,1/n+1+1/(2n(n+1)))}(\cdot)$, LWR–Greenshields velocity $V(\xi) \coloneqq 1-\xi$, and exponential weight, i.e., $\gamma_\alpha(\cdot) \coloneqq \alpha^{-1}\mathbb{1}_{(-\infty,0]}(\cdot)\exp(\cdot/\alpha)$.

CHAPTER 5

Long-time convergence of a nonlocal Burgers equation toward the local N-wave

This Chapter deals with the study of the long-time asymptotics for the nonlocal regularization of the Burgers equation

$$\begin{cases} \partial_t \rho(t,x) + \partial_x\big(W[\rho](t,x)\rho(t,x)\big) = 0, & (t,x) \in (0,+\infty)\times\mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{5.0.1}$$

with

$$W[\rho](t,x) := \int_{-\infty}^{x} \exp(y-x)\rho(t,y)\,\mathrm{d}y, \qquad (t,x)\in(0,+\infty)\times\mathbb{R}, \tag{5.0.2}$$

which also satisfies the identity

$$\partial_x W[\rho](t,x) = \rho(t,x) - W[\rho](t,x), \qquad (t,x)\in(0,+\infty)\times\mathbb{R}. \tag{5.0.3}$$

We assume that the initial data satisfies

$$\rho_0 \in L^1(\mathbb{R};\mathbb{R}_{\geq 0}) \cap L^\infty(\mathbb{R};\mathbb{R}_{\geq 0}) \tag{5.0.4}$$

and introduce the notation $M := \int_{\mathbb{R}} \rho_0(x)\,\mathrm{d}x$ for its L^1-mass.

Our main theorem can be stated as follows.

Theorem 5.0.1 (Long-time asymptotics). *Let ρ_0 satisfy assumption* (5.0.4)*. Let ρ be the unique weak solution of the nonlocal Burgers equation* (5.0.1) *and let W be the corresponding nonlocal term. Then, for $p\in[1,+\infty)$, we have*

$$\begin{aligned} &t^{\frac{1}{2}\left(1-\frac{1}{p}\right)}\|\rho(t,\cdot)-w(t,\cdot)\|_{L^p(\mathbb{R})} \to 0, \\ &t^{\frac{1}{2}\left(1-\frac{1}{p}\right)}\|W(t,\cdot)-w(t,\cdot)\|_{L^p(\mathbb{R})} \to 0 \qquad \text{as } t\to+\infty, \end{aligned} \tag{5.0.5}$$

where w denotes the unique entropy solution (N-wave solution) of the Burgers equation

$$\begin{cases} \partial_t w(t,x) + \partial_x(w^2(t,x)) = 0, & (t,x)\in(0,+\infty)\times\mathbb{R}, \\ w(0,x) = M\delta_{\{x=0\}}, & x\in\mathbb{R}, \end{cases} \tag{5.0.6}$$

which is given explicitly by

$$w(t,x) = \begin{cases} \dfrac{x}{2t} & \text{if } x \in (0,\sqrt{4Mt}), \\ 0 & \text{otherwise.} \end{cases} \tag{5.0.7}$$

As outlined in the introductory Chapter 1, for a given $\lambda > 0$, we consider the rescaled function

$$\rho_\lambda(t,x) := \lambda\rho(\lambda^2 t, \lambda x), \tag{5.0.8}$$

which solves

$$\begin{cases} \partial_t \rho_\lambda(t,x) + \partial_x\big(W_\lambda[\rho_\lambda](t,x)\rho_\lambda(t,x)\big) = 0, & (t,x)\in(0,+\infty)\times\mathbb{R}, \\ \rho_\lambda(0,x) = \rho_{0,\lambda}(x) := \lambda\rho_0(\lambda x), & x\in\mathbb{R} \end{cases} \tag{5.0.9}$$

with

$$W_\lambda[\rho_\lambda](t,x) := \lambda\int_{-\infty}^{x} \exp(\lambda(y-x))\rho_\lambda(t,y)\,\mathrm{d}y, \qquad (t,x)\in(0,+\infty)\times\mathbb{R}. \tag{5.0.10}$$

We recall the following well-posedness result and some fundamental properties of the solution from the previous Chapters.

THEOREM 5.0.2 (Existence and uniqueness of weak solutions and maximum principle). *Let assumptions* (5.0.4) *hold. Then, for every $\lambda > 0$, there exists a unique weak solution $\rho_\lambda \in C\big([0,+\infty); L^1(\mathbb{R})\big) \cap L^\infty((0,+\infty); L^\infty(\mathbb{R}))$ of the nonlocal Burgers equation* (5.0.9) *and the following maximum principle holds:*

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}} \rho_{0,\lambda}(x) \leq \rho_\lambda(t,x) \leq \|\rho_{0,\lambda}\|_{L^\infty(\mathbb{R})}, \quad (t,x) \in (0,+\infty)\times\mathbb{R}. \tag{5.0.11}$$

Moreover, for the nonlocal impact W_λ, the following properties hold:

(1) $W_\lambda \in W^{1,\infty}\left((0,+\infty)\times\mathbb{R}\right)$ *and* $\operatorname*{ess\,inf}_{x\in\mathbb{R}} \rho_{0,\lambda}(x) \leq W_\lambda \leq \|\rho_{0,\lambda}\|_{L^\infty(\mathbb{R})}$;

(2) $W_\lambda \in C^0\left((0,+\infty); L^1(\mathbb{R})\right)$; *in particular, if* $\|\rho_\lambda(t,\cdot)\|_{L^1(\mathbb{R})} = M$, *then* $\|W_\lambda(t,\cdot)\|_{L^1(\mathbb{R})} = M$;

(3) *if* $\rho_{0,\lambda} \in C^k(\mathbb{R})$, *then* $W_\lambda \in C^{k+1}\left((0,+\infty)\times\mathbb{R}\right)$ *for* $k \geq 0$.

Furthermore, W_λ solves the following Cauchy problem in the strong sense:

$$\begin{cases} \partial_t W_\lambda(t,x) + W_\lambda(t,x)\partial_x W_\lambda(t,x) \\ \quad = \lambda \displaystyle\int_{-\infty}^{x} \exp(\lambda(y-x)) W_\lambda(t,y)\partial_y W_\lambda(t,y)\,\mathrm{d}y, & (t,x)\in(0,+\infty)\times\mathbb{R}, \\ W_\lambda(0,x) = \lambda \displaystyle\int_{-\infty}^{x} \exp(\lambda(y-x))\rho_{0,\lambda}(y)\,\mathrm{d}y, & x\in\mathbb{R}. \end{cases} \tag{5.0.12}$$

For the limit problem (5.0.6), we rely on a more general well-posedness result from [**192**, Theorem 1.1 & Remark 1.1].

THEOREM 5.0.3 (Non-negative entropy solutions with measure initial data). *Let us consider the local conservation law*

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u(t,x)) = 0, & (t,x)\in(0,+\infty)\times\mathbb{R}, \\ u(0,x) = \mu, & x\in\mathbb{R}. \end{cases} \tag{5.0.13}$$

Let us assume that $f : \mathbb{R}\to\mathbb{R}$ is locally Lipschitz continuous with $f(0) = 0$ and $f([0,\infty)) \subset [0,\infty)$ and that μ is a non-negative finite measure on $\mathbb{R}$. Then there exists at most one non-negative solution $u \in C\left((0,+\infty); L^1(\mathbb{R})\right) \cap L^\infty((\tau,+\infty)\times\mathbb{R})$, for all $\tau\in(0,+\infty)$, which satisfies the Kružkov entropy condition, i.e.

$$\begin{gathered} \forall k\in\mathbb{R},\ \forall\varphi\in C_c^\infty((0,+\infty)\times\mathbb{R};\mathbb{R}_+): \\ \int_0^{+\infty}\int_{\mathbb{R}} \left(|u-k|\partial_t\varphi + \operatorname{sign}(u-k)(f(u)-f(k))\partial_x\varphi\right)\mathrm{d}x\,\mathrm{d}t \geq 0, \end{gathered}$$

and achieves the initial datum in the narrow (or weak) sense of measures[1],

$$\lim_{t\to 0} u(t,\cdot) = \mu \qquad \textit{narrowly in } \mathbb{R}.$$

In particular, in our setting, Theorem 5.0.3 yields the uniqueness of the N-wave entropy solution (5.0.7) of (5.0.6).

REMARK 5.0.1 (Non-negativity condition and uniqueness). *As noted in* [**192**, Remark 1.2], *the uniqueness result in Theorem 5.0.3 fails without the assumption of non-negativity for the solutions. This hypothesis can, however, be replaced by taking $f(\xi) := \operatorname{sign}(\xi)|\xi|^q$ (with $q > 1$) or by $f(\xi) := |\xi|^q$ and assuming that the initial datum is achieved in a stronger sense (as shown in* [**192**, Theorem 1.2] *and* [**192**, Theorem 1.3] *respectively).*

In Section 5.1, we obtain the key and a priori estimates on W_λ needed to prove Theorem 5.0.1. Then, in Section 5.2, we combine them and establish the convergence of $\{W_\lambda\}_{\lambda>0}$ and $\{\rho_\lambda\}_{\lambda>0}$ to the N-wave solution of the local Burgers equation as $\lambda\to+\infty$; or, equivalently, of $\{W(t,\cdot)\}_{t>0}$ and $\{\rho(t,\cdot)\}_{t>0}$ as $t\to+\infty$. This convergence result is illustrated by several numerical simulations in Section 5.3.

[1]A sequence of signed Radon measures $\{\mu_n\}_{n\in\mathbb{N}}$ on $\mathbb{R}$ *converges narrowly* (or *in the weak sense*) to μ if, for all bounded and continuous test functions $\varphi\in C_b(\mathbb{R})$, we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}} \varphi\,\mathrm{d}\mu_n = \int_{\mathbb{R}} \varphi\,\mathrm{d}\mu.$$

See [**37**, Chapter 8].

5.1. A priori estimates

Before presenting our key a priori estimates, let us recall the following stability result of the nonlocal conservation law (5.0.1) with respect to the initial datum.

LEMMA 5.1.1 (Stability of the nonlocal term with respect to the initial datum). *Let $\rho_{0,1}$, $\rho_{0,2} \in L^1(\mathbb{R})$ be given and denote by W_1, $W_2 \in L^\infty((0,T);W^{1,\infty}(\mathbb{R}))$ the nonlocal terms associated with the corresponding solutions of* (5.0.9). *Then, the following stability result holds: for all $t \in [0,T]$,*

$$\|W_1(t,\cdot)-W_2(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq C(\lambda, \|\rho_{0,1}\|_{L^\infty(\mathbb{R})}, \|\rho_{0,2}\|_{L^\infty(\mathbb{R})}, \|\rho_{0,1}\|_{L^1(\mathbb{R})}, \|\rho_{0,2}\|_{L^1(\mathbb{R})})\|\rho_{0,1}-\rho_{0,2}\|_{L^1(\mathbb{R})},$$

where C is a suitable constant that depends only on the quantities mentioned above.

PROOF. From the results in [**167**] (see also CHAPTER 2), we know that the solution of (5.0.9) can be written as

$$\rho_1(t,x) = \rho_{0,1}\left(\xi_{W_1}(t,x;0)\right)\partial_2\xi_{W_1}(t,x;0) \quad \text{and} \quad \rho_2(t,x) = \rho_{0,2}\left(\xi_{W_2}(t,x;0)\right)\partial_2\xi_{W_2}(t,x;0),$$

where ξ_{W_1} and ξ_{W_2} solve the characteristic ODEs

$$\begin{aligned} \xi_{W_1}(t,x;\tau) &= x + \int_t^\tau W_1(s,\xi_{W_1}(t,x;s))\,\mathrm{d}s, \quad \tau \in [0,T], \\ \xi_{W_2}(t,x;\tau) &= x + \int_t^\tau W_2(s,\xi_{W_2}(t,x;s))\,\mathrm{d}s, \quad \tau \in [0,T]. \end{aligned} \tag{5.1.1}$$

In particular, we recall that the nonlocal terms corresponding to the initial data $\rho_{0,1}$ and $\rho_{0,2}$ satisfy the following fixed-point equations for $(t,x) \in (0,T)\times\mathbb{R}$:

$$\begin{aligned} W_1(t,x) &= \lambda \int_{-\infty}^x \exp(\lambda(y-x))\rho_1(t,y)\,\mathrm{d}y \\ &= \lambda \int_{-\infty}^x \exp(\lambda(y-x))\rho_{0,1}(\xi_{W_1}(t,y;0))\partial_2\xi_{W_1}(t,y;0)\,\mathrm{d}y \\ &= \lambda \int_{-\infty}^{\xi_{W_1}(t,x;0)} \exp\Big(\lambda(\xi_{W_1}(0,z;t)-x)\Big)\rho_{0,1}(z)\,\mathrm{d}z; \\ W_2(t,x) &= \lambda \int_{-\infty}^{\xi_{W_2}(t,x;0)} \exp\Big(\lambda(\xi_{W_2}(0,z;t)-x)\Big)\rho_{0,2}(z)\,\mathrm{d}z. \end{aligned}$$

Taking the absolute value of the difference, we have

$$\begin{aligned} &\lambda^{-1}\big|W_1(t,x)-W_2(t,x)\big| \\ &= \left| \int_{-\infty}^{\xi_{W_1}(t,x;0)} \exp\big(\lambda(\xi_{W_1}(0,z;t)-x)\big)\rho_0(z)\,\mathrm{d}z \right. \\ &\qquad\qquad \left. - \int_{-\infty}^{\xi_{W_2}(t,x;0)} \exp\big(\lambda(\xi_{W_2}(0,z;t)-x)\big)\rho_{0,2}(z)\,\mathrm{d}z \right| \\ &\leq \int_{\min\{\xi_{W_1}(t,x;0),\xi_{W_2}(t,x;0)\}}^{\max\{\xi_{W_1}(t,x;0),\xi_{W_2}(t,x;0)\}} \big(|\rho_{0,1}(y)|+|\rho_{0,2}(y)|\big)\,\mathrm{d}y \\ &\quad + \int_{-\infty}^{\min\{\xi_{W_1}(t,x;0),\xi_{W_2}(t,x;0)\}} \Big(\exp\big(\lambda(\xi_{W_1}(0,z;t)-x)\big)\rho_{0,1}(z) \\ &\qquad\qquad - \exp\big(\lambda(\xi_{W_2}(0,z;t)-x)\big)\rho_{0,2}(z)\Big)\,\mathrm{d}z \\ &\leq |\xi_{W_1}(t,x;0)-\xi_{W_2}(t,x;0)|\big(\|\rho_{0,1}\|_{L^\infty(\mathbb{R})}+\|\rho_{0,2}\|_{L^\infty(\mathbb{R})}\big) \\ &\quad + \lambda\|\xi_{W_1}(0,\cdot;t)-\xi_{W_2}(0,\cdot;t)\|_{L^\infty(\mathbb{R})}\big(\|\rho_{0,1}\|_{L^1(\mathbb{R})}+\|\rho_{0,2}\|_{L^1(\mathbb{R})}\big)+\|\rho_{0,1}-\rho_{0,2}\|_{L^1(\mathbb{R})}. \end{aligned} \tag{5.1.2}$$

To conclude, we need to study the stability of the characteristics with respect to W_1 and W_2. For $(t,x,\tau) \in (0,T)\times\mathbb{R}\times(0,T)$, we compute

$$\big|\xi_{W_1}(t,x;\tau)-\xi_{W_2}(t,x;\tau)\big| = \left|\int_t^\tau W_1(s,\xi_{W_1}(t,x;s))-W_2(s,\xi_{W_2}(t,x;s))\,\mathrm{d}s\right|$$

$$
\begin{aligned}
&= \left| \int_t^\tau W_1(s, \xi_{W_1}(t,x;s)) - W_2(s, \xi_{W_1}(t,x;s))\,\mathrm{d}s \right| \\
&\quad + \left| \int_t^\tau W_2(s, \xi_{W_1}(t,x;s)) - W_2(s, \xi_{W_2}(t,x;s))\,\mathrm{d}s \right| \\
&\leq \int_{\min\{t,\tau\}}^{\max\{t,\tau\}} \|W_1(s,\cdot) - W_2(s,\cdot)\|_{L^\infty(\mathbb{R})}\,\mathrm{d}s \\
&\quad + \|\partial_x W_2\|_{L^\infty((0,T);L^\infty(\mathbb{R}))} \int_{\min\{t,\tau\}}^{\max\{t,\tau\}} \|\xi_{W_1}(t,\cdot;s) - \xi_{W_2}(t,\cdot;s)\|_{L^\infty(\mathbb{R})}\,\mathrm{d}s.
\end{aligned}
$$

Gronwall's inequality yields

$$
\begin{aligned}
&\left\|\xi_{W_1}(t,\cdot;\tau) - \xi_{W_2}(t,\cdot;\tau)\right\|_{L^\infty(\mathbb{R})} \\
&\leq \int_{\min\{t,\tau\}}^{\max\{t,\tau\}} \|W_1(s,\cdot) - W_2(s,\cdot)\|_{L^\infty(\mathbb{R})}\,\mathrm{d}s\ \exp\left(|t-\tau| \|\partial_x W_2\|_{L^\infty((0,T);L^\infty(\mathbb{R}))}\right).
\end{aligned}
$$

Plugging this into (5.1.2), we get

$$
\begin{aligned}
&\|W_1(t,\cdot) - W_2(t,\cdot)\|_{L^\infty(\mathbb{R})} \\
&\leq \lambda\big(\|\rho_{0,1}\|_{L^\infty(\mathbb{R})} + \|\rho_{0,2}\|_{L^\infty(\mathbb{R})}\big) \int_{\min\{t,\tau\}}^{\max\{t,\tau\}} \|W_1(s,\cdot) - W_2(s,\cdot)\|_{L^\infty(\mathbb{R})}\,\mathrm{d}s \\
&\qquad \times \exp\left(|t-\tau| \|\partial_x W_2\|_{L^\infty((0,T);L^\infty(\mathbb{R}))}\right) \\
&\quad + \lambda^2\big(\|\rho_{0,1}\|_{L^1(\mathbb{R})} + \|\rho_{0,2}\|_{L^1(\mathbb{R})}\big) \int_t^\tau \|W_1(s,\cdot) - W_2(s,\cdot)\|_{L^\infty(\mathbb{R})}\,\mathrm{d}s \\
&\qquad \times \exp\left(|t-\tau| \|\partial_x W_2\|_{L^\infty((0,T);L^\infty(\mathbb{R}))}\right) \\
&\quad + \|\rho_{0,1} - \rho_{0,2}\|_{L^1(\mathbb{R})}.
\end{aligned}
$$

Applying again Gronwall's inequality on $W_1 - W_2$ and recalling that

$$
\partial_x W_2 = \lambda(\rho_2 - W_2) \implies \|\partial_x W_2\|_{L^\infty((0,T);L^\infty(\mathbb{R}))} \leq 2\lambda \|\rho_{0,2}\|_{L^\infty(\mathbb{R})} \tag{5.1.3}
$$

(thanks to the maximum principle in Theorem 5.0.2), we conclude the proof. ☐

As a first step, we prove an Oleĭnik-type inequality on the nonlocal term W_λ. The result is essentially contained in CHAPTER 4. We present the proof below for the sake of completeness.

THEOREM 5.1.1 (Oleĭnik-type inequality for W_λ). *Given ρ_0 such that* (5.0.4) *holds, the solution W_λ of* (5.0.12) *satisfies*

$$
\frac{W_\lambda(t,x) - W_\lambda(t,y)}{x-y} \leq \frac{1}{t}, \qquad t > 0, \quad x, y \in \mathbb{R},\, x \neq y, \tag{5.1.4}
$$

for all $\lambda > 0$.

PROOF OF THEOREM 5.1.1. We consider a smoothed initial datum $\rho_{0,\lambda}^\varepsilon$ (for $\varepsilon > 0$) and call the corresponding smooth nonlocal term W_λ^ε. We then compute, differentiating the PDE in (5.0.12) with respect to x,

$$
\begin{aligned}
\partial_{tx}^2 W_\lambda^\varepsilon = &- W_\lambda^\varepsilon \partial_{xx}^2 W_\lambda^\varepsilon - (\partial_x W_\lambda^\varepsilon)^2 - \lambda W_\lambda^\varepsilon \partial_x W_\lambda^\varepsilon \\
&+ \lambda^2 \int_{-\infty}^x \exp(\lambda(y-x)) W_\lambda^\varepsilon(t,y) \partial_y W_\lambda^\varepsilon(t,y)\,\mathrm{d}y.
\end{aligned} \tag{5.1.5}
$$

For $t > 0$ fixed, considering $m(t) = \sup_{y\in\mathbb{R}} \partial_y W_\lambda^\varepsilon(t,y)$ and assuming—without loss of generality—that $m(t) \geq 0$, we estimate the right-hand side of (5.1.5) as follows:

$$
\partial_{tx}^2 W_\lambda^\varepsilon = -W_\lambda^\varepsilon \partial_{xx}^2 W_\lambda^\varepsilon - (\partial_x W_\lambda^\varepsilon)^2 - \lambda W_\lambda^\varepsilon \partial_x W_\lambda^\varepsilon + \lambda^2 \int_{-\infty}^x \exp(\lambda(y-x)) W_\lambda^\varepsilon(t,y) \partial_y W_\lambda^\varepsilon(t,y)\,\mathrm{d}y
$$

$$
\begin{aligned}
&= -W_\lambda^\varepsilon \partial_{xx}^2 W_\lambda^\varepsilon - (\partial_x W_\lambda^\varepsilon)^2 - \lambda W_\lambda^\varepsilon \partial_x W_\lambda^\varepsilon \\
&\qquad + \lambda^2 \int_{-\infty}^x \exp(\lambda(y-x))\Big(\rho_\lambda(t,y) - \lambda^{-1}\partial_y W_\lambda^\varepsilon(t,y)\Big)\partial_y W_\lambda^\varepsilon(t,y)\,\mathrm{d}y \\
&= -W_\lambda^\varepsilon \partial_{xx}^2 W_\lambda^\varepsilon - (\partial_x W_\lambda^\varepsilon)^2 - \lambda W_\lambda^\varepsilon \partial_x W_\lambda^\varepsilon \\
&\qquad + \lambda^2 \int_{-\infty}^x \exp(\lambda(y-x))\rho_\lambda^\varepsilon(t,y)\partial_y W_\lambda^\varepsilon(t,y)\,\mathrm{d}y \underbrace{-\lambda \int_{-\infty}^x \exp(\lambda(y-x))|\partial_y W_\lambda^\varepsilon(t,y)|^2\,\mathrm{d}y}_{\le 0} \\
&\le -W_\lambda^\varepsilon \partial_{xx}^2 W_\lambda^\varepsilon - (\partial_x W_\lambda^\varepsilon)^2 - \lambda W_\lambda^\varepsilon \partial_x W_\lambda^\varepsilon + m(t)\lambda^2 \underbrace{\int_{-\infty}^x \exp(\lambda(y-x))\rho_\lambda^\varepsilon(t,y)\,\mathrm{d}y}_{=\lambda^{-1}W_\lambda^\varepsilon(t,x)}
\end{aligned}
$$

We have that, for every $t > 0$, there exists a maximum point of $\partial_y W_\lambda^\varepsilon(t,y)$ (by choosing, e.g., a compactly supported $\rho_{0,\lambda}^\varepsilon$ and relying on the regularity results of [**167**]). Using [**98**, Theorem 2.1], we consider $\bar{x}(t) \in \mathbb{R}$ such that $m(t) = \partial_x W_\lambda^\varepsilon(t, \bar{x}(t))$, evaluate the previous expression at $x = \bar{x}(t)$, and compute

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t) \le -m^2(t).$$

Since $\widetilde{m}(t) = 1/t$ is a solution of the above Riccati-type differential inequality and $\widetilde{m}(0) = \infty$, we use the comparison principle for ODEs to conclude that $m(t) \le 1/t$ and thus

$$\frac{W_\lambda^\varepsilon(t,x) - W_\lambda^\varepsilon(t,y)}{x-y} = \frac{1}{x-y}\int_y^x \partial_x W_\lambda^\varepsilon(t,\xi)\,\mathrm{d}\xi \le \frac{1}{t}, \qquad t>0, \quad x,y \in \mathbb{R},\, x \ne y.$$

Taking the limit $\varepsilon \to 0^+$, thanks to Lemma 5.1.1, we conclude the proof. ☐

As a byproduct of (5.1.4), we prove (arguing as in [**127**, Lemma 1.2]) that an L^∞-bound holds for all $t > 0$ (which blows up as $t \to 0^+$).

Lemma 5.1.2 (L^∞-bound on W_λ). *The following L^∞-bounds on W_λ and ρ_λ hold:*

$$0 \le W_\lambda(t,x) \le \sqrt{\frac{2M}{t}}, \qquad (t,x) \in (0,+\infty)\times\mathbb{R}, \tag{5.1.6}$$

$$0 \le \rho_\lambda(t,x) \le \sqrt{\frac{2M}{t}} + \frac{1}{\lambda t}, \qquad (t,x) \in (0,+\infty)\times\mathbb{R}. \tag{5.1.7}$$

Proof. The fact that, for all $t > 0$, $W_\lambda(t,\cdot), \rho_\lambda(t,\cdot) \ge 0$ holds is contained in point (1) of Theorem 5.0.2. To prove the upper-bound in (5.1.6), let us fix a time $t > 0$ and a point $\bar{x} \in \mathbb{R}$. By Lemma 5.1.1, we have

$$W_\lambda(t,x) \ge W_\lambda(t,\bar{x}) - \frac{1}{t}(\bar{x} - x), \qquad \text{for all } x \le \bar{x},$$

i.e.,

$$W_\lambda(t,x) \ge \frac{1}{t}(x - (\bar{x} + W_\lambda(t,\bar{x})t)), \qquad \text{for all } 0 \le x - (\bar{x} + W_\lambda(t,\bar{x})t) \le W_\lambda(t,\bar{x})t.$$

Integrating over $\mathbb{R}$, we deduce

$$
\begin{aligned}
M = \int_{\mathbb{R}} W_\lambda(t,x)\,\mathrm{d}x &\ge \int_{\mathbb{R}\cap\{x \ge \bar{x} + W_\lambda(t,\bar{x})t\}} \frac{x - (\bar{x} + W_\lambda(t,\bar{x})t)}{t}\,\mathrm{d}x \\
&\ge \int_0^{W_\lambda(t,\bar{x})t} \left(\frac{x}{t}\right)\,\mathrm{d}x = \frac{1}{2}W_\lambda^2(t,\bar{x})\,t,
\end{aligned}
$$

which implies

$$W_\lambda(t,\bar{x}) \le \sqrt{\frac{2M}{t}} \qquad \text{for all } t>0,\ \bar{x} \in \mathbb{R}.$$

The bound (5.1.7) follows from (5.1.6) and Theorem 5.1.1. Indeed, by (5.0.10), we have

$$\begin{aligned} 0 \le \rho_\lambda(t,x) &= W_\lambda(t,x) + \frac{1}{\lambda}\partial_x W_\lambda(t,x) \\ &\le \sqrt{\frac{2M}{t}} + \frac{1}{\lambda t}. \end{aligned}$$

□

As a second corollary, from (5.1.4), we deduce the following $\mathrm{BV}_{\mathrm{loc}}$-regularization result (see [**38**, Eq. (4.3)] and [**39**, Lemma 2.2 (ii) & Remark 2.3] and Lemma 4.2.1).

Corollary 5.1.1 ($\mathrm{BV}_{\mathrm{loc}}$-regularization effect). *The function $W_\lambda(t,\cdot)$ belongs to $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ for every $t>0$ and uniformly with respect to $\lambda>0$: namely, for every compact interval $K \Subset \mathbb{R}$,*

$$|W_\lambda(t,\cdot)|_{\mathrm{TV}(K)} \le 2\left(\frac{|K|}{t} + \|W_\lambda(t,\cdot)\|_{L^\infty(K)}\right), \quad t>0. \tag{5.1.8}$$

Proof. Let $K := [a,b] \Subset \mathbb{R}$ be a compact interval of $\mathbb{R}$ and fix $t>0$. Since $W_\lambda(t,\cdot) \in L^\infty(K) \subset L^1(K)$, we only need to prove, thanks to the characterization of BV functions in [**228**, Lemma 37.4] (see also [**8**, Remark 2.5 & Exercise 3.3] or [**184**, Corollary 2.17]), that there exists $C>0$ such that

$$\int_{K_h} \frac{|W_\lambda(x+h)-W_\lambda(x)|}{h}\,\mathrm{d}x \le C, \qquad \forall h>0, \quad \text{where } K_h := \{x\in K : x+h \in K\}.$$

Taking Lemma 5.1.1 into account, we note that

$$\frac{W_\lambda(t,x+h)-W_\lambda(t,x)}{h} = \frac{1}{t} - \underbrace{\left(\frac{1}{t} - \frac{W_\lambda(t,x+h)-W_\lambda(t,x)}{h}\right)}_{\ge 0}, \tag{5.1.9}$$

which implies

$$\frac{W_\lambda(t,x+h)-W_\lambda(t,x)}{h} \le \frac{1}{t} + \left(\frac{1}{t} - \frac{W_\lambda(t,x+h)-W_\lambda(t,x)}{h}\right).$$

Integrating over K_h and taking the absolute values on both sides yields

$$\begin{aligned} \int_{K_h} \frac{|W_\lambda(t,x+h)-W_\lambda(t,x)|}{h}\,\mathrm{d}x &\le \int_{K_h} \left(\frac{1}{t} + \left(\frac{1}{t} - \frac{W_\lambda(t,x+h)-W_\lambda(t,x)}{h}\right)\right)\mathrm{d}x \\ &= 2\int_{K_h} \frac{1}{t}\,\mathrm{d}x + \int_{\mathbb{R}} W_\lambda(t,x)\left(\frac{\mathbb{1}_K(x+h)-\mathbb{1}_K(x)}{h}\right)\mathrm{d}x \\ &\le 2\int_{K_h} \frac{1}{t}\,\mathrm{d}x + \|W_\lambda(t,\cdot)\|_{L^\infty(K)} \underbrace{\int_{\mathbb{R}} \left(\frac{|\mathbb{1}_K(x+h)-\mathbb{1}_K(x)|}{h}\right)\mathrm{d}x}_{=\mathrm{TV}(\mathbb{1}_K)=2} \\ &= 2\left(\frac{|K|}{t} + \|W_\lambda(t,\cdot)\|_{L^\infty(K)}\right). \end{aligned}$$

□

5.2. Long-time behavior

As a first step toward finishing the proof of Theorem 5.0.1, we show that $\{W_\lambda\}_{\lambda>0}$ is compact in the canonical $C\big([t_0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$ topology. We note that the time-interval does not include $t=0$ because the L^∞ estimate from Lemma 5.1.2 blows up as $t\to 0^+$.

Lemma 5.2.1 (Compactness of $\{W_\lambda\}_{\lambda>0}$ in $C\big([t_0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$). *Let $t_0,\ T>0$ be fixed. The set $\{W_\lambda\}_{\lambda>0} \subseteq C\big([t_0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$ of solutions to (5.0.1) is compactly embedded into $C\big([t_0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$, i.e.*

$$\Big\{W_\lambda \in C\big([t_0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big) : \ W_\lambda \text{ satisfies (5.0.10)},\ \lambda>0\Big\} \overset{c}{\hookrightarrow} C\big([t_0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big).$$

Proof. Arguing as Chapter 3, we shall apply the compactness result in [**224**, Lemma 1]: given a Banach space B, a set $F \subset C([t_0,T];B)$ is relatively compact in $C([t_0,T];B)$ if

1. $F(t) := \{f(t) \in B : f \in F\}$ is relatively compact in B for all $t \in [t_0, T]$;
2. F is uniformly equi-continuous, i.e.

$$\forall \sigma > 0\ \exists \delta > 0 \text{ s. t. } \forall f \in F\ \forall (t_1, t_2) \in [t_0, T]^2 : |t_1 - t_2| \leq \delta \implies \|f(t_1) - f(t_2)\|_B \leq \sigma.$$

In our case, let us fix a compact interval $K \Subset \mathbb{R}$ and define $B := L^1(K)$ and $F(t) := \{W_\lambda(t, \cdot) \in L^1(K) : \lambda > 0\}$.

Thanks to Lemma 5.1.1, we know that $W_\lambda(t, \cdot)$ has a uniform total variation bound and, by Helly's compactness theorem (see [**184**, Theorem 13.35]),

$$F(t) \overset{\mathrm{c}}{\hookrightarrow} L^1(K), \qquad t \in [t_0, T].$$

It remains to show the second point, the uniform equi-continuity. To this end, we again smooth the initial datum $\rho_{0,\lambda}$ as $\rho^{\varepsilon}_{0,\lambda}$, with $\varepsilon > 0$, and call the corresponding smooth nonlocal term W^{ε}_λ. Then, we can estimate

$$\begin{aligned}
\left\|W^{\varepsilon}_\lambda(t_1, \cdot) - W^{\varepsilon}_\lambda(t_2, \cdot)\right\|_{L^1(\mathbb{R})} &= \left\|\int_{t_2}^{t_1} \partial_s W^{\varepsilon}_\lambda(s, \cdot)\,\mathrm{d}s\right\|_{L^1(\mathbb{R})} \\
&\leq \left\|\int_{t_2}^{t_1} W^{\varepsilon}_\lambda(s, \cdot)\partial_2 W^{\varepsilon}_\lambda(s, \cdot)\,\mathrm{d}s\right\|_{L^1(\mathbb{R})} \\
&\quad + \left\|\int_{t_2}^{t_1} \lambda \int_*^{\infty} \exp(\lambda(* - y))\partial_y W^{\varepsilon}_\lambda(s, y) W^{\varepsilon}_\lambda(s, y)\,\mathrm{d}y\,\mathrm{d}s\right\|_{L^1(\mathbb{R})} \\
&\leq \|W^{\varepsilon}_\lambda\|_{L^\infty((0,T);L^\infty(\mathbb{R}))} |W^{\varepsilon}_\lambda|_{L^\infty((0,T);\mathrm{TV}(\mathbb{R}))} |t_1 - t_2| \\
&\quad + \|W^{\varepsilon}_\lambda\|_{L^\infty((0,T);L^\infty(\mathbb{R}))} |W^{\varepsilon}_\lambda|_{L^\infty((0,T);\mathrm{TV}(\mathbb{R}))} |t_1 - t_2|,
\end{aligned}$$

where we used Fubini–Tonelli's theorem to exchange the order of integration and estimate the last term. Thanks to Lemmas 5.1.1 and 5.1.2, we have that this is a uniform bound in $\lambda > 0$ and $\varepsilon > 0$. This yields the uniform equi-continuity so that we obtain indeed the claimed compactness.

□

We can now complete the proof of Theorem 5.0.1 arguing as in [**127**, Section 2].

Proof of Theorem 5.0.1. The core of the proof consists in showing that the family $\{\rho_\lambda\}_{\lambda>0}$ converges to the N-wave defined in (5.0.7). We shall divide the argument of this theorem in several steps.

Step 1. *Compactness of the family* $\{W_\lambda\}_{\lambda>0}$ *in* $C\big([t_0, T]; L^1_{\mathrm{loc}}(\mathbb{R})\big)$. For any $0 < t_0 < T$, by Lemma 5.2.1, we have that W_λ converges (up to extracting a subsequence) to a limit point w^* strongly in $C\big([t_0, T]; L^1_{\mathrm{loc}}(\mathbb{R})\big)$; hence, we also have $W_\lambda(t, \cdot) \to w^*(t, \cdot)$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ for all $t \in [t_0, T]$ and $W_\lambda \to w^*$ pointwise (again up to subsequences) for all $t \in [t_0, T]$ and a.e. $x \in \mathbb{R}$.

Thanks to (5.0.10), we can deduce that ρ_λ also converges to w^* along the same subsequence. Indeed, first we observe that

$$\|W_\lambda(t, \cdot) - \rho_\lambda(t, \cdot)\|_{L^1(\mathbb{R})} = \lambda^{-1} |W_\lambda(t, \cdot)|_{\mathrm{TV}(\mathbb{R})}$$

and thus we also obtain

$$\lim_{\lambda \to +\infty} \|\rho_\lambda - w\|_{C([t_0,T];L^1_{\mathrm{loc}}(\mathbb{R}))} = 0.$$

Step 2a. *Tail control and convergence of the family* $\{\rho_\lambda\}_{\lambda>0}$ *in* $C([t_0, T], L^1(\mathbb{R}))$. In order to pass from the convergence $\rho_\lambda \to w^*$ strongly in $C\big([t_0, T]; L^1_{\mathrm{loc}}(\mathbb{R})\big)$ to the convergence in $C\big([t_0, T]; L^1(\mathbb{R})\big)$, we need a uniform bound on the "tail" of the functions $\{\rho_\lambda\}_{\lambda>1}$. We shall prove that there exists a constant $C = C(M)$ such that

$$\int_{\{|x|>2R\}} \rho_\lambda(t, x)\,\mathrm{d}x \leq \int_{\{|x|>R\}} \rho_0(x)\,\mathrm{d}x + \frac{C(M)}{R} t^{1/2}, \qquad t > 0. \tag{5.2.1}$$

Since $\rho_0 \in L^1(\mathbb{R})$, the right-hand side of (5.2.1) can be made arbitrarily small choosing R large enough. Then, from (5.2.1), the convergence

$$\rho_\lambda \to w^* \qquad \text{strongly in } C([t_0, T], L^1(\mathbb{R})) \quad \text{as } \lambda \to +\infty$$

follows by considering the splitting

$$\int_{\mathbb{R}} |\rho_\lambda(t,x) - w^*(t,x)|\,\mathrm{d}x = \int_{\{x<2R\}} |\rho_\lambda(t,x) - w^*(t,x)|\,\mathrm{d}x + \int_{\{x>2R\}} |\rho_\lambda(t,x) - w^*(t,x)|\,\mathrm{d}x.$$

In order to prove (5.2.1), let us consider a test function $\varphi \in C^\infty(\mathbb{R})$ such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ for $|x| > 2$, and $\varphi \equiv 0$ for $|x| \le 1$; we consider the rescaling $\varphi_R := \varphi(\cdot/R)$ which satisfies $\|\partial_x \varphi_R\|_{L^\infty(\mathbb{R})} \le C/R$ for some $C > 0$. Let us multiply the PDE in (5.0.9) by φ_R, integrate in $(0,t)\times\mathbb{R}$ (for some $t > 0$), and perform an integration by parts (to rigorously justify this computation, we can use a smoothing argument based on Lemma 5.1.1):

$$\int_{\mathbb{R}} \rho_\lambda(t,x)\varphi_R(x)\,\mathrm{d}x = \int_{\mathbb{R}} \rho_\lambda(0,x)\varphi_R(x)\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}} \rho_\lambda(s,x) W_\lambda(s,x)\partial_x\varphi_R(x)\,\mathrm{d}x\,\mathrm{d}s.$$

We remark that

$$\begin{aligned}\int_{\mathbb{R}} \rho_\lambda(0,x)\varphi_R(x)\,\mathrm{d}x &= \int_{\{|x|\ge R\}} \rho_\lambda(0,x)\,\mathrm{d}x \\ &= \int_{\{|x|\ge \lambda R\}} \rho(0,x)\,\mathrm{d}x \le \int_{\{|x|>R\}} \rho(0,x)\,\mathrm{d}x,\end{aligned}$$

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} \rho_\lambda(s,x)W_\lambda(s,x)\partial_x\varphi_R(x)\,\mathrm{d}x\,\mathrm{d}s &\le \|\partial_x\varphi_R\|_{L^\infty(\mathbb{R})} \int_0^t \|\rho_\lambda(s,\cdot)\|_{L^1(\mathbb{R})}\|W_\lambda(s,\cdot)\|_{L^\infty(\mathbb{R})}\,\mathrm{d}s \\ &\le \frac{C}{R}\int_0^t M\sqrt{\frac{2M}{s}}\,\mathrm{d}s \le \frac{C}{R}\frac{1}{\sqrt{2}}M^{3/2}t^{1/2},\end{aligned}$$

where we used Lemma 5.1.2 in the last line.

Step 2b. *Tail control and convergence of the family $\{W_\lambda\}_{\lambda>0}$ in $C([t_0,T],L^1(\mathbb{R}))$.* Since

$$\int_{\{|x|>2R\}} W_\lambda(t,x)\,\mathrm{d}x = \lambda \int_{\{|x|>2R\}} \int_{-\infty}^x \exp(\lambda(y-x))\rho(t,y)\,\mathrm{d}y\,\mathrm{d}x,$$

we use Fubini–Tonelli's theorem to deduce

$$\int_{\{|x|>2R\}} W_\lambda(t,x)\,\mathrm{d}x \le \int_{\{|x|>2R\}} \rho_\lambda(t,x)\,\mathrm{d}x, \qquad t > 0,$$

which yields, thanks to (5.2.1),

$$\int_{\{|x|>2R\}} W_\lambda(t,x)\,\mathrm{d}x \le \int_{\{|x|>R\}} \rho_0(x)\,\mathrm{d}x + \frac{C(M)}{R}t^{1/2}, \qquad t > 0. \tag{5.2.2}$$

As a byproduct of Steps 1 and 2, we note that the limit point w^* satisfies

$$w^* \in C((0,+\infty); L^1(\mathbb{R};\mathbb{R}_{\ge 0})) \cap L^\infty((\tau,+\infty); L^\infty(\mathbb{R};\mathbb{R}_{\ge 0})) \text{ for all } \tau > 0, \qquad \int_{\mathbb{R}} w^*(t,x)\,\mathrm{d}x = M.$$

Step 3. *Identification of the initial condition.* We now identify the initial datum taken by the limit point w^*, i.e., we verify that the initial condition $M\delta_0$ is achieved in the weak sense of non-negative measures on $\mathbb{R}$. We need to prove that, for all $\varphi \in C_b(\mathbb{R})$,

$$\lim_{t\to 0^+} \int_{\mathbb{R}} w^*(t,x)\varphi(x)\,\mathrm{d}x = M\varphi(0).$$

To this end, arguing as in [**127**, pp. 52–54], we shall split the argument into two steps. First, we consider a smaller class of test functions $\varphi \in C_c^\infty(\mathbb{R};[0,1])$ and, secondly, $\varphi \in C_b(\mathbb{R})$.

We start by estimating, for a test function $\varphi \in C_c^\infty(\mathbb{R};[0,1])$,

$$\begin{aligned}&\left|\int_{\mathbb{R}} \rho_\lambda(t,x)\varphi(x)\,\mathrm{d}x - \int_{\mathbb{R}} \rho_\lambda(0,x)\varphi(x)\,\mathrm{d}x\right| \\ &\le \left|\int_0^t \int_{\mathbb{R}} \partial_x\varphi(x)W_\lambda(s,x)\rho_\lambda(s,x)\,\mathrm{d}x\,\mathrm{d}s\right| \\ &\le \|\partial_x\varphi\|_{L^\infty(\mathbb{R})} \int_0^t \|\rho_\lambda(s,\cdot)\|_{L^1(\mathbb{R})}\|W_\lambda(s,\cdot)\|_{L^\infty(\mathbb{R})}\,\mathrm{d}s\end{aligned}$$

$$\leq \|\partial_x\varphi\|_{L^\infty(\mathbb{R})} C(M) \int_0^t s^{-1/2}\,\mathrm{d}s \leq C(M)\|\partial_x\varphi\|_{L^\infty(\mathbb{R})}\sqrt{t}.$$

Then, letting $\lambda \to +\infty$, we obtain

$$\left|\int_{\mathbb{R}} w^*(t,x) - M\varphi(0)\right| \leq C(M)\|\partial_x\varphi\|_{L^\infty(\mathbb{R})}\sqrt{t},$$

which, in turn, goes to zero as $t \to 0^+$.

As a second step, let us consider the case of a bounded continuous function $\varphi \in C_b(\mathbb{R})$. We shall rely on an approximation argument and on the tail control of ρ_λ in (5.2.1). Let us consider a regularized test function φ_ε obtained as $\varphi_\varepsilon := \varphi * \eta_\varepsilon$ (where η_ε denotes a standard mollifier; see [**128**, Appendix C.4]), such that $\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R})} \leq \|\varphi\|_{L^\infty(\mathbb{R})}$, $\varphi_\varepsilon \to \varphi$ uniformly on compact sets of $\mathbb{R}$ as $\varepsilon \to 0^+$, and $\|\varphi_\varepsilon\|_{W^{1,\infty}(\mathbb{R})} \leq C(\varepsilon)$. We then write

$$\begin{aligned}\left|\int_{\mathbb{R}} \rho_\lambda(t,x)\varphi(x)\,\mathrm{d}x - M\varphi(0)\right| &\leq \left|\int_{\mathbb{R}} \rho_\lambda(t,x)\varphi_\varepsilon(x)\,\mathrm{d}x - M\varphi(0)\right| \\ &\quad + \left|\int_{\{|x|>2R\}} \rho_\lambda(t,x)\left(\varphi(x)-\varphi_\varepsilon(x)\right)\mathrm{d}x\right| + \left|\int_{\{|x|<2R\}} \rho_\lambda(t,x)\left(\varphi(x)-\varphi_\varepsilon(x)\right)\mathrm{d}x\right|.\end{aligned}$$

The control of the first term follows by the same argument developed above. For the second and third term, we estimate

$$\begin{aligned}\left|\int_{\{|x|>2R\}} \rho_\lambda(t,x)\left(\varphi(x)-\varphi_\varepsilon(x)\right)\mathrm{d}x\right| &\leq 2\|\varphi\|_{L^\infty(\mathbb{R})}\left(\int_{\{|x|>R\}} \rho_0(x)\,\mathrm{d}x + \frac{C(M)}{R}t^{1/2}\right), \\ \left|\int_{\{|x|<2R\}} \rho_\lambda(t,x)\left(\varphi(x)-\varphi_\varepsilon(x)\right)\mathrm{d}x\right| &\leq \|\varphi-\varphi_\varepsilon\|_{L^\infty(\{|x|<2R\})}\int_{\mathbb{R}} \rho_\lambda(t,x)\,\mathrm{d}x \\ &= M\|\varphi-\varphi_\varepsilon\|_{L^\infty(\{|x|<2R\})},\end{aligned}$$

which can both be made arbitrarily small provided that $\varepsilon > 0$ is small enough and $R > 0$ is large enough.

A similar argument can be used for $\{W_\lambda\}_{\lambda>0}$. Indeed, for $\varphi \in C_c^\infty(\mathbb{R};[0,1])$, we estimate

$$\begin{aligned}&\left|\int_{\mathbb{R}} W_\lambda(t,x)\varphi(x)\,\mathrm{d}x - \int_{\mathbb{R}} W_\lambda(0,x)\varphi(x)\,\mathrm{d}x\right| = \left|\int_0^t\int_{\mathbb{R}} \partial_s W_\lambda(s,x)\varphi(x)\,\mathrm{d}x\,\mathrm{d}s\right| \\ &\leq \underbrace{\int_0^t\int_{\mathbb{R}} \partial_x\varphi(x)\left|W_\lambda(s,x)\right|^2\mathrm{d}x\,\mathrm{d}s}_{=:I_1} \\ &\qquad + \underbrace{\lambda\left|\int_0^t\int_{\mathbb{R}} |\varphi(x)|\int_{-\infty}^x \exp(\lambda(y-x))W_\lambda(s,x)\left(\partial_x W_\lambda(s,x) - W_\lambda(s,y)\partial_y W_\lambda(s,y)\right)\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s\right|}_{=:I_2}.\end{aligned}$$

For the term I_1, we compute

$$\begin{aligned}I_1 &\leq \|\partial_x\varphi\|_{L^\infty(\mathbb{R})}\int_0^t \|W_\lambda(s,\cdot)\|_{L^1(\mathbb{R})}\|W_\lambda(s,\cdot)\|_{L^\infty(\mathbb{R})}\,\mathrm{d}s \\ &\leq \|\partial_x\varphi\|_{L^\infty(\mathbb{R})}C(M)\int_0^t s^{-1/2}\,\mathrm{d}s \leq C(M)\|\partial_x\varphi\|_{L^\infty(\mathbb{R})}\sqrt{t},\end{aligned}$$

where, in the last line, we used Lemma 5.1.2.

For I_2, using Fubini–Tonelli's theorem,we compute

$$\begin{aligned}I_2 = \Bigg|&\int_0^t\int_{\mathbb{R}} \varphi(x)W_\lambda(s,x)\partial_x W_\lambda(s,x)\,\mathrm{d}x\,\mathrm{d}s \\ &- \lambda\int_0^t\int_{\mathbb{R}} \varphi(x)\int_{-\infty}^x \exp(\lambda(y-x))W_\lambda(s,y)\partial_y W_\lambda(s,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s\Bigg|\end{aligned}$$

$$= \left| \int_0^t \int_{\mathbb{R}} \varphi(x) W_\lambda(s,x) \partial_x W_\lambda(s,x) \,\mathrm{d}x \,\mathrm{d}s \right.$$
$$\left. - \lambda \int_0^t \int_{\mathbb{R}} W_\lambda(s,y) \partial_y W_\lambda(s,y) \int_y^\infty \varphi(x) \exp(\lambda(y-x)) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}s \right|;$$

integrating by parts on the term $x \mapsto \exp(\lambda(y-x))$ yields

$$I_2 = \left| \int_0^t \int_{\mathbb{R}} W_\lambda(s,y) \partial_y W_\lambda(s,y) \int_y^\infty \partial_x \varphi(x) \exp(\lambda(y-x)) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}s \right|;$$

integrating by parts on the term $y \mapsto W_\lambda(s,y)\partial_y W_\lambda(s,y)$ and using the fact that $\lim_{x\to\pm\infty} W_\lambda(t,\cdot) = 0$ (which is a consequence of the fact that $W_\lambda(t,\cdot) \in L^1(\mathbb{R}) \cap \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ for $t > 0$ and $\lambda > 0$), we then get

$$\begin{aligned}
I_2 &= \left| -\lambda \frac{1}{2} \int_0^t \int_{\mathbb{R}} |W_\lambda(s,y)|^2 \int_y^\infty \partial_x \varphi(x) \exp(\lambda(y-x)) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}s + \frac{1}{2} \int_0^t \int_{\mathbb{R}} |W_\lambda(s,y)|^2 \partial_y \varphi(y) \,\mathrm{d}y \,\mathrm{d}s \right| \\
&\le \frac{\lambda}{2} \|\partial_x \varphi\|_{L^\infty(\mathbb{R})} \int_0^t \int_{\mathbb{R}} |W_\lambda(s,y)|^2 \int_y^\infty \exp(\lambda(y-x)) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}s \\
&\quad + \frac{1}{2} \|\partial_x \varphi\|_{L^\infty(\mathbb{R})} \int_0^t \int_{\mathbb{R}} |W_\lambda(s,y)|^2 \,\mathrm{d}y \,\mathrm{d}s \\
&= \|\partial_x \varphi\|_{L^\infty(\mathbb{R})} \int_0^t \int_{\mathbb{R}} |W_\lambda(s,y)|^2 \,\mathrm{d}y \,\mathrm{d}s \\
&\le \|\partial_x \varphi\|_{L^\infty(\mathbb{R})} \int_0^t \|W_\lambda(s,\cdot)\|_{L^1(\mathbb{R})} \|W_\lambda(s,\cdot)\|_{L^\infty(\mathbb{R})} \,\mathrm{d}s \\
&\le \|\partial_x \varphi\|_{L^\infty(\mathbb{R})} C(M) \int_0^t s^{-1/2} \,\mathrm{d}s \le C(M) \|\partial_x \varphi\|_{L^\infty(\mathbb{R})} \sqrt{t},
\end{aligned}$$

where, in the last line, we used Lemma 5.1.2. Thus, for any $\varepsilon > 0$, we can choose $\tau > 0$ and $\lambda_0 > 0$ such that

$$\left| \int_{\mathbb{R}} W_\lambda(t,x) \varphi(x) \,\mathrm{d}x - M\varphi(0) \right| \le \varepsilon \qquad \text{for all } 0 < t < \tau,\ \lambda > \lambda_0.$$

The rest of the argument for $\varphi \in C_b(\mathbb{R})$ goes through as above.

Step 4. *Entropy admissibility of the limit point.* The limit point w^* is actually the unique entropy admissible N-wave solution w of the Burgers equation (5.0.6) defined in (5.0.7). This follows immediately from passing to the limit pointwise in the Oleĭnik inequality (5.1.4). Owing to Urysohn's subsequence principle, from the uniqueness of the entropy solution of (5.0.6), we also deduce that the whole families $\{\rho_\lambda\}_{\lambda>0}$ and $\{W_\lambda\}_{\lambda>0}$ converge to w (not just up to extracting a subsequence).

Step 5. *Conclusion of the proof.* From the steps above, we have that

$$\|W_\lambda(t,\cdot) - w(t,\cdot)\|_{L^1(\mathbb{R})} \to 0 \qquad \text{as } \lambda \to +\infty,$$

where w denotes the N-wave solution entropy of (5.0.6). For $p = 1$, (5.0.5) is a consequence of the fact that

$$\begin{aligned}
\rho_\lambda(1,x) - w(1,x) &= \lambda\rho(\lambda^2, \lambda x) - w(1,x), \\
W_\lambda(1,x) - w(1,x) &= \lambda W(\lambda^2, \lambda x) - w(1,x),
\end{aligned}$$

(and that the same would hold true replacing $t = 1$ by any fixed $\bar{t} > 0$), i.e., letting $\lambda \to +\infty$ for a fixed time $\bar{t} > 0$ is equivalent to fixing $\lambda = 1$ and letting $t \to +\infty$.

To prove the result also for $p \in (1, +\infty)$, we argue by interpolation. Indeed, we have that, for $t > 0$, $\{\rho_\lambda\}_{\lambda>0}$ and $\{W_\lambda\}_{\lambda>0}$ are also uniformly bounded in $L^q(\mathbb{R})$ (being in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for every $t > 0$) and $w \in L^q(\mathbb{R})$, with $q \in (1, +\infty)$. Then, Hölder's inequality yields

$$\|\rho_\lambda(t,\cdot) - w(t,\cdot)\|_{L^p(\mathbb{R})} \leq \|\rho_\lambda(t,\cdot) - w(t,\cdot)\|_{L^1(\mathbb{R})}^{\frac{1}{2p-1}} \Big(\|\rho_\lambda(t,\cdot)\|_{L^{2p}(\mathbb{R})} + \|w(t,\cdot)\|_{L^{2p}(\mathbb{R})}\Big)^{\frac{2(p-1)}{2p-1}},$$

$$\|W_\lambda(t,\cdot) - w(t,\cdot)\|_{L^p(\mathbb{R})} \leq \|W_\lambda(t,\cdot) - w(t,\cdot)\|_{L^1(\mathbb{R})}^{\frac{1}{2p-1}} \Big(\|W_\lambda(t,\cdot)\|_{L^{2p}(\mathbb{R})} + \|w(t,\cdot)\|_{L^{2p}(\mathbb{R})}\Big)^{\frac{2(p-1)}{2p-1}},$$

from which the result follows. □

5.3. Numerical experiments

In this Section, we showcase the result in Theorem 5.0.1 numerically. For the nonlocal problem, we rely on a non-dissipative solver based on characteristics (see [**173**] and [**213**, Chapter 3]). More precisely, the simulations illustrate the convergence

$$\bar{\rho}(t,\cdot) \to \bar{w} \quad \text{in } L^1(\mathbb{R}) \quad \text{as } t \to +\infty,$$

for the rescaled variables

$$\bar{\rho} := \sqrt{t}\rho, \quad \bar{w} := \sqrt{t}w, \quad y := x/\sqrt{t},$$

in which the N-wave is stationary (i.e., time-independent) and given by

$$\bar{w}(y) = \begin{cases} \dfrac{y}{2} & \text{if } y \in (0, \sqrt{4M}), \\ 0 & \text{otherwise.} \end{cases}$$

To start with, in Figure 5.1, we present the evolution of the solution of (5.0.1) on long time horizons for the following initial data:

(1) $\rho_0(x) := \mathbb{1}_{[0,1]}(x)$, **(2)** $\rho_0(x) := 2x\,\mathbb{1}_{[0,1]}(x)$,
(3) $\rho_0(x) := 6x(1-x)\mathbb{1}_{[0,1]}(x)$, **(4)** $\rho_0(x) := 2x\,\mathbb{1}_{[0,0.5]}(x) + \mathbb{1}_{[0.5,1]}(x)$,

for $x \in \mathbb{R}$. In all cases, we observe the convergence toward the N-wave profile of the (local) Burgers equation (5.0.6).

For (left) continuous initial data (as is the case in **(2)**,**(3)**, and **(4)**), the N-wave is also approximated by left-continuous functions. This is a well-known fact, as nonlocal conservation laws preserve regularity [**167**, Corollary 5.3] (see also Theorem 5.0.2). In particular, for case **(4)**, there are two jumps downwards in the initial datum and the first one is damped out over time (still observable for $t = 10$ at $x \approx 1$). This can be understood when recalling that around $x \approx 1$ the velocity of the dynamics is smaller than for $x < 1$ so that the density increases between both points and the jump decreases (which is visible in particular for $t = 1$ and $t = 10$).

Secondly, in Figure 5.2, we consider $\gamma(\cdot) = \mathbb{1}_{(0,1)}(\cdot)$ instead of an exponential weight in (5.0.2), i.e. we study

$$W[\rho](t,x) := \int_{x-1}^{x} \rho(t,y)\,\mathrm{d}y, \qquad (t,x) \in (0,T) \times \mathbb{R}.$$

The numerical simulation shows that, even in this case (which is not covered by the results of the present paper or by the ones on the singular limit problem contained in Chapters 3 and 4), a convergence result can be observed. However, the convergence seems to occur "less regularly" as the constant kernel generates more and more points where the solution is not differentiable. Indeed, in contrast to the exponential kernel case, the regularity of the solution for piece-wise constant kernels depends points-wise and locally (on the trace of backward characteristics) on initial data, kernel, and their interplay.

Finally, we present some simulations illustrating the case of a more general power-type velocity: namely,

$$\begin{cases} \partial_t\rho(t,x) + \partial_x(W^{q-1}(t,x)\rho(t,x)) = 0, & (t,x) \in (0,+\infty) \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{5.3.1}$$

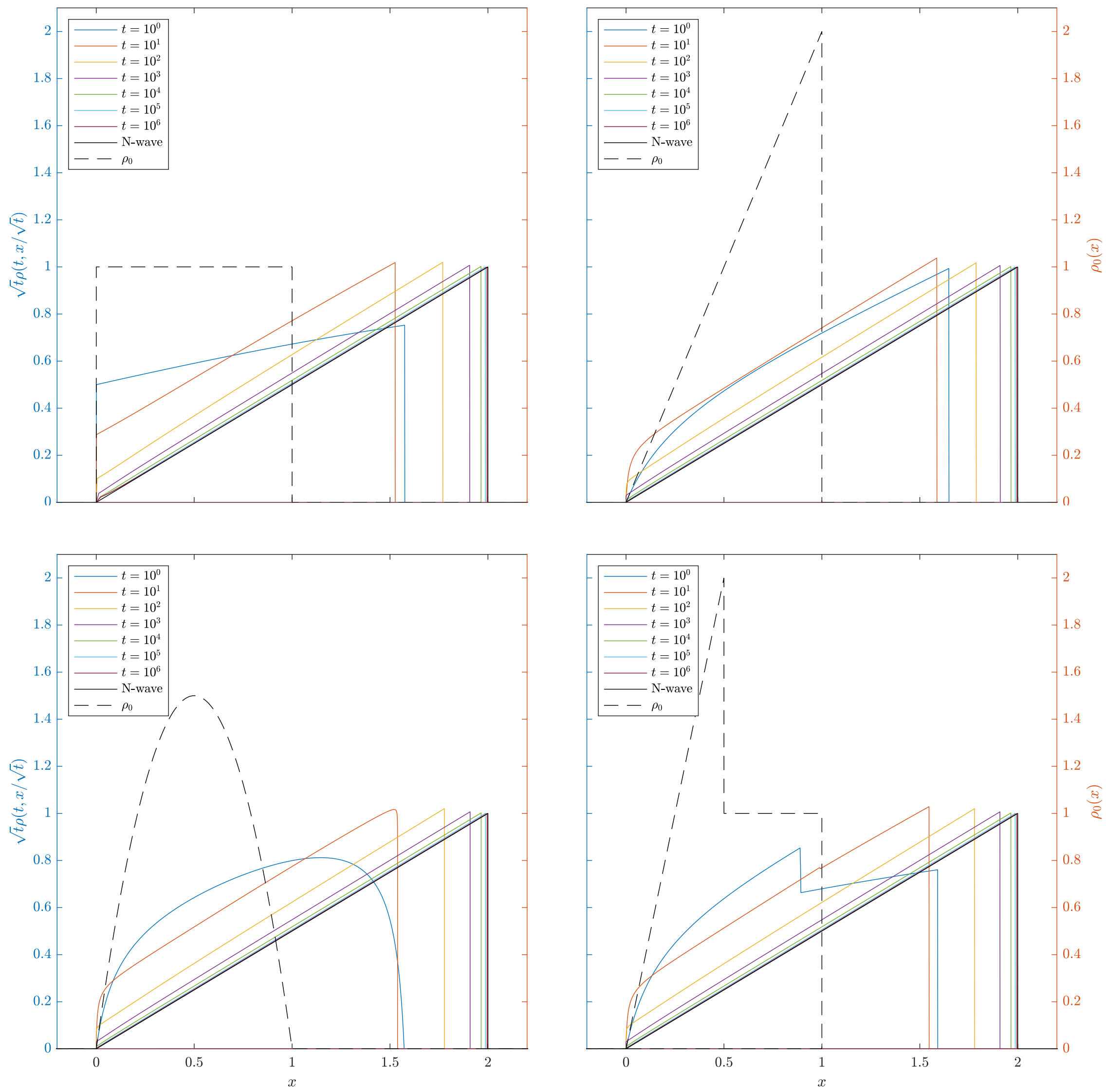

FIGURE 5.1. Convergence to the N-wave profile for the nonlocal regularization of the Burgers equation. TOP LEFT: $\rho_0(x) \coloneqq \mathbb{1}_{[0,1]}(x)$. TOP RIGHT: $\rho_0(x) \coloneqq 2x\,\mathbb{1}_{[0,1]}(x)$. BOTTOM LEFT: $\rho_0(x) \coloneqq 6x(1-x)\mathbb{1}_{[0,1]}(x)$. BOTTOM RIGHT: $\rho_0(x) \coloneqq 2x\,\mathbb{1}_{[0,0.5]}(x) + \mathbb{1}_{[0.5,1]}(x)$.

for some for $q \geq 2$. In this case, the explicit N-wave solution of the corresponding local conservation law

$$\begin{cases} \partial_t \rho(t,x) + \partial_x(\rho^q(t,x)) = 0, & (t,x) \in (0,+\infty) \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{5.3.2}$$

is given by

$$w_q(t,x) = \begin{cases} \left(\dfrac{x}{qt}\right)^{\frac{1}{q-1}} & \text{if } x \in \left(0,\ q\left(\frac{M}{q-1}\right)^{\frac{q-1}{q}} t^{\frac{1}{q}}\right), \\ 0 & \text{otherwise.} \end{cases}$$

that is, in the rescaled variables

$$\bar{w}_q \coloneqq t^{1/q} w_q, \quad y \coloneqq x t^{-1/q},$$

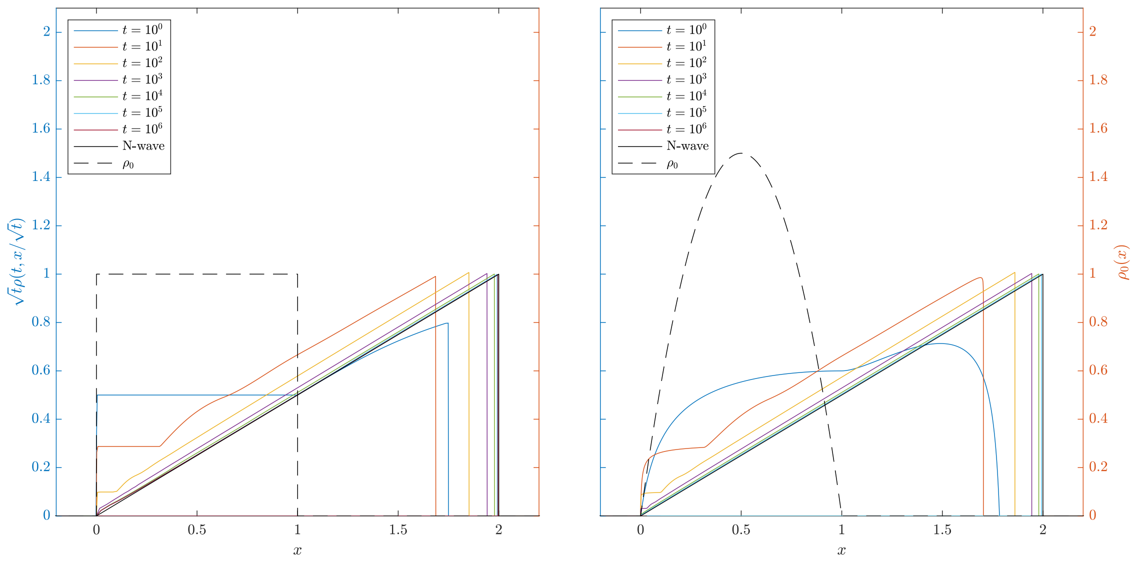

FIGURE 5.2. Convergence to the N-wave profile for the nonlocal regularization of the Burgers equation with the piecewise-constant weight $\gamma(\cdot) := \mathbb{1}_{(0,1)}(\cdot)$. LEFT: $\rho_0(x) := \mathbb{1}_{[0,1]}(x)$. RIGHT: $\rho_0(x) := 6x(1-x)\mathbb{1}_{[0,1]}(x)$.

$$\bar{w}_q(y) = \begin{cases} \left(\dfrac{y}{q}\right)^{\frac{1}{q-1}} & \text{if } y \in \left(0,\ q\left(\frac{M}{q-1}\right)^{\frac{q-1}{q}}\right), \\ 0 & \text{otherwise}. \end{cases}$$

(see [**192**, Eq. (2.1)]). In particular, in Figure 5.3 (for $q = 3$), the convergence result seems to hold.

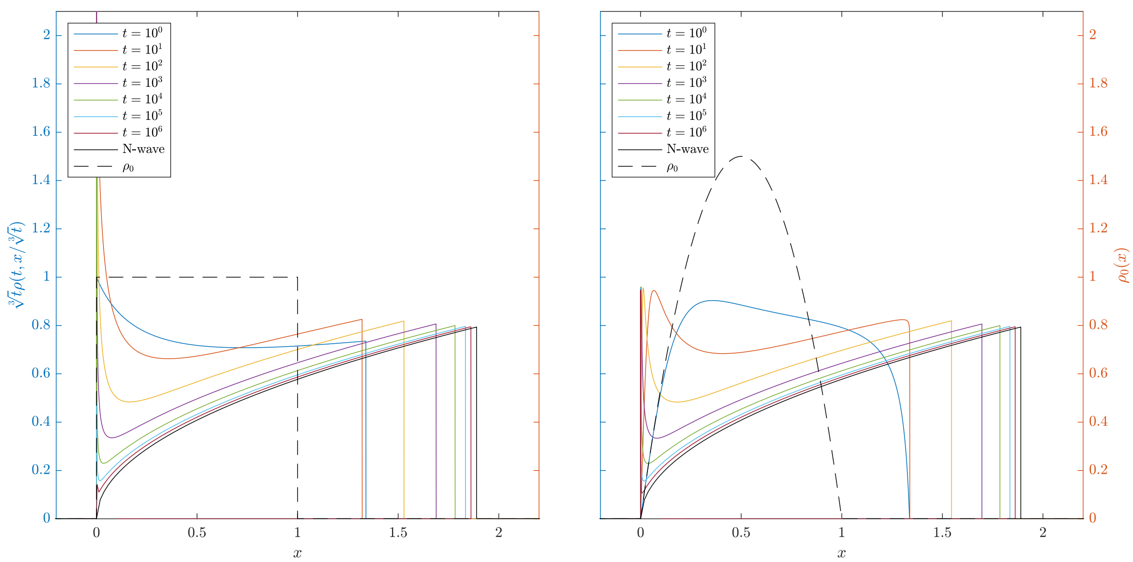

FIGURE 5.3. Convergence to the N-wave profile for the nonlocal regularization of $\partial_t \rho + \partial_x \rho^3 = 0$ with exponential weight. LEFT: $\rho_0(x) := \mathbb{1}_{[0,1]}(x)$. RIGHT: $\rho_0(x) := 6x(1-x)\mathbb{1}_{[0,1]}(x)$.

In this case, none of the previously established results holds. However, the numerical experiments point to the fact that we may still observe the L^1-convergence to the N-wave profile. The behavior of the rescaled solution, which explodes at $x = 0$ is particularly noteworthy. It can be explained as follows. For the conservation law

$$\partial_t \rho(t,x) + \partial_x\big(W[\rho]^2(t,x)\rho(t,x)\big) = 0, \quad (t,x) \in (0,T) \times \mathbb{R},$$

we can compute, along characteristics,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\rho(t,\xi(0,x;t)) &= \partial_t \rho(t,\xi(0,x;t)) + \partial_2 \rho(t,\xi(0,x;t))\partial_3 \xi(0,x;t) \\
&= -\partial_2 \rho(t,\xi(0,x;t)) W[\rho]^2(t,\xi(0,x;t)) \\
&\quad - 2\rho(t,\xi(0,x;t)) W[\rho](t,\xi(0,x;t)) \partial_2 W[\rho](t,\xi(0,x;t)) \\
&\quad + \partial_2 \rho(t,\xi(0,x;t)) W[\rho]^2(t,\xi(0,x;t)) \\
&= -2\rho(t,\xi(0,x;t)) W[\rho](t,\xi(0,x;t)) \partial_2 W[\rho](t,\xi(0,x;t)).
\end{aligned}$$

As W "looks" to the left and the solution vanishes on the left half space for all time $t > 0$, we have that $W[\rho](t,0) = 0$ for all $t > 0$; thus, the value of the solution at $x = 0$ never changes, i.e. $\lim_{x \to 0^+} \rho(t,x) = \rho_0(0)$ for all $t > 0$, which yields the long-time behavior at $x = 0$ observed in Figure 5.3 upon rescaling.

CHAPTER 6

Nonlocal–to–local singular limit problem with artificial viscosity

In this Chapter, we consider the nonlocal problem with artificial viscosity

$$
\begin{cases} \partial_t \rho_{\alpha,\nu}(t,x) + \partial_x(V(W_\alpha[\rho_{\alpha,\nu}](t,x))\rho_{\alpha,\nu}(t,x)) = \nu \partial^2_{xx}\rho_{\alpha,\nu}(t,x), & (t,x)\in(0,T)\times\mathbb{R},\\ \rho_{\alpha,\nu}(0,x) = \rho_{0,\nu}(x), & x\in\mathbb{R}, \end{cases} \tag{6.0.1}
$$

with

$$
W_\alpha[\rho_{\alpha,\nu}](t,x) \coloneqq \frac{1}{\alpha}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right)\rho_{\alpha,\nu}(t,y)\,\mathrm{d}y, \qquad (t,x)\in(0,T)\times\mathbb{R}. \tag{6.0.2}
$$

We assume the following natural conditions to be satisfied:

$$
\rho_0 \in L^1(\mathbb{R})\cap L^\infty(\mathbb{R}), \quad 0\le\rho_0\le 1; \tag{6.0.3}
$$

$$
V\in W^{1,\infty}(\mathbb{R})\cap C^2(\mathbb{R}), \quad V\ge 0, \quad V'\le 0; \tag{6.0.4}
$$

$$
f:\xi\mapsto \xi V(\xi) \text{ is genuinely nonlinear, i.e. } \mathcal{L}(\{(\xi V(\xi))''=0\})=0, \tag{6.0.5}
$$

where $\mathcal{L}$ denotes the (one-dimensional) Lebesgue measure.

We smooth initial datum in the following way:

$$
\{\rho_{0,\nu}\}_{\nu>0}\subset C^\infty_{\mathrm{c}}(\mathbb{R}), \tag{6.0.6}
$$

$$
\rho_{0,\nu} \overset{\nu\to 0}{\longrightarrow} \rho_0 \text{ a.e. and in } L^p_{\mathrm{loc}}(\mathbb{R}),\ p\in[1,\infty), \tag{6.0.7}
$$

$$
0\le\rho_{0,\nu}\le 1, \quad \nu>0, \tag{6.0.8}
$$

$$
\|\rho_{0,\nu}\|_{L^2(\mathbb{R})}\le C, \tag{6.0.9}
$$

where $C>0$ is a constant independent of $\alpha,\nu>0$.

As outlined in Chapter 1, the reformulation of the problem as

$$
\begin{cases} \partial_t W_{\alpha,\nu} + \partial_x(V(W_{\alpha,\nu})W_{\alpha,\nu}) \\ \quad = \alpha\partial^2_{tx}W_{\alpha,\nu} + \nu\partial^2_{xx}W_{\alpha,\nu} + \alpha\partial_x(V(W_{\alpha,\nu})\partial_x W_{\alpha,\nu}) - \alpha\nu\partial^3_{xxx}W_{\alpha,\nu}, & (t,x)\in(0,T)\times\mathbb{R},\\ W_{\alpha,\nu}(0,x) = \frac{1}{\alpha}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right)\rho_{0,\nu}(y)\,\mathrm{d}y, & x\in\mathbb{R}, \end{cases} \tag{6.0.10}
$$

is the key to obtaining the a priori estimates needed in the proof of our main result: we show that, when the nonlocal term together with the viscosity approaches zero, the family $\{\rho_{\alpha,\nu}\}_{\alpha,\nu>0}$ converges to the entropy solution of the local conservation law

$$
\begin{cases} \partial_t\rho(t,x) + \partial_x\big(V(\rho(t,x))\rho(t,x)\big) = 0, & (t,x)\in(0,T)\times\mathbb{R},\\ \rho(0,x)=\rho_0(x), & x\in\mathbb{R}. \end{cases} \tag{6.0.11}
$$

Theorem 6.0.1 (Nonlocal–to–local limit). *Let $\{\rho_{\alpha,\nu}\}_{\alpha,\nu}$ be a family of classical solutions of the Cauchy problem* (6.0.1)*. Then, for all*

$$
(\alpha,\nu)\subset\mathbb{R}^2_{>0} \ \text{ such that } \ (\alpha,\nu)\to(0,0) \ \text{ and } \ \frac{\alpha}{\nu}\to 0, \tag{6.0.12}
$$

there exists $\rho\in L^\infty((0,+\infty)\times\mathbb{R})$ such that

$$
\rho_{\alpha,\nu}\to\rho \quad \text{a.e. and in } L^p_{\mathrm{loc}}((0,T)\times\mathbb{R}), \text{ with } p\in[1,\infty),
$$

and ρ is the entropy solution of the Cauchy problem (6.0.11)*.*

In Section 6.1, we prove the well-posedness of (6.0.1)—which, in turn, implies the rigorous equivalence between (6.0.1) and (6.0.10)—and the a priori estimates required for the study of the singular limit. More specifically, we establish L^∞-bounds on $\rho_{\alpha,\nu}$ and $W_{\alpha,\nu}$ and an L^2-estimate on $W_{\alpha,\nu}(t,\cdot)$ which also involves the H^2-norm of $W_{\alpha,\nu}(t,\cdot)$.

In Section 6.2, we use the previous estimates to prove that the family $\{\rho_{\alpha,\nu}\}_{\alpha,\nu>0}$ is compact in L^p. To this end, we rely on Tartar's compensated compactness technique and show that the family $\{\partial_t\eta(\rho_{\alpha,\nu})+\partial_x q(\rho_{\alpha,\nu})\}_{\alpha,\nu>0}$, for every convex entropy–entropy-flux pair, is compact in $H^{-1}_{\mathrm{loc}}((0,+\infty)\times\mathbb{R})$. Finally, we check that the limit function ρ is the entropy solution of the local conservation law (6.0.11). In the compactness estimates and the verification of the entropy condition, the assumption $\alpha = o(\nu)$ is crucial.

6.1. A priori estimates

We start by proving the well-posedness of classical solutions of (6.0.1), their non-negativity, and an upper-bound in terms of the L^∞ norm of the initial data. This, in turn, implies an L^∞-estimate on $W_{\alpha,\nu}$.

LEMMA 6.1.1 (Well-posedness and L^∞-estimate). *For every $\alpha,\nu>0$, there exists a unique non-negative smooth solution $\rho_{\alpha,\nu}\in C^\infty([0,T)\times\mathbb{R})\cap W^{2,2}((0,T)\times\mathbb{R})$ of the Cauchy problem* (6.0.1) *such that*

$$0\le \rho_{\alpha,\nu},\, W_{\alpha,\nu}\le 1.$$

PROOF. Since $\rho_{0,\nu}\in W^{2,2}(\mathbb{R})$, the existence and uniqueness of smooth solutions of (6.0.1) can be proven arguing similarly to [**90**, Theorem 2.1] or [**82, 75, 67, 180**]: although in our case the kernel is not smooth and compactly supported, a fixed-point argument based on the Duhamel's formula yields the well-posedness result. We focus on showing the L^∞-bound on the solutions (these are well-known in the literature as well; but we present a simple proof for the sake of completeness). To prove $\rho_{\alpha,\nu}\ge 0$, we consider the function

$$\eta(\xi) = -\xi\mathbb{1}_{(-\infty,0]}(\xi),\qquad \xi\in\mathbb{R},$$

which satisfies

$$\eta'(\xi) = -\mathbb{1}_{(-\infty,0]}(\xi),\quad \eta''(\xi)=\delta_{\{\xi=0\}}\ge 0,\qquad \xi\in\mathbb{R}. \tag{6.1.1}$$

Multiplying (6.0.1) by $\eta'(\rho_{\alpha,\nu})$, integrating over $\mathbb{R}$, and using [**27**, Lemma 2] yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\eta(\rho_{\alpha,\nu})\,\mathrm{d}x &= \int_{\mathbb{R}}\partial_t\rho_{\alpha,\nu}\eta'(\rho_{\alpha,\nu})\,\mathrm{d}x\\
&=\nu\int_{\mathbb{R}}\partial^2_{xx}\rho_{\alpha,\nu}\eta'(\rho_{\alpha,\nu})\,\mathrm{d}x-\int_{\mathbb{R}}\partial_x(V(W_{\alpha,\nu})\rho_{\alpha,\nu}))\eta'(\rho_{\alpha,\nu})\,\mathrm{d}x\\
&=-\nu\int_{\mathbb{R}}\underbrace{(\partial_x\rho_{\alpha,\nu})^2\eta''(\rho_{\alpha,\nu})}_{\ge 0}\,\mathrm{d}x+\int_{\mathbb{R}}V(W_{\alpha,\nu})\partial_x\rho_{\alpha,\nu}\underbrace{\rho_{\alpha,\nu}\eta''(\rho_{\alpha,\nu})}_{=0\,(\text{see (6.1.1)})}\,\mathrm{d}x\\
&\le 0.
\end{aligned}$$

Integrating over $(0,t)$ and using (6.0.8) and (6.1.1), we compute

$$0\le\int_{\mathbb{R}}\eta(\rho_{\alpha,\nu}(t,x))\,\mathrm{d}x\le\int_{\mathbb{R}}\eta(\rho_{0,\nu}(x))\,\mathrm{d}x=0.$$

Therefore, $\eta(\rho_{\alpha,\nu})\equiv 0$ and thus

$$\rho_{\alpha,\nu}(t,x)\ge 0. \tag{6.1.2}$$

To prove $\rho_{\alpha,\nu}\le 1$, we follow the argument in [**172**, Corollary 5.9]. For $t\ge 0$, let

$$X_{\max}(t) \coloneqq \left\{x\in\mathbb{R} : \rho_{\alpha,\nu}(t,x)=\|\rho_{\alpha,\nu}(t,\cdot)\|_{L^\infty(\mathbb{R})}\right\}.$$

For $x\in X_{\max}(t)$ and a.e. $t>0$, we have (owing to [**98**, Theorem 2.1])

$$\begin{aligned}
\partial_t\rho_{\alpha,\nu}(t,x) = &-\underbrace{\partial_x\rho_{\alpha,\nu}(t,x)V(W_{\alpha,\nu}(t,x))}_{=0}-\frac{\rho_{\alpha,\nu}(t,x)}{\alpha}V'(W_{\alpha,\nu}(t,x))W_{\alpha,\nu}(t,x)\\
&+\frac{\rho^2_{\alpha,\nu}(t,x)}{\alpha}V'(W_{\alpha,\nu}(t,x))+\underbrace{\nu\partial^2_{xx}\rho_{\alpha,\nu}(t,x)}_{\le 0}
\end{aligned}$$

$$\leq \underbrace{\frac{\rho_{\alpha,\nu}(t,x)}{\alpha} V'(W_{\alpha,\nu}(t,x))(\rho_{\alpha,\nu}(t,x) - W_{\alpha,\nu}(t,x))}_{\leq 0},$$

where we have used (6.1.2), (6.0.4), and the fact that, for $x \in X_{\max}(t)$,

$$\partial_x \rho_{\alpha,\nu}(t,x) = 0,$$
$$W_{\alpha,\nu}(t,x) - \rho_{\alpha,\nu}(t,x) = \frac{1}{\alpha}\int_x^{\infty} \exp\left(\frac{x-y}{\alpha}\right)\big(\rho_{\alpha,\nu}(t,y) - \rho_{\alpha,\nu}(t,x)\big)\,\mathrm{d}y \leq 0.$$

We have thus shown that, for all points $x \in X_{\max}(t)$,

$$\partial_t \rho_{\alpha,\nu}(t,x) \leq 0,$$

which implies

$$\|\rho_{\alpha,\nu}(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \|\rho_{\alpha,\nu}(0,\cdot)\|_{L^\infty(\mathbb{R})} \leq 1.$$

The L^∞-estimate for $W_{\alpha,\nu}$ then follows from the one for $\rho_{\alpha,\nu}$ thanks to (6.0.2). □

From the regularity of $\rho_{\alpha,\nu}$, we deduce that problems (6.0.1), (1.1.6), and (6.0.10) are indeed equivalent and $W_{\alpha,\nu}$ is also smooth.

Relying on (6.0.10), we obtain an energy estimate for $W_{\alpha,\nu}$. In the proof, a key role is played by the assumption (6.0.12) on the ratio α/ν.

Lemma 6.1.2 (Energy estimate). *If $W_{\alpha,\nu}$ is the solution of (6.0.10), then the following estimate holds:*

$$\begin{aligned}\|W_{\alpha,\nu}(t,\cdot)\|^2_{L^2(\mathbb{R})} &+ \alpha^2 \|\partial_x W_{\alpha,\nu}(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ &+ \nu \int_0^t \|\partial_x W_{\alpha,\nu}(s,\cdot)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s + \alpha^2\nu \int_0^t \left\|\partial^2_{xx} W_{\alpha,\nu}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \leq C,\end{aligned} \tag{6.1.3}$$

for some constant $C > 0$ independent from α and ν and for every $t \geq 0$. In particular,

$\{W_{\alpha,\nu}\}_{\alpha,\nu>0}$, $\{\alpha\partial_x W_{\alpha,\nu}\}_{\alpha,\nu>0}$ are bounded in $L^\infty((0,+\infty);L^2(\mathbb{R}))$,
$\{\sqrt{\nu}\partial_x W_{\alpha,\nu}\}_{\alpha,\nu>0}$, $\{\alpha\sqrt{\nu}\partial^2_{xx} W_{\alpha,\nu}\}_{\alpha,\nu>0}$ are bounded in $L^2((0,+\infty)\times\mathbb{R})$.

Proof. We differentiate the L^2-norm $\frac{1}{2}\|W_{\alpha,\nu}(t,\cdot)\|_{L^2(\mathbb{R})}$ with respect to time and, using (6.0.10), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \frac{W^2_{\alpha,\nu}}{2}\,\mathrm{d}x &= \int_{\mathbb{R}} W_{\alpha,\nu}\partial_t W_{\alpha,\nu}\,\mathrm{d}x \\ &= -\int_{\mathbb{R}} \partial_x(V(W_{\alpha,\nu})W_{\alpha,\nu})W_{\alpha,\nu}\,\mathrm{d}x + \alpha\int_{\mathbb{R}} W_{\alpha,\nu}\partial^2_{tx}W_{\alpha,\nu}\,\mathrm{d}x \\ &\quad + \nu\int_{\mathbb{R}} W_{\alpha,\nu}\partial^2_{xx}W_{\alpha,\nu}\,\mathrm{d}x + \alpha\int_{\mathbb{R}} W_{\alpha,\nu}\partial_x(V(W_{\alpha,\nu})\partial_x W_{\alpha,\nu})\,\mathrm{d}x - \alpha\nu\int_{\mathbb{R}} W_{\alpha,\nu}\partial^3_{xxx}W_{\alpha,\nu}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} V(W_{\alpha,\nu})W_{\alpha,\nu}\partial_x W_{\alpha,\nu}\,\mathrm{d}x + \alpha\int_{\mathbb{R}} W_{\alpha,\nu}\partial^2_{tx}W_{\alpha,\nu}\,\mathrm{d}x \\ &\quad + \nu\int_{\mathbb{R}} W_{\alpha,\nu}\partial^2_{xx}W_{\alpha,\nu}\,\mathrm{d}x + \alpha\int_{\mathbb{R}} W_{\alpha,\nu}\partial_x(V(W_{\alpha,\nu})\partial_x W_{\alpha,\nu})\,\mathrm{d}x + \alpha\nu\int_{\mathbb{R}} \partial_x W_{\alpha,\nu}\partial^2_{xx}W_{\alpha,\nu}\,\mathrm{d}x \\ &= \underbrace{\int_{\mathbb{R}} \partial_x\left(\int_0^{W_{\alpha,\nu}} V(\xi)\xi\,\mathrm{d}\xi\right)\mathrm{d}x}_{=0} - \alpha\int_{\mathbb{R}} \partial_t W_{\alpha,\nu}\partial_x W_{\alpha,\nu}\,\mathrm{d}x \\ &\quad - \nu\int_{\mathbb{R}} (\partial_x W_{\alpha,\nu})^2\,\mathrm{d}x - \alpha\int_{\mathbb{R}} V(W_{\alpha,\nu})(\partial_x W_{\alpha,\nu})^2\,\mathrm{d}x + \alpha\nu\underbrace{\int_{\mathbb{R}} \partial_x\left(\frac{(\partial_x W_{\alpha,\nu})^2}{2}\right)\mathrm{d}x}_{=0} \\ &= -\alpha\int_{\mathbb{R}} \partial_t W_{\alpha,\nu}\partial_x W_{\alpha,\nu}\,\mathrm{d}x - \nu\int_{\mathbb{R}} (\partial_x W_{\alpha,\nu})^2\,\mathrm{d}x - \alpha\int_{\mathbb{R}} V(W_{\alpha,\nu})(\partial_x W_{\alpha,\nu})^2\,\mathrm{d}x.\end{aligned}$$

Using again (6.0.10),

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \frac{W_{\alpha,\nu}^2}{2}\,\mathrm{d}x + \nu\int_{\mathbb{R}} (\partial_x W_{\alpha,\nu})^2\,\mathrm{d}x \\
&= \alpha\int_{\mathbb{R}} \partial_x(V(W_{\alpha,\nu})W_{\alpha,\nu})\partial_x W_{\alpha,\nu}\,\mathrm{d}x - \alpha^2\int_{\mathbb{R}} \partial_x W_{\alpha,\nu}\partial_{tx}^2 W_{\alpha,\nu}\,\mathrm{d}x \\
&\quad - \underbrace{\alpha\nu\int_{\mathbb{R}} \partial_x W_{\alpha,\nu}\partial_{xx}^2 W_{\alpha,\nu}\,\mathrm{d}x}_{=0} - \alpha^2\int_{\mathbb{R}} \partial_x W_{\alpha,\nu}\partial_x(V(W_{\alpha,\nu})\partial_x W_{\alpha,\nu})\,\mathrm{d}x \\
&\quad + \alpha^2\nu\int_{\mathbb{R}} \partial_x W_{\alpha,\nu}\partial_{xxx}^3 W_{\alpha,\nu}\,\mathrm{d}x - \alpha\int_{\mathbb{R}} V(W_{\alpha,\nu})(\partial_x W_{\alpha,\nu})^2\,\mathrm{d}x \\
&= \alpha\int_{\mathbb{R}} V(W_{\alpha,\nu})(\partial_x W_{\alpha,\nu})^2\,\mathrm{d}x + \alpha\int_{\mathbb{R}} V'(W_{\alpha,\nu})W_{\alpha,\nu}(\partial_x W_{\alpha,\nu})^2\,\mathrm{d}x \\
&\quad - \alpha^2\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \frac{(\partial_x W_{\alpha,\nu})^2}{2}\,\mathrm{d}x + \alpha^2\int_{\mathbb{R}} V(W_{\alpha,\nu})\partial_x W_{\alpha,\nu}\partial_{xx}^2 W_{\alpha,\nu}\,\mathrm{d}x \\
&\quad - \alpha^2\nu\int_{\mathbb{R}} (\partial_{xx}^2 W_{\alpha,\nu})^2\,\mathrm{d}x - \alpha\int_{\mathbb{R}} V(W_{\alpha,\nu})(\partial_x W_{\alpha,\nu})^2\,\mathrm{d}x.
\end{aligned}$$

In the computation above, the decay of the solution at infinity and their regularity make the boundary terms in the integration by parts vanish. Using the L^∞-bound established in Lemma 6.1.1 and Young's inequality, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \frac{W_{\alpha,\nu}^2 + \alpha^2(\partial_x W_{\alpha,\nu})^2}{2}\,\mathrm{d}x + \nu\int_{\mathbb{R}} (\partial_x W_{\alpha,\nu})^2\,\mathrm{d}x + \alpha^2\nu\int_{\mathbb{R}} (\partial_{xx}^2 W_{\alpha,\nu})^2\,\mathrm{d}x \\
&= \alpha\int_{\mathbb{R}} V'(W_{\alpha,\nu})W_{\alpha,\nu}(\partial_x W_{\alpha,\nu})^2\,\mathrm{d}x + \alpha^2\int_{\mathbb{R}} V(W_{\alpha,\nu})\partial_x W_{\alpha,\nu}\partial_{xx}^2 W_{\alpha,\nu}\,\mathrm{d}x \\
&\le \alpha\int_{\mathbb{R}} V'(W_{\alpha,\nu})W_{\alpha,\nu}(\partial_x W_{\alpha,\nu})^2\,\mathrm{d}x \\
&\quad + \frac{\alpha^2}{2\nu}\int_{\mathbb{R}} (V(W_{\alpha,\nu})\partial_x W_{\alpha,\nu})^2\,\mathrm{d}x + \frac{\alpha^2\nu}{2}\int_{\mathbb{R}} (\partial_{xx}^2 W_{\alpha,\nu})^2\,\mathrm{d}x \\
&\le \alpha\left(\|V'\|_{L^\infty(0,1)} + \frac{\alpha}{\nu}\|V\|_{L^\infty(0,1)}^2\right)\int_{\mathbb{R}} (\partial_x W_{\alpha,\nu})^2\,\mathrm{d}x + \frac{\alpha^2\nu}{2}\int_{\mathbb{R}} (\partial_{xx}^2 W_{\alpha,\nu})^2\,\mathrm{d}x.
\end{aligned}$$

Thanks to (6.0.12), when α and ν are small, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \frac{W_{\alpha,\nu}^2 + \alpha^2(\partial_x W_{\alpha,\nu})^2}{2}\,\mathrm{d}x + \frac{\nu}{2}\int_{\mathbb{R}} (\partial_x W_{\alpha,\nu})^2\,\mathrm{d}x + \frac{\alpha^2\nu}{2}\int_{\mathbb{R}} (\partial_{xx}^2 W_{\alpha,\nu})^2\,\mathrm{d}x \le 0.$$

Finally, due to (6.0.6), (6.0.7), and (6.0.8), we conclude the proof by integrating over $(0,t)$. We note that the constant C in (6.1.3) is independent of $\alpha, \nu > 0$, because (6.0.9) implies $\|\alpha\partial_x W_{\alpha,\nu}(0,\cdot)\|_{L^2(\mathbb{R})} = \|W_{\alpha,\nu}(0,\cdot) - \rho_{0,\nu}\|_{L^2(\mathbb{R})} \le C$. □

6.2. Compensated compactness framework and proof of the convergence result

In this Section, we use Tartar's compensated compactness method (see [**197, 227**] and [**116**, Lemma 17.4.1]) to obtain strong convergence of a subsequence of solutions of (6.0.1) to the unique entropy solution of (6.0.11).

LEMMA 6.2.1 (Tartar's compensated compactness). *Let $f \in C^2(\mathbb{R})$ be a genuinely nonlinear function, i.e. $\mathcal{L}(\{f'' = 0\}) = 0$, and $\{\rho_\delta\}_{\delta>0}$ be a measurable family of functions defined on $\mathbb{R}_+ \times \mathbb{R}$ such that*

$$\|\rho_\delta\|_{L^\infty((0,T)\times\mathbb{R})} \le M_T, \qquad T,\,\delta > 0,$$

and the family

$$\{\partial_t\eta(\rho_\delta) + \partial_x q(\rho_\delta)\}_{\delta>0}$$

is compact in $H^{-1}_{\mathrm{loc}}(\mathbb{R}_+\times\mathbb{R})$, *for every convex* $\eta\in C^2(\mathbb{R})$ *and* $q'=f'\eta'$. *Then there exist a sequence* $\{\delta_n\}_{n\in\mathbb{N}}\subset(0,+\infty)$, $\delta_n\to 0$, *and a map* $\rho\in L^\infty((0,T)\times\mathbb{R})$, $T>0$, *such that*

$$\rho_{\delta_n}\longrightarrow\rho \quad \textit{a.e. and in } L^p_{\mathrm{loc}}((0,+\infty)\times\mathbb{R}),\ 1\le p<\infty.$$

To check that the family $\{\rho_{\alpha,\nu}\}_{\alpha,\nu>0}$ satisfies the assumptions of Lemma 6.2.1, we rely on Murat's compact embedding (see [**200**] and [**116**, Lemma 17.2.2]).

LEMMA 6.2.2 (Murat's compact embedding). *Let* Ω *be a bounded open subset of* $\mathbb{R}^N$, $N\ge 2$. *Let us suppose that a sequence* $\{\Lambda_n\}_{n\in\mathbb{N}}$ *of distributions is bounded in* $W^{-1,p}(\Omega)$, *for some* $2<p\le\infty$, *and that*

$$\Lambda_n=\Lambda_{1,n}+\Lambda_{2,n},$$

where $\{\Lambda_{1,n}\}_{n\in\mathbb{N}}$ *lies in a compact subset of* $H^{-1}(\Omega)$ *and* $\{\Lambda_{2,n}\}_{n\in\mathbb{N}}$ *lies in a bounded subset of* $L^1_{\mathrm{loc}}(\Omega)$. *Then,* $\{\Lambda_n\}_{n\in\mathbb{N}}$ *lies in a compact subset of* $H^{-1}_{\mathrm{loc}}(\Omega)$.

PROOF OF THEOREM 6.0.1. First, we observe that

$$\begin{aligned}\partial_t\rho_{\alpha,\nu}+\partial_x(V(\rho_{\alpha,\nu})\rho_{\alpha,\nu})&=\nu\partial^2_{xx}\rho_{\alpha,\nu}+\partial_x\big((V(\rho_{\alpha,\nu})-V(W_{\alpha,\nu}))\rho_{\alpha,\nu}\big)\\&=\nu\partial^2_{xx}\rho_{\alpha,\nu}+\partial_x\big(b(t,x)(\rho_{\alpha,\nu}-W_{\alpha,\nu})\rho_{\alpha,\nu}\big)\\&=\nu\partial^2_{xx}\rho_{\alpha,\nu}-\alpha\partial_x\big(b(t,x)\partial_xW_{\alpha,\nu}\rho_{\alpha,\nu}\big),\end{aligned}\tag{6.2.1}$$

where

$$b(t,x):=\int_0^1V'(\theta\rho_{\alpha,\nu}(t,x)+(1-\theta)W_{\alpha,\nu}(t,x))\,\mathrm{d}\theta,\qquad (t,x)\in(0,T)\times\mathbb{R}.\tag{6.2.2}$$

Let $\eta,q:\mathbb{R}\to\mathbb{R}$ be a C^2 convex entropy–entropy-flux pair for the conservation law (6.0.11), i.e. $\eta,q\in C^2(\mathbb{R})$, $\eta''\ge0$, $\eta'f'=q'$, where $f:\xi\mapsto V(\xi)\xi$. Multiplying (6.2.1) by $\eta'(\rho_{\alpha,\nu})$ yields

$$\begin{aligned}\partial_t\eta(\rho_{\alpha,\nu})+\partial_xq(\rho_{\alpha,\nu})&=\nu\eta'(\rho_{\alpha,\nu})\partial^2_{xx}\rho_{\alpha,\nu}-\alpha\eta'(\rho_{\alpha,\nu})\partial_x(b\,\partial_xW_{\alpha,\nu}\rho_{\alpha,\nu})\\&=\nu\partial^2_{xx}\eta(\rho_{\alpha,\nu})-\nu\eta''(\rho_{\alpha,\nu})(\partial_x\rho_{\alpha,\nu})^2-\alpha\partial_x(\eta'(\rho_{\alpha,\nu})b\,\partial_xW_{\alpha,\nu}\rho_{\alpha,\nu})\\&\quad+\alpha\eta''(\rho_{\alpha,\nu})b\,\partial_xW_{\alpha,\nu}\rho_{\alpha,\nu}\partial_x\rho_{\alpha,\nu}.\end{aligned}$$

To apply Tartar's compensated compactness, we show that the right-hand side is compact in $H^{-1}_{\mathrm{loc}}((0,+\infty)\times\mathbb{R})$. By Lemma 6.1.2 and Lemma 6.1.1, we have, for $T>0$,

$$\begin{aligned}\|\nu\eta'(\rho_{\alpha,\nu})\partial_x\rho_{\alpha,\nu}\|_{L^2((0,T)\times\mathbb{R})}&=\|\nu\eta'(\rho_{\alpha,\nu})\partial_xW_{\alpha,\nu}-\alpha\nu\eta'(\rho_{\alpha,\nu})\partial^2_{xx}W_{\alpha,\nu}\|_{L^2((0,T)\times\mathbb{R})}\\&\le\sqrt{\nu}\,\big\|\eta'(\rho_{\alpha,\nu})\big\|_{L^\infty((0,T)\times\mathbb{R})}\|\sqrt{\nu}\partial_xW_{\alpha,\nu}-\alpha\sqrt{\nu}\partial^2_{xx}W_{\alpha,\nu}\|_{L^2((0,T)\times\mathbb{R})}\\&\le\sqrt{\nu}\,c_T\to0.\end{aligned}$$

Additionally, we obtain

$$\begin{aligned}&\|\nu\eta''(\rho_{\alpha,\nu})(\partial_x\rho_{\alpha,\nu})^2\|_{L^1((0,T)\times\mathbb{R})}\\&=\left\|\nu\eta''(\rho_{\alpha,\nu})\Big(\partial_xW_{\alpha,\nu}-\alpha\partial^2_{xx}W_{\alpha,\nu}\Big)^2\right\|_{L^1((0,T)\times\mathbb{R})}\\&=\left\|\eta''(\rho_{\alpha,\nu})\Big(\nu(\partial_xW_{\alpha,\nu})^2-2\alpha\nu\partial_xW_{\alpha,\nu}\partial^2_{xx}W_{\alpha,\nu}+\nu\alpha^2(\partial^2_{xx}W_{\alpha,\nu})^2\Big)\right\|_{L^1((0,T)\times\mathbb{R})}\\&\le c_T\end{aligned}$$

as well as

$$\begin{aligned}&\|\alpha\eta'(\rho_{\alpha,\nu})b(t,x)\partial_xW_{\alpha,\nu}\rho_{\alpha,\nu}\|_{L^2((0,T)\times\mathbb{R})}\\&\le\frac{\alpha}{\sqrt{\nu}}\|\eta'(\rho_{\alpha,\nu})b(t,x)\rho_{\alpha,\nu}\|_{L^\infty((0,T)\times\mathbb{R})}\|\sqrt{\nu}\partial_xW_{\alpha,\nu}\|_{L^2((0,T)\times\mathbb{R})}\\&\le\frac{\alpha}{\sqrt{\nu}}c_T\to0\end{aligned}$$

and

$$
\begin{aligned}
&\|\alpha\eta''(\rho_{\alpha,\nu})b\,\partial_x W_{\alpha,\nu}\rho_{\alpha,\nu}\partial_x\rho_{\alpha,\nu}\|_{L^1((0,T)\times\mathbb{R})}\\
&\quad= \|\alpha\eta''(\rho_{\alpha,\nu})b\,\rho_{\alpha,\nu}\partial_x W_{\alpha,\nu}\Big(\partial_x W_{\alpha,\nu}-\alpha\partial^2_{xx}W_{\alpha,\nu}\Big)\|_{L^1((0,T)\times\mathbb{R})}\\
&\quad= \|\eta''(\rho_{\alpha,\nu})b\,\rho_{\alpha,\nu}\Big(\alpha(\partial_x W_{\alpha,\nu})^2-\alpha^2\partial_x W_{\alpha,\nu}\partial^2_{xx}W_{\alpha,\nu}\Big)\|_{L^1((0,T)\times\mathbb{R})}\\
&\quad\le \|\eta''(\rho_{\alpha,\nu})b\,\rho_{\alpha,\nu}\Big(\alpha(\partial_x W_{\alpha,\nu})^2+\alpha^2|\partial_x W_{\alpha,\nu}\partial^2_{xx}W_{\alpha,\nu}|\Big)\|_{L^1((0,T)\times\mathbb{R})}\\
&\quad\le \|\eta''(\rho_{\alpha,\nu})b\,\rho_{\alpha,\nu}\|_{L^\infty((0,T)\times\mathbb{R})}\left\|\frac{\alpha}{\nu}\nu(\partial_x W_{\alpha,\nu})^2+\frac{\alpha^2\nu}{2}(\partial^2_{xx}W_{\alpha,\nu})^2+\frac{\alpha^2}{2\nu^2}\nu(\partial_x W_{\alpha,\nu})^2\right\|_{L^1((0,T)\times\mathbb{R})}\\
&\quad\le c_T.
\end{aligned}
$$

Then, by Lemma 6.2.2, we deduce that $\{\partial_t\eta(\rho_{\alpha,\nu})+\partial_x q(\rho_{\alpha,\nu})\}_{\alpha,\nu>0}$ is compact in $H^{-1}_{\mathrm{loc}}((0,+\infty)\times\mathbb{R})$. Therefore, by Lemma 6.2.1, we conclude that, given (6.0.12), there exists a function $\rho\in L^\infty((0,T)\times\mathbb{R})$, $T>0$, such that

$$
\rho_{\alpha_n,\nu_n}\longrightarrow\rho \quad\text{in } L^p_{\mathrm{loc}}((0,+\infty)\times\mathbb{R}),\ p\in[1,\infty),\text{ and a.e. in }(0,+\infty)\times\mathbb{R}.
$$

By Lebesgue's dominated convergence theorem, we have that ρ is a weak solution of (6.0.11). It remains to show that ρ is an entropy solution. We start by observing that

$$
\begin{aligned}
\partial_t\eta(\rho_{\alpha_n,\nu_n})+\partial_x q(\rho_{\alpha_n,\nu_n}) &= \nu_n\partial^2_{xx}\eta(\rho_{\alpha_n,\nu_n})-\underbrace{\nu_n\eta''(\rho_{\alpha_n,\nu_n})(\partial_x\rho_{\alpha_n,\nu_n})^2}_{\ge 0}\\
&\quad-\alpha_n\partial_x(\eta'(\rho_{\alpha_n,\nu_n})b(t,x)\partial_x W_{\alpha_n,\nu_n}\rho_{\alpha_n,\nu_n})\\
&\quad+\alpha_n\eta''(\rho_{\alpha_n,\nu_n})b(t,x)\partial_x W_{\alpha_n,\nu_n}\rho_{\alpha_n,\nu_n}\partial_x\rho_{\alpha_n,\nu_n}\\
&\le \nu_n\partial^2_{xx}\eta(\rho_{\alpha_n,\nu_n})-\alpha_n\partial_x(\eta'(\rho_{\alpha_n,\nu_n})b(t,x)\partial_x W_{\alpha_n,\nu_n}\rho_{\alpha_n,\nu_n})\\
&\quad+\alpha_n\eta''(\rho_{\alpha_n,\nu_n})b(t,x)\partial_x W_{\alpha_n,\nu_n}\rho_{\alpha_n,\nu_n}\partial_x\rho_{\alpha_n,\nu_n}.
\end{aligned}
$$

Let us consider a non-negative test function $\varphi\in C_c^\infty([0,\infty)\times\mathbb{R};\mathbb{R}_+)$. Then,

$$
\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}}\Big(\eta(\rho_{\alpha_n,\nu_n})\partial_t\varphi+q(\rho_{\alpha_n,\nu_n})\partial_x\varphi\Big)\,\mathrm{d}t\,\mathrm{d}x+\int_{\mathbb{R}}\eta(\rho_{0,\nu_n}(x))\varphi(0,x)\,\mathrm{d}x\\
&\ge \nu_n\int_0^\infty\int_{\mathbb{R}}\eta(\rho_{\alpha_n,\nu_n})\partial^2_{xx}\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\alpha_n\int_0^\infty\int_{\mathbb{R}}\Big(\eta'(\rho_{\alpha_n,\nu_n})b(t,x)\partial_x W_{\alpha_n,\nu_n}\rho_{\alpha_n,\nu_n}\Big)\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\alpha_n\int_0^\infty\int_{\mathbb{R}}\eta''(\rho_{\alpha_n,\nu_n})b(t,x)\rho_{\alpha_n,\nu_n}\partial_x W_{\alpha_n,\nu_n}\Big(\partial_x W_{\alpha_n,\nu_n}-\alpha_n\partial^2_{xx}W_{\alpha_n,\nu_n}\Big)\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&=\underbrace{\nu_n\int_0^\infty\int_{\mathbb{R}}\eta(\rho_{\alpha_n,\nu_n})\partial^2_{xx}\varphi\,\mathrm{d}x\,\mathrm{d}t}_{=:I_1}\\
&\quad+\underbrace{\alpha_n\int_0^\infty\int_{\mathbb{R}}\Big(\eta'(\rho_{\alpha_n,\nu_n})b(t,x)\partial_x W_{\alpha_n,\nu_n}\rho_{\alpha_n,\nu_n}\Big)\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t}_{=:I_2}\\
&\quad+\underbrace{\alpha_n\int_0^\infty\int_{\mathbb{R}}\eta''(\rho_{\alpha_n,\nu_n})b(t,x)\rho_{\alpha_n,\nu_n}(\partial_x W_{\alpha_n,\nu_n})^2\,\mathrm{d}x}_{=:I_3}\\
&\quad\underbrace{-\frac{\alpha_n}{\nu_n}\int_0^\infty\int_{\mathbb{R}}\eta''(\rho_{\alpha_n,\nu_n})b(t,x)\rho_{\alpha_n,\nu_n}\sqrt{\nu_n}\partial_x W_{\alpha_n,\nu_n}\alpha_n\sqrt{\nu_n}\partial^2_{xx}W_{\alpha_n,\nu_n}\varphi\,\mathrm{d}x\,\mathrm{d}t}_{=:I_4}.
\end{aligned}
$$

Due to the estimates done in the first part of the proof, we have

$$
I_1\le \nu_n\|\eta\|_{L^\infty(\mathbb{R})}\|\partial^2_{xx}\varphi\|_{L^1(\mathbb{R}^2)};
$$

$$
\begin{aligned}
I_2 &\leq \frac{\alpha_n}{\sqrt{\nu_n}} \|\eta'(\rho_{\alpha_n,\nu_n}) b\, \rho_{\alpha_n,\nu_n}\|_{L^\infty((0,T)\times\mathbb{R})} \|\sqrt{\nu_n}\partial_x W_{\alpha_n,\nu_n}\|_{L^2((0,T)\times\mathbb{R})} \leq \frac{\alpha_n}{\sqrt{\nu_n}} c_T;\\
I_3 &\leq \frac{\alpha_n}{\nu_n} \|\eta''(\rho_{\alpha_n,\nu_n}) b\, \rho_{\alpha_n,\nu_n}\|_{L^\infty((0,T)\times\mathbb{R})} \left\|\nu_n (\partial_x W_{\alpha_n,\nu_n})^2\right\|_{L^1((0,T)\times\mathbb{R})} \leq \frac{\alpha_n}{\nu_n} c_T;\\
I_4 &\leq \frac{\alpha_n}{\nu_n} \|\eta''(\rho_{\alpha_n,\nu_n}) b\, \rho_{\alpha_n,\nu_n}\|_{L^\infty((0,T)\times\mathbb{R})} \|\sqrt{\nu_n}\partial_x W_{\alpha_n,\nu_n}\|_{L^2((0,T)\times\mathbb{R})}\\
&\qquad \times \|\alpha_n \sqrt{\nu_n}\partial^2_{xx} W_{\alpha_n,\nu_n}\|_{L^2((0,T)\times\mathbb{R})} \|\varphi\|_{L^\infty(\mathbb{R}^2)} \leq \frac{\alpha_n}{\nu_n} c_T.
\end{aligned}
$$

Passing to the limit, owing to assumption (6.0.12) and to Lebesgue's dominated convergence theorem, we conclude that ρ is the entropy solution of the local conservation law (6.0.11).

The fact that, in the statement of the theorem, the family $\{\rho_{\alpha,\nu}\}_{\alpha,\nu>0}$ converges to ρ and not just up to subsequences follows from the uniqueness of entropy solutions of (6.0.11) and from Urysohn's subsequence principle, i.e. $\rho_{\alpha,\nu} \to \rho$ if and only if for all subsequences $\{\rho_{\alpha_n,\nu_n}\}_{n\in\mathbb{N}}$, there exists a subsubsequence $\{\rho_{\alpha_{n_k},\nu_{n_k}}\}_{k\in\mathbb{N}}$ such that $\rho_{\alpha_{n_k},\nu_{n_k}} \to \rho$ as $k \to +\infty$. $\square$

CHAPTER 7

Boundary controllability and asymptotic stabilization of a nonlocal traffic model

In this Chapter, we investigate the boundary controllability and stabilization for the following nonlocal conservation law:

$$
\begin{cases} \partial_t \rho(t,x) + \partial_x(V(\mathcal{W}[\rho](t,x))\rho(t,x)) = 0, & (t,x) \in \Omega_T, \\ \rho(0,x) = \rho_0(x), & x \in (0,1), \\ V(\mathcal{W}[\rho](t,0))\rho(t,0) = V(\mathcal{W}[\rho](t,0))u_\ell(t), & t \in (0,T), \end{cases} \tag{7.0.1}
$$

with $\Omega_T := (0,T) \times (0,1)$, supplemented by the nonlocal operator

$$
\mathcal{W}[\rho](t,x) := \frac{1}{\alpha} \int_x^\infty \exp\left(\frac{x-y}{\alpha}\right) \left(\begin{cases} \rho(t,y) & \text{if } y < 1 \\ u_r(t) & \text{if } y \geq 1 \end{cases}\right) \mathrm{d}y, \quad (t,x) \in \Omega_T. \tag{7.0.2}
$$

Here, $\rho : \overline{\Omega_T} \to [0,1]$ is the traffic density, $\rho_0 : (0,1) \to [0,1]$ is the initial datum, $u_\ell : (0,T) \to [0,1]$ is the in-flux boundary datum at $x = 0$, $u_r : (0,T) \to [0,1]$ is the (nonlocal) right-hand side boundary datum, $V : [0,1] \to \mathbb{R}$ is the velocity, and $\alpha > 0$ is the nonlocal average parameter.

The choice of an exponential weight enables the boundary condition prescribed on the flux in (7.0.1) to be given directly in terms of density, i.e. as

$$
\rho(t,0) = u_\ell(t), \quad t \in [0,T], \tag{7.0.3}
$$

provided $\min\{\|\rho_0\|_{L^1((0,1))}, u_r(t)\} < 1$ for all $t \in [0,T]$. Indeed, in this case, the velocity V is never zero at the boundary (or anywhere else), so the boundary datum always enters the domain and is thus always attained.

In Section 7.1, we recall some preliminary results on well-posedness for the IBVP (7.0.1).

In Section 7.2, we prove that any end-state can be reached from appropriately defined initial and boundary datum on a sufficiently small time-horizon.

In Section 7.3, we discuss the exact controllability to a given end-state or out-flux of the nonlocal model with boundary controls on the left (in-flux) and on the right (out-flux) of the domain. We prove that this is equivalent to the existence of a solution of the corresponding backward-in-time nonlocal conservation law.

Section 7.4 centers on the long-time behavior of solutions when constant boundary conditions are prescribed and the initial condition is suitably chosen. We show that the solution converges to the corresponding constant steady state. Some numerical simulations verify our results and suggest that they should hold for every initial datum.

Finally, in Section 7.5, we state the existence and uniqueness of steady-state solutions for constant u_ℓ and u_r.

7.1. Preliminaries

We first recall some well-known results on the existence and uniqueness of solutions to the IBVP (7.0.1). To this end, we introduce the following (regularity) assumptions.

Assumption 1 (Assumption on the data of the IBVP). *For $T > 0$, we assume*

$$
\begin{aligned}
&\rho_0 \in L^\infty((0,1);[0,1]); \\
&V \in W^{1,\infty}((0,1);\mathbb{R}_{\geq 0}): \ V' \leq 0, \quad V' \not\equiv 0, \quad \big(V(\xi) = 0 \iff \xi = 1\big); \\
&(u_\ell, u_r) \in L^\infty((0,T);[0,1])^2.
\end{aligned}
$$

Following [**172**, Definition 2.4], we adopt the following notion of solution.

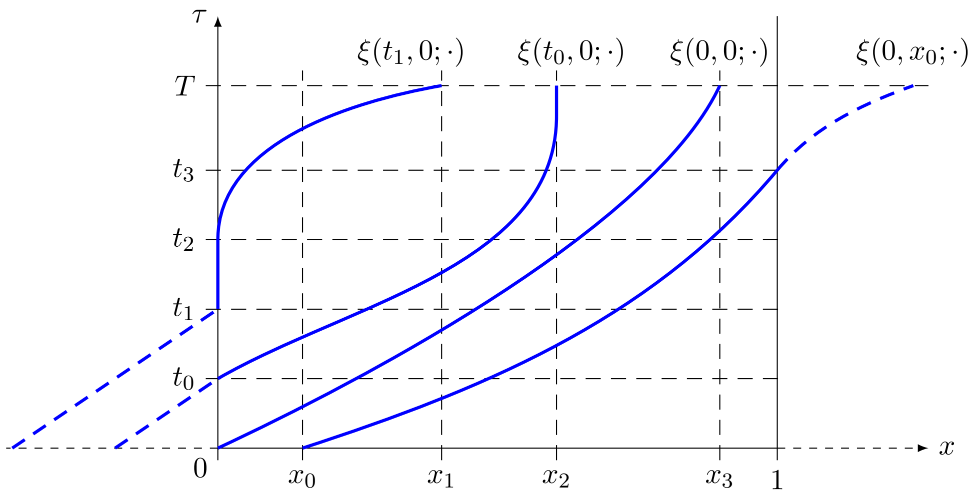

FIGURE 7.1. Characteristics in a bounded domain. Cf. [**172**, Figure 2].

DEFINITION 7.1.1 (Weak solutions for the IBVP). *We say that* $\rho \in C\big([0,T]; L^1((0,1))\big) \cap L^\infty((0,T); L^\infty((0,1)))$ *is a* weak solution *to the IBVP* (7.0.1) *if, for every* $\varphi \in W^{1,\infty}((0,T)\times(0,1))$ *with* $\varphi(T,\cdot) = 0$ *and* $\varphi(\cdot,1) = 0$*, we have*

$$\begin{aligned} 0 = &\iint_{\Omega_T} \Big(\partial_t\varphi(t,x)\rho(t,x) + V(\mathcal{W}[\rho](t,x))\rho(t,x)\partial_x\varphi(t,x)\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^1 \rho_0(x)\varphi(0,x)\,\mathrm{d}x + \int_0^T \varphi(t,0)V(\mathcal{W}[\rho])(t,0)u_\ell(t)\,\mathrm{d}t. \end{aligned} \tag{7.1.1}$$

The existence and uniqueness of weak solutions were investigated in [**172**]. We recall the principal well-posedness result in the following theorem.

THEOREM 7.1.1 (Existence, uniqueness, and maximum principle). *Given Assumption 1, the IBVP* (7.0.1) *admits a unique weak solution* $\rho \in C\big([0,T]; L^1((0,1))\big) \cap L^\infty((0,T); L^\infty((0,1)))$ *in the sense of Definition 7.1.1. Moreover, the solution can be written in terms of characteristics, for* $(t,x) \in \Omega_T$*, as*

$$\rho(t,x) = \begin{cases} \rho_0(\xi_{w^*}(t,x;0))\,\partial_x\xi_{w^*}(t,x;0), & x \geq \xi_{w^*}(0,0;t), \\ u(\xi_{w^*}[t,x]^{-1}_{\max}(0))\,\partial_2\xi_{w^*}(t,x;\xi_{w^*}[t,x]^{-1}_{\max}(0)), & x \leq \xi_{w^*}(0,0;t), \end{cases} \tag{7.1.2}$$

where $\xi : [0,T]\times[0,1]\times[0,T] \to \mathbb{R}_{\geq 0}$ *is the characteristic curve that satisfies the Volterra-type integral equation*

$$\xi_{w^*}(t,x;\tau) = x + \int_t^\tau V(w^*(s,\xi_{w^*}(t,x;s)))\,\mathrm{d}s, \quad (t,x,\tau) \in \overline{\Omega_T}\times[0,T], \tag{7.1.3}$$

$\xi^{-1}_{\max}[t,x]$ *denotes the time-inverted characteristics tracing back the points* $(t,x) \in \{(t,x) \in \Omega_T : x \leq \xi_{w^*}(0,0;t)\}$ *to the boundary (*[**172**, Definition 2.5, Eq. (2.3)]*) and* $w^* \in L^\infty\big((0,T); W^{1,\infty}((0,1))\big)$ *is the unique solution of a fixed-point equation on* $(t,x) \in \Omega_T$ *given in* [**172**, Theorem 3.1, Eq. (3.2)]*. In addition, the following maximum principle holds:*

$$0 \leq \rho(t,x) \leq \max\{\|\rho_0\|_{L^\infty((0,1))}, \|u_\ell\|_{L^\infty((0,T))}, \|u_r\|_{L^\infty((0,T))}\}, \qquad (t,x) \in \Omega_T. \tag{7.1.4}$$

PROOF. For a compactly supported nonlocal weight that is monotonically decreasing, the proof can be found in [**172**, Theorem 3.1, Theorem 4.2, and Corollary 5.9]. The exponential weight considered in (7.0.2) actually simplifies the analysis and the proof can be obtained analogously. We omit the details. ∎

7.2. Reachability for sufficiently small times

In this Section, we show that, for any given function in $L^\infty((0,1);[0,1])$, we can select a suitable boundary and initial datum so that the solution of the corresponding nonlocal conservation law reaches the target at a (sufficiently small) time $T>0$. The key idea behind the proof is to consider the backward-in-time problem, whose solvability is equivalent to the controllability of the given forward problem. Owing to the results presented in [**167**], the backward problem is solvable for any terminal data for a sufficiently small time-horizon. This is because the nonlocal velocity function is Lipschitz continuous for a small time (independent of the specific nonlocal weight and area of integration provided the initial datum is essentially bounded).

This result differs from that of local conservation laws, where the attainable set necessarily needs to satisfy an Oleĭnik inequality [**205, 205**], also for an arbitrarily small time.

THEOREM 7.2.1 (Exact controllability on a small time-horizon). *For every $\rho_{des} \in L^\infty((0,1);[0,1))$ with $\|\rho_{des}\|_{L^\infty((0,1))} < 1$, there exists a time $T>0$, controls $u_\ell, u_r \in L^\infty((0,T);[0,1))$ and initial datum $\rho_0 \in L^\infty((0,1);[0,1))$ such that the corresponding weak solution*

$$\rho \in C\big([0,T];L^1((0,1))\big) \cap L^\infty\left((0,T);L^\infty((0,1))\right)$$

to the IBVP (7.0.1) *satisfies*

$$\rho(T,x) \coloneqq \rho_{des}(x), \qquad x \in (0,1).$$

Here, ρ_{des} stands for the ***des****ired state we want to achieve.*

PROOF. For $u_r \equiv c \in [0,1)$, as shown in [**167**, Theorem 2.20], there exists a sufficiently small time-horizon $T>0$ such that the auxiliary end-value problem

$$\begin{cases} \partial_t p(t,x) + \partial_x\big(V(\mathcal{W}[p](t,x))p(t,x)\big) = 0, & (t,x) \in (0,T)\times\mathbb{R}, \\ p(T,x) = \rho_{\text{des}}(x), & x \in (0,1), \\ p(T,x) = c, & x \in \mathbb{R}\setminus(0,1), \end{cases} \tag{7.2.1}$$

with

$$\mathcal{W}[p](t,x) \coloneqq \frac{1}{\alpha}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right) p(t,y)\,\mathrm{d}y, \tag{7.2.2}$$

admits a unique solution $p \in C\big([0,T];L^1_{\text{loc}}(\mathbb{R})\big)$. Moreover, by [**167**, Lemma 2.6, Item 2], there exists $d>0$ (depending on α,ρ_{des},c, and V) such that

$$\|p(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \max\{\|\rho_{\text{des}}\|_{L^\infty((0,1))}\,,\,c\}e^{d(T-t)}.$$

The key idea of interpreting the control problem as a Cauchy problem on $\mathbb{R}$ backward in time is illustrated in Figure 7.2. Thus, for

$$T \le \frac{1}{d}\log\big(\max\{\|\rho_{\text{des}}\|_{L^\infty((0,1))}\,,\,c\}^{-1}\big),$$

we obtain $\|p(t,\cdot)\|_{L^\infty(\mathbb{R})} \le 1$ for all $t\in[0,T]$. Consequently, by choosing

$$\begin{aligned} u_\ell(t) &= p(t,0), && t\in(0,T),\\ u_r(t) &= c, && t\in(0,T),\\ \rho_0(x) &= p(0,x), && x\in(0,1), \end{aligned}$$

the boundary and initial data are admissible and the solution to the corresponding problem (7.0.1) satisfies $\rho(T,\cdot) = \rho_{\text{des}}$ on $(0,1)$. We note that u_ℓ is given by $p(\cdot,0)$, which can be evaluated as an L^1 function at $x=0$ as the backward "velocity" is not zero.

□

REMARK 7.2.1 (Surjectivity of state to control map on a small time-horizon). *The statement in Theorem 7.2.1 amounts to*

$$\bigcup_{t\in(0,T]}\ \bigcup_{\substack{u_\ell\in L^\infty((0,T);[0,1))\\ u_r\in L^\infty((0,T);[0,1))\\ \rho_0\in L^\infty((0,1);[0,1))}} \rho[\rho_0,u_\ell,u_r](t,\cdot) = L^\infty((0,1);[0,1)),$$

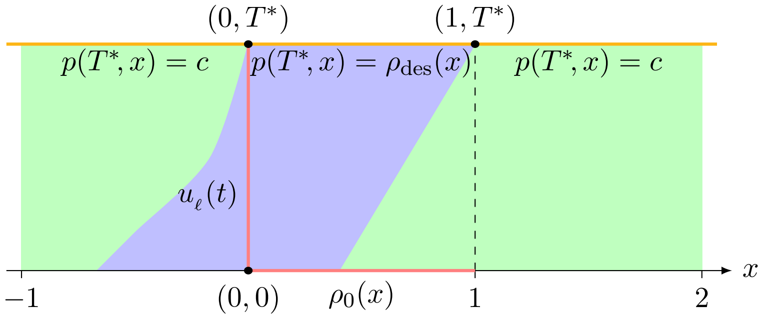

FIGURE 7.2. Transformation of the end boundary value problem into a backward-in-time Cauchy problem on $\mathbb{R}$. The desired state and the "boundary" data are in gold; the corresponding ρ_0 and u_ℓ are in red, yielding—forward in time—the desired state ρ_{des}.

where $\rho[\rho_0, u_\ell, u_r] \in C\big([0,T]; L^1((0,1))\big) \cap L^\infty((0,T); L^\infty((0,1)))$ *denotes the weak solution of the IBVP* (7.0.1)*, with initial datum* ρ_0*, left-hand side boundary datum* u_ℓ*, and nonlocal right-hand side boundary datum* u_r*.*

EXAMPLE 7.2.1 (Numerical example for exact controllability on a sufficiently small time-horizon). *We consider a target function*

$$\rho_{des} \coloneqq \frac{1}{2} + \frac{1}{4}\mathbb{1}_{(\frac{1}{4},\frac{1}{2})} - \frac{1}{4}\mathbb{1}_{(\frac{1}{2},\frac{3}{4})}.$$

We verify numerically that we can find suitable ρ_0*,* u_ℓ*, and* u_r *such that* $\rho(T,\cdot) = \rho_{des}$ *for the sufficiently small time-horizon* $T = 0.6$ *(see Figure 7.3). We note that, for local conservation laws, Oleĭnik's entropy condition would prevent the reachability of this state. The important role of the nonlocal parameter* $\alpha > 0$ *can also be observed. The smaller the* α *in the given example (here* $\alpha \in \{1, 0.9, 0.8\}$*), the more the solution increases backward in time. This is illustrated in the first three rows of Figure 7.3 and, in particular, in the boundary datum; so, for* $\alpha = 0.8$*, the backward solution has already exceeded* 1 *and is thus not admissible for* $T = 0.6$*. The fourth row in Figure 7.3 represents the solution and control for a sufficiently small* α *(namely,* $\alpha = 0.1$*). Here, the final time needs to be chosen to be much smaller,* $T = 0.05$*, and even then the backward solution reaches the bound* 1 *and would cease to exist if we were to consider it on a larger time-horizon. Due to the short time-horizon in the fourth row, the significant changes in the desired datum* ρ_{des} *are tackled mainly through the initial datum, while the boundary datum is taken almost constant.*

7.3. Exact boundary controllability and time-inverted dynamics

In this Section, we consider two control problems:

1. steering a given initial state toward a prescribed target end-state;
2. achieving a prescribed out-flux on the right-hand side of the road.

In both cases, we show that exact controllability holds if and only if the corresponding backward-in-time dynamics admit a weak solution satisfying some bounds. This result is essentially due to the fact that, for nonlocal conservation laws, there is no loss of information (with respect to initial and boundary data).

Our approach is reminiscent of the strategy used to obtain an exact controllability result for the linear transport equation (see [**104**, Section 2.1]):

$$\begin{cases} \partial_t \rho(t,x) + \partial_x \rho(t,x) = 0, & (t,x) \in (0,T)\times(0,1),\\ \rho(0,x) = \rho_0(x), & x \in (0,1),\\ \rho(t,0) = u(t), & t \in (0,T). \end{cases}$$

Namely, given $\rho_0 \in L^p((0,1))$ with $p \in \mathbb{R}_{\geq 1} \cup \{\infty\}$ and a target profile $\rho_{\text{des}} \in L^p((0,1))$, a control $u \in L^p((0,1))$ exists so that $\rho(T,\cdot) = \rho_{\text{des}}$ if and only if $T \geq 1$. The key to the proof is observing

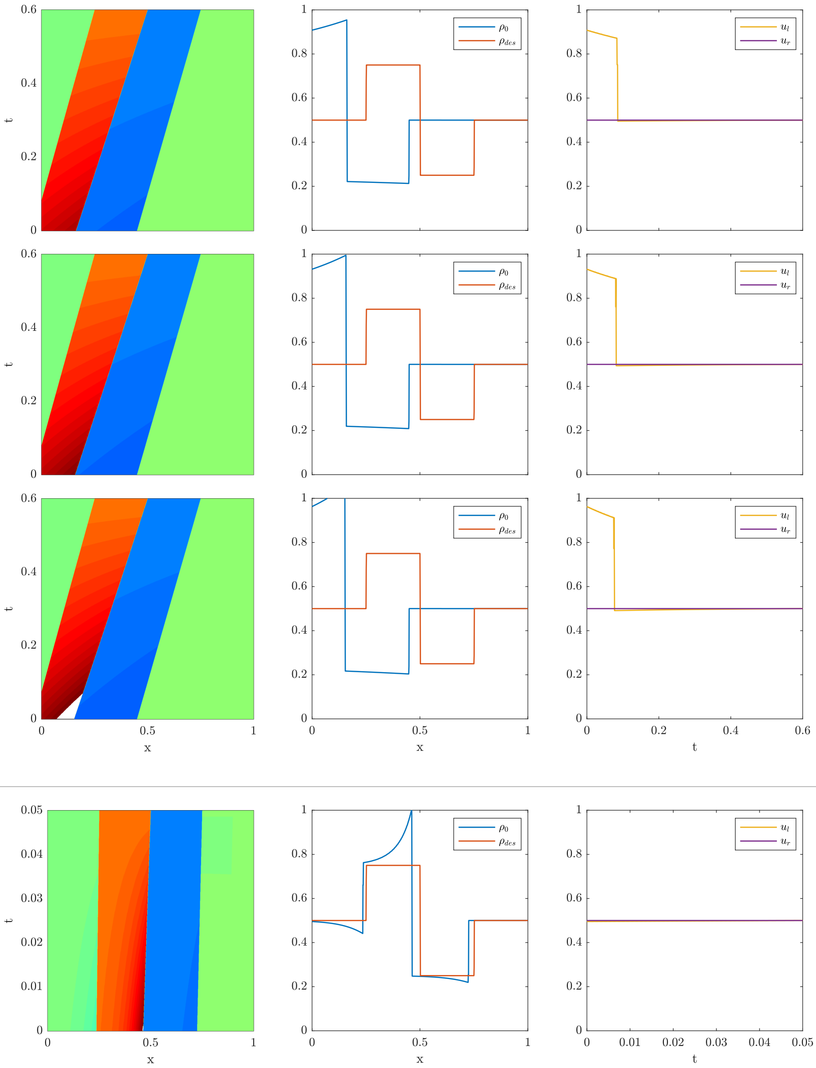

FIGURE 7.3. Illustration of Example 7.2.1 for different $\alpha \in \{0.8, 0.9, 1, 0.1\}$. LEFT: The solution with the proper boundary and initial datum to reach the desired state $\rho_{\text{des}} := \frac{1}{2} + \frac{1}{4}\mathbb{1}_{(0.25,0.5)} - \frac{1}{4}\mathbb{1}_{(0.5,0.75)}$ for $T = 0.6$. MIDDLE: Desired state ρ_{des} and the corresponding initial state ρ_0 to steer the system to ρ_{des}. RIGHT: Boundary data, i.e. u_ℓ and u_r, to steer the system to ρ_{des}. COLOR BAR: 0 1

that the solution of the IBVP is given explicitly by

$$\rho(t, x) = \begin{cases} \rho_0(x - t), & (t, x) \in (0, 1) \times (0, T),\ t \leq x, \\ u(t - x), & (t, x) \in (0, 1) \times (0, T),\ t > x; \end{cases}$$

therefore, if $T \geq 1$, we can choose

$$u(t) := \begin{cases} \rho_{\text{des}}(T - t), & t \in (T - 1, T), \\ 0, & t \in (0, T - 1), \end{cases}$$

and the solution then satisfies $\rho(T, x) = u(T - x) = \rho_{\text{des}}(x)$ for $x \in (0, 1)$. In other words, after the initial data (which moves along characteristics) leaves the domain, we can inject the solution of the backward-in-time problem having ρ_{des} as data into the left-hand side boundary. Since the waves of hyperbolic equations have a finite speed of propagation and the control is applied at the boundary, an exact controllability result requires that the time-horizon T must be sufficiently large.

In the study of our nonlocal model, the first crucial step is to know that the initial state leaves the domain in a finite time as well. This seems very natural when prescribing a density $u_r \in [0, 1)$ as the right-hand side boundary datum, which "pulls out" the initial data for non-zero velocities.

However, in contrast to the linear case, for the nonlocal conservation law considered here, the initial datum—even after leaving the domain—still influences the solution.

The result regarding the initial datum leaving the domain is detailed in the following lemma and illustrated in Figure 7.4.

Lemma 7.3.1 (Initial datum leaving domain in finite time). *Given Assumption 1 and a large enough $T > 0$, let us assume that $\|u_r\|_{L^\infty((0,T))} < 1$. Then, the initial datum, evolving with the dynamics in* (7.0.1)*, leaves the domain in finite time, i.e. the corresponding characteristics ξ emanating from* $(0,0)$ *(as defined in* (7.1.3)*) satisfy*

$$\exists! T^* \in (0,T] \text{ s. t. } \xi_{w^*}(0,0;T^*) = 1 \text{ with } T^* \leq V\left(1 - \frac{1 - \|u_r\|_{L^\infty((0,T))}}{\varepsilon}\right)^{-1}. \tag{7.3.1}$$

Proof. We show that the zero characteristics move with a positive speed bounded away from zero. To this end, we use the maximum principle in Theorem 7.1.1 and estimate the nonlocal operator in (7.0.2) as follows for $(t,x) \in \Omega_T$:

$$\begin{aligned}
\mathcal{W}[q](t,x) &= \frac{1}{\alpha}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right)\left(\begin{cases}\rho(t,y) & \text{if } y < 1\\ u_r(t) & \text{if } y \geq 1\end{cases}\right)\mathrm{d}y\\
&\leq \frac{1}{\alpha}\int_0^\infty \exp\left(-\frac{y}{\alpha}\right)\left(\begin{cases}1 & \text{if } y < 1\\ u_r(t) & \text{if } y \geq 1\end{cases}\right)\mathrm{d}y\\
&= \frac{1}{\alpha}\int_0^1 \exp\left(-\frac{y}{\alpha}\right)\mathrm{d}y + \frac{u_\ell(t)}{\alpha}\int_1^\infty \exp\left(-\frac{y}{\alpha}\right)\mathrm{d}y\\
&= 1 - e^{-1} + u_r(t)e^{-1} = 1 - \frac{1-u_r(t)}{e} \leq 1 - \frac{1-\|u_r\|_{L^\infty((0,T))}}{e}.
\end{aligned}$$

Using this estimate, which is uniform in $(t,x) \in \Omega_T$, and the monotonicity of V, we can bound the zero characteristics in (7.1.3) from below:

$$\begin{aligned}
\xi_{w^*}(0,0;t) &= \int_0^t V(\mathcal{W}[q](s,\xi(0,0;s)))\,\mathrm{d}s\\
&\geq \int_0^t V\Big(1 - \frac{1-\|u_r\|_{L^\infty((0,T))}}{e}\Big)\,\mathrm{d}s = tV\Big(1 - \frac{1-\|u_r\|_{L^\infty((0,T))}}{e}\Big).
\end{aligned} \tag{7.3.2}$$

As V is non-zero at $1 - \frac{1-\|u_r\|_{L^\infty((0,T))}}{e}$, we have the upper-bound

$$\widetilde{T} = V\left(1 - \frac{1-\|u_r\|_{L^\infty((0,T))}}{e}\right)^{-1} \tag{7.3.3}$$

on the time when the initial datum has necessarily left the domain Ω_T. This also explains the assumption of T being sufficiently large, as we require $T \geq \widetilde{T}$. As $\xi_{w^*}(0,0;\cdot) \in C([0,T])$, a time $T^* \in (0,T]$ satisfying $\xi_{w^*}(0,0;T^*) = 1$ indeed exists. As $t \mapsto \xi_{w^*}(0,0;t)$ is also strictly monotone, such a T^* is unique.

□

Remark 7.3.1 (Improved upper-bounds on T^* for linear velocities). *In particular, in the case of the LWR–Greenshields velocity function, i.e. $V(\xi) := 1 - \xi$, we obtain an improved estimate on the bound in* (7.3.3)*. We make the same ansatz as in* (7.3.2) *and compute*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\xi_{w^*}(0,0;t)\\
&= V\big(\mathcal{W}[\rho,u_r](t,\xi_{w^*}(0,0;t))\big) = 1 - \mathcal{W}[\rho,u_r](t,\xi_{w^*}(0,0;t))\\
&= 1 - \frac{1}{\alpha}\int_{\xi_{w^*}(0,0;t)}^\infty \exp\left(\frac{\xi_{w^*}(0,0;t)-y}{\alpha}\right)\left(\begin{cases}\rho(t,y) & \textit{if } y \leq 1\\ u_r(t) & \textit{if } y \geq 1\end{cases}\right)\mathrm{d}y
\end{aligned}$$

taking advantage of the maximum principle (7.1.4) *in Theorem 7.1.1*

$$
\begin{aligned}
&\geq 1-\frac{1}{\alpha}\int_{\xi_{w^*}(0,0;t)}^{\infty}\exp\left(\frac{\xi_{w^*}(0,0;t)-y}{\alpha}\right)\left(\begin{cases}\max\{\|\rho_0\|_{L^\infty((0,1))},u_r(t)\} & \textit{if } y\leq 1\\ u_r(t) & \textit{if } y\geq 1\end{cases}\right)\mathrm{d}y\\
&=1-\frac{\max\{\|\rho_0\|_{L^\infty((0,1))},u_r(t)\}}{\alpha}\int_{\xi_{w^*}(0,0;t)}^{1}\exp\left(\frac{\xi_{w^*}(0,0;t)-y}{\alpha}\right)\mathrm{d}y\\
&\quad-\frac{u_r(t)}{\alpha}\int_1^{\infty}\exp\left(\frac{\xi_{w^*}(0,0;t)-y}{\alpha}\right)\mathrm{d}y\\
&=1+\max\{\|\rho_0\|_{L^\infty((0,1))},u_r(t)\}\Big(\exp\left(\frac{\xi_{w^*}(0,0;t)-1}{\alpha}\right)-1\Big)\\
&\quad-u_r(t)\exp\left(\frac{\xi_{w^*}(0,0;t)-1}{\alpha}\right)\\
&=\exp\left(\frac{\xi_{w^*}(0,0;t)-1}{\alpha}\right)\big(\max\{\|\rho_0\|_{L^\infty((0,1))},u_r(t)\}-u_r(t)\big)\\
&\quad+1-\max\{\|\rho_0\|_{L^\infty((0,1))},u_r(t)\}\\
&\geq\exp\left(\frac{\xi_{w^*}(0,0;t)-1}{\alpha}\right)\max\{\|\rho_0\|_{L^\infty((0,1))}-\|u_r\|_{L^\infty((0,T))},0\}\\
&\quad+1-\max\{\|\rho_0\|_{L^\infty((0,1))},\|u_r\|_{L^\infty((0,T))}\}.
\end{aligned}
$$

Recalling that $\xi(0,0;0)=0$ *and solving the previous differential inequality in the case of equality, we obtain the following expression for the corresponding solution:*

$$
\begin{aligned}
y_\alpha(t)&=1+bt-\alpha\ln\left(a\left(1-e^{\frac{bt}{\alpha}}\right)+be^{\frac{1}{\alpha}}\right)+\alpha\ln(b),\\
a&:=\max\{\|\rho_0\|_{L^\infty((0,1))}-\|u_r\|_{L^\infty((0,T))},0\},\\
b&:=1-\max\{\|\rho_0\|_{L^\infty((0,1))},\|u_r\|_{L^\infty((0,T))}\}.
\end{aligned}
\tag{7.3.4}
$$

Solving for $T^*>0$ *such that* $y(T^*)=1$ *gives an upper-bound on* T^**:*

$$
T^*_{\textit{improved}}(\alpha)=\frac{\alpha}{b}\ln\left(\frac{a+b\exp\left(\frac{1}{\alpha}\right)}{a+b}\right).
\tag{7.3.5}
$$

Let us compare the results in Lemma 7.3.1 with the improved estimate in this remark. If we let $V(\xi):=1-\xi,\ \xi\in[0,1],\ \rho_0:=\frac{1}{2}$, *and* $u_r:=0$, *then, for the estimate in* (7.3.1)*, we obtain an upper-bound on* T^* *(which we call* T_1^**) given by*

$$T_1^*=\frac{1}{1-\left(1-\frac{1}{e}\right)}=e.$$

From (7.3.5)*, we obtain, for* $\alpha>0$*,*

$$T^*_{\textit{improved}}(\alpha)\leq 2\alpha\ln\left(\frac{1}{2}\left(1+e^{\frac{1}{\alpha}}\right)\right),$$

which is illustrated in Figure 7.4 for $\alpha\in(0,2)$*. The improved estimate on* T^**, i.e.* $T^*_{\textit{improved}}(\alpha)$ *for* $\alpha>0$*, is sharper. It also depends on the nonlocal parameter* $\alpha>0$*. As the right-hand side boundary datum is minimal here, it is expected that the nonlocal impact* $\mathcal{W}[\rho,u_r](t,\xi(0,0;t))$ *becomes smaller with larger* α *as the nonlocal right-hand side datum* u_r *has an increasingly powerful influence on the nonlocal impact. Thus, the velocity is higher and the upper-bound on the time when the initial datum has left becomes smaller.*

REMARK 7.3.2 (Limit $\alpha\to 0^+$). *The upper-bound* $T^*_{\textit{improved}}(\alpha)$ *(see* (7.3.5)*) on* T^**, the time when the initial datum has left the domain (defined in* (7.3.1)*), is a function of* $\alpha>0$*. For* $\alpha\to 0^+$*, we formally obtain the local conservation law. Although the results from* CHAPTERS *3 and 4 cannot be easily extended to the IBVP, it is still interesting to calculate the limit for* $\alpha\to 0^+$ *of the*

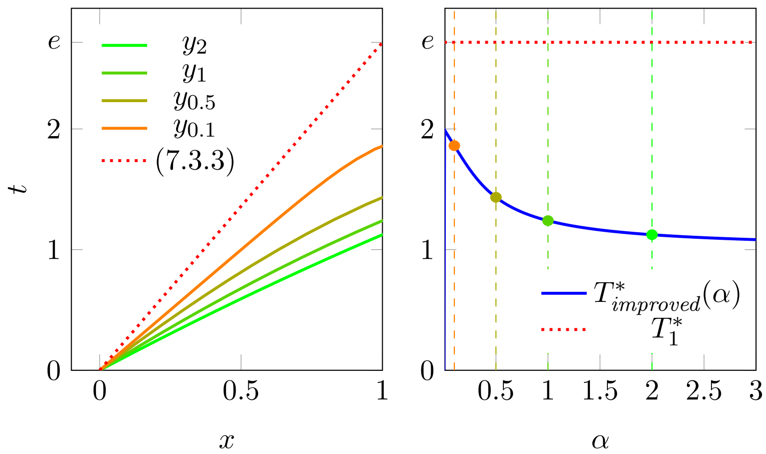

FIGURE 7.4. Illustration of the different upper-bounds for the initial datum leaving the domain as defined in (7.3.4). Here, we chose $u_r := 0$ and $\rho_0 := \frac{1}{2}$. LEFT: The different upper-bounds for the zero characteristics $t \mapsto \xi(0,0;t)$. The dashed red line is the—rather coarse—estimate uniform in α given in Lemma 7.3.1. The solid lines, which represent the improved upper-bounds on T^* for the LWR–Greenshields velocity functions, exhibit higher accuracy. RIGHT: The improved bounds on T^* for different values of the parameter α. As α becomes larger, the upper-bound becomes smaller. This is because we have initialized the right-hand side u_r as zero—so, for larger α, this zero becomes "more and more dominant", leading to an increased velocity approaching 1 in the limit. We also have $\lim_{\alpha\to 0^+} T^*(\alpha) = 1$.

upper-bound $T^*_{improved}(\alpha)$. We obtain

$$\lim_{\alpha\to 0^+} T^*_{improved}(\alpha) = \lim_{\alpha\to 0} \frac{\alpha}{b} \ln\left(\frac{a + b\exp(\frac{1}{\alpha})}{a+b}\right) = \frac{1}{b},$$

(compare also Figure 7.4 for $b = \frac{1}{2}$). This is then an upper-bound for the time the local conservation law needs to transport the mass of the initial datum out of the domain.

Having shown that, for a reasonable right-hand side boundary value u_r, the initial datum leaves the domain in finite time, we can state our main result in this Section.

THEOREM 7.3.1 (Equivalence controllability/time-inverted dynamics). *Let us suppose that Assumption 1 holds, $\rho_0 \in L^\infty((0,1);[0,1])$, $u_r \in L^\infty(\mathbb{R}_{>0};[0,c])$, $c \in [0,1)$, $\rho_{des} \in L^\infty((0,1);[0,1))$, and $\rho_r \in L^\infty((0,\infty);[0,1])$. Let us define*

$$T^* := T^*_{\rho_0,u_r} := \operatorname*{arg\,min}_{t>0} \{\xi[\rho_0,u_r](0,0;t) = 1\}, \tag{7.3.6}$$

$$\Xi_{\rho_0,u_r} := \{(t,x) \in \Omega_{T^*} : \xi[\rho_0,u_r](0,0;t) < x < 1\}, \tag{7.3.7}$$

$$v[\rho,u_r](t,x) := \begin{cases} \rho(t,x) & \textit{if } (t,x) \in \Xi_{\rho_0,u_r}, \\ u_r(t) & \textit{if } x > 1, \\ 0 & \textit{otherwise}, \end{cases} \qquad (t,x) \in \Xi \cup [0,T^*] \times \mathbb{R}_{>1}, \tag{7.3.8}$$

$$\widetilde{\mathcal{W}}[p,v](t,x) := \frac{1}{\alpha}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right)\left(\begin{cases} p(t,y) & \textit{if } (t,y) \in \Omega_{T^*}\setminus\Xi_{\rho_0,u_r} \\ v[\rho,u_r](t,y) & \textit{otherwise} \end{cases}\right) \mathrm{d}y, \qquad (t,x) \in \Omega_{T^*}. \tag{7.3.9}$$

Then, the following two results hold.

(1) *There exists* $u_\ell \in L^\infty((0,T^*);[0,1])$ *such that* $\rho(T^*,\cdot) = \rho_{des}$ *if and only if the backward nonlocal conservation law*

$$\partial_t p(t,x) = -\partial_x\big(V(\widetilde{\mathcal{W}}[p, v[\rho, u_r]](t,x)p(t,x)\big), \qquad (t,x) \in \Omega_{T^*} \setminus \Xi_{\rho_0,u_r}, \tag{7.3.10}$$

$$p(T^*,x) = \rho_{des}(x), \qquad x \in (0,1), \tag{7.3.11}$$

with $v[\rho,u_r]$ *as in* (7.3.8) *and* $\tilde{\mathcal{W}}$ *as in* (7.3.9), *admits a solution satisfying* $\|p\|_{L^\infty((0,T^*);L^\infty((0,1)))} \leq 1$.

(2) *There exists* $T \in [T^*,\infty)$ *and* $u_\ell \in L^\infty((0,T);[0,1])$ *such that* $\rho(t,1) \equiv \rho_r(t)$ *for a.e.* $t \in (T^*,T)$ *if and only if the backward nonlocal conservation law*

$$\partial_t p(t,x) = -\partial_x\big(V(\widetilde{\mathcal{W}}[p, v[\rho, u_r]](t,x)p(t,x)\big) \tag{7.3.12}$$

$$p(t,1) = \rho_r(t), \qquad t \in (T^*,T), \tag{7.3.13}$$

$$p(T,x) = 0, \qquad x \in [0,1], \tag{7.3.14}$$

admits a solution, satisfying $\|p\|_{L^\infty((0,T);L^\infty((0,1)))} \leq 1$.

Proof. First, we mention that T^* as in (7.3.6) exists and is unique owing to Lemma 7.3.1. We prove only Claim (1) as the second result can be obtained analogously.

Let us assume that we can control the system to the desired end-state/boundary-state. Then, we can time-invert the dynamics; the solution to the corresponding backward-in-time system exists and satisfies (7.3.10)–(7.3.11).

Conversely, let us assume that the backward system admits a weak solution. Then, we can evaluate the solution at $x = 0$ to obtain the proper boundary data, which indeed serves as a control to steer the system toward the desired state. The regularity required for this to hold is $C\big([0,1];L^1((0,T))\big)$. Although such regularity is not guaranteed in general (compare also [**172**, Remark 5.6]), it does hold provided the corresponding velocity is bounded away from zero. This is true in the underlying case, as also illustrated in Lemma 7.3.1, as long as $\|u_r\|_{L^\infty((0,T))} < 1$. □

The red lines indicate the data to be fitted, the blue areas illustrate the prescribed initial datum and the right-hand side nonlocal impact. The backward problem is—in both cases—considered on the grey area/domain.

Remark 7.3.3 (Controlling to target state and out-flux simultaneously). *It is straightforward to generalize the previous result to the case where we seek a left-hand side boundary datum* u_ℓ *and nonlocal right-hand side datum* u_r *such that, for a large time* $T > T^*$, *the end-state satisfies*

$$\rho(T,\cdot) = \rho_{des}$$

and the boundary value at $x = 1$ *is*

$$\rho(\cdot,1) = \rho_r.$$

We do not go into details.

As the previous result is not explicit in the sense that we cannot "a priori" determine which final states we can control the system to, we show in the following that a constant state can always be reached in a sufficiently large time when also controlling u_r.

Proposition 7.3.1 (Controllability to constant state). *Let* $\rho_{des} \equiv c \in [0,1)$ *and* $\rho_0 \in L^\infty((0,1);[0,1])$ *be given. Then, there exists* $T > 0$ *and* $(u_\ell, u_r) \in L^\infty((0,T);[0,1])^2$ *such that* $\rho(T,\cdot) \equiv \rho_{des}$, *where* ρ *denotes the solution of the IBVP* (7.0.1) *with boundary datum* u_ℓ, *right-hand side datum* u_r, *and initial datum* ρ_0.

Proof. We prove this result by introducing different steps in which we control the solution to a specific target.

Step 1. *Control to zero.* First, from Lemma 7.3.1, there exists $T^* > 0$ such that, if $u_\ell(t) = u_r(t) = 0$ for all $t \in [0,T^*]$, then the initial datum leaves the domain:

$$\rho(T^*,\cdot) \equiv 0,$$

i.e. the road is fully evacuated at $t = T^*$.

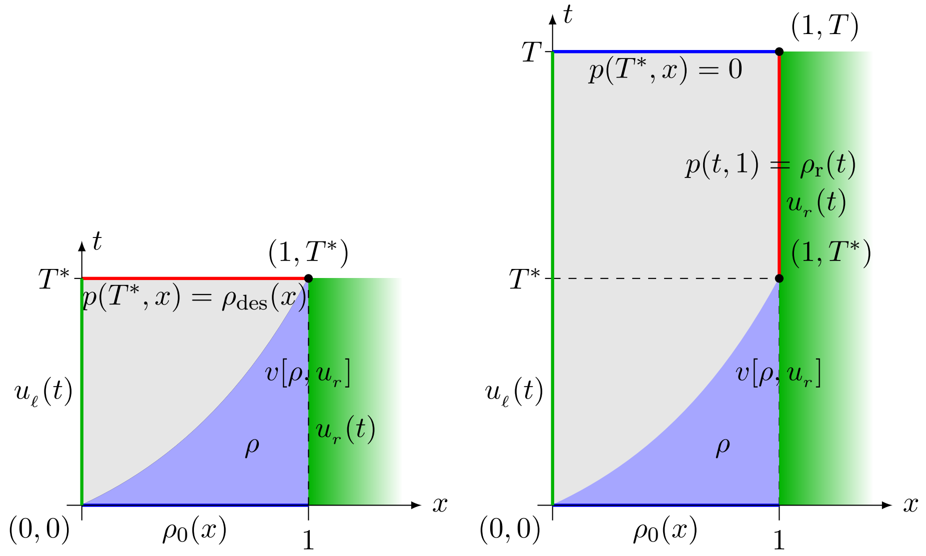

FIGURE 7.5. LEFT: Illustration of the statement in Theorem 7.3.1, (Claim (1)). Red shows the desired end-value we wish to control the system to and blue shows the known quantities. The green color indicates the boundary controls that we use. The backward-in-time equation is considered in the grey area. RIGHT: Illustration of the statement in Theorem 7.3.1 (Claim (2)). Red again indicates (here) the out-flux we would like to achieve and in blue we have the quantities that are given (in particular, the end-value can be chosen arbitrarily). Green indicates the quantity we can use to control the system, i.e. left and right-hand side boundary data. The backward system is considered on the grey area with explicit boundary conditions from $(1,T^*)$ to $(1,T)$.

Step 2. *Control to a small constant target and iteration.* Second, we show that the zero initial state can be controlled in finite time to the steady state $\varepsilon>0$ (for ε sufficiently small) and iterate this process until we have reached the target constant state. We take advantage of Theorem 7.3.1 and consider the following sequences of surrogate backward-in-time problems on $(t,x)\in\Omega_{T_n^*}\setminus\Xi_{\varepsilon(n-1),\varepsilon(n-1)}$:

$$\begin{aligned}\partial_t p_n(t,x)&=-\partial_x\big(V(\widetilde{\mathcal{W}}[p_n,v[\varepsilon(n-1),\varepsilon(n-1)]](t,x)p_n(t,x)\big),\\ p_n(T_n^*,x)&=n\varepsilon,\end{aligned}\tag{7.3.15}$$

where $n\in\mathbb{N}_{\geq1}$ and $T_n^*:=\sum_{k=1}^n T^*_{(k-1)\varepsilon,(k-1)\varepsilon}$ as in (7.3.6). As we will stop when we find $n^*\in\mathbb{N}_{\geq1}$ such that $n^*\varepsilon=c$ (we can always choose ε so that this holds), we can immediately provide a uniform upper-bound on T_n^* by invoking Lemma 7.3.1:

$$T^*_{(n-1)\varepsilon,(n-1)\varepsilon}\leq\frac{1}{V\big(1-\frac{1-c}{\varepsilon}\big)}\quad\text{and}\quad T_n^*\leq\frac{n}{V\big(1-\frac{1-c}{\varepsilon}\big)},\qquad n\in\mathbb{N}_{\geq1},\tag{7.3.16}$$

thanks to the monotonicity of V. Now, we show that, for a sufficiently small ε, the system (7.3.15) admits a solution on the entire time-horizon $\frac{1}{V(1-\frac{1-c}{\varepsilon})}$. To this end, we examine how at a given space-time point $(t,x)\in(T_{n-1}^*,T_n^*)\setminus\Xi_{\varepsilon(n-1),\varepsilon(n-1)}$ a maximum evolves backward in time. Assuming that at such a (t,x) the solution is maximal, parametrized on the characteristics, i.e., $p_n(t,\xi(T^*,x;t))=\|p_n(t,\cdot)\|_{L^\infty(\mathbb{R})}$ (and thus also $\partial_2p_n(t,\xi(T^*,x;t))=0$), we estimate (recalling the definition of the operator v in (7.3.8))

$$\begin{aligned}&-\frac{\mathrm{d}}{\mathrm{d}t}p_n(t,\xi(T_n^*,x;t))\\ &\quad=V'(\widetilde{\mathcal{W}}[p_n,v[\varepsilon(n-1),\varepsilon(n-1)]](t,\xi(T_n^*,x;t))\\ &\qquad\times\partial_2\widetilde{\mathcal{W}}[p,v[\varepsilon(n-1),\varepsilon(n-1)]](t,\xi(T_n^*,x;t))\end{aligned}$$

$$
\begin{aligned}
&= V'(\widetilde{\mathcal{W}}[p_n, v[\varepsilon(n-1), \varepsilon(n-1)]](t, \xi(T_n^*, x; t)) \\
&\qquad \times \frac{1}{\alpha}(\widetilde{\mathcal{W}}[p_n, v[\varepsilon(n-1), \varepsilon(n-1)]](t, \xi(T_n^*, x; t)) - p_n(t, \xi(T_n^*, x; t))) \\
&\leq \frac{1}{\alpha}\|V'\|_{L^\infty((0,1))} p_n(t, \xi(T_n^*, x; t))^2.
\end{aligned}
$$

Integrating the previous differential inequality backward in time from T_n^* to t yields the upper-bound

$$
\|p_n(t, \cdot)\|_{L^\infty((0,1))} \leq \frac{1}{\frac{n}{\varepsilon} - \frac{1}{\alpha}\|V'\|_{L^\infty((0,1))}(T_n^* - t)} \overset{!}{\leq} 1.
$$

For admissibility, we need to ensure that $\|p_n(t, \cdot)\|_{L^\infty((0,1))} \leq 1$, which is satisfied if

$$
t \geq T_n^* - \alpha \frac{1 - n\varepsilon}{\varepsilon \|V'\|_{L^\infty((0,1))}} \geq T_n^* - \alpha \frac{1 - \kappa}{\varepsilon \|V'\|_{L^\infty((0,1))}}.
$$

For $\varepsilon > 0$ sufficiently small, we obtain the well-posedness of the backward system (7.3.15) on every time-horizon and thus, particularly, on the time-horizon required for the initial datum to leave, i.e. (7.3.16). As the estimates are uniform in $n \in \mathbb{N}_{\geq 1}$, we can then pick as many sequences as needed to iteratively control the zero initial datum to ε, 2ε,... until we have reached the constant state c. □

Remark 7.3.4 (Extensions of Proposition 7.3.1). *Proposition 7.3.1 can be generalized. For instance, the solution can also be steered to a monotonically increasing ρ_{des} by first controlling it to the sufficiently large constant state $\mathbb{R} \ni c \geq \|\rho_{des}\|_{L^\infty((0,1))}$ and then showing that the backward-in-time system does not blow up (due to the assumed monotonicity). Another extension might consist of slightly perturbing the constant ρ_{des} with respect to the L^∞-norm and still achieving controllability (compare also Remark 7.2.1). We do not go into details.*

Example 7.3.1 (Controllability and lack of controllability in minimal time). *We present some examples related to the state controllability result in Theorem 7.3.1. In Figure 7.6, we consider three cases: $\rho_{des}(x) := \frac{1}{2}(1-x)$, $\rho_{des}(x) := \frac{1}{2}x$, and $\rho_{des} := \frac{1}{2}$, with initial and right boundary data given by $\rho_0(x) := \frac{1}{2}(1+x)$ and $u_r := \frac{1}{2}$. Figure 7.7 also shows the left-hand side boundary datum u_ℓ to achieve the desired final state ρ_{des} in minimal time. In the three pictures on the left-hand side in Figure 7.6, the initial datum leaves faster. This is due to the fact that α is larger, meaning that the nonlocal right-hand side boundary datum $u_r = \frac{1}{2}$ has a higher impact on the velocity of the entire road. Another noteworthy point is that, for smaller α and end-datum $\rho_{des}(x) = \frac{1}{2}x$ (see the fifth pictures in Figure 7.6 or the maximum of the yellow dotted curve in Figure 7.7), the solution actually becomes larger than the desired state and then decreases. This indicates that, in general, not every end-state can be tracked, as the corresponding control could exceed 1 and therefore would not be admissible.*

Finally, all the images indicate that the solution below the characteristics emanating from $(0,0)$, i.e. the solution which only depends on the initial datum and the right-hand side nonlocal impact u_r, stays the same regardless of the boundary datum. Thus, the time when the initial datum has left is the same.

7.4. Long-time behavior

In this Section, we are concerned with the long-time behavior of the solution to the IBVP (7.0.1) when prescribing constant boundary data $(u_\ell, u_r) \in [0,1)^2$. Under the assumption that the initial datum is uniformly less than or equal to, or greater than or equal to, $u_\ell = u_r$, we can show that the solution converges to a given constant.

Theorem 7.4.1 (Long-time behavior). *Let us suppose that $\kappa \in (0,1)$ is given, Assumption 1 holds, $\alpha > 0$, $u_r \equiv \kappa$, $u_\ell \equiv \kappa$, and $V'(\xi) < 0$ for $\xi \in [0,1)$. In addition, let us assume that*

$$
\Big(\rho_0 \geq \kappa \text{ on } (0,1)\Big) \quad \text{or} \quad \Big(\rho_0 \leq \kappa \text{ on } (0,1)\Big). \tag{7.4.1}
$$

Then, the corresponding solution ρ converges exponentially in time to κ:

$$
\|\rho(t, \cdot) - \kappa\|_{L^1((0,1))} \leq \alpha \left(\exp\left(\frac{\|\rho_0 - \kappa\|_{L^1((0,1))}}{\alpha}\right) - 1\right) \exp\left(\frac{K(\alpha)}{\alpha} t\right), \qquad t \geq 0,
$$

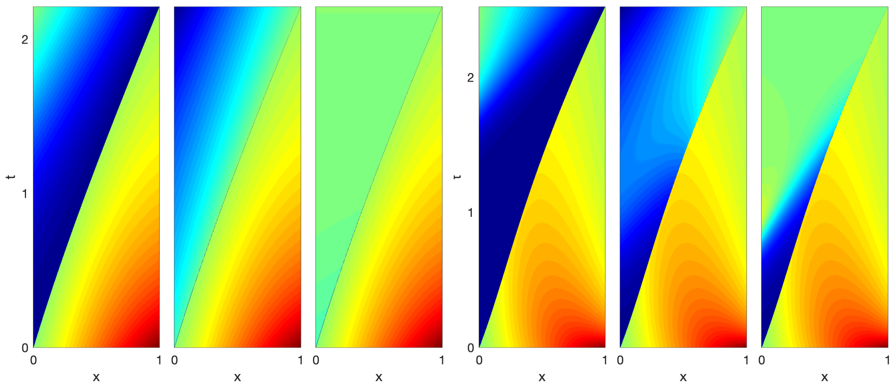

FIGURE 7.6. The three images on the left correspond to $\alpha = 1$, the three on the right to $\alpha = 0.1$. In the leftmost images of both triples, $\rho_{\mathrm{des}}(x) := \frac{1}{2}(1-x)$; in the middle, $\rho_{\mathrm{des}}(x) := \frac{1}{2}x$; and, in the right images, $\rho_{\mathrm{des}}(x) := \frac{1}{2}$. In all images, the initial datum is given by $\rho_0(x) := \frac{1}{2}(1+x)$ and the right-hand side boundary datum by $u_r(t) := \frac{1}{2}$. COLOR BAR: 0 1

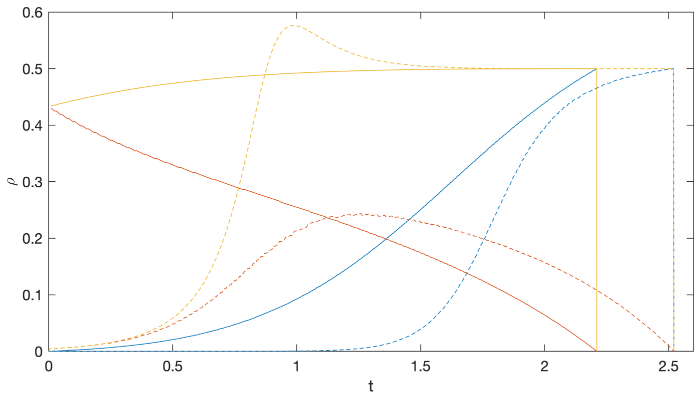

FIGURE 7.7. Corresponding to Figure 7.6 in Example 7.3.1, we illustrate the left-hand side boundary datum u_ℓ to achieve the desired final state ρ_{des} in minimal time. Solid lines represent the boundary datum for $\alpha = 1$, dashed lines for $\alpha = 0.1$. The colors represent the related desired state ρ_{des} that we wish to achieve: for $x \in [0,1]$, we have in red $\rho_{\mathrm{des}}(x) := \frac{1}{2}x$, in blue $\rho_{\mathrm{des}}(x) := \frac{1}{2}(1-x)$, and in yellow $\rho_{\mathrm{des}}(x) := \frac{1}{2}$.

where

$$\bar{\kappa} := (1-\kappa)\left(1-\exp\left(-\alpha^{-1}\right)\right),$$
$$K(\alpha) := (1-\kappa)\kappa \sup_{\xi\in\langle\kappa,\bar{\kappa}\rangle} V'(\xi)\exp\left(-\alpha^{-1}\right) < 0,$$
$$\langle a,b\rangle := \left(\min\{a,b\},\max\{a,b\}\right),\ (a,b)\in\mathbb{R}^2.$$

PROOF. Let us define the difference between $\rho(t,\cdot)$ and κ in the integral sense for $t\in[0,T]$:

$$M(t) := \int_0^1 (\rho(t,x)-\kappa)\,\mathrm{d}x.$$

As we want to compute the time-derivative of $M(t)$, we first need to show that $t \mapsto M(t)$ is differentiable. This can be achieved by taking advantage of the solution formula in terms of the characteristics in (7.1.2). Assuming that $T^* > 0$ and $\xi(0,0;T^*) = 1$, we can write, for $t\in[0,T^*]$,

$$M(t) = \int_0^{\xi(0,0;t)} u(\xi_{w^*}[t,x]^{-1}_{\max}(0))\,\partial_2\xi_{w^*}(t,x;\xi_{w^*}[t,x]^{-1}_{\max}(0))\,\mathrm{d}x$$

$$
\begin{aligned}
&+\int_{\xi(0,0;t)}^{1} \rho_0(\xi(t,x;0))\partial_2\xi(t,x;0)\,\mathrm{d}x-\kappa\\
&=\int_0^t u(z)V(\mathcal{W}[\rho](z,0))\,\mathrm{d}z+\int_0^{\xi(t,1;0)}\rho_0(z)\,\mathrm{d}z-\kappa,
\end{aligned}
$$

which is differentiable with respect to t sufficiently small. Taking the time derivative, we obtain

$$
\begin{aligned}
M'(t)&=u(t)V(\mathcal{W}[\rho](t,0))+\rho_0(\xi(t,1;0))\partial_1\xi(t,1;0)\\
&=\kappa V(\mathcal{W}[\rho](t,0))-\rho(t,1)V(\mathcal{W}[\rho](t,1)),
\end{aligned}
\tag{7.4.2}
$$

which actually holds for every time $t>0$.

Let us assume for now that $\rho_0\geq\kappa$. Then, owing to the maximum principle (7.1.4),

$$
\rho(t,x)\geq\kappa,\qquad (t,x)\in[0,T]\times(0,1),
\tag{7.4.3}
$$

which yields

$$
1\geq M(t)\geq 0,\qquad t\in[0,T].
\tag{7.4.4}
$$

The upper-bound 1 is a consequence of the maximum principle and the assumption $\kappa\in[0,1]$. Then, we estimate the nonlocal impact as follows:

$$
\begin{aligned}
\mathcal{W}[\rho](t,0)&:=\frac{1}{\alpha}\int_0^\infty \exp\Big(-\frac{s}{\alpha}\Big)\left(\begin{cases}\rho(t,s) & s<1\\ \kappa & s\geq 1\end{cases}\right)\mathrm{d}s,\\
&\overset{(7.4.3)}{\geq}\frac{\kappa}{\alpha}\int_0^{1-M(t)}\exp\Big(-\frac{s}{\alpha}\Big)\,\mathrm{d}s+\frac{1}{\alpha}\int_{1-M(t)}^{1}\exp\Big(-\frac{s}{\alpha}\Big)\,\mathrm{d}s+\frac{\kappa}{\alpha}\int_1^\infty\exp\Big(-\frac{s}{\alpha}\Big)\,\mathrm{d}s\\
&=\frac{\kappa}{\alpha}\int_0^\infty\exp\Big(-\frac{s}{\alpha}\Big)\,\mathrm{d}s+\frac{1-\kappa}{\alpha}\int_{1-M(t)}^{1}\exp\Big(-\frac{s}{\alpha}\Big)\,\mathrm{d}s\\
&=\kappa+(1-\kappa)\left(\exp\Big(-\frac{1-M(t)}{\alpha}\Big)-\exp\Big(-\frac{1}{\alpha}\Big)\right)\\
&=\kappa+\exp\Big(-\frac{1}{\alpha}\Big)(1-\kappa)\left(\exp\Big(\frac{M(t)}{\alpha}\Big)-1\right),
\end{aligned}
$$

from which we can continue the estimate on $M(t)$ in (7.4.2). Recalling that $\rho(t,1)\geq\kappa$ for all $t\in[0,T]$ and that $V'\leq 0$, we get

$$
M'(t)\leq V\Big(\kappa+\exp\Big(-\frac{1}{\alpha}\Big)(1-\kappa)\Big(\exp\Big(\frac{M(t)}{\alpha}\Big)-1\Big)\Big)\kappa-\underbrace{V(\kappa)\rho(t,1)}_{\geq V(\kappa)\kappa}.
$$

Using the mean-value theorem and defining $\bar{\kappa}:=(1-\kappa)(1-\exp(-\alpha^{-1}))<1$, we deduce

$$
\begin{aligned}
M'(t)&\leq\sup_{\xi\in(\kappa,\bar{\kappa})}V'(\xi)\Big(\exp\Big(-\frac{1}{\alpha}\Big)(1-\kappa)\Big(\exp\Big(\frac{M(t)}{\alpha}\Big)-1\Big)\Big)\kappa\\
&\leq(1-\kappa)\kappa\sup_{\xi\in(\kappa,\bar{\kappa})}V'(\xi)\exp\Big(-\frac{1}{\alpha}\Big)\Big(\exp\Big(\frac{M(t)}{\alpha}\Big)-1\Big).
\end{aligned}
$$

We solve the previous differential inequality for equality, call the solution $M_{=}(t)$, and obtain

$$
\begin{aligned}
M_{=}(t)&=-\alpha\ln\left(\left(\exp\Big(-\frac{M(0)}{\alpha}\Big)-1\right)\exp\Big(\frac{K(\alpha)t}{\alpha}\Big)+1\right),\\
K(\alpha)&:=(1-\kappa)\kappa\sup_{\xi\in(\kappa,\bar{\kappa})}V'(\xi)\exp\Big(-\frac{1}{\alpha}\Big)<0.
\end{aligned}
\tag{7.4.5}
$$

From this, we conclude

$$
0\leq M(t)\leq M_{=}(t),\qquad t>0.
\tag{7.4.6}
$$

Using the fact that $\ln(\xi) \leq \xi - 1$ for $\xi > 0$, we compute

$$M_{=}(t) = \alpha \ln\left(\left(1 - \left(1 - \exp\left(-\frac{M(0)}{\alpha}\right)\right)\exp\left(\frac{K(\alpha)t}{\alpha}\right)\right)^{-1}\right)$$

$$\leq \alpha \frac{\left(1 - \exp\left(-\frac{M(0)}{\alpha}\right)\right)\exp\left(\frac{K(\alpha)t}{\alpha}\right)}{1 - \left(1 - \exp\left(-\frac{M(0)}{\alpha}\right)\right)\exp\left(\frac{K(\alpha)t}{\alpha}\right)} \leq \alpha\left(\exp\left(\frac{M(0)}{\alpha}\right) - 1\right)\exp\left(\frac{K(\alpha)}{\alpha}t\right).$$

For initial datum $\rho_0(x) \leq \kappa$ for a.e. $x \in (0,1)$, the results follow by performing similar estimates with the opposite sign.

□

Remark 7.4.1 (Theorem 7.4.1 for $\kappa = 0$ and $\kappa = 1$). *The previous result does not provide exponential stability for $\kappa \in \{0,1\}$ as then $K(\alpha) = 0$ for $\alpha > 0$.*

However, for $\kappa = 0$, the boundary contribution to the solution is zero and, by Lemma 7.3.1, we know that the initial data leaves the domain in finite time $T^* > 0$. Afterwards, the solution remains identically zero so we have the stability to the zero solution in finite time and, in particular, exponentially.

For $\kappa = 1$, the situation is slightly more delicate. We look at time-evolution of the L^1-norm of the solution:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\rho(t,\cdot)\|_{L^1((0,1))} &= -\int_0^1 \partial_x\big(\rho(t,y)V(\mathcal{W}[\rho,u_r](t,y))\big)\,\mathrm{d}y \\
&= \rho(t,0)V(\mathcal{W}[\rho,u_r](t,0)) - \rho(t,1)V(\mathcal{W}[\rho,u_r](t,1)) \\
&= V(\mathcal{W}[\rho,u_r](t,0)) \\
&\geq V\left(\frac{1}{\alpha}\int_0^{\|\rho(t,\cdot)\|_{L^1((0,1))}} 1\cdot\exp\left(\frac{-y}{\alpha}\right)\mathrm{d}y + \exp\left(-\frac{1}{\alpha}\right)\right) \\
&= V\left(1 - \exp\left(\frac{-\|\rho(t,\cdot)\|_{L^1((0,1))}}{\alpha}\right) + \exp\left(-\frac{1}{\alpha}\right)\right)
\end{aligned} \tag{7.4.7}$$

using the mean-value theorem, there exists $\zeta \in (0,1)$ such that

$$\begin{aligned}
&= V(1) - V'(\zeta)\left(\exp\left(\frac{-\|\rho(t,\cdot)\|_{L^1((0,1))}}{\alpha}\right) - \exp\left(-\frac{1}{\alpha}\right)\right) \\
&\geq -\sup_{\xi\in(0,1)} V'(\xi)\left(\exp\left(\frac{-\|\rho(t,\cdot)\|_{L^1((0,1))}}{\alpha}\right) - \exp\left(-\frac{1}{\alpha}\right)\right)
\end{aligned}$$

and, consequently,

$$\|\rho(t,\cdot)\|_{L^1((0,1))} \geq 1 + \alpha\log\left(\left(\exp\left(\frac{\|\rho_0\|_{L^1((0,1))} - 1}{\alpha}\right) - 1\right)\exp\left(t\sup_{\xi\in(0,1)} V'(\xi)\frac{\exp\left(-\frac{1}{\alpha}\right)}{\alpha}\right) + 1\right).$$

As $\ln(\xi+1) \geq \xi(\xi+1)^{-1}$ for $\xi > -1$, we can continue our estimate as

$$\geq 1 + \alpha\frac{\left(\exp\left(\frac{\|\rho_0\|_{L^1((0,1))}-1}{\alpha}\right) - 1\right)\exp\left(t\sup_{\xi\in(0,1)} V'(\xi)\frac{\exp\left(-\frac{1}{\alpha}\right)}{\alpha}\right)}{\left(\exp\left(\frac{\|\rho_0\|_{L^1((0,1))}-1}{\alpha}\right) - 1\right)\exp\left(t\sup_{\xi\in(0,1)} V'(\xi)\frac{\exp\left(-\frac{1}{\alpha}\right)}{\alpha}\right) + 1}$$

$$\geq 1 + \frac{\alpha}{2}\left(\exp\left(\frac{\|\rho_0\|_{L^1((0,1))} - 1}{\alpha}\right) - 1\right)\exp\left(t\sup_{\xi\in(0,1)} V'(\xi)\frac{\exp\left(-\frac{1}{\alpha}\right)}{\alpha}\right).$$

This is the exponential convergence to the steady-state solution in the case that $\kappa = 1$, i.e., in the case the road is blocked on the right-hand side and $u_\ell \equiv 1$. For the statement to hold, we require

$$\sup_{\xi\in(0,1)} V'(\xi) < 0.$$

In case this assumption does not hold and only Assumption 1 applies, we can still show that the solution converges to the 1 *solution for* $t \to \infty$*, but without any order of convergence. The convergence is then due to the fact that the mapping* $t \mapsto \|\rho(t,\cdot)\|_{L^1((0,1))}$ *is monotonically increasing in* t *and bounded from above by* 1*. Then, we know that a limit point for this sequence exists, i.e., there exists* $A \in (0,1]$ *such that* $\lim_{t\to+\infty} \|\rho(t,\cdot)\|_{L^1((0,1))} = A$*. Thanks to* (7.4.7)*, the time derivative of* $\|\rho(t,\cdot)\|_{L^1((0,1))}$ *is non-negative and only zero for* $\|\rho(t,\cdot)\|_{L^1((0,1))} = 1$ *which implies* $A = 1$*.*

EXAMPLE 7.4.1 (Long-time behavior and comparison to steady-state solutions). *In Figure 7.8, we present some numerical simulations related to Theorem 7.4.1. We assume that*

$$V(\xi) := 1 - \xi,\ u_r := \frac{1}{2},\ u_\ell \in \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\},\ \alpha \in \{0.1, 1\},\ \rho_0 := \frac{1}{2}\mathbb{1}_{(\frac{1}{3},\frac{2}{3})}.$$

One remarkable feature which can be seen in all images is the fact that, after the initial datum has left, the solution no longer changes substantially and appears to become stationary. Although we are not able to show this in the general case, it appears that all solutions converge to the corresponding steady state, anticipating the existence and uniqueness of steady-state solutions in Theorem 7.5.1. Indeed, this is also illustrated in Figure 7.9: in the image on the left-hand side, the solutions are plotted at $t \in \{2,4,8\}$*; in the image on the right-hand side, we see the steady-state solution in comparison to the corresponding solution at time* $t = 8$*.*

It is worth mentioning the impact of the size of the nonlocal parameter $\alpha > 0$*. As the* L^1*-mass of the initial datum is smaller than* $u_r = \frac{1}{2}$*, in the present case, the initial datum leaves more quickly when* α *is larger.*

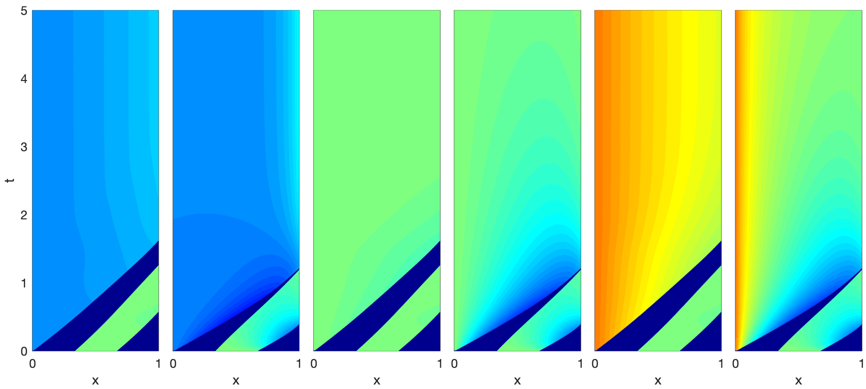

FIGURE 7.8. The images are ordered from left to right. FIRST: $u_\ell \equiv \frac{1}{4}$, $u_r \equiv \frac{1}{2}$, $\alpha = 1$. SECOND: $u_\ell \equiv \frac{1}{4}, u_r \equiv \frac{1}{2}, \alpha = 0.1$. THIRD: $u_\ell \equiv \frac{1}{2}$, $u_r \equiv \frac{1}{2}$, $\alpha = 1$. FOURTH: $u_\ell \equiv \frac{1}{2}$, $u_r \equiv \frac{1}{2}$, $\alpha = 0.1$. FIFTH: $u_\ell \equiv \frac{3}{4}$, $u_r \equiv \frac{1}{2}$, $\alpha = 1$. SIXTH: $u_\ell \equiv \frac{3}{4}$, $u_r \equiv \frac{1}{2}$, $\alpha = 0.1$. The initial datum is $\rho_0 := \frac{1}{2}\mathbb{1}_{(\frac{1}{3},\frac{2}{3})}$ in every case. COLOR BAR: 0 1

7.5. Steady states

In the following theorem, we prove the existence and uniqueness of steady-state solutions on a bounded domain when prescribing a constant left-hand side boundary datum and a constant nonlocal right-hand side boundary datum.

THEOREM 7.5.1 (Steady-state solutions on bounded domains). *For every* $u_\ell \equiv$ *const* $\in [0,1], u_r \equiv$ *const* $\in [0,1)$*, there exists a unique and monotone* $\bar\rho \in W^{1,\infty}((0,1); [\min\{u_\ell, u_r\}, \max\{u_\ell, u_r\}])$ *satisfying*

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\bar\rho(x) V(\mathcal{W}[\bar\rho, u_r](x))\Big) = 0, \qquad x \in [0,1], \tag{7.5.1}$$

$$\mathcal{W}[\bar\rho, u_r](x) = \frac{1}{\alpha}\int_x^\infty \exp\left(\frac{x-y}{\alpha}\right)\left(\begin{cases}\bar\rho(y) & \text{if } y < 1\\ u_r & \text{if } y \geq 1\end{cases}\right)\mathrm{d}y, \qquad x \in [0,1], \tag{7.5.2}$$

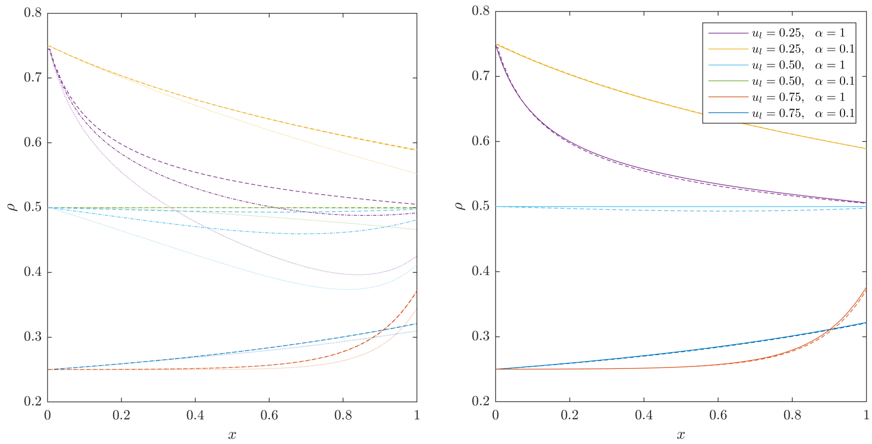

FIGURE 7.9. LEFT: Illustrations of the evolution of solutions at different time snippets $t \in \{2,4,8\}$ (dotted $t = 2$, dash-dotted $t = 4$, and dashed $t = 8$). The different colors represent the six different cases in Figure 7.8 for different u_ℓ and α (as described in the legend) and $u_r \equiv 0.5$. RIGHT: Comparison of the different solutions at $t = 8$ and the corresponding steady-state solutions as in Theorem 7.5.1.

$$\bar{\rho}(0) = u_\ell. \tag{7.5.3}$$

In addition, if $V \in C^\infty([0,1])$, then $\bar{\rho} \in C^\infty([0,1])$. The function $\bar{\rho}$ is called the steady state *corresponding to the boundary data u_ℓ and u_r.*

For $u_\ell \equiv u_r$ a solution is given by $\bar{\rho} \equiv u_r$, which can be checked by substituting it into (7.5.1)–(7.5.3). However, we need to prove that this is the only solution and that one and only one solution exists in the case $u_r \neq u_\ell$.

PROOF. **Step 1.** *Existence.* As a first step, we show the existence of solutions using a Schauder-type fixed-point argument. A solution of (7.5.1)–(7.5.3) can be interpreted as a fixed-point of the mapping

$$\boldsymbol{F} : \begin{cases} \Omega & \to \Omega, \\ \bar{\rho} & \mapsto \Big(x \mapsto u_\ell \frac{V(\mathcal{W}[\bar{\rho},u_r](0))}{V(\mathcal{W}[\bar{\rho},u_r](x))}\Big), \end{cases} \tag{7.5.4}$$

with a suitable $\Omega \subset C([0,1])$ yet to be defined. We distinguish two different cases: $u_\ell \geq u_\ell$ and $u_\ell \leq u_r$.

If $u_\ell \geq u_r$, we define

$$\begin{aligned} \Omega &:= \big\{\bar{\rho} \in W^{1,\infty}((0,1)) : \big(u_r \leq \bar{\rho}(x) \leq u_\ell\big) \wedge \big(\mathcal{A} \leq \bar{\rho}'(x) \leq 0\big)\ \forall x \in [0,1]\big\}, \\ \mathcal{A} &:= -u_\ell \frac{V(u_r)\|V'\|_{L^\infty(u_r,u_\ell)}(u_\ell - u_r)}{\alpha V(u_\ell)^2}, \end{aligned} \tag{7.5.5}$$

and show that $\boldsymbol{F}$ is a self-mapping on Ω, i.e. $\boldsymbol{F}[\Omega] \subseteq \Omega$. To this end, we take $\bar{\rho} \in \Omega$ and compute, for $x \in [0,1]$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{F}[\bar{\rho}](x) = -u_\ell \frac{V(\mathcal{W}[\bar{\rho},u_r](0))}{V(\mathcal{W}[\bar{\rho},u_r](x))^2} V'(\mathcal{W}[\bar{\rho},u_r](x))\partial_x \mathcal{W}[\bar{\rho},u_r](x). \tag{7.5.6}$$

Since $V' \leq 0$, we need to show that $\partial_x \mathcal{W}[\bar{\rho},u_r] \leq 0$, which we do with the following manipulations, for $x \in [0,1]$,

$$\partial_x \mathcal{W}[\bar{\rho},u_r](x) = \partial_x\Big(\frac{1}{\alpha}\int_x^1 \exp\Big(\frac{x-y}{\alpha}\Big)\bar{\rho}(y)\,\mathrm{d}y\Big) + \partial_x\Big(\frac{1}{\alpha}u_r\int_1^\infty \exp\Big(\frac{x-y}{\alpha}\Big)\,\mathrm{d}y\Big) \tag{7.5.7}$$

$$= -\frac{1}{\alpha}\bar{\rho}(x) + \frac{1}{\alpha^2}\int_x^1 \exp\Big(\frac{x-y}{\alpha}\Big)\bar{\rho}(y)\,\mathrm{d}y + \frac{1}{\alpha^2}u_r\int_1^\infty \exp\Big(\frac{x-y}{\alpha}\Big)\,\mathrm{d}y \tag{7.5.8}$$

$$= \frac{1}{\alpha}\Big(\mathcal{W}[\bar\rho, u_r](x) - \bar\rho(x)\Big). \tag{7.5.9}$$

As $\bar\rho$ is monotonically decreasing and $\bar\rho \geq u_r$, we obtain that $\mathcal{W}[\bar\rho, u_r] \leq \bar\rho$ and, consequently,

$$\partial_x \mathcal{W}[\bar\rho, u_r] \leq 0; \tag{7.5.10}$$

thus $\partial_x \boldsymbol{F}[\bar\rho] \leq 0$. From the monotonicity, it also follows that $\boldsymbol{F}[\bar\rho] \leq u_\ell$ as, by the very definition in (7.5.6), it holds that $\boldsymbol{F}[\bar\rho](0) = u_\ell$.

Next, we show that $\boldsymbol{F}[\bar\rho] \geq u_r$. To this end, let us assume, for the sake of finding a contradiction, that there exists $x^* \in (0,1)$ such that $\boldsymbol{F}[\bar\rho](x^*) < u_r$. As $\boldsymbol{F}[\bar\rho]$ is monotonically decreasing, we know that $\boldsymbol{F}[\bar\rho](x) < u_r$ for all $x \in (x^*, 1]$. For $x = 1$, it holds $\mathcal{W}[\boldsymbol{F}[\bar\rho], u_r](1) = u_r$, but $\mathcal{W}[\boldsymbol{F}[\bar\rho], u_r](x) < u_r$ for all $x \in [x^*, 1)$; this is a contradiction to the monotonicity of $\mathcal{W}[\bar\rho, u_r]$ as stated in (7.5.10). Thus, we conclude that

$$\boldsymbol{F}[\bar\rho](x) \geq u_r, \qquad x \in [0,1].$$

Next, we prove the lower-bound on $\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{F}[\bar\rho]$. Recalling (7.5.6), we estimate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{F}[\bar\rho](x) &\geq -u_\ell \frac{V(\mathcal{W}[u_r, u_r](0))\|V'\|_{L^\infty(u_r,u_\ell)}(u_\ell - \mathcal{W}[u_r, u_r](0))}{\alpha V(\mathcal{W}[u_\ell, u_r](0))^2}\\ &= -u_\ell \frac{V(u_r)\|V'\|_{L^\infty(u_r,u_\ell)}(u_\ell - u_r)}{\alpha V(u_\ell)^2} = \mathcal{A}.\end{aligned}$$

Thus, we have shown that $\boldsymbol{F}[\Omega] \subset \Omega$. To prove the existence of solutions, we apply Schauder's fixed-point theorem (see [**232**, Corollary 2.13]), which requires the following assumptions to be satisfied.

1. $\boldsymbol{F} : \Omega \to \Omega$ is continuous in a suitable topology. By choosing $C([0,1])$ with the natural maximum norm, $\boldsymbol{F}$ is indeed continuous.
2. The set Ω is closed in $C([0,1])$ and it is convex. We have closedness as we have uniform constraints on $\bar\rho$ in the definition of Ω; the convexity is evident.
3. Ω is compact in $C([0,1])$. We have this as the derivatives of functions in Ω have 0 as upper-bound and $\mathcal{A}$ as lower-bound, which is uniform. Therefore, the functions in Ω are uniformly Lipschitz continuous with Lipschitz constant $\mathcal{A}$. Thus, they are also equi-continuous and we can apply Ascoli–Arzelà's theorem [**48**, Theorem 4.25], which guarantees the claimed compactness, i.e. $\Omega \overset{\mathrm{c}}{\hookrightarrow} C([0,1])$.

Using Schauder's fixed-point theorem, we conclude that there exists a solution of (7.5.4) lying in Ω as defined in (7.5.5).

If $u_r \leq u_\ell$, the proof of existence is almost identical to the case $u_r \geq u_\ell$ when exchanging the monotonicity in Ω from decreasing to increasing. We do not go into details.

Step 2. *Uniqueness.* For the uniqueness, we reformulate the steady-state equation in (7.5.1) as a system of ODEs: introducing $g = \mathcal{W}[\bar\rho, u_r]$, we write

$$\begin{aligned}\bar\rho'(x) &= -\frac{1}{\alpha}\frac{\bar\rho(x)V'(g(x))(g(x)-\bar\rho(x))}{V(g(x))}, \quad && \bar\rho(0) = u_\ell, \qquad && x \in (0,1),\\ g'(x) &= \frac{1}{\alpha}(g(x) - \bar\rho(x)), && g(1) = u_r, && x \in (0,1).\end{aligned} \tag{7.5.11}$$

Let us consider now instead the following end-value problem for $s \in [0,1]$

$$\rho'(x) = -\frac{1}{\alpha}\frac{\rho(x)V'(g(x))(g(x)-\rho(x))}{V(g(x))}, \qquad x \in (0,1), \tag{7.5.12}$$

$$g'(x) = \frac{1}{\alpha}(g(x) - \rho(x)), \qquad x \in (0,1), \tag{7.5.13}$$

$$\rho(1) = s, \tag{7.5.14}$$

$$g(1) = u_r. \tag{7.5.15}$$

From the Lipschitz continuity of the right-hand side, we deduce that this end value problem has a unique solution of corresponding regularity.

Let us assume that $u_r < 1$ and consider the following end-value problem in $\bar{\rho}$:

$$\bar{\rho}_s(x) = \frac{sV(u_r)}{V(W[\bar{\rho}_s, u_r](x))}, \qquad x \in [0,1]. \tag{7.5.16}$$

We claim that the left-hand side boundary datum (which we want to prescribe) is strictly monotone with respect to $s \in [0,1]$. The differentiability of $\bar{\rho}_s$ with respect to s follows by the implicit function theorem (see [**232**, Theorem 4.B]); to obtain an expression for the derivative, we differentiate the fixed-point problem in (7.5.16) with respect to $s \in [0,1]$ and deduce, for $x \in [0,1]$,

$$\begin{aligned}\partial_s \bar{\rho}_s(x) &= \frac{V(u_r)}{V(W[\bar{\rho}_s, u_r](x))} - \frac{sV(u_r)}{V(W[\bar{\rho}_s, u_r](x))^2}\partial_s V(W[\bar{\rho}_s, u_r](x)) \\ &= \frac{V(u_r)}{V(W[\bar{\rho}_s, u_r](x))} - \frac{sV(u_r)}{V(W[\bar{\rho}_s, u_r](x))^2}V'(W[\bar{\rho}_s, u_r](x))W[\partial_s\bar{\rho}_s, 0](x).\end{aligned}$$

This is a Volterra integral equation of the second kind in $\partial_s\bar{\rho}_s$ (where $\bar{\rho}_s$ is given) and admits a unique solution by classical fixed-point methods. Moreover, owing to the specific structure of the right-hand side, we have $\partial_s\bar{\rho}_s > 0$ on $[0,1]$; thus, we can conclude that left-hand side boundary datum (which we would like to prescribe in our original problem (7.5.11)) is strictly monotone with respect to s. As we have shown previously (by the Schauder-type argument) that we can achieve all left-hand side boundary data, we have the existence and uniqueness of steady-state solutions for $u_r < 1$.

A similar proof can be made for $u_\ell \geq u_r$ by changing the IBVP to the corresponding end-value problem and again using the existence of solutions as obtained by the previous Schauder-type argument.

Step 3. *Regularity.* To establish the higher regularity of solutions, we recall the fixed-point problem in (7.5.4) which has a unique solution by the argument above. Differentiating gives

$$\bar{\rho}'(x) = -u_\ell \frac{V(\mathcal{W}[\bar{\rho}, u_r](0))}{V(\mathcal{W}[\bar{\rho}, u_r](x))^2} V'(\mathcal{W}[\bar{\rho}, u_r](x))\partial_x \mathcal{W}[\bar{\rho}, u_r](x), \quad x \in [0,1]. \tag{7.5.17}$$

We know that $\bar{\rho} \in W^{1,\infty}((0,1))$; moreover, as $\partial_x\mathcal{W}[\bar{\rho}, u_r]$ is Lipschitz continuous by (7.5.7)–(7.5.9), the entire right-hand side of (7.5.17) is Lipschitz continuous and thus also $\bar{\rho}'$ is. This argument can be iterated to deduce the claimed regularity. $\square$

REMARK 7.5.1 (The case $u_r \equiv 1$ in Theorem 7.5.1). *In Theorem 7.5.1, we assumed $u_r \neq 1$ to avoid the need to write the boundary condition on the left-hand side in terms of flux and instead of density (see* (7.0.3) *and the original description of the boundary values in* (7.0.1)*). For $u_r \equiv 1$, the boundary condition for the steady-state solution in* (7.5.3) *needs to be formulated in terms of flux, i.e.*

$$V(\mathcal{W}[\bar{\rho}, 1](0))\bar{\rho}(0) = V(\mathcal{W}[\bar{\rho}, 1](0))u_\ell \tag{7.5.18}$$

(and $V(\mathcal{W}[\bar{\rho}, 1](0))$ might be zero). In this case, we compute

$$\bar{\rho}(x)V(\mathcal{W}[\bar{\rho}, 1](x)) = \bar{\rho}(1)V(\mathcal{W}[\bar{\rho}, 1](1)) = \bar{\rho}(1)V(1) = 0, \qquad x \in [0,1].$$

For this equation to hold, the following needs to be satisfied:

$$\bar{\rho}(x) = 0 \quad \textit{or} \quad \mathcal{W}[\bar{\rho}, 1](x) = 1, \quad x \in [0,1].$$

As this also has to hold at $x = 0$, we deduce

$$\bar{\rho}(0) = 0 \quad \textit{or} \quad \mathcal{W}[\bar{\rho}, 1](0) = 0.$$

This can only hold if $u_\ell \equiv 0$ (first case) or $\bar{\rho} \equiv 1$ for all $u_\ell \in (0,1]$ (second case, the left-hand side boundary datum is not necessarily attained, but the flux condition (7.5.18) *holds). However, in the first case, $u_\ell \equiv 0$, the solution $\bar{\rho}$ is not uniquely determined on $(0,1)$ and all solutions can be parametrized, for $a \in [0,1]$, by*

$$\bar{\rho} = \mathbb{1}_{[a,1]}, \qquad u_\ell \equiv 0, \quad u_r \equiv 1.$$

In the description of traffic flow, this can be interpreted as a red light at the end of the road and no entering cars. A traffic jam of any length at the traffic light is then a stationary solution.

Part 2

Conservation laws on networks

CHAPTER 8

Controllability of advection-diffusion equations on networks and singular limits

In this Chapter, we prove the following results.

1. By leveraging the method of characteristics, we control the hyperbolic problem (1.2.2) to zero (for sufficiently long times) by acting on the "input" boundary vertices and discuss the optimal time for which (1.2.2) is null-controllable.
2. We estimate the cost of controllability for the parabolic problem (1.2.1), which depends on the time-horizon: for small times, we prove the (exponential) blow-up of the cost of controllability; for sufficiently long time horizon, we prove its decay.

In Section 8.1, we introduce some preliminary information on the function spaces used throughout the Chapter and present the known results on the well-posedness of problems (1.2.2) and (1.2.1) and on the convergence of (1.2.1) to (1.2.2).

In Section 8.2, we state our main theorems and present some pathological cases to illustrate the sharpness of our results.

In Section 8.3, we prove the controllability result for the transport equation on a tree-shaped network by relying on the classical method of characteristics: thanks to the flux conservation condition in (1.2.2), we are able to argue analogously to the case of a bounded interval, where it suffices to take zero boundary data as control.

Sections 8.4 and 8.5 are dedicated to the singular limit problem. In the first one, we prove the blow-up of the cost of controllability for the parabolic problem (1.2.1) and, in the second one, we prove the decay (for sufficently long time-horizon).

8.1. Preliminaries

8.1.1. Function spaces on a network and parametrization of the edges. As in [**125**], we use the following notation for the space of square-integrable functions:

$$L^2(\mathcal{E}) \coloneqq L^2(e_1) \times \cdots \times L^2(e_m) = \{w : w^e \in L^2(e),\ e \in \mathcal{E}\},$$

$m = |\mathcal{E}|$, with the norm and scalar product

$$\|w\|^2_{L^2(\mathcal{E})} \coloneqq \sum_{e\in\mathcal{E}} \|w^e\|^2_{L^2(e)} \quad \text{and} \quad (w_1, w_2)_{L^2(\mathcal{E})} \coloneqq \sum_{e\in\mathcal{E}} (w_1^e, w_2^e)_{L^2(e)}.$$

Sometimes, we write also $\int_{\mathcal{E}} w_1 w_2 \,\mathrm{d}x \coloneqq \sum_{e\in\mathcal{E}} \int_e w_1^e w_2^e \,\mathrm{d}x$. We also use the (piecewise) Sobolev space

$$H^s_{\mathrm{pw}}(\mathcal{E}) \coloneqq \{w \in L^2(\mathcal{E}) : w^e \in H^s(e),\ e \in \mathcal{E}\},$$

with

$$\|w\|^2_{H^s_{\mathrm{pw}}(\mathcal{E})} \coloneqq \sum_{e\in\mathcal{E}} \|w^e\|^2_{H^s(e)} \quad \text{and} \quad (w_1, w_2)_{H^s_{\mathrm{pw}}(\mathcal{E})} \coloneqq \sum_{e\in\mathcal{E}} (w_1^e, w_2^e)_{H^s(e)}.$$

Similarly, we define the spaces of (piecewise) k-times differentiable functions $C^k_{\mathrm{pw}}(\mathcal{E})$ and the Sobolev spaecs $W^{1,p}_{\mathrm{pw}}(\mathcal{E})$. For $s > \frac{1}{2}$, the functions in $H^s_{\mathrm{pw}}(\mathcal{E})$ are continuous on $e \in \mathcal{E}$, but may be discontinuous across the junction; we then denote by $H^s(\mathcal{E})$ the subspace of functions belonging to $H^s_{\mathrm{pw}}(\mathcal{E})$ which are also continuous across the junction. Every $w \in H^1(\mathcal{E})$ has a unique value $w(v)$ at every vertex $v \in \mathcal{V}$ and we use $\ell^2(\mathcal{V})$ to denote the set of possible vertex values. That is, for a function

$w : \mathcal{V} \to \mathbb{R}$, we define the space $\ell^2(\mathcal{V})$ endowed with the norm and scalar product

$$\|w\|_{\ell^2(\mathcal{V})} := \sqrt{\sum_{v\in\mathcal{V}} |w(v)|^2} \quad \text{and} \quad (w_1, w_2)_{\ell^2(\mathcal{V})} := \sum_{v\in\mathcal{V}} w_1(v)w_2(v).$$

Also, we define the distance between vertices and layers of a tree-shaped network as follows.

DEFINITION 8.1.1 (Distance and layers on a graph). *We define the* distance $\operatorname{dist}(v_1, v_2)$ *between two vertices v_1 and v_2 in the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as the minimum number of edges contained in a path joining them (if any exists—otherwise,* $\operatorname{dist}(v_1, v_2) = \infty$*). In addition, if $\mathcal{G}$ is a tree-shaped network, we fix a* root vertex $v \in V_\partial$ *and call* i-th layer of the tree-shaped network (with respect to v) *the subset vertices at a distance i from v.*

For future use, let us note that, given a piecewise-continuous function on a tree-shaped network, we may make it continuous by adding a piecewise-constant function to it.

LEMMA 8.1.1 (Continuity of functions in trees). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree-shaped network and g a piecewise-continuous function. Then, for every $e \in \mathcal{E}$, there exists $\mathfrak{c}^e \in \mathbb{R}$ such that the function $\tilde{g}$, defined on each edge by $\tilde{g}^e := g^e + \mathfrak{c}^e$, is continuous on $\overline{\mathcal{E}}$.*

PROOF. We argue by induction on the number of vertices. The base case, a tree-shaped network with two vertices (i.e., one edge), is trivial. Let us then assume that the property is true for a tree-shaped network with N vertices and prove that it holds for a tree-shaped network with $N+1$ vertices. We have that there exists at least one vertex u with degree 1 and with an edge $\tilde{e}$ incident to some vertex $v \in \mathcal{V} \setminus \{u\}$. Then the graph $(\mathcal{V} \setminus \{u\}, \mathcal{E} \setminus \{\tilde{e}\})$ satisfies the inductive hypothesis; thus, we can define the constants $\mathfrak{c}^e$ for all $e \in \mathcal{E} \setminus \{\tilde{e}\}$. It just remains to find a suitable constant in $\tilde{e}$; to this end, it suffices to consider

$$\mathfrak{c}^{\tilde{e}} = -g^e(v) + \tilde{g}_{|(\mathcal{V}\setminus\{u\},\, \mathcal{E}\setminus\{\tilde{e}\})}(v).$$

$\square$

8.1.2. Well-posedness of the parabolic and hyperbolic problems. We start by recalling a well-posedness result for the parabolic problem (1.2.1) (see [**125**, Theorem 3]), which follows from Lumer-Phillips' theorem (see [**210**, Chapter 1.4]).

THEOREM 8.1.1 (Well-posedness for the parabolic problem). *For any $y_0 \in H^1(\mathcal{E}) \cap H^2_{\mathrm{pw}}(\mathcal{E})$ and $u_\varepsilon \in C^2([0,T]; \ell_2(\mathcal{V}_\partial))$, the parabolic problem* (1.2.1) *has a unique classical solution*

$$y_\varepsilon \in C^1([0,T]; L^2(\mathcal{E})) \cap C^0([0,T]; H^1(\mathcal{E}) \cap H^2_{\mathrm{pw}}(\mathcal{E})).$$

Similarly, in [**125**, Theorem 6], a well-posedness result for the transport problem was obtained.

THEOREM 8.1.2 (Well-posedness for the hyperbolic problem). *For any $y_0 \in H^1_{\mathrm{pw}}(\mathcal{E})$ and $u_\varepsilon \in C^2([0,T]; \ell_2(\mathcal{V}^{\mathrm{in}}_\partial))$, the hyperbolic problem* (1.2.2) *has a unique classical solution*

$$y \in C^1([0,T]; L^2(\mathcal{E})) \cap C^0([0,T]; H^1_{\mathrm{pw}}(\mathcal{E})).$$

As the viscosity parameter tends to zero, the solution of (1.2.1) converges to the one of (1.2.2) (see [**125**, Theorem 10]).

THEOREM 8.1.3 (Error estimate for the vanishing viscosity approximation). *For any $y_0 \in H^1(\mathcal{E}) \cap H^2_{\mathrm{pw}}(\mathcal{E})$ and $u \in C^2([0,T]; \ell^2(\mathcal{V}_\partial))$, let*

$$y_\varepsilon \in C^1([0,T]; L^2(\mathcal{E})) \cap C^0([0,T]; H^1(\mathcal{E}) \cap H^2_{\mathrm{pw}}(\mathcal{E}))$$

be the solution of the parabolic problem (1.2.1) *with $u_\varepsilon = u$ and*

$$y \in C^1([0,T]; L^2(\mathcal{E})) \cap C^0([0,T]; H^1_{\mathrm{pw}}(\mathcal{E}))$$

be the solution of the hyperbolic problem (1.2.2) *with boundary data $u_{|\mathcal{V}^{\mathrm{in}}_\partial}$. Let us suppose that* (1.2.3) *holds. Then*

$$\|y_\varepsilon - y\|_{L^\infty((0,T); L^2(\mathcal{E}))} \le C\sqrt{\varepsilon}, \tag{8.1.1}$$

where the constant C depends on the time-horizon T but is independent of the diffusion parameter $\varepsilon \in (0,1]$.

REMARK 8.1.1 (L^2 data). *If we only assume* $y_0 \in L^2(\mathcal{E})$ *and* $u_\varepsilon \in L^2((0,T);\ell^2(\mathcal{V}))$*, we can still define the solution of* (1.2.1) *by transposition (as in* [**104**, Section 2.5]*), show that it belongs to* $C^0([0,T];L^2(\mathcal{E}))$*, and prove a vanishing viscosity convergence result.*

8.2. Main results

8.2.1. Control of the hyperbolic problem. Our first result concerns the null-controllability of the system (1.2.2) on a tree-shaped network: in particular, we are interested in controlling the flow across the network using controls placed at the inflow vertices $\mathcal{V}_\partial^{\text{in}}$.

To this end, we define an upper-bound and a lower-bound on the time in which information propagates across the network (i.e., the maximal and minimal travel time of the characteristics across the network).

DEFINITION 8.2.1 (Propagation time on a network). *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *be a tree-shaped network. We define recursively the functions* $\widehat{T}, \widetilde{T} : \mathcal{V} \to \mathbb{R}_+$ *as follows:*

$$
\begin{aligned}
&\widehat{T}(v) := 0, \quad \widetilde{T}(v) := 0 && \textit{if } v \in \mathcal{V}_\partial^{\text{in}},\\
&\widehat{T}(v) := \max_{e=(v^e,v)\in\mathcal{E}^{\text{in}}(v)} \left(\widehat{T}(v^e) + \frac{a^e\ell^e}{b^e}\right) && \textit{if } v \in \mathcal{V}_0 \cup \mathcal{V}_\partial^{\text{out}},\\
&\widetilde{T}(v) := \min_{e=(v^e,v)\in\mathcal{E}^{\text{in}}(v)} \left(\widetilde{T}(v^e) + \frac{a^e\ell^e}{b^e}\right) && \textit{if } v \in \mathcal{V}_0 \cup \mathcal{V}_\partial^{\text{out}}.
\end{aligned}
$$

The times $\widehat{T}(v)$ *and* $\widetilde{T}(v)$ *are respectively the* maximal *and* minimal propagation time *required for information to reach* $v \in \mathcal{V}$ *from a node in* $\mathcal{V}_\partial^{\text{in}}$*.*

Since the graph $\mathcal{G}$ has no loops (being a tree), we can prove inductively that $\widehat{T}$ and $\widetilde{T}$ are well-defined.

EXAMPLE 8.2.1 (Propagation times). *Let us consider the graph in Figure 8.1 with vertices* $\mathcal{V} = \{v_1, v_2, v_3, v_4\}$ *and edges* $e_1 = (v_1, v_3) \simeq (0, 2)$*,* $e_2 = (v_2, v_3) \simeq (0, 1)$*, and* $e_3 = (v_3, v_4) \simeq (0, 2)$*. We consider the system* (1.2.2) *with* $a^{e_1} = a^{e_2} = a^{e_3} = b^{e_1} = b^{e_2} = 1$ *and* $b^{e_3} = 2$*. We can compute the maximal travel time to reach* $v \in \mathcal{V}$ *as follows:*

$$
\begin{aligned}
&\widehat{T}(v_1) = \widehat{T}(v_2) = 0, \quad \widehat{T}(v_3) = \max_{i\in\{1,2\}} \left(\widehat{T}(v_i) + \ell^{e_i}\right) = \max\{1,2\} = 2,\\
&\widehat{T}(v_4) = \widehat{T}(v_3) + \frac{\ell^{e_3}}{b^3} = 2 + 1 = 3.
\end{aligned}
$$

Moreover, we compute the minimal travel time to reach $v \in \mathcal{V}$ *as follows:*

$$
\begin{aligned}
&\widetilde{T}(v_1) = \widetilde{T}(v_2) = 0, \quad \widetilde{T}(v_3) = \min_{i\in\{1,2\}} \left(\widetilde{T}(v_i) + \ell^{e_i}\right) = \min\{1,2\} = 1,\\
&\widetilde{T}(v_4) = \widetilde{T}(v_3) + \frac{\ell^{e_3}}{b^3} = 1 + 1 = 2.
\end{aligned}
$$

By relying on this notion of propagation time and on the classical method of characteristics, we can prove the following controllability result.

THEOREM 8.2.1 (Null-controllability for the hyperbolic problem). *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *be a tree-shaped network and let* y *be the solution of* (1.2.2) *for* $u = 0$*.*

(1) *For all* $T \geq \max_{v\in\mathcal{V}_\partial^{\text{out}}} \widehat{T}(v)$*, we have* $y(T, \cdot) = 0$*. More precisely,* $y^e(T, x) = 0$ *for all* $x \in e = (v_1, v_2) \simeq (0, \ell^e)$ *and* $T \geq \widehat{T}(v_1) + \frac{a^e x}{b^e}$*.*
(2) *For* $T < \max_{v\in\mathcal{V}_\partial^{\text{out}}} \widetilde{T}(v)$*, system* (1.2.2) *is not null-controllable.*

The proof of Theorem 8.2.1 is given in Section 8.3.

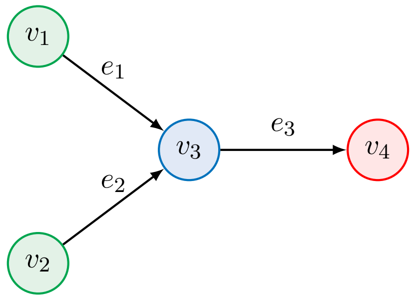

FIGURE 8.1. Star-shaped graph with edges $e_1 = (v_1, v_3)$, $e_2 = (v_2, v_3)$, $e_3 = (v_3, v_4)$; inner vertex $\mathcal{V}_0 = \{v_3\}$ (blue), and boundary vertices $\mathcal{V}_\partial = \{v_1, v_2, v_4\}$. We split the set of boundary vertices into inflow and outflow vertices: $\mathcal{V}_\partial^{\text{in}} = \{v_1, v_2\}$ (green) and $\mathcal{V}_\partial^{\text{out}} = \{v_4\}$ (red), respectively.

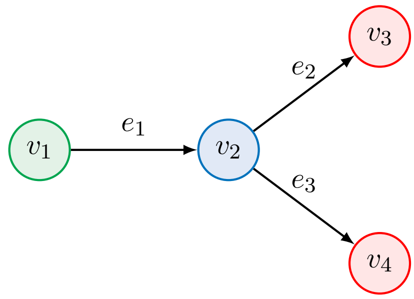

FIGURE 8.2. Star-shaped graph with edges $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, $e_3 = (v_2, v_4)$; inner vertex $\mathcal{V}_0 = \{v_2\}$ (blue), and boundary vertices $\mathcal{V}_\partial = \{v_1, v_3, v_4\}$. We split the set of boundary vertices into inflow and outflow vertices: $\mathcal{V}_\partial^{\text{in}} = \{v_1\}$ (green) and $\mathcal{V}_\partial^{\text{out}} = \{v_3, v_4\}$ (red), respectively.

8.2.2. Control of the parabolic problem. Our next theorem provides the controllability of the parabolic system (1.2.1) on tree-shaped networks by acting on the external vertices (except at most one) without any further geometric constraint.

THEOREM 8.2.2 (Controllability of parabolic systems of networks). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree-shaped network and $y_0 \in L^2(\mathcal{E})$. Given $T > 0$, there exists $u \in L^2((0,T); \ell^2(\mathcal{V}_\partial))$ such that the solution of* (1.2.1) *satisfies $y(T, \cdot) = 0$.*

Let us point out that it does not suffice to act on $\mathcal{V}_\partial^{\text{in}}$ to drive the system (1.2.1) to zero (see Proposition 8.2.2).

The proof of Theorem 8.2.2 essentially follows by using a Carleman inequality similar to [**164**, Proposition 3.1]. However, in [**164**], the authors do not keep track of the viscosity parameter. We can also prove Theorem 8.2.2 as a direct byproduct of our study on the cost of the controllability (see Theorem 8.2.3 and Remark 8.5.1 below).

8.2.3. Cost of controllability in the singular limit. Our final main theorem provides estimates on the cost of controllability of (1.2.1).

THEOREM 8.2.3 (Estimates on the cost of controllability). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree-shaped network and let us assume that* (1.2.3) *holds.*

(1) *There exist $\overline{T}, c > 0$ such that, for ε small enough and all $T < \overline{T}$, the following lower-bound holds:*

$$K(\varepsilon, T, a^e, b^e, \mathcal{G}) \geq c e^{c/\varepsilon}. \tag{8.2.1}$$

(2) *There exist $T_0, c, C > 0$ such that, for ε small enough and all $T \geq T_0$, the following upper-bound holds:*

$$K(\varepsilon, T, a^e, b^e, \mathcal{G}) \leq C e^{-c/\varepsilon}. \tag{8.2.2}$$

The proof of Theorem 8.2.3 is given in Sections 8.4 and 8.5 and is based on estimating the cost of observability of the adjoint variable. The duality between the cost of controllability and the cost of observability is summarized in the following lemma (see [**104**, Chapter 2.3]).

LEMMA 8.2.1 (Cost of observability). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree-shaped network. Let us suppose that*

$$\int_0^T \sum_{v \in V_\partial} |\partial_{n^e(v)} \varphi_\varepsilon^e(t, v)|^2 \, \mathrm{d}t \neq 0, \qquad \forall \varphi_T \neq 0, \tag{8.2.3}$$

and

$$\widetilde{K}(\varepsilon,T,a^e,b^e,\mathcal{G}) \coloneqq \frac{1}{\sqrt{\varepsilon}} \sup_{\varphi_T\in L^2(\mathcal{E})\setminus\{0\}} \frac{\|a\,\varphi_\varepsilon(0,\cdot)\|_{L^2(\mathcal{E})}}{\left(\int_0^T \sum_{v\in V_\partial} |\partial_{n^e(v)}\varphi_\varepsilon(t,v)|^2\,\mathrm{d}t\right)^{1/2}} < +\infty, \tag{8.2.4}$$

where φ_ε is solution of (1.2.4) *with datum φ_T. Then* (1.2.1) *is null-controllable with cost*

$$K(\varepsilon,T,a^e,b^e,\mathcal{G}) = \widetilde{K}(\varepsilon,T,a^e,b^e,\mathcal{G}). \tag{8.2.5}$$

In addition, if, for some $\varphi_T \in L^2(\mathcal{E})$, we have

$$\int_0^T \sum_{v\in V_\partial} |\partial_{n^e(v)}\varphi^e_\varepsilon(t,v)|^2\,\mathrm{d}t = 0, \tag{8.2.6}$$

then system (1.2.1) *is not approximately controllable.*

We recall that (1.2.1) is *approximately controllable in $L^2(\mathcal{E})$* at time $T>0$ if the range of the application $u\in\mathcal{U}\mapsto y_\varepsilon(T,\cdot)\in L^2(\mathcal{E})$ is dense in $L^2(\mathcal{E})$.

To simplify some computations in what follows, we define the function

$$z^e_\varepsilon(t,x) \coloneqq \varphi^e_\varepsilon(t,x)e^{(xb^e+\mathfrak{c}^e)/2\varepsilon}, \tag{8.2.7}$$

where $\mathfrak{c}^e$ the constants given in Lemma 8.1.1 that makes the function $xb^e+\mathfrak{c}^e$ continuous on $\overline{\mathcal{E}}$. This function satisfies the following symmetrized system (see [**26**, Section 2.1]):

$$\begin{cases} -a^e\partial_t z^e_\varepsilon(t,x) - \varepsilon\partial^2_{xx}z^e_\varepsilon(t,x) + \frac{|b^e|^2}{4\varepsilon}z^e_\varepsilon(t,x) = 0, & (t,x)\in(0,T)\times e,\ \ \forall e\in\mathcal{E},\\ z^e_\varepsilon(t,v)=0, & t\in(0,T),\ v\in\mathcal{V}_\partial,\\ z^{e_1}_\varepsilon(t,v) = z^{e_2}_\varepsilon(t,v), & t\in(0,T),\ v\in\mathcal{V}_0,\ \forall e_1,e_2\in\mathcal{E}(v),\\ \sum_{e\in\mathcal{E}(v)}\varepsilon\partial_{n^e(v)}z^e_\varepsilon(t,v)=0, & t\in(0,T),\ v\in\mathcal{V}_0,\\ z^e_\varepsilon(T,x) = z^e_T(x) \coloneqq \varphi^e_T(x)e^{(xb^e+\mathfrak{c}^e)/2\varepsilon}, & x\in e,\ \forall e\in\mathcal{E}. \end{cases} \tag{8.2.8}$$

Here, we relied on (1.2.3) to obtain $(8.2.8)_4$.

8.2.4. Pathological examples and further remarks. For particular graphs and choices of the coefficients in (1.2.1) and (1.2.2), we can build several pathological examples to illustrate the scope of our controllability results.

Remark 8.2.1 (Counterexample to exact controllability of (1.2.2) to any target state $y(T,\cdot)\in C^0_{\mathrm{pw}}(\mathcal{E})$). *While we are able to prove null-controllability, and thus controllability to trajectories because of linearity, we may not have exact controllability to any $y\in C^0_{\mathrm{pw}}(\mathcal{E})$—namely, when $|\mathcal{V}^{\mathrm{out}}_\partial| > |\mathcal{V}^{\mathrm{in}}_\partial|$, due to symmetry constraints. For example, let us consider the graph in Figure 8.2, made of the vertices $\mathcal{V}=\{v_1,\ v_2,\ v_3,\ v_4\}$, the edges $e_1=(v_1,v_2)\simeq(0,1)$, $e_2=(v_2,v_3)\simeq(0,1)$, and $e_3=(v_2,v_4)\simeq(0,1)$, and with $a^{e_i}=b^{e_i}=1$ for $i\in\{1,2,3,4\}$. In* (1.2.2)*, we take $a^{e_1}=a^{e_2}=a^{e_3}=b^{e_1}=1$, $b^{e_2}=1/2$, and $b^{e_3}=1/2$. Then, for any $y_0\in L^2(\mathcal{E})$ and $u\in C^2([0,T];\ell^2(v_1))$, the solution y of* (1.2.2) *satisfies $y^{e_2}(t,x)=y^{e_3}(t,x)$ for $t>1$.*

Next, we illustrate some issues arising from networks with loops.

Remark 8.2.2 (Networks with loops and controls). *Let us consider a graph with vertices $\mathcal{V}=\{v_1,$ $v_2,\ v_3,\ v_4\}$ and edges $e_1=(v_1,v_2)$, $e_2=(v_2,v_3)$, $e_3=(v_3,v_4)$, and $e_4=(v_4,v_2)$ (see the left-side picture in Figure 8.3). In this case, for the free system (i.e., with Dirichlet boundary condition $u_1\equiv 0$ at v_1), we can prove that the total mass is constant, as*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{E}} ay\,\mathrm{d}x = b^{e_1}\partial_{n^{e_1}(v_1)}y^{e_1}(v_1) = 0.$$

Here, the first equality is a consequence of $(1.2.2)_1$ *and* $(1.2.2)_4$*; the second one holds because y^{e_1} is null in a neighborhood of v_1 for all $t>0$ by the method of characteristics. And yet, we can use the method of characteristics to prove that the hyperbolic system is null-controllable with a control that is non-zero.*

A similar example consists of the same graph with an additional output vertex v_5 and $e_5=(v_3,v_5)$ (see the right-side picture in Figure 8.3). In that case, the mass is not conserved, but the zero-control *does not take the system to equilibrium. For example, if* $y_0 \equiv 1$*, we can prove by contradiction that* $y(t,x) > 0$ *for all* $t \geq 0$ *and* $x \in e_2 \cup e_3 \cup e_4$*. Moreover, as in the previous example, we can use the method of characteristics to prove that the hyperbolic system is null controllable, with a control that is non-zero.*

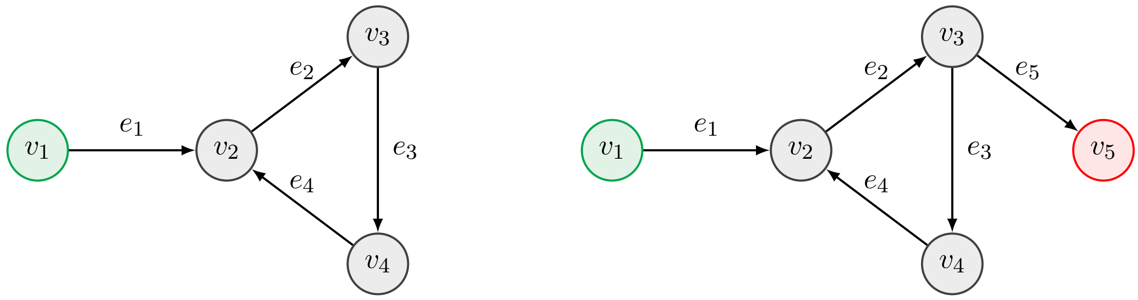

FIGURE 8.3. Graphs with loops used in Remark 8.2.2. LEFT: graph with one input vertex v_1 (green) and loop made of vertices v_2, v_3, and v_4 (gray). RIGHT: graph with input vertex v_1 (green), output vertex v_5 (red) and loop made of vertices v_2, v_3, and v_4 (gray).

We note that system (1.2.1) cannot be controlled for any $\varepsilon > 0$ by acting on fewer boundary vertices. In the case without advection, this result can be found in [**164**, Remark 3.2].

PROPOSITION 8.2.1 (Lack of null-controllability with fewer controls). *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *be the graph in Figure 8.2, made of the vertices* $\mathcal{V} = \{v_1,\ v_2,\ v_3,\ v_4\}$ *and the edges* $e_1 = (v_1, v_2) \simeq (0,1)$*,* $e_2 = (v_2, v_3) \simeq (0,1)$*, and* $e_3 = (v_2, v_4) \simeq (0,1)$ *Then, system* (1.2.1)*, with coefficients* $a^{e_1} = a^{e_2} = a^{e_3} = 1$ *and* $b^{e_1} = b^{e_2} = b^{e_3} = 0$*, is not approximately controllable by acting only on* v_1 *(i.e., if* $u^{v_3} \equiv u^{v_4} \equiv 0$*).*

Heuristically, the motivation for such a result is that, by symmetry, the effect of the control on e_2 and e_3 is identical, so we cannot control both y^{e_2} and y^{e_3} simultaneously (unless some irrationality condition on the length of the edges holds; compare [**117**, Corollary 8.6]).

In a similar way, we can prove the claim also for the system (1.2.1) with advection terms.

PROPOSITION 8.2.2 (Lack of null-controllability with fewer controls). *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *be the graph in Figure 8.2, made of the vertices* $\mathcal{V} = \{v_1,\ v_2,\ v_3,\ v_4\}$ *and the edges* $e_1 = (v_1, v_2) \simeq (0,1)$*,* $e_2 = (v_2, v_3) \simeq (0,1)$*, and* $e_3 = (v_2, v_4) \simeq (0,1)$*. Then, system* (1.2.1) *(with coefficients* $b^{e_1} = a^{e_1} = a^{e_2} = a^{e_3} = 1$ *and* $b^{e_2} = b^{e_3} = \frac{1}{2}$*) is not approximately controllable in* $L^2(\mathcal{E})$ *by acting only on* v_1 *(i.e., if* $u^{v_3} \equiv u^{v_4} \equiv 0$*) for any* $\varepsilon > 0$*.*

PROOF. By duality (see [**104**, Theorem 2.43]), it suffices to show that there are non-zero solutions of (1.2.4) satisfying $\partial_{n^e(v_1)}\varphi(\cdot, v_1) = 0$. Considering (8.2.7), this is equivalent to showing that there are non-zero solutions of (8.2.8) satisfying $\partial_{n^e(v_1)} z(\cdot, v_1) = 0$. From the spectral decomposition of the Laplacian on the graph, we can construct such a solution as follows:

$$z^{e_1} = 0, \qquad z^{e_2} = \exp\left[\left(\varepsilon\pi^2 + \frac{1}{16\varepsilon}\right)t\right]\sin(\pi x), \qquad z^{e_3} = -\exp\left[\left(\varepsilon\pi^2 + \frac{1}{16\varepsilon}\right)t\right]\sin(\pi x).$$

□

Finally, we note the relevance of (1.2.3) for our results.

REMARK 8.2.3 (On the balance relation (1.2.3)). *Without* (1.2.3)*, we must replace* $(8.2.8)_4$ *by*

$$\sum_{e\in\mathcal{E}(v)} \varepsilon \partial_{n^e(v)} z^e_\varepsilon(t, v) = -\sum_{e\in\mathcal{E}(v)} \frac{n^e(v)b^e}{2} z^e_\varepsilon(t, v). \tag{8.2.9}$$

or, equivalently, $(1.2.4)_4$ *by*

$$\sum_{e\in\mathcal{E}(v)} \varepsilon \partial_{n^e(v)} \varphi^e_\varepsilon(t, v) = -\sum_{e\in\mathcal{E}(v)} \frac{n^e(v)b^e}{2} \varphi^e_\varepsilon(t, v). \tag{8.2.10}$$

Also, without condition (1.2.3)*, the system* (1.2.2) *may not be dissipative at the junctions. Indeed, from* (1.2.2)*, we compute*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{E}} ay^2(t,x)\,\mathrm{d}x = -\sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)} n^e(v)b^e|y^e|^2(t,v), \tag{8.2.11}$$

which is not non-positive in general. For example, let us consider a simple 1–to–1 junction modeled as follows:

$$\begin{cases} \partial_t y^-(t,x) + b^-\partial_x y^-(t,x) = 0, & t\in(0,T),\ x\in(-1,0),\\ \partial_t y^+(t,x) + b^+\partial_x y^+(t,x) = 0, & t\in(0,T),\ x\in(0,1),\\ y^-(t,-1) = 0, & t>0,\\ b^-y^-(t,0) = b^+y^+(t,0), & t>0,\\ y^-(0,x) = y_0^-(x), & x\in(-1,0),\\ y^+(0,x) = y_0^+(x), & x\in(0,1), \end{cases} \tag{8.2.12}$$

for $b^-,b^+>0$*. Here, the term in* (8.2.11) *is given by*

$$\begin{aligned} -\sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)} n^e(v)b^e(y^e)^2(v) &= -1\cdot b^-|y^-(t,0)|^2 - (-1)b^+|y^+(t,0)|^2\\ &= b^+|y^+(t,0)|^2 - \frac{(b^+)^2}{b^-}|y^+(t,0)|^2\\ &= b^+\left(1-\frac{b^+}{b^-}\right)|y^+(t,0)|^2. \end{aligned}$$

In the second equation, we have used $(8.2.12)_4$*. Consequently, if* $b^->b^+$*, energy is added at the junctions and the problem is not dissipative.*

8.3. Controllability of the transport problem

We prove Theorem 8.2.1 by means of the method of characteristics, following [**104**, Chapter 2.1.2., p. 30]. More specifically, we use an induction argument over the layers of the tree.

Proof of Theorem 8.2.1. **Step 1.** *Null-controllability.* The proof of Claim (1) is based on an induction of the distance of the vertex v_1 to the exterior vertices (see Definition 8.1.1).

The base case is $v_1\in\mathcal{V}_\partial^{\mathrm{in}}$. The equality $y(v_1)=0$ is satisfied because of the boundary condition (recalling that $u\equiv 0$). Moreover, within each edge $e\in\mathcal{E}^{\mathrm{out}}(v_1)$, the function y^e behaves like the solution to a classical transport equation. Consequently, as in [**104**, Chapter 2.1.2., p. 30], $y^e(t,x)=0$ for all $x\in e=(v_1,v_2)\simeq(0,\ell^e)$ and $t\geq\frac{a^e x}{b^e}$, and in particular, $y^e(t,v_2)=0$ for all $t\geq\widehat{T}(v_2)$, as $\widehat{T}(v_2)\geq\frac{a^e\ell^e}{b^e}$.

Let us now continue with the inductive case $v_1\in\mathcal{V}_0$. For all $e\in\mathcal{E}^{\mathrm{in}}(v_1)$, the equality $y^e(t,v_1)=0$ is satisfied for $t\geq\widehat{T}(v_1)$ by the inductive hypothesis. From the transmission conditions $(1.2.2)_3$ and $(1.2.2)_4$, we deduce that, for $e\in\mathcal{E}^{\mathrm{out}}(v_1)$, $y^e(t,v_1)=0$ as well, for all $t\geq\widehat{T}(v_1)$. Furthermore, within the edge $e\in\mathcal{E}^{\mathrm{out}}(v_1)$, the function y^e behaves like a transport equation in a segment; thus, $y^e(t,x)=0$ for $x\in(0,\ell^e)$, where $e\simeq(0,\ell^e)$ and $t\geq\widehat{T}(v_1)+\frac{a^e x}{b^e}$, and in particular, $y^e(t,v_2)=0$ for all $t\geq\widehat{T}(v_2)$, as $\widehat{T}(v_2)\geq\widehat{T}(v_1)+\frac{a^e\ell^e}{b^e}$. As a byproduct of the same argument, we can also deduce the claim for $v\in\mathcal{V}_\partial^{\mathrm{out}}$.

Step 2. *Minimal propagation time.* To prove Claim (2), we consider the solution of (1.2.2) with initial value $y_0\equiv 1$. Then, an inductive argument (as in the proof of Claim (1)) shows that, for any control u, there exist $v\in\mathcal{V}_0\cup\mathcal{V}_\partial^{\mathrm{out}}$ and $e\in\mathcal{E}(v)$ such that $y^e(t,v)>0$ on $[0,\widetilde{T}(v))$.

□

Remark 8.3.1 (Positivity assumption on a and b when they depend on the space variable). *The proof of Theorem 8.2.1 remains valid when* a^e *and* b^e *depend on the space variable, assuming* $a\in C^1_{\mathrm{pw}}([0,T]\times\mathcal{E})$, $\min_{\mathcal{E}} a>0$, $b\in C^1_{\mathrm{pw}}([0,T]\times\mathcal{E})$, *and* $\min_{\mathcal{E}} b>0$*. The positivity is needed because, if the transport term vanishes at some point, then the characteristics may not leave the domain.*

8.4. Blow-up of the cost of controllability

In this Section, we prove Claim (1) of Theorem 8.2.3.

8.4.1. Agmon-type inequality. We start by proving an Agmon-type inequality (see [**5**, Theorem 5.9]), which gives an exponentially weighted energy estimate.

LEMMA 8.4.1 (Agmon-type inequality). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree-shaped network and let us assume that* (1.2.3) *holds. Let $\zeta \in H^2_{\mathrm{pw}}(\mathcal{E}) \cap H^1(\mathcal{E})$ satisfy*

$$a^e \partial_t \zeta^e + b^e \partial_x \zeta^e - |\partial_x \zeta^e|^2 \geq 0, \quad t \in (0,T),\ x \in e,\ \forall e \in \mathcal{E}. \tag{8.4.1}$$

Then the solution φ_ε of the adjoint system (1.2.4) *satisfies the following Agmon-type inequality: for $t \in (0,T)$,*

$$\begin{aligned}&\sum_{e\in\mathcal{E}} \frac{a^e}{2} \int_e |e^{\zeta^e(t,x)/\varepsilon} \varphi^e_\varepsilon(t,x)|^2 \,\mathrm{d}x + \varepsilon \sum_{e\in\mathcal{E}} \int_t^T \int_e |\partial_x (e^{\zeta^e(s,x)/\varepsilon} \varphi^e_\varepsilon(s,x))|^2 \,\mathrm{d}x \,\mathrm{d}s \\ &\leq \sum_{e\in\mathcal{E}} \frac{a^e}{2} \int_e |e^{\zeta^e(T,x)/\varepsilon} \varphi^e_\varepsilon(T,x)|^2 \,\mathrm{d}x.\end{aligned}$$

PROOF. Let $e \in \mathcal{E}$. Then,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} \frac{a^e}{2} \int_e |e^{\zeta^e/\varepsilon} \varphi^e_\varepsilon|^2 \,\mathrm{d}x \\ &= a^e \int_e e^{\zeta^e/\varepsilon} \partial_t e^{\zeta^e/\varepsilon} |\varphi^e_\varepsilon|^2 \,\mathrm{d}x + a^e \int_e e^{2\zeta^e/\varepsilon} \varphi \partial_t \varphi^e_\varepsilon \,\mathrm{d}x \\ &= a^e \int_e e^{\zeta^e/\varepsilon} \partial_t e^{\zeta^e/\varepsilon} |\varphi^e_\varepsilon|^2 \,\mathrm{d}x + \int_e e^{2\zeta^e/\varepsilon} \varphi^e_\varepsilon \big(- b^e \partial_x \varphi^e_\varepsilon - \varepsilon \partial^2_{xx} \varphi^e_\varepsilon \big) \,\mathrm{d}x \\ &= a^e \int_e e^{\zeta^e/\varepsilon} \partial_t e^{\zeta^e/\varepsilon} |\varphi^e_\varepsilon|^2 \,\mathrm{d}x + \frac{b^e}{2} \int_e (\varphi^e_\varepsilon)^2 \partial_x e^{2\zeta^e/\varepsilon} \,\mathrm{d}x - \frac{b^e}{2} [(\varphi^e_\varepsilon)^2 e^{2\zeta^e/\varepsilon}]_e \\ &\quad + \varepsilon \int_e (\partial_x \varphi^e_\varepsilon)^2 e^{2\zeta^e/\varepsilon} \,\mathrm{d}x + \varepsilon \int_e \partial_x \varphi^e_\varepsilon \varphi^e_\varepsilon \partial_x e^{2\zeta^e/\varepsilon} \,\mathrm{d}x - \varepsilon [\partial_x \varphi^e_\varepsilon \varphi^e_\varepsilon e^{2\zeta^e/\varepsilon}]_e \\ &= \frac{1}{\varepsilon} \int_e e^{2\zeta^e/\varepsilon} (\varphi^e_\varepsilon)^2 \Big(a^e \partial_t \zeta^e + b^e \partial_x \zeta^e - |\partial_x \zeta^e|^2 \Big) \,\mathrm{d}x + \varepsilon \int_e (\partial_x (\varphi^e_\varepsilon e^{\zeta^e/\varepsilon}))^2 \,\mathrm{d}x \\ &\quad - \frac{b^e}{2} [(\varphi^e_\varepsilon)^2 e^{2\zeta^e/\varepsilon}]_e - \varepsilon [\varphi^e_\varepsilon \partial_x \varphi^e_\varepsilon e^{2\zeta^e/\varepsilon}]_e.\end{aligned}$$

Summing up over $e \in \mathcal{E}$ and using the continuity of ζ^e, the Dirichlet boundary conditions, the junction conditions, and the flux balance condition (1.2.3), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \sum_{e\in\mathcal{E}} \frac{a^e}{2} \int_e |e^{\zeta^e/\varepsilon} \varphi^e_\varepsilon|^2 \,\mathrm{d}x &= \frac{1}{\varepsilon} \sum_{e\in\mathcal{E}} \int_e |e^{\zeta^e/\varepsilon} \varphi^e_\varepsilon(t,x)|^2 \underbrace{\Big(a^e \partial_t \zeta^e + b^e \partial_x \zeta^e - |\partial_x \zeta^e|^2 \Big)}_{\geq 0} \,\mathrm{d}x \\ &\quad + \varepsilon \sum_{e\in\mathcal{E}} \int_e (\partial_x (\varphi^e_\varepsilon e^{\zeta^e/\varepsilon}))^2 \,\mathrm{d}x \\ &\geq \varepsilon \sum_{e\in\mathcal{E}} \int_e (\partial_x (\varphi^e_\varepsilon e^{\zeta^e/\varepsilon}))^2 \,\mathrm{d}x.\end{aligned}$$

We conclude the proof by integrating this expression on (t,T). □

8.4.2. Non-degeneracy of the solution. As a second preliminary tool, we show that the mass of $\varphi_\varepsilon(0,\cdot)$ is bounded away from zero for small times.

LEMMA 8.4.2 (Non-degeneracy of the solution). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree-shaped network and let us assume that* (1.2.3) *holds and consider $\tilde{e} \in \mathcal{E}$ such that $\tilde{e} \simeq (0, \ell^{\tilde{e}}) \in \mathcal{E}$. Let $\varphi_T \geq 0$ be a non-null $C^\infty_{\mathrm{pw}}(\mathcal{E}) \cap C^0(\overline{\mathcal{E}})$ function such that $\mathrm{supp}(\varphi_T) \Subset \tilde{e}$. Then, there exists $c > 0$ such that, for all $\varepsilon \in (0,1]$ and T small enough,*

$$\|\varphi_\varepsilon(0,\cdot)\|_{L^2(\mathcal{E})} \geq c, \tag{8.4.2}$$

where φ_ε is the solution of (1.2.4) *with final value* φ_T.

Proof. By Hölder's inequality, there exists $C > 0$, depending on the length of the edges of the graph, such that

$$\|\varphi_\varepsilon(0,\cdot)\|_{L^1(\mathcal{E})} \le C\|\varphi_\varepsilon(0,\cdot)\|_{L^2(\mathcal{E})}.$$

Therefore, it suffices to prove that there exists $c > 0$ such that, for all $\varepsilon \in (0,1]$,

$$\int_{\tilde{e}} |\varphi_\varepsilon^{\tilde{e}}(0,x)|\,\mathrm{d}x \ge c. \tag{8.4.3}$$

First, because of [**125**, Lemma 7], we have that $\varphi_\varepsilon \ge 0$ for all $t \in (0,T)$. To prove (8.4.3), it suffices to use a comparison result with the solution of the following PDE posed on the edge $\tilde{e}$:

$$\begin{cases} -a^{\tilde{e}}\partial_t\tilde{\varphi}_\varepsilon(t,x) - b^{\tilde{e}}\partial_x\tilde{\varphi}_\varepsilon(t,x) - \varepsilon\partial^2_{xx}\tilde{\varphi}_\varepsilon(t,x) = 0, & t\in(0,T),\ x\in\tilde{e},\\ \tilde{\varphi}_\varepsilon(t,0) = \tilde{\varphi}_\varepsilon(t,\ell^{\tilde{e}}) = 0, & t\in(0,T),\\ \tilde{\varphi}_\varepsilon(T,x) = \varphi_T^{\tilde{e}}(x) & x\in\tilde{e}. \end{cases} \tag{8.4.4}$$

Indeed, by the non-negativity of φ_ε, we have, for $\hat{\varphi}_\varepsilon = \varphi_\varepsilon^{\tilde{e}} - \tilde{\varphi}$,

$$\begin{cases} -a^{\tilde{e}}\partial_t\hat{\varphi}_\varepsilon(t,x) - b^{\tilde{e}}\partial_x\hat{\varphi}_\varepsilon(t,x) - \varepsilon\partial^2_{xx}\hat{\varphi}_\varepsilon(t,x) = 0, & t\in(0,T),\ x\in\tilde{e},\\ \hat{\varphi}_\varepsilon(t,0) \ge 0, & t\in(0,T),\\ \hat{\varphi}_\varepsilon(t,\ell^{\tilde{e}}) \ge 0, & t\in(0,T),\\ \hat{\varphi}_\varepsilon(T,x) = 0, & x\in\tilde{e}. \end{cases}$$

This yields $\varphi_\varepsilon^{\tilde{e}} \ge \tilde{\varphi}_\varepsilon$ in $\tilde{e}$. Thus, it suffices to prove that there exists some $c > 0$ such that

$$\int_{\tilde{e}} \tilde{\varphi}_\varepsilon(0,x)\,\mathrm{d}x \ge c > 0$$

for all $\varepsilon \in (0,1]$. Since $\varepsilon \mapsto \tilde{\varphi}_\varepsilon(0,\cdot)$ is continuous from $(0,1]$ to $L^1(\tilde{e})$, it suffices to check that the limit is not null when $\varepsilon \to 0^+$. This follows from the $C^0([0,T];L^2(\mathcal{E}))$ convergence result stated in Theorem 8.1.3 and from the fact that, for the transport equation obtained passing to the limit in (8.4.4), this holds true by finite speed of propagation as long as T is small enough ($T < \frac{a^{\tilde{e}}}{b^{\tilde{e}}}(\ell^{\tilde{e}} - \inf\{x : \varphi_T^{\tilde{e}}(x) > 0\})$).

□

8.4.3. Proof of Claim (1) **of Theorem 8.2.3.** Using the tools developed in the previous Sections, we complete the proof of Claim (1) of Theorem 8.2.3. The main ideas of the proof are as follows: choosing an initial datum supported away from the boundary vertices, we establish an exponentially growing lower-bound on the cost of observability in (8.2.4) by relying on Lemmas 8.4.1 and 8.4.2.

Proof of Theorem 8.2.3, Claim (1). Let $\tilde{e} \simeq (0,\ell^{\tilde{e}}) \in \mathcal{E}$ and let φ_T be a non-zero smooth function such that

$$\operatorname{supp}(\varphi_T) \subset \tilde{e} \cap \left(\frac{\ell^{\tilde{e}}}{4}, \frac{3\ell^{\tilde{e}}}{4}\right) \Subset \tilde{e}. \tag{8.4.5}$$

Thanks to Lemma 8.4.2, we know that (8.4.2) is satisfied uniformly in ε; thus, it suffices to prove that the observed mass decays exponentially. To this end, we define the auxiliary function

$$\zeta^{\tilde{e}}(t,x) := b^{\tilde{e}}\left(x - \frac{\ell^{\tilde{e}}}{2}\right)^2 - \frac{(b^{\tilde{e}})^2}{a^{\tilde{e}}}(T-t)(\ell^{\tilde{e}} + (\ell^{\tilde{e}})^2), \quad (t,x)\in(0,T)\times\tilde{e},$$

and ζ, which is its extension by two constant functions outside the edge $\tilde{e}$ (namely, $\zeta^{\tilde{e}}(t,0)$ on the left and by $\zeta^{\tilde{e}}(t,\ell^{\tilde{e}})$ on the right). We then compute

$$\begin{aligned} a^{\tilde{e}}\partial_t\zeta^{\tilde{e}}(t,x) + b^{\tilde{e}}\partial_x\zeta^{\tilde{e}}(t,x) - |\partial_x\zeta^{\tilde{e}}(t,x)|^2 &= (b^{\tilde{e}})^2(\ell^{\tilde{e}} + (\ell^{\tilde{e}})^2) + (b^{\tilde{e}})^2\left(2x - \ell^{\tilde{e}}\right) - (b^{\tilde{e}})^2\left(2x - \ell^{\tilde{e}}\right)^2\\ &\ge (b^{\tilde{e}})^2(\ell^{\tilde{e}} + (\ell^{\tilde{e}})^2) - (b^{\tilde{e}})^2\ell^{\tilde{e}} - (b^{\tilde{e}})^2\left(\ell^{\tilde{e}}\right)^2\\ &= 0, \qquad (t,x)\in(0,T)\times\tilde{e}, \end{aligned}$$

and

$$a^e\partial_t\zeta^e(t,x)+b^e\partial_x\zeta^e(t,x)-|\partial_x\zeta^e(t,x)|^2=a^e\frac{(b^{\tilde e})^2}{a^{\tilde e}}(\ell^{\tilde e}+(\ell^{\tilde e})^2)\geq 0,\qquad (t,x)\in(0,T)\times(\mathcal{E}\setminus\{\tilde e\}).$$

Using the assumption (8.4.5), we deduce the following estimate:

$$\int_{\mathcal{E}}|e^{\zeta(T,x)/\varepsilon}\varphi_T(x)|^2\,\mathrm{d}x\leq\exp\left(\frac{b^{\tilde e}(\ell^{\tilde e})^2}{8\varepsilon}\right)\int_{\tilde e}|\varphi_T^{\tilde e}|^2\,\mathrm{d}x.$$

We define a smooth cut-off function $\chi\in C^\infty(\mathcal{E})$ as follows: for every $e=(u,v)\simeq(0,\ell^e)\in\mathcal{E}$,

– if $u,v\in\mathcal{V}_\partial$, we define χ^e as a function whose value is 1 in $[0,\frac{\ell^e}{8}]\cup[\frac{7\ell^e}{8},\ell^e]$ and 0 in $[\frac{\ell^e}{4},\frac{3\ell^e}{4}]$;
– if $u\in\mathcal{V}_\partial$ and $v\notin\mathcal{V}_\partial$, we define χ^e as a function whose value is 1 in $[0,\frac{\ell^e}{8}]$ and 0 in $[\frac{\ell^e}{4},\ell^e]$;
– if $v\in\mathcal{V}_\partial$ and $u\notin\mathcal{V}_\partial$, we define χ^e as a function whose value is 1 in $[\frac{7\ell^e}{8},\ell^e]$ and 0 in $[0,\frac{3\ell^e}{4}]$;
– if $u,v\in\mathcal{V}_0$, we let $\chi^e=0$.

Then, $\psi_\varepsilon^e:=\chi\varphi_\varepsilon^e e^{\zeta^e/\varepsilon}$ satisfies the following system:

$$\begin{cases}-a^e\partial_t\psi_\varepsilon^e(t,x)-b^e\partial_x\psi_\varepsilon^e(t,x)-\varepsilon\partial_{xx}^2\psi_\varepsilon^e(t,x)=F[\chi,\varphi_\varepsilon^e,e^{\zeta^e/\varepsilon}], & t\in(0,T),\ x\in e,\ \forall e\in\mathcal{E},\\ \psi_\varepsilon^e(t,v)=0, & t\in(0,T),\ v\in\mathcal{V}_\partial,\ \forall e\in\mathcal{E}(v),\\ \psi_\varepsilon^e(t,v)=0, & t\in(0,T),\ v\in\mathcal{V}_0,\ \forall e\in\mathcal{E}(v),\\ \psi_\varepsilon^e(T,x)=0, & x\in e,\ \forall e\in\mathcal{E},\end{cases}$$

where we introduced

$$\begin{aligned}F[\chi,\varphi_\varepsilon^e,e^{\zeta^e/\varepsilon}]=&-b^e\partial_x\chi\varphi_\varepsilon^e e^{\zeta^e/\varepsilon}-\varepsilon\partial_{xx}^2\chi\varphi_\varepsilon^e e^{\zeta^e/\varepsilon}-2\partial_x\chi\partial_x\zeta\psi_\varepsilon^e\\&-\frac{a^e}{\varepsilon}\partial_t\zeta^e\psi_\varepsilon^e-\frac{b^e}{\varepsilon}\partial_x\zeta^e\psi_\varepsilon^e-\frac{1}{\varepsilon}(\partial_x\zeta^e)^2\psi_\varepsilon^e-\partial_{xx}^2\zeta^e\psi_\varepsilon^e\\&-2\varepsilon\partial_x\varphi_\varepsilon^e\partial_x\chi e^{\zeta^e/\varepsilon}-2\partial_x\zeta^e\partial_x\varphi_\varepsilon^e\chi e^{\zeta^e/\varepsilon}\\=&-b^e\partial_x\chi\varphi_\varepsilon^e e^{\zeta^e/\varepsilon}-\varepsilon\partial_{xx}^2\chi\varphi_\varepsilon^e e^{\zeta^e/\varepsilon}-2\partial_x\chi\partial_x\zeta\psi_\varepsilon^e\\&-\frac{a^e}{\varepsilon}\partial_t\zeta^e\psi_\varepsilon^e-\frac{b^e}{\varepsilon}\partial_x\zeta^e\psi_\varepsilon^e-\frac{1}{\varepsilon}(\partial_x\zeta^e)^2\psi_\varepsilon^e-\partial_{xx}^2\zeta^e\psi_\varepsilon^e\\&+2\varepsilon\varphi_\varepsilon^e\partial_x(\partial_x\chi e^{\zeta^e/\varepsilon})+2\varphi_\varepsilon^e\partial_x(\partial_x\zeta^e\chi e^{\zeta^e/\varepsilon})\\&-2\varepsilon\partial_x(\varphi_\varepsilon^e\partial_x\chi e^{\zeta^e/\varepsilon})-2\partial_x(\partial_x\zeta^e\varphi_\varepsilon^e\chi e^{\zeta^e/\varepsilon}).\end{aligned}$$

Moreover, using the Dirichlet boundary conditions and the fact that $\chi=1$ in a neighborhood of $\mathcal{V}_\partial$, we deduce

$$\partial_{n^e(v)}\psi_\varepsilon^e(t,v)=e^{\zeta/\varepsilon}\partial_{n^e(v)}\varphi_\varepsilon^e(t,v),\quad\forall v\in\mathcal{V}_\partial.$$

From classical regularity estimates for the heat equation (applied in every edge with an end in $\mathcal{V}_\partial$), we obtain

$$\begin{aligned}\int_0^T\sum_{v\in\mathcal{V}_\partial}e^{2\zeta^e(t,v)/\varepsilon}|\partial_{n^e(v)}\varphi_\varepsilon^e(t,v)|^2\,\mathrm{d}t&=\int_0^T\sum_{v\in\mathcal{V}_\partial}|\partial_{n^e(v)}\psi_\varepsilon^e(t,v)|^2\,\mathrm{d}t\\&\leq C\varepsilon^{-2}\|F[\chi,\varphi_\varepsilon,e^{\zeta/\varepsilon}]\|^2_{L^2((0,T);H^{-1}(\mathcal{E}))}\\&\leq C\varepsilon^{-4}\sum_{e\in\mathcal{E}}\int_0^T\int_e\left|e^{\zeta^e(t,x)/\varepsilon}\varphi_\varepsilon^e(t,x)\right|^2\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

which yields

$$\begin{aligned}&\exp\left(\frac{b^{\tilde e}(\ell^{\tilde e})^2}{2\varepsilon}-\frac{2(b^{\tilde e})^2T(\ell^{\tilde e}+(\ell^{\tilde e})^2)}{a^{\tilde e}\varepsilon}\right)\int_0^T\sum_{v\in\mathcal{V}_\partial}|\partial_{n^e(v)}\varphi_\varepsilon^e(t,v)|^2\,\mathrm{d}t\\&\leq\int_0^T\sum_{v\in\mathcal{V}_\partial}e^{2\zeta^e(t,v)/\varepsilon}|\partial_{n^e(v)}\varphi_\varepsilon^e(t,v)|^2\,\mathrm{d}t\\&\leq\frac{C}{\varepsilon^4}\sum_{e\in\mathcal{E}}\int_0^T\int_e|e^{\zeta^e(t,x)/\varepsilon}\varphi_\varepsilon^e(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Combining these with Lemma 8.4.1, we deduce

$$
\begin{aligned}
&\int_0^T \sum_{v\in\mathcal{V}_\partial} |\partial_{n^e(v)}\varphi_\varepsilon^e(t,v)|^2\,\mathrm{d}t \\
&\le \frac{CT}{\varepsilon^4}\exp\left(-\frac{3b^{\tilde e}(\ell^{\tilde e})^2}{8\varepsilon}+\frac{2(b^{\tilde e})^2T(\ell^{\tilde e}+(\ell^{\tilde e})^2)}{a^{\tilde e}\varepsilon}\right)\|\varphi_T\|^2_{L^2(\mathcal{E})} \\
&= \exp\left(\ln(CT)+4\ln(\varepsilon^{-1})-\frac{3b^{\tilde e}(\ell^{\tilde e})^2}{8\varepsilon}+\frac{2(b^{\tilde e})^2T(\ell^{\tilde e}+(\ell^{\tilde e})^2)}{a^{\tilde e}\varepsilon}\right)\|\varphi_T\|^2_{L^2(\mathcal{E})}.
\end{aligned}
\tag{8.4.6}
$$

Plugging (8.4.6) into (8.2.5), we conclude that (8.2.1) holds for T small enough.

□

8.5. Decay of the cost of controllability

8.5.1. The decay property for the parabolic problem. In order to prove Claim (2) of Theorem 8.2.3, we start by deducing a decay property for the solution of (1.2.1).

PROPOSITION 8.5.1 (Decay property). *Let $\mathcal{G}=(\mathcal{V},\mathcal{E})$ be a tree-shaped network and let us assume that* (1.2.3) *holds. Then, there exist c, $C>0$ such that the solution of the parabolic problem* (1.2.4) *satisfies the following decay property for all $\varphi_T\in L^2(\mathcal{E})$, $\varepsilon\in(0,1)$, and $t\in(0,T)$:*

$$
\|\varphi_\varepsilon(t,\cdot)\|^2_{L^2(\mathcal{E})}\,\mathrm{d}x \le \exp\left(\frac{C-c(T-t)}{\varepsilon}\right)\|\varphi_T\|^2_{L^2(\mathcal{E})}. \tag{8.5.1}
$$

PROOF. In order to prove (8.5.1), we first obtain a decay property for the symmetrized system (8.2.8). Multiplying the PDEs in (8.2.8) by z_ε^e, integrating by parts, and summing up over all the edges, we obtain (by using $(8.2.8)_3$) that

$$
-\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left(\int_{\mathcal{E}} a|z_\varepsilon(t,x)|^2\,\mathrm{d}x\right)+\varepsilon\int_{\mathcal{E}}|\partial_x z_\varepsilon(t,x)|^2\,\mathrm{d}x+\int_{\mathcal{E}}\frac{|b|^2}{4\varepsilon}|z_\varepsilon(t,x)|^2\,\mathrm{d}x=0.
$$

Consequently,

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathcal{E}} a|z_\varepsilon(t,x)|^2\,\mathrm{d}x\right) &= -2\varepsilon\int_{\mathcal{E}}|\partial_x z_\varepsilon(t,x)|^2\,\mathrm{d}x-\int_{\mathcal{E}}\frac{|b|^2}{2\varepsilon}|z_\varepsilon(t,x)|^2\,\mathrm{d}x \\
&\le -\int_{\mathcal{E}}\frac{|b|^2}{2\varepsilon}|z_\varepsilon(t,x)|^2\,\mathrm{d}x = -\int_{\mathcal{E}}\frac{|b|^2}{2a\varepsilon}a|z_\varepsilon(t,x)|^2\,\mathrm{d}x.
\end{aligned}
$$

Then, by using $\min_{e\in\mathcal{E}} a^e$ and $\min_{e\in\mathcal{E}} b^e>0$,

$$
-\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathcal{E}} a|z_\varepsilon(t,x)|^2\,\mathrm{d}x\right)\le -\frac{c}{\varepsilon}\int_{\mathcal{E}} a|z_\varepsilon(t,x)|^2\,\mathrm{d}x,
$$

for $c:=\frac{1}{2}\min_{e\in\mathcal{E}}\frac{|b^e|^2}{a^e}>0$. Using backward Gronwall's inequality on (t,T) yields

$$
\begin{aligned}
\min_{e\in\mathcal{E}} a^e\int_{\mathcal{E}}|z_\varepsilon(t,x)|^2\,\mathrm{d}x \le \int_{\mathcal{E}} a|z_\varepsilon(t,x)|^2\,\mathrm{d}x &\le \exp\left(-\frac{c(T-t)}{\varepsilon}\right)\int_{\mathcal{E}} a|z_T(x)|^2\,\mathrm{d}x \\
&\le \max_{e\in\mathcal{E}} a^e\exp\left(-\frac{c(T-t)}{\varepsilon}\right)\int_{\mathcal{E}}|z_T(x)|^2\,\mathrm{d}x.
\end{aligned}
$$

Defining

$$
\widetilde{C}:=\ln\left(\frac{\max_{e\in\mathcal{E}} a^e}{\min_{e\in\mathcal{E}} a^e}\right)\ge 0,
$$

we obtain that

$$
\int_{\mathcal{E}}|z_\varepsilon(t,x)|^2\,\mathrm{d}x \le \exp\left(\frac{\widetilde{C}}{\varepsilon}-\frac{c(T-t)}{\varepsilon}\right)\int_{\mathcal{E}}|z_T(x)|^2\,\mathrm{d}x.
$$

Finally, reverting the change of variables (8.2.7), we obtain the decay property (8.5.1) for $c>0$ defined as before and

$$
C:=\max_{e\in\mathcal{E}}\sup_{x\in e}(xb^e+\mathfrak{c}^e)-\min_{e\in\mathcal{E}}\inf_{x\in e}(xb^e+\mathfrak{c}^e)+\widetilde{C}.
$$

□

8.5.2. A Carleman inequality. The main technical tool needed for the proof of the decay of the cost of controllability is a Carleman inequality. The difficulty in the proof arises from the boundary terms at the junctions. To suitably deal with them, we define the *Fursikov–Imanuvilov weights* (see [**136**]) with a piecewise C^2 auxiliary function. Piecewise-C^2 weights were first used for proving Carleman inequalities for the heat equation with discontinuous coefficients in [**31**]; more recently, similar functions were used to study coupled systems with Kirchhoff-type conditions in [**164**] and in [**34**]. We cannot use the results in [**164**, Proposition 3.1] directly to deduce Proposition 8.5.2 because we need to keep track of the dependence of s and τ on the viscosity parameter ε. To this end, we also propose a more general construction of the auxiliary functions in the Fursikov–Imanuvilov weights.

We define an auxiliary function $\eta \in C^2_{\mathrm{pw}}(\mathcal{E}) \cap C^0(\overline{\mathcal{E}})$ recursively by the edges joining the i-th and $(i+1)$-th layer of the tree. In the construction of η, a parameter $\delta > 0$ intervenes. This parameter is sufficiently small (depending on $(a^e)_{e\in\mathcal{E}}$ and $(b^e)_{e\in\mathcal{E}}$), and its exact value will be chosen later on in Step 2 of the proof of the Carleman inequality. For now, the only assumption is $\delta \in (0, 1]$.

To construct the function, let us start with the base case: the edge $e = (v_1, v_2) \simeq (0, \ell^e)$, where v_1 is the root of the tree:

$$\eta^e(x) := 1 + x - \frac{1-\delta}{2\ell^e} x^2.$$

As for the inductive case, let us consider $\tilde{e} = (\tilde{v}_1, \tilde{v}_2) \simeq (x^{\tilde{e}}, x^{\tilde{e}} + \ell^{\tilde{e}})$ with $\tilde{v}_1$ is on the i-th layer and $\tilde{v}_2$ on the $(i+1)$-th layer for $i \geq 1$. Given k, the value taken by η at $\tilde{v}_1$, we define $\eta^{\tilde{e}}$ as follows:

$$\eta^{\tilde{e}}(x) := k + (x - x^{\tilde{e}}) - \frac{1-\delta}{2\ell^e}(x - x^{\tilde{e}})^2.$$

An example on the construction of η can be found in Figure 8.4. We highlight that the function η satisfies the following properties:

(1) $\eta \in C^2_{\mathrm{pw}}(\mathcal{E}) \cap C^0(\overline{\mathcal{E}})$; moreover, $\|\eta\|_{W^{2,\infty}_{\mathrm{pw}}(\mathcal{E})}$ is bounded uniformly in δ for all $\delta \in (0, 1)$;
(2) $\partial^2_{xx}\eta^e = -\frac{1-\delta}{2\ell^e}$ and $|\partial_x \eta| \in [\delta, 1]$ on $\overline{\mathcal{E}}$;
(3) given a vertex $v \in \mathcal{V}_0$, there exists an edge $\tilde{e} \in \mathcal{E}(v)$ such that $\partial_{n^{\tilde{e}}(v)}\eta^{\tilde{e}}(v) = \delta$ and $\partial_{n^e(v)}\eta^e(v) = -1$ for all $e \in \mathcal{E}(v) \setminus \{\tilde{e}\}$.

This auxiliary function allows us to define usual Fursikov–Imanuvilov weights:

$$\alpha(t,x) := \frac{e^{8\tau\|\eta\|_\infty} - e^{\tau(6\|\eta\|_\infty + \eta(x))}}{t(T-t)}, \quad \xi(t,x) := \frac{e^{\tau(6\|\eta\|_\infty + \eta(x))}}{t(T-t)}, \tag{8.5.2}$$

where $\tau \in \mathbb{R}$ is a fixed parameter (and in particular independent of the edge) that will be chosen later. For future use, we note that there exists $C > 0$ such that, for sufficiently large τ,

$$|\partial_t \alpha| \leq T\xi^2, \quad |\partial^2_{xt}\alpha| \leq CT\tau\xi^2, \quad |\partial^2_{tt}\alpha| \leq 2(\xi^2 + T^2\xi^3) \leq 4T^2\xi^3, \quad |\partial_t \xi| \leq T\xi^2. \tag{8.5.3}$$

Now we are ready to state the following Carleman inequality.

PROPOSITION 8.5.2 (Carleman inequality). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree-shaped network and let us assume that* (1.2.3) *holds. Let z_ε be the solution of* (8.2.8). *Then, for δ small enough, there exists a positive constant $C = C(\mathcal{G}, a, b, \delta)$ such that, for all $z_T \in L^2(\mathcal{E})$, the solution z_ε of* (8.2.8) *satisfies the following Carleman inequality:*

$$\begin{aligned} &s\tau^2 \iint_Q e^{-2s\alpha}\xi|\partial_x z_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t + s^3\tau^4 \iint_Q e^{-2s\alpha}\xi^3|z_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\leq C \sum_{v\in\mathcal{V}_\partial} s\tau \int_0^T e^{-2s\alpha}\xi(t,v)|\partial_{n^e(v)} z^e_\varepsilon(t,v)|^2 \,\mathrm{d}t, \end{aligned} \tag{8.5.4}$$

where $Q := (0,T) \times \mathcal{E}$, α, and ξ are the Fursikov–Imanuvilov weights *defined in* (8.5.2), *$\varepsilon \in (0, 1]$, $\tau \geq C$, and $s \geq C(T + T^2)\varepsilon^{-1}$.*

Throughout the proof, to simplify the notation, all the constants may change from line to line and will be denoted by $C_\delta > 0$ when they depend on δ and by $C > 0$ when they do not.

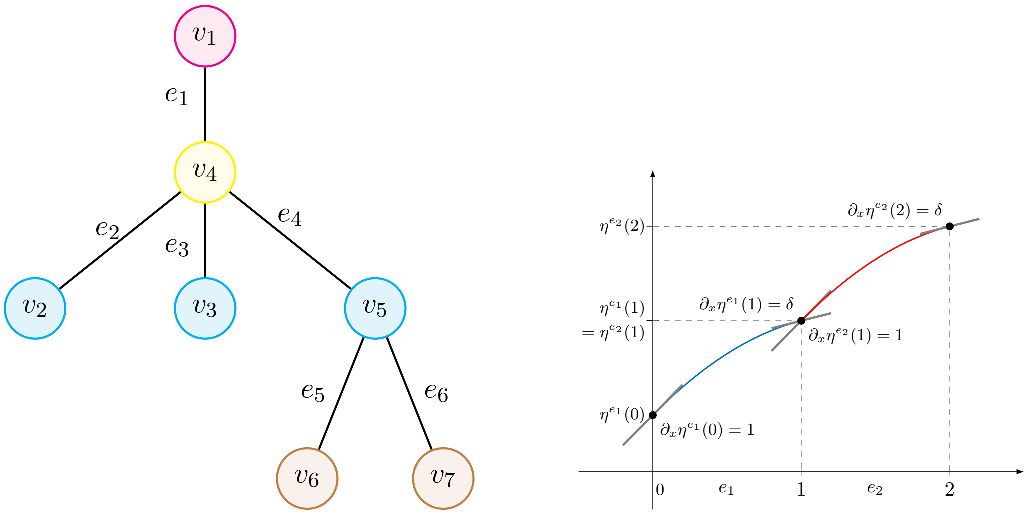

FIGURE 8.4. Construction of the auxiliary function $\eta \in C^2_{\text{pw}}(\mathcal{E}) \cap C^0(\bar{\mathcal{E}})$. In the tree-shaped network in the LEFT picture, we distinguish four layers: the first (magenta) is $\{v_1\}$; the second (yellow) is $\{v_4\}$, the third (cyan) is $\{v_2, v_3, v_5\}$; and the fourth (brown) is $\{v_6, v_7\}$. Let us identify $e_1 := (v_1, v_4) \simeq (0, 1)$, $e_2 := (v_4, v_2) \simeq (1, 2)$, $e_3 := (v_4, v_3) \simeq (1, 2)$, $e_4 := (v_4, v_5) \simeq (1, 2)$, $e_5 := (v_5, v_6) \simeq (2, 3)$, $e_6 := (v_5, v_6) \simeq (2, 5/2)$. The auxiliary function η may be defined as follows: $\eta^{e_1}(x) = 1 + x - \frac{1-\delta}{2}x^2$, $\eta^{e_2}(x) = \eta^{e_3}(x) = \eta^{e_4}(x) = \frac{3+\delta}{2} + (x-1) - \frac{1-\delta}{2}(x-1)^2$, $\eta^{e_5}(x) = \eta^{e_4}(2) + (x-2) - \frac{1-\delta}{2}(x-2)^2$, $\eta^{e_6}(x) = \eta^{e_4}(2) + (x-2) - (1-\delta)(x-2)^2$. As an example, in the RIGHT picture, we plot η^{e_1} and η^{e_2} with $\delta = \frac{1}{4}$. In the proof of Proposition 8.5.2, if $\mathcal{G}$ is the graph illustrated above, we observe the vertex v_1 with v_4; v_4 with v_2, v_3, v_5; and v_5 with v_6, v_7. So, by transitivity, we actually only need to observe using v_2, v_3, v_6, and v_7 (i.e., we do not need the vertex in the 1st layer).

PROOF. **Step 0.** *Strategy of the proof and choice of the auxiliary functions.* The main idea is to observe the nodes of the i-th layer with the nodes of the $(i+1)$-th layer (see Definition 8.1.1).

With the weights defined in (8.5.2) we consider the change of variable $\psi = e^{-s\alpha} z_\varepsilon$, for z_ε given in (8.2.8). It is important to remark that

$$\psi(T, x) = \psi(0, x) = \partial_x \psi(0, x) = \partial_x \psi(T, x) = 0, \tag{8.5.5}$$

which will allow us to integrate by parts in the time variable without having to worry about the boundary terms.

From (8.2.8), we obtain that ψ satisfies

$$L_1\psi + L_2\psi = L_3\psi, \tag{8.5.6}$$

where

$$\begin{cases} L_1\psi := -2\varepsilon s\tau^2 a^{-1/2}|\partial_x\eta|^2\xi\psi - 2\varepsilon s\tau a^{-1/2}\xi\partial_x\eta\partial_x\psi + a^{1/2}\partial_t\psi, \\ L_2\psi := \varepsilon s^2\tau^2 a^{-1/2}|\partial_x\eta|^2\xi^2\psi + \varepsilon a^{-1/2}\partial^2_{xx}\psi + sa^{1/2}\partial_t\alpha\psi - \frac{|b|^2}{4\varepsilon}a^{-1/2}\psi, \\ L_3\psi := \varepsilon s\tau a^{-1/2}\partial^2_{xx}\eta\xi\psi - \varepsilon s\tau^2 a^{-1/2}|\partial_x\eta|^2\xi\psi. \end{cases} \tag{8.5.7}$$

Indeed, from (8.2.8), we compute

$$\begin{aligned} &a\partial_t\psi + \varepsilon\partial^2_{xx}\psi - \frac{|b|^2}{4\varepsilon}\psi \\ &\quad = -as\partial_t\alpha\psi + \varepsilon s\tau\partial^2_{xx}\eta\xi\psi + \varepsilon s\tau^2|\partial_x\eta|^2\xi\psi + \varepsilon s^2\tau^2|\partial_x\eta|^2\xi^2\psi + 2\varepsilon s\tau\partial_x\eta\xi e^{-s\alpha}\partial_x z. \end{aligned} \tag{8.5.8}$$

Combining (8.5.8) with the fact that

$$2\varepsilon s\tau\partial_x\eta\xi e^{-s\alpha}\partial_x z = 2\varepsilon s\tau\partial_x\eta\xi\partial_x\psi - 2\varepsilon s^2\tau^2|\partial_x\eta|^2\xi^2\psi,$$

we deduce that ψ satisfies (8.5.6).

We now argue as in [**26, 150**], but paying extra attention to keep track of the boundary terms at junctions. In what follows, we use the notation $(L_i\psi)_j$ for the j-th term in the expression of $L_i\psi$ given above. From (8.5.6), we have

$$\|L_1\psi + L_2\psi\|^2_{L^2(Q)} = \|L_1\psi\|^2_{L^2(Q)} + \|L_2\psi\|^2_{L^2(Q)} + 2(L_1\psi, L_2\psi)_{L^2(Q)} = \|L_3\psi\|^2_{L^2(Q)}.$$

In the next two steps, we estimate the product

$$(L_1\psi, L_2\psi)_{L^2(Q)} = \sum_{e\in\mathcal{E}}((L_1\psi)^e, (L_2\psi)^e)_{L^2((0,T)\times e)}.$$

In particular, we show that, for a suitable choice of the parameters τ and s, the choice of the weights in (8.5.2) makes it positive up to a term depending on the normal derivative.

Step 1. *Estimates in the interior.* In this step, we perform integrations by parts in the spirit of [**26, 150**], but keeping track of the boundary terms appearing at the vertices of the graph.

First, we note that

$$((L_1\psi)_1, (L_2\psi)_1)_{L^2(Q)} = -2\varepsilon^2 s^3\tau^4 \iint_Q a^{-1}|\partial_x\eta|^4\xi^3|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{8.5.9}$$

Secondly, we compute

$$\begin{aligned}
((L_1\psi)_2,&(L_2\psi)_1)_{L^2(Q)}\\
&= -2\varepsilon^2 s^3\tau^3 \iint_Q a^{-1}|\partial_x\eta|^2\partial_x\eta\xi^3\psi\partial_x\psi \,\mathrm{d}x\,\mathrm{d}t\\
&= 3\varepsilon^2 s^3\tau^4 \iint_Q a^{-1}|\partial_x\eta|^4\xi^3|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \varepsilon^2 s^3\tau^3 \iint_Q a^{-1}\partial_x((\partial_x\eta)^3)\xi^3|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \varepsilon^2 s^3\tau^3 \sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)} \int_0^T (a^e)^{-1}|\partial_{n^e(v)}\eta^e|^2\partial_{n^e(v)}\eta^e(\xi^e)^3|\psi^e|^2(t,v)\,\mathrm{d}t\\
&= 3\varepsilon^2 s^3\tau^4 \iint_Q a^{-1}|\partial_x\eta|^4\xi^3|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t\\
&\quad + o\left(\varepsilon^2 s^3\tau^4 \iint_Q \xi^3|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t\right)\\
&\quad \underbrace{-\varepsilon^2 s^3\tau^3 \sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)} \int_0^T (a^e)^{-1}|\partial_{n^e(v)}\eta^e|^2\partial_{n^e(v)}\eta^e(\xi^e)^3|\psi^e|^2(t,v)\,\mathrm{d}t}_{=:J_1}.
\end{aligned} \tag{8.5.10}$$

Here, we introduced the $o(\cdot)$ notation because, for all $\varepsilon > 0$, we have that, if $\tau \geq \varepsilon^{-1}\max_{e\in\mathcal{E}}\{3(a^e)^{-1}\}\|\partial^2_{xx}\eta(\partial_x\eta)^2\|_{L^\infty(\mathcal{E})}$, then

$$\varepsilon^2 s^3\tau^3 \iint_Q a^{-1}\partial_x((\partial_x\eta)^3)\xi^3|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \varepsilon\varepsilon^2 s^3\tau^4 \iint_Q \xi^3|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t$$

Thirdly, integration by parts (with respect to the time variable) yields, using (8.5.5), for $\tau \geq C$ and $s \geq C(T+T^2)\varepsilon^{-1}$,

$$\begin{aligned}
((L_1\psi)_3, (L_2\psi)_1)_{L^2(Q)} &= \varepsilon s^2\tau^2 \iint_Q |\partial_x\eta|^2\xi^2\psi\partial_t\psi \,\mathrm{d}x\,\mathrm{d}t\\
&= -\varepsilon s^2\tau^2 \iint_Q |\partial_x\eta|^2\partial_t\xi\xi|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t\\
&= o\left(\varepsilon^2 s^3\tau^4 \iint_Q \xi^3|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t\right).
\end{aligned} \tag{8.5.11}$$

Indeed, because of $|\partial_x \eta| \in [\delta, 1]$ and the properties of the weights, we have that $s \geq \varepsilon^{-1}\varepsilon^{-1}T$ and $\tau \geq 1$ imply

$$\varepsilon s^2\tau^2 \iint_Q |\partial_x\eta|^2 |\partial_t \xi| \xi |\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \varepsilon\varepsilon^2 s^3\tau^4 \iint_Q \xi^3 |\psi|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Next, we have

$$\begin{aligned}
((L_1\psi)_1, (L_2\psi)_2)_{L^2(Q)} &= -2\varepsilon^2 s\tau^2 \iint_Q a^{-1}|\partial_x\eta|^2 \xi\psi\partial_{xx}^2\psi \,\mathrm{d}x\,\mathrm{d}t \\
&= 2\varepsilon^2 s\tau^2 \iint_Q a^{-1}|\partial_x\eta|^2\xi|\partial_x\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 2\varepsilon^2 s\tau^2 \iint_Q a^{-1}\partial_x(|\partial_x\eta|^2\xi)\psi\partial_x\psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \underbrace{- \sum_{v\in\mathcal{V}} \sum_{e\in\mathcal{E}(v)} 2\varepsilon^2 s\tau^2 \int_0^T (a^e)^{-1}|\partial_{n^e(v)}\eta^e|^2\xi^e\psi^e\partial_{n^e(v)}\psi^e(t,v)\,\mathrm{d}t}_{=:J_2}\,.
\end{aligned} \tag{8.5.12}$$

Using Cauchy-Schwarz' inequality, owing to the properties of η and the weights, we obtain that

$$\begin{aligned}
\left|2\varepsilon^2 s\tau^2 \iint_Q a^{-1}\partial_x(|\partial_x\eta|^2\xi)\psi\partial_x\psi \,\mathrm{d}x\,\mathrm{d}t\right| &= \Big|2\varepsilon^2 s\tau^3 \iint_Q a^{-1}|\partial_x\eta|^2\partial_x\eta\xi\psi\partial_x\psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 2\varepsilon^2 s\tau^2 \iint_Q a^{-1}\partial_x(|\partial_x\eta|^2)\xi\psi\partial_x\psi \,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\leq C\varepsilon^2\tau^2 \iint_Q |\partial_x\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + C\varepsilon^2 s^2\tau^4 \iint_Q \xi^2|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t;
\end{aligned}$$

and thus

$$\begin{aligned}
&2\varepsilon^2 s\tau^2 \iint_Q a^{-1}\partial_x(|\partial_x\eta|^2\xi)\psi\partial_x\psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = o\left(\varepsilon^2 s\tau^2 \iint_Q \xi|\partial_x\psi|^2 \,\mathrm{d}x\,\mathrm{d}t\right) + o\left(\varepsilon^2 s^3\tau^4 \iint_Q \xi^3|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t\right),
\end{aligned} \tag{8.5.13}$$

where we have used the fact $1 \leq T^2\xi$.

In addition, we compute

$$\begin{aligned}
&((L_1\psi)_2, (L_2\psi)_2)_{L^2(Q)} = -2\varepsilon^2 s\tau \iint_Q a^{-1}\partial_x\eta\xi\partial_{xx}^2\psi\partial_x\psi \,\mathrm{d}x\,\mathrm{d}t \\
&= \varepsilon^2 s\tau^2 \iint_Q a^{-1}|\partial_x\eta|^2\xi|\partial_x\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + o\left(\varepsilon^2 s\tau^2 \iint_Q \xi|\partial_x\psi|^2 \,\mathrm{d}x\,\mathrm{d}t\right) \\
&\underbrace{- \sum_{v\in\mathcal{V}} \sum_{e\in\mathcal{E}(v)} \varepsilon^2 s\tau \int_0^T (a^e)^{-1}\partial_{n^e(v)}\eta^e\xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t}_{=:J_3}\,.
\end{aligned} \tag{8.5.14}$$

Moreover, with (8.5.5), we can prove that

$$\begin{aligned}
((L_1\psi)_3, (L_2\psi)_2)_{L^2(Q)} &= \varepsilon \iint_Q \partial_{xx}^2\psi\partial_t\psi \,\mathrm{d}x\,\mathrm{d}t \\
&= -\frac{\varepsilon}{2} \iint_Q \partial_t(|\partial_x\psi|^2) + \underbrace{\sum_{v\in\mathcal{V}} \sum_{e\in\mathcal{E}(v)} \varepsilon \int_0^T \partial_{n^e(v)}\psi^e\partial_t\psi^e(t,v)\,\mathrm{d}t}_{=:J_4} \\
&= J_4.
\end{aligned} \tag{8.5.15}$$

Finally, with analogous computations, using (8.5.3), we obtain that

$$
\begin{aligned}
&(L_1\psi,(L_2\psi)_3+(L_2\psi)_4)_{L^2(Q)}\\
&= o\left(\varepsilon^2 s^3\tau^4 \iint_Q \xi^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right)\\
&\quad+\underbrace{\sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)} s\tau\int_0^T \xi^e\partial_{n^e(v)}\eta^e\left(s\varepsilon\partial_t\alpha^e-\frac{|b^e|^2}{4}\right)|\psi^e|^2(t,v)\,\mathrm{d}t}_{=:J_5}.
\end{aligned}
\tag{8.5.16}
$$

Summing up, we have proved that

$$
\begin{aligned}
&3\varepsilon^2 s\tau^2\iint_Q a^{-1}|\partial_x\eta|^2\xi|\partial_x\psi|^2\,\mathrm{d}x\,\mathrm{d}t+\varepsilon^2 s^3\tau^4\iint_Q a^{-1}|\partial_x\eta|^4\xi^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t+\sum_{i=1}^5 J_i\\
&\quad=(L_1\psi,L_2\psi)_{L^2(Q)}+o\left(\varepsilon^2 s^3\tau^4\iint_Q\xi^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right)+o\left(\varepsilon^2 s\tau^2\iint_Q\xi|\partial_x\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right).
\end{aligned}
$$

From here, we obtain that, for all $\varepsilon>0$, $\lambda\geq C_\delta$, and $s\geq C_\delta(T+T^2)\varepsilon^{-1}$,

$$
C_\delta^{-1}\left(\varepsilon^2 s\tau^2\iint_Q\xi|\partial_x\psi|^2\,\mathrm{d}x\,\mathrm{d}t+\varepsilon^2 s^3\tau^4\iint_Q\xi^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right)+\sum_{i=1}^5 J_i\leq(L_1\psi,L_2\psi)_{L^2(Q)}.
\tag{8.5.17}
$$

Step 2. *Estimation of the boundary terms.* In this part of the proof, we estimate the boundary terms $J_1,\dots,J_5$. In particular, we need to make a distinction between *exterior vertices*, which can be treated as in [**26, 150**] (since they correspond to the boundary terms appearing in a classical IBVP), and *junctions*, which require new more precise computations. As we are going to see, the terms corresponding to the exterior vertices $v\in\mathcal{V}_\partial$ either vanish (due to the zero Dirichlet boundary condition in (8.2.8)) or can be moved to the right-hand side of the Carleman estimate (corresponding to the "classical" boundary terms that appear in [**150**]). The interior junction terms at $v\in\mathcal{V}_0$ are more critical: they are on the left-hand side of the Carleman estimate and we need to show that they are non-negative. To this end, we will rely on the properties of the auxiliary function η and on the Kirchhoff junction condition $(8.2.8)_4$ (which, in turn, was formulated thanks to (1.2.3)). At the end of the computations, all these boundary terms at the junction can be absorbed into the expression on the right-hand side of (8.5.26), which is non-negative.

To begin with, let us deal with the boundary term J_1 in (8.5.10). If $v\in\mathcal{V}_\partial$, we get that $\psi(t,v)=0$ from the Dirichlet boundary conditions. Otherwise, for each interior node $v\in\mathcal{V}_0$, we use property (3) of η and choose δ small enough to get

$$
\begin{aligned}
&-\varepsilon^2 s^3\tau^3\int_0^T\left(\sum_{e\in\mathcal{E}(v)}(a^e)^{-1}|\partial_{n^e(v)}\eta^e|^2\partial_{n^e(v)}\eta^e\right)(\xi^e)^3|\psi^e|^2(t,v)\,\mathrm{d}t\\
&\geq C^{-1}\varepsilon^2 s^3\tau^3\int_0^T(\xi^e)^3|\psi^e|^2(t,v)\,\mathrm{d}t.
\end{aligned}
\tag{8.5.18}
$$

Taking into account (8.5.3) and $\xi T^2\geq 1$, we obtain

$$
J_5=o\left(\varepsilon^2 s^3\tau^3\int_0^T(\xi^e)^3|\psi^e|^2(t,v)\,\mathrm{d}t\right).
$$

Next, let us study the boundary term J_3 given in (8.5.14) for each $v\in\mathcal{V}$, i.e.

$$
-\varepsilon^2 s\tau\int_0^T\sum_{e\in\mathcal{E}(v)}(a^e)^{-1}\xi^e\partial_{n^e(v)}\eta^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t.
$$

If $v\in\mathcal{V}_\partial$, then there exists only one edge $e\in\mathcal{E}(v)$ and

$$
-\varepsilon^2 s\tau\int_0^T\xi^e(a^e)^{-1}\partial_{n^e(v)}\eta^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t\geq -C\varepsilon^2 s\tau\int_0^T\xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t.
$$

On the other hand, if $v \in \mathcal{V}_0$, by the construction of η, there exists an edge $\tilde{e} \in \mathcal{E}(v)$ (the edge joining the previous layer to v) for which $\partial_{n_{\tilde{e}}} \eta^{\tilde{e}}(v) = \delta$ and $\partial_{n^e} \eta^e(v) = -1$ for all $e \in \mathcal{E}(v) \setminus \{\tilde{e}\}$. Then,

$$\begin{aligned}
&- \varepsilon^2 s\tau \int_0^T \sum_{e\in\mathcal{E}(v)} (a^e)^{-1} \xi^e \partial_{n^e(v)} \eta^e |\partial_{n^e(v)} \psi^e|^2(t,v)\,\mathrm{d}t \\
&= \varepsilon^2 s\tau \int_0^T \sum_{e\in\mathcal{E}(v)\setminus\{\tilde{e}\}} (a^e)^{-1} \xi^e |\partial_{n^e(v)} \psi^e|^2(t,v)\,\mathrm{d}t - \delta\varepsilon^2 s\tau \left(a^{\tilde{e}}\right)^{-1} \int_0^T \xi^{\tilde{e}} |\partial_{n^{\tilde{e}}} \psi^{\tilde{e}}|^2(t,v)\,\mathrm{d}t.
\end{aligned} \tag{8.5.19}$$

We then have to absorb the boundary term of the edge $\tilde{e}$; to this end, we use the fact that $\psi = z_\varepsilon e^{-s\alpha}$ to get

$$\begin{aligned}
&\delta\varepsilon^2 s\tau \left(a^{\tilde{e}}\right)^{-1} \int_0^T \xi^{\tilde{e}} |\partial_{n^{\tilde{e}}(v)} \psi^{\tilde{e}}|^2(t,v)\,\mathrm{d}t \\
&\leq 2\delta\varepsilon^2 s^3\tau \left(a^{\tilde{e}}\right)^{-1} \int_0^T \xi^{\tilde{e}} |\partial_{n^{\tilde{e}}(v)} \alpha^{\tilde{e}}|^2 |\psi^{\tilde{e}}|^2(t,v)\,\mathrm{d}t \\
&\qquad + 2\delta s\tau \left(a^{\tilde{e}}\right)^{-1} \int_0^T \xi^{\tilde{e}} e^{-2s\alpha^{\tilde{e}}} |\varepsilon \partial_{n^{\tilde{e}}(v)} z_\varepsilon^{\tilde{e}}|^2(t,v)\,\mathrm{d}t \\
&= o\left(\varepsilon^2 s^3\tau^3 \int_0^T (\xi^{\tilde{e}})^3 |\psi^{\tilde{e}}|^2(t,v)\,\mathrm{d}t\right) \\
&\qquad + 2\delta s\tau \left(a^{\tilde{e}}\right)^{-1} \int_0^T \xi^{\tilde{e}} e^{-2s\alpha^{\tilde{e}}} |\varepsilon \partial_{n^{\tilde{e}}(v)} z_\varepsilon^{\tilde{e}}|^2(t,v)\,\mathrm{d}t.
\end{aligned} \tag{8.5.20}$$

Since $\partial_{n^e(v)} z_\varepsilon^e = \partial_{n^e(v)} \psi^e e^{s\alpha} + s\partial_{n^e(v)} \alpha^e \psi^e e^{s\alpha^e}$, we compute (using the fact that α and ξ are continuous at the junctions)

$$\begin{aligned}
&\delta\varepsilon^2 s\tau \int_0^T \xi^{\tilde{e}} e^{-2s\alpha^{\tilde{e}}} \left| \sum_{e\in\mathcal{E}(v)\setminus\{\tilde{e}\}} (a^e)^{-1} \partial_{n^e(v)} z_\varepsilon^e \right|^2 (t,v)\,\mathrm{d}t \\
&\leq \delta\varepsilon^2 s\tau \int_0^T \sum_{e\in\mathcal{E}(v)\setminus\{\tilde{e}\}} (a^e)^{-1} \xi^e \left|\partial_{n^e(v)} \psi^e\right|^2 (t,v)\,\mathrm{d}t + C\delta\varepsilon^2 s^3\tau^3 \int_0^T (\xi^e)^3 |\psi^e|^2(t,v)\,\mathrm{d}t.
\end{aligned} \tag{8.5.21}$$

Summing up, taking δ small enough, we obtain

$$\begin{aligned}
J_1 + J_3 + J_5 \geq{}& C^{-1} \sum_{v\in\mathcal{V}_0} \varepsilon^2 s^3\tau^3 \int_0^T (\xi^e)^3 |\psi^e|^2(t,v)\,\mathrm{d}t \\
&+ C^{-1} \sum_{v\in\mathcal{V}_0} \varepsilon^2 s\tau \int_0^T \sum_{e\in\mathcal{E}(v)\setminus\{\tilde{e}\}} \xi^e |\partial_{n^e(v)} \psi^e|^2(t,v)\,\mathrm{d}t \\
&- C \sum_{v\in\mathcal{V}_\partial} \varepsilon^2 s\tau \int_0^T \xi^e |\partial_{n^e(v)} \psi^e|^2(t,v)\,\mathrm{d}t.
\end{aligned} \tag{8.5.22}$$

With these considerations, we can also absorb J_2, using (8.5.22) by Cauchy-Schwarz' inequality, for $s \geq C(T+T^2)\varepsilon^{-1}$ large enough and $\delta > 0$ small enough. Notably, we have to estimate

$$\int_0^T (a^e)^{-1} |\partial_{n^e(v)} \eta^e|^2 \xi^e \psi^e \partial_{n^e(v)} \psi^e(t,v)\,\mathrm{d}t,$$

for all $v \in \mathcal{V}$ and $e \in \mathcal{E}(v)$. If $v \in \mathcal{V}_0$, this term is null by the Dirichlet boundary conditions. If $v \in \mathcal{V}_\partial$, we take into account that, by the property (3) of η, there exists an edge $\tilde{e} \in \mathcal{E}(v)$ (the edge joining the previous layer to v) for which $\partial_{n_{\tilde{e}}} \eta^{\tilde{e}}(v) = \delta$ and such that $\partial_{n^e} \eta^e(v) = -1$ for all $e \in \mathcal{E}(v) \setminus \{\tilde{e}\}$. If $e \in \mathcal{E}(v) \setminus \{\tilde{e}\}$, as $T^2\xi \geq 1$, we have

$$2\varepsilon^2 s\tau^2 \int_0^T (a^e)^{-1} |\partial_x \eta^e|^2 \xi^e \psi^e \partial_{n^e(v)} \psi^e(t,v)\,\mathrm{d}t$$

$$
\begin{aligned}
&\leq C\varepsilon^2\tau\int_0^T|\partial_{n^e(v)}\psi^e(t,v)|^2\,\mathrm{d}t+C\varepsilon^2s^2\tau^3\int_0^T(\xi^e)^2|\psi^e(t,v)|^2\,\mathrm{d}t\\
&=o\left(\varepsilon^2s\tau\int_0^T\xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t\right)+o\left(\varepsilon^2s^3\tau^3\int_0^T(\xi^e)^3|\psi^e|^2(t,v)\,\mathrm{d}t\right).
\end{aligned}
$$

Moreover, as $\delta\in(0,1]$, using the continuity at the junctions, we estimate the boundary term of the edge $\tilde{e}$ as follows:

$$
\begin{aligned}
&2\varepsilon^2s\tau^2\int_0^T(a^{\tilde{e}})^{-1}|\partial_x\eta^{\tilde{e}}|^2\xi^{\tilde{e}}\psi^{\tilde{e}}\partial_{n^{\tilde{e}}(v)}\psi^{\tilde{e}}(t,v)\,\mathrm{d}t=2\delta^2\varepsilon^2s\tau^2\int_0^T(\tilde{a}^e)^{-1}\xi^{\tilde{e}}\psi^{\tilde{e}}\partial_{n^{\tilde{e}}(v)}\psi^{\tilde{e}}(t,v)\,\mathrm{d}t\\
&=o\left(\varepsilon^2s^3\tau^3\int_0^T(\xi^e)^3|\psi^e|^2(t,v)\,\mathrm{d}t\right)+o\left(\varepsilon^2s\tau\int_0^T\sum_{e\in\mathcal{E}(v)\setminus\{\tilde{e}\}}\xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t\right).
\end{aligned}
$$

This is done with Cauchy-Schwarz' inequality, being the normal derivative estimated with the help of (8.5.20) and (8.5.21). Consequently, for δ small enough,
(8.5.23)

$$
J_2=o\left(\sum_{v\in\mathcal{V}_0}\varepsilon^2s^3\tau^3\int_0^T\xi^3|\psi^e|^2(t,v)\,\mathrm{d}t\right)+o\left(\sum_{v\in\mathcal{V}_0}\varepsilon^2s\tau\int_0^T\sum_{e\in\mathcal{E}(v)\setminus\{\tilde{e}\}}\xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t\right).
$$

To conclude, let us study the boundary term J_4 in (8.5.15). If $v\in\mathcal{V}_\partial$, then $\partial_t\psi=0$ because of the Dirichlet boundary conditions. Otherwise, if $v\in\mathcal{V}_0$, from $(8.2.8)_4$, we obtain

$$
\begin{aligned}
&\varepsilon\sum_{e\in\mathcal{E}(v)}\int_0^T\partial_{n^e(v)}\psi^e\partial_t\psi^e(t,v)\,\mathrm{d}t\\
&=\varepsilon s\sum_{e\in\mathcal{E}(v)}\int_0^T\partial_{n^e(v)}\xi^ee^{-s\alpha^e}z_\varepsilon^e\partial_t(e^{-s\alpha^e}z_\varepsilon^e)(t,v)\,\mathrm{d}t\\
&=-\varepsilon s^2\sum_{e\in\mathcal{E}(v)}\int_0^T\partial_{n^e(v)}\xi^e\partial_t\alpha^ee^{-2s\alpha^e}|z_\varepsilon^e|^2(t,v)\,\mathrm{d}t\\
&\qquad+\varepsilon s\sum_{e\in\mathcal{E}(v)}\int_0^T\partial_{n^e(v)}\xi^ee^{-2s\alpha^e}\frac{\partial_t(|z_\varepsilon^e|^2)}{2}(t,v)\,\mathrm{d}t\\
&=-\varepsilon s^2\sum_{e\in\mathcal{E}(v)}\int_0^T\partial_{n^e(v)}\xi^e\partial_t\alpha^e|\psi^e|^2(t,v)\,\mathrm{d}t\\
&\qquad-\varepsilon s\sum_{e\in\mathcal{E}(v)}\int_0^T\left[\partial_t(\partial_{n^e(v)}\xi^e)-2s\partial_t\alpha^e\right]\frac{|\psi^e|^2}{2}(t,v)\,\mathrm{d}t\\
&=o\left(\varepsilon^2s^3\tau^3\int_0^T(\xi^e)^3|\psi^e|^2(t,v)\,\mathrm{d}t\right).
\end{aligned}
\tag{8.5.24}
$$

To sum up the results of this step, we have proved that, for $\varepsilon>0$, $\tau\geq C$, $s\geq C(T+T^2)\varepsilon^{-1}$, and δ small enough, the following estimate holds:

$$
\begin{aligned}
\sum_{\ell=1}^5J_\ell\geq{}&C^{-1}\sum_{v\in\mathcal{V}_0}\varepsilon^2s^3\tau^3\int_0^T(\xi^e)^3|\psi^e|^2(t,v)\,\mathrm{d}t\\
&+C^{-1}\sum_{v\in\mathcal{V}_0}\varepsilon^2s\tau\int_0^T\sum_{e\in\mathcal{E}(v)\setminus\{\tilde{e}\}}\xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t\\
&-C\varepsilon^2s\tau\int_0^T\sum_{v\in\mathcal{V}_\partial}\xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t.
\end{aligned}
\tag{8.5.25}
$$

Thus, if we fix $\delta = \delta_0$ sufficiently small such that (8.5.25) holds, combining (8.5.17) and (8.5.25), we get

$$
\begin{aligned}
&\varepsilon^2 s^3\tau^4 \iint_Q \xi^3|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s\tau^2 \iint_Q \xi|\partial_x\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \sum_{v\in\mathcal{V}_0} \varepsilon^2 s^3\tau^3 \int_0^T (\xi^e)^3|\psi^e|^2(t,v)\,\mathrm{d}t + \sum_{v\in\mathcal{V}_0} \varepsilon^2 s\tau \int_0^T \sum_{e\in\mathcal{E}(v)\setminus\{\tilde{e}\}} \xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t \\
&\leq C_{\delta_0}\left((L_1\psi, L_2\psi)_{L^2(Q)} + \varepsilon^2 s\tau \int_0^T \sum_{v\in\mathcal{V}_\partial} \xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t \right).
\end{aligned}
\tag{8.5.26}
$$

Step 3. *Conclusion of the proof.* From (8.5.26), it is classical to obtain (8.5.4) as in [**26, 150**]: we add $\frac{C}{2}(\|L_1\psi\|^2_{L^2(Q)} + \|L_2\psi\|^2_{L^2(Q)})$ to both sides of (8.5.26); we write $\|L_1\psi + L_2\psi\|^2_{L^2(Q)} = \|L_3\psi\|^2_{L^2(Q)}$, whose right-hand side can be estimated as

$$
\begin{aligned}
\|L_3\psi\|^2_{L^2(Q)} &= \iint_Q \left|\varepsilon s\tau a^{-1/2}\partial^2_{xx}\eta\xi\psi - \varepsilon s\tau^2 a^{-1/2}|\partial_x\eta|^2\xi\psi\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&= o\left(\varepsilon^2 s^3\tau^4 \iint_Q \xi^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right);
\end{aligned}
$$

and we thus deduce that

$$
\varepsilon^2 s^3\tau^4 \iint_Q \xi^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s\tau^2 \iint_Q \xi|\partial_x\psi|^2\,\mathrm{d}x\,\mathrm{d}t \leq C_{\delta_0}\varepsilon^2 s\tau \int_0^T \sum_{v\in\mathcal{V}_\partial} \xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t.
$$

From this inequality, by recalling the identity $z_\varepsilon = e^{s\alpha}\psi$ and the Dirichlet boundary conditions satisfied by z_ε, we conclude

$$
\begin{aligned}
&\varepsilon^2 s^3\tau^4 \iint_Q e^{-2s\alpha}\xi^3|z_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s\tau^2 \iint_Q e^{-2s\alpha}\xi|\partial_x z_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t + o\left(\varepsilon^2 s^3\tau^4 \iint_Q e^{-2s\alpha}\xi^3|z_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\right) \\
&\leq C_{\delta_0} \sum_{v\in\mathcal{V}_\partial} \varepsilon^2 s\tau \int_0^T e^{-2s\alpha^e}\xi^e|\partial_{n^e(v)}z^e_\varepsilon|^2(t,v)\,\mathrm{d}t,
\end{aligned}
$$

which yields (8.5.4). $\square$

As a consequence of Proposition 8.5.2, we deduce the following observability inequality.

COROLLARY 8.5.1 (Observability inequality in one unit of time). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree-shaped network and let us assume that* (1.2.3) *holds. Let us fix $T > 1$. Then, there exists a constant $C > 0$ (independent of T) such that, for all $\varphi_T \in L^2(\mathcal{E})$, the solution φ_ε of* (1.2.4) *satisfies*

$$
\|\varphi_\varepsilon(T-1,\cdot)\|^2_{L^2(\mathcal{E})} \leq e^{C/\varepsilon} \sum_{v\in V_\partial} \int_{T-1}^T |\partial_{n^e(v)}\varphi^e_\varepsilon(t,v)|^2\,\mathrm{d}t. \tag{8.5.27}
$$

PROOF. It suffices to prove that, for all $T > 1$, there exists $C > 0$ (independent of T) such that, for all $z_T \in L^2(\mathcal{E})$, the solution z_ε of (8.2.8) satisfies

$$
\|z_\varepsilon(T-1,\cdot)\|^2_{L^2(\mathcal{E})} \leq e^{C/\varepsilon} \sum_{v\in V_\partial} \int_{T-1}^T |\partial_{n^e(v)}z^e_\varepsilon(t,v)|^2\,\mathrm{d}t.
$$

Indeed, taking into account (8.2.7) and the Dirichlet boundary conditions of φ_ε, we can then compute

$$
\begin{aligned}
\|\varphi_\varepsilon(T-1,\cdot)\|^2_{L^2(\mathcal{E})} &\leq e^{C_1/\varepsilon}\|z_\varepsilon(T-1,\cdot)\|^2_{L^2(\mathcal{E})} \\
&\leq e^{(C_1+C)/\varepsilon} \sum_{v\in V_\partial} \int_{T-1}^T |\partial_{n^e(v)}z^e_\varepsilon(t,v)|^2\,\mathrm{d}t \\
&\leq e^{(C_1+C+C_2)/\varepsilon} \sum_{v\in V_\partial} \int_{T-1}^T |\partial_{n^e(v)}\varphi^e_\varepsilon(t,v)|^2\,\mathrm{d}t.
\end{aligned}
$$

Let χ be a smooth cut-off function supported in $(-\infty, 3/4]$ such that $\chi(\xi) = 1$ for all $\xi \leq 1/4$. The function $\psi_\varepsilon(t, \cdot) = z_\varepsilon(t, \cdot)\chi(t - T + 1)$ satisfies

$$\begin{cases} -a^e \partial_t \psi_\varepsilon^e(t, x) - \varepsilon \partial_{xx}^2 \psi_\varepsilon^e(t, x) + \frac{|b^e|^2}{4\varepsilon} \psi_\varepsilon^e(t, x) & \\ \quad = -a^e z_\varepsilon(t) \chi'(t - T + 1), & t \in (0, T),\ x \in e,\ \forall e \in \mathcal{E}, \\ \psi_\varepsilon^e(t, v) = 0, & t \in (0, T),\ v \in \mathcal{V}_\partial, \\ \psi_\varepsilon^{e_1}(t, v) = \psi_\varepsilon^{e_2}(t, v), & t \in (0, T),\ v \in \mathcal{V}_0,\ \forall e_1, e_2 \in \mathcal{E}(v), \\ \sum_{e \in \mathcal{E}(v)} \varepsilon \partial_{n^e(v)} \psi_\varepsilon^e(t, v) = 0, & t \in (0, T),\ v \in \mathcal{V}_0, \\ \psi_\varepsilon^e(T, x) = 0, & x \in e,\ e \in \mathcal{E}. \end{cases}$$

Thus, multiplying the first equation by ψ_ε^e and integrating in $(T-1, T) \times \mathcal{E}$ yields

$$\begin{aligned} \frac{1}{2} \int_{\mathcal{E}} a |\psi_\varepsilon(T - 1, x)|^2 \, \mathrm{d}x + \iint_{(T-1,T)\times\mathcal{E}} \frac{|b|^2}{4\varepsilon} |\psi_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t + \varepsilon \iint_{(T-1,T)\times\mathcal{E}} |\partial_x \psi_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t \\ = - \iint_{(T-1,T)\times\mathcal{E}} a |z_\varepsilon|^2 \chi'(t - T + 1) \chi(t - T + 1) \, \mathrm{d}t \, \mathrm{d}x. \end{aligned}$$

Consequently, since $\psi_\varepsilon(T - 1, \cdot) = z_\varepsilon(T - 1, \cdot)\chi(0) = z_\varepsilon(T - 1, \cdot)$ and the support of χ' is in $\left[\frac{1}{4}, \frac{3}{4}\right]$, we deduce

$$\begin{aligned} \int_{\mathcal{E}} a |\psi_\varepsilon(T - 1, x)|^2 \, \mathrm{d}x &\leq -2 \iint_{(T-3/4,T-1/4)\times\mathcal{E}} a |z_\varepsilon|^2 \chi'(t - T + 1) \chi(t - T + 1) \, \mathrm{d}x \, \mathrm{d}t \\ &\leq C \iint_{(T-3/4,T-1/4)\times\mathcal{E}} |z_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t. \end{aligned}$$

It thus remains to prove that

$$\begin{aligned} \iint_{(T-3/4,T-1/4)\times\mathcal{E}} |z_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t &= \iint_{(1/4,3/4)\times\mathcal{E}} |z_\varepsilon(\tilde{t} + T - 1, x)|^2 \, \mathrm{d}x \, \mathrm{d}\tilde{t} \\ &\leq e^{C/\varepsilon} \sum_{v \in V_\partial} \int_0^1 |\partial_{n^e(v)} z_\varepsilon^e(\tilde{t} + T - 1, v)|^2 \, \mathrm{d}\tilde{t} \\ &= e^{C/\varepsilon} \sum_{v \in V_\partial} \int_{T-1}^T |\partial_{n^e(v)} z_\varepsilon^e(t, v)|^2 \, \mathrm{d}t, \end{aligned}$$

which follows from Proposition 8.5.2—used with $T = 1$, δ sufficiently small, $\tau = \widetilde{C}$, and $s = \widetilde{C}\varepsilon^{-1}$ (where $\widetilde{C}$ is a sufficiently large constant). $\square$

8.5.3. Proof of Claim (2) **of Theorem 8.2.3.** Using the observability inequality in Corollary 8.5.1, we now conclude the proof of the main result.

Proof of Theorem 8.2.3, Claim (2). This claim is a direct consequence of Corollary 8.5.1 and Proposition 8.5.1. In fact, combining the decay estimate (8.5.1) and the observability inequality (8.5.27), we conclude that

$$\|\varphi_\varepsilon(0, \cdot)\|_{L^2(\mathcal{E})}^2 \leq e^{(C-cT)/\varepsilon} \|\varphi_\varepsilon(T - 1, \cdot)\|_{L^2(\mathcal{E})}^2 \leq \sum_{v \in V_\partial} e^{(C-cT)/\varepsilon} \int_{T-1}^T |\partial_{n^e(v)} \varphi_\varepsilon^e(t, v)|^2 \, \mathrm{d}t.$$

(recalling that the constants $C > 0$ may change from term to term). As a consequence, we obtain that (8.2.2) holds for a sufficiently large time $T > 0$. $\square$

Remark 8.5.1 (On the controllability of (1.2.1)). *We may prove the controllability of* (1.2.1) *as a byproduct of Proposition 8.5.2. Reasoning as Corollary 8.5.1, we obtain that there exists a constant $C_T > 0$, depending on the time variable, such that*

$$\|\varphi_\varepsilon(0, \cdot)\|_{L^2(\mathcal{E})}^2 \leq e^{C_T/\varepsilon} \sum_{v \in V_\partial} \int_0^T |\partial_{n^e(v)} \varphi_\varepsilon^e(t, v)|^2 \, \mathrm{d}t,$$

where φ_ε is the of (1.2.4) *with $\varphi_T \in L^2(\mathcal{E})$. With this observability inequality, Lemma 8.2.1 yields the claimed controllability result.*

Remark 8.5.2 (The case of non-constant coefficients). *The same techniques that we have developed in this Chapter apply to prove analogous results if the coefficients depend on the time and space variables as long as* $a \in C^1_{\mathrm{pw}}([0,T] \times \mathcal{E})$*,* $\min_{\mathcal{E}} a > 0$*,* $b \in C^1_{\mathrm{pw}}([0,T] \times \mathcal{E})$*,* $\min_{\mathcal{E}} b > 0$*, and* $\sum_{e \in \mathcal{E}(v)} n^e(v) b^e(v) = 0$ *for all* $v \in \mathcal{V}_0$ *(see Remark 8.3.1 for the necessity of the positivity of* b*). In fact, the decay property is proved with the transformation*

$$z^e_\varepsilon := \varphi^e_\varepsilon \exp\left(\frac{\displaystyle\int_0^x b^e(\xi)\,\mathrm{d}\xi + c^e}{2\varepsilon}\right), \tag{8.5.28}$$

where c^e *are the right constants given by Lemma 8.1.1 so that* $\int_0^x b(\xi)\,\mathrm{d}\xi + c^e$ *is continuous and we used the parametrization of each edge* e *as a segment. With this change of variables, we obtain the system*

$$\begin{cases} -a^e \partial_t z^e_\varepsilon(t,x) + \left(\frac{|b^e|^2}{4\varepsilon} - \frac{\partial_x b^e}{2}\right) z^e_\varepsilon(t,x) & \\ \quad = \varepsilon \partial^2_{xx} z^e_\varepsilon(t,x), & t \in (0,T),\ x \in e,\ \forall e \in \mathcal{E}, \\ z^e_\varepsilon(t,v) = 0, & t \in (0,T),\ v \in \mathcal{V}_\partial, \\ z^{e_1}_\varepsilon(t,v) = z^{e_2}_\varepsilon(t,v), & t \in (0,T),\ v \in \mathcal{V}_0,\ \forall e_1, e_2 \in \mathcal{E}^{\mathrm{out}}(v), \\ \sum_{e \in \mathcal{E}(v)} \varepsilon \partial_{n^e(v)} z^e_\varepsilon(t,v) = 0 & t \in (0,T),\ v \in \mathcal{V}_0, \\ z_\varepsilon(0,x) = z_0(x), & x \in e,\ \forall e \in \mathcal{E}. \end{cases} \tag{8.5.29}$$

Then, the computations for decay and Carleman estimates are still valid; indeed, we just get some lower-order terms that can be easily absorbed.

CHAPTER 9

Controllability of entropy solutions of scalar conservation laws at a junction via Lyapunov methods

The main result of this Chapter concerns the controllability of entropy solutions of (1.1.18) to a prescribed trajectory by means of a Lyapunov-type approach based on [**122**].

Theorem 9.0.1 (Controllability of entropy solutions on star-shaped graphs). *Let us assume that hypotheses (F1)–(F3) are satisfied and let $v = (v_1, \dots, v_{n+m})$ be the entropy solutions of* (1.2.5) *(in the sense of Definition 9.1.1) with initial data $v_{0,\ell} \in L^\infty(I_\ell; \mathbb{R}_+)$ for $\ell \in \{1, \dots, n+m\}$ and boundary data $v_{b,i} \in L^\infty((0, +\infty); \mathbb{R}_+)$ for $i \in \{1, \dots, n\}$ (v is a* target trajectory*). Let us consider any other initial data $u_{0,\ell} \in L^\infty(I_\ell; \mathbb{R}_+)$ for $\ell \in \{1, \dots, n+m\}$. Then, the entropy solution $u = (u_1, \dots, u_{n+m})$ of* (1.2.5) *corresponding to initial data $u_{0,\ell}$ and the in-flux boundary data of v, i.e. $u_{b,i} \equiv v_{b,i}$ for all $i \in \{1, \dots, n\}$, satisfies*

$$u_\ell(t, x) = v_\ell(t, x), \quad t > \widehat{T}, \text{ a.e. } x \in I_\ell, \ \forall \ell \in \{1, \dots, n+m\},$$

where the control time $\widehat{T}$ is given by $\widehat{T} := \max_{i \in \{1,\dots,n\}}\{L_i/c_i\} + \max_{j \in \{n+1,\dots,n+m\}}\{L_j/c_j\}$.

Remark 9.0.1 (Null-controllability). *If we assume $\sum_{i=1}^n f_i(0) = \sum_{j=n+1}^{n+m} f_j(0)$ (or, alternatively, $f_\ell(0) = 0$ for all $\ell \in \{1, \dots, n+m\}$), then 0 is an admissible entropy solution of* (1.2.5) *with $u_0 \equiv 0$ and $u_{b,i} \equiv f_i(0)$ (or $u_{b,i} \equiv 0$, respectively). Then, considering $v(t, \cdot) \equiv 0$ for all $t \geq 0$, Theorem 9.0.1 can be seen as a null-controllability result: we steer the system to the zero state by considering the boundary control $u_{b,i} \equiv f_i(0)$ (or $u_{b,i} \equiv 0$, respectively).*

Remark 9.0.2 (Controllability of entropy solutions on tree-shaped graphs). *We can prove a similar result on a* tree-shaped network *arguing by induction as in* Chapter *8. In that case, in the statement of Theorem 9.0.1, we need to introduce a suitable notion of* maximal propagation time *required for information to flow out of the tree (similarly to* Chapter *8).*

Finally, we prove a stabilization result that provides some robustness estimate for Theorem 9.0.1 and is the first step towards the analysis of the cost of controllability for conservation laws on networks in the vanishing viscosity singular limit (cf. Chapter 8 for the corresponding result in the linear setting).

Theorem 9.0.2 (Exponential stabilization for the viscous problem). *Let us assume that hypotheses (F1) and (F3) are satisfied and $n \leq m$. Let $u_\varepsilon = (u_{\varepsilon,1}, \dots, u_{\varepsilon,n+m})$ and $v_\varepsilon = (v_{\varepsilon,1}, \dots, v_{\varepsilon,n+m})$ be classical solutions of* (1.2.6) *(in the sense of* [**79**, Theorem 1.2]*) with initial data $u_{0,\varepsilon,\ell} \in C^\infty(I_\ell; \mathbb{R}_+)$ and $v_{0,\varepsilon,\ell} \in C^\infty(I_\ell; \mathbb{R}_+)$, respectively, and same boundary data $u_{b,\ell} \equiv v_{b,\ell} \in L^\infty((0, +\infty); \mathbb{R}_+)$ for all $\ell \in \{1, \dots, n+m\}$. Then,*

$$\sum_{\ell=1}^{n+m} \|u_{\varepsilon,\ell}(t, \cdot) - v_{\varepsilon,\ell}(t, \cdot)\|_{L^1(I_\ell)} \leq e^{-\frac{c\alpha}{2\varepsilon}\left((1-\frac{\alpha}{2})ct - L\right)} \sum_{\ell=1}^{n+m} \|u_{\varepsilon,0,\ell} - v_{\varepsilon,0,\ell}\|_{L^1(I_\ell)}, \quad t > 0,$$

for any $\alpha \in (0, 1]$, $c := \min_{\ell \in \{1,\dots,n+m\}} c_\ell$, and $L := \max_{i \in \{1,\dots,n\}} L_i + \max_{j \in \{n+1,\dots,n+m\}} L_j$.

We remark that the role of the assumption $n \leq m$ in the energy dissipation mechanism for viscous conservation laws at a junction is also discussed in [**57**].

9.1. Entropy admissible solutions for scalar conservation laws on networks

In this Section, following [**19**], we review some known results on the entropy formulation for conservation laws at a junction. We remark that the theory of [**19**] was developed in the case of bell-shaped fluxes; however, the results still apply under the assumption (F3), which is the setting of the more recent works [**202, 129**].

Let us start by considering an IBVP on the half-line for a scalar conservation law with Lipschitz continuous flux:

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u(t,x)) = 0, & t>0, x>0, \\ u(0,x) = u_0(x), & x>0, \\ u(t,0) = u_b(t), & t>0, \end{cases} \tag{9.1.1}$$

We say that u is an entropy solution of (9.1.1) if it is a Kružkov entropy solution in the interior of the half-plane $\mathbb{R}_+ \times \mathbb{R}_+$, i.e.

$$\partial_t |u-k| + \partial_x \left(\operatorname{sign}(u-k)(f(u)-f(k))\right) \le 0$$

holds in the sense of distributions for every $k \in \mathbb{R}$, and if it satisfies the boundary condition *in the sense of Bardos–LeRoux–Nédélec* (see [**27, 22**]), i.e. the strong trace $u(t,0+)$ satisfies

$$f(u(t,0+)) = G(u_b(t), u(t,0+)),$$

where G denotes the *Godunov numerical flux associated to* f (see [**161**, Eq. (3.8)]), which is given by

$$G(a,b) := \begin{cases} \min\limits_{\xi\in[a,b]} f(\xi) & \text{if } a \le b, \\ \max\limits_{\xi\in[b,a]} f(\xi) & \text{if } a \ge b. \end{cases}$$

Due to the results in [**209, 231**], for a Lipschitz continuous flux f such that f' is not identically zero on any interval (cf. assumptions (F1)–(F2)), the function $u(t,\cdot)$ possesses one-sided limits; in particular, we can define the strong trace of u on $\mathbb{R}_+ \times \{0\}$ which is mentioned above. The Bardos–LeRoux–Nédélec condition is generally recognized as the correct interpretation of the Dirichlet boundary condition for hyperbolic conservation laws. This is justified in particular by convergence of vanishing viscosity or numerical approximations of the boundary value problem: indeed, it may happen that the limit (hyperbolic) problem satisfies an effective boundary condition that may differ from the formal boundary condition prescribed for the approximation level due to viscous or numerical boundary layer effects (see [**22, 219**] for a more detailed discussion of boundary conditions for hyperbolic conservation laws).

With these preliminary notions, we can present the notions of entropy-admissible solutions for conservation laws on networks studied in [**19**]. We remark that there the authors considered $I_i = \mathbb{R}_-$ and $I_j = \mathbb{R}_+$, so we need to extend [**19**, Definition 1.2] slightly to deal with the case of I_ℓ being segments. On the other hand, we restrict ourselves to the setting of assumptions (F1)–(F3).

DEFINITION 9.1.1 (Entropy admissible solution: formulation using Godunov fluxes at the junction). *Given $u_{0,\ell} \in L^\infty(I_\ell;\mathbb{R}_+)$ and $u_{b,i} \in L^\infty((0,+\infty);\mathbb{R}_+)$, we say that $u = (u_1,\dots,u_{n+m})$ is an* entropy solution *of* (1.2.5) *if $u_\ell \in L^\infty((0,+\infty)\times I_\ell)$ for all $\ell \in \{1,\dots,n+m\}$ and the following conditions are satisfied.*

(1) *For all $\ell \in \{1,\dots,n+m\}$, the function u_ℓ is an entropy solution of the conservation law in the interior of I_ℓ, i.e. for all non-negative test functions $\varphi_\ell \in C_c^\infty([0,+\infty)\times I_\ell;\mathbb{R}_+)$ and for any constant $k \in \mathbb{R}$, there holds*

$$\int_0^\infty \int_{I_\ell} \left(\eta(u_\ell,k)\partial_t\varphi_\ell + q_\ell(u_\ell,k)\partial_x\varphi_\ell\right) \mathrm{d}x\,\mathrm{d}t + \int_{I_\ell} \eta(u_{0,\ell},k)\varphi_\ell(0,x)\,\mathrm{d}x \ge 0,$$

where $\eta(u_\ell,k) := |u_\ell - k|$ and $q_\ell(u_\ell,k) := \operatorname{sign}(u_\ell-k)(f_\ell(u_\ell)-f_\ell(k))$.

(2) *The boundary condition in the exterior vertices of the network is satisfied in the sense of Bardos–LeRoux–Nédélec, i.e.*

$$f_i(u_i(t,-L_i)) = G_i(u_{b,i}(t), u_i(t,-L_i)), \qquad a.e.\ t>0, \quad i \in \{1,\dots,n\},$$

where G_i is the Godunov flux associated with f_i.

(3) *The junction condition is satisfied in the following sense: there exists a function* $p \in L^\infty((0,+\infty);\mathbb{R}_+)$ *such that*

$$\begin{aligned} f_i(u_i(t,0-)) &= G_i(u_i(t,0-),p(t)), \quad \text{a.e. } t>0, \quad i\in\{1,\dots,n\},\\ f_j(u_j(t,0+)) &= G_j(p(t),u_j(t,0+)), \quad \text{a.e. } t>0, \quad j\in\{n+1,\dots,n+m\}, \end{aligned}$$

and the conservativity condition

$$\sum_{i=1}^{n} G_i(u_i(t,0-),p(t)) = \sum_{j=n+1}^{n+m} G_j(p(t),u_j(t,0+)), \quad \text{for a.e. } t>0$$

holds.

Remark 9.1.1 (The case of monotone fluxes). *Under hypothesis (F3), the flux function is strictly increasing and the Godunov flux is given by* $G_i(a,b) = f_i(a)$. *As a consequence, in Point (2) of Definition 9.1.1, we cannot impose a boundary condition at* $x = L_j$, *but only at* $x = -L_i$ *for* $i \in \{1,\dots,n\}$, *which is given by*

$$f_i(u_i(t,-L_i)) = f_i(u_{b,i}(t)), \quad i\in\{1,\dots,n\}.$$

We also note that, being the flux invertible, we can equivalently write

$$u_i(t,-L_i) = u_{b,i}(t), \quad i\in\{1,\dots,n\}.$$

Moreover, Point (3) reduces to

$$\begin{aligned} f_j(u_j(t,0+)) &= f_{n+1}(u_{n+1}(t,0+)), \quad j\in\{n+1,\dots,n+m\},\\ \sum_{i=1}^{n} f_i(u_i(t,0-)) &= \sum_{j=n+1}^{n+m} f_j(u_j(t,0+)). \end{aligned}$$

The second line indicates the conservation of mass; the first one indicates that the entropy-admissibility condition amounts to requiring an equi-distribution of the flux coming out of the junction.

Definition 9.1.1 can be equivalently reformulated in terms of an *adapted entropy inequality* (see [**19**, Definition 2.10]).

Definition 9.1.2 (Entropy admissible solution: formulation using adapted entropies at the junction). *Given* $u_{0,\ell} \in L^\infty(I_\ell;\mathbb{R}_+)$ *and* $u_b \in L^\infty((0,+\infty);\mathbb{R}_+)$, *we say that* $u = (u_1,\dots,u_{n+m})$ *is an* entropy solution *of* (1.2.5) *if* $u_\ell \in L^\infty((0,+\infty)\times I_\ell)$ *for all* $\ell\in\{1,\dots,n+m\}$ *and the following conditions are satisfied.*

(1) *Points (1) and (2) of Definition 9.1.1 hold.*
(2) *For any* $k = (k_1,\dots,k_{n+m}) \in \mathcal{G}_{VV}$, u_ℓ *satisfies the* adapted entropy inequality *on the network, i.e. for all non-negative test functions* $\varphi_\ell \in C_c^\infty((0,+\infty)\times \bar{I}_\ell;\mathbb{R}_+)$ *such that* $\varphi_\ell(t,0) = \varphi_1(t,0)$, *there holds*

$$\sum_{\ell=1}^{n+m}\int_0^\infty\int_{I_\ell}\big(\eta(u_\ell,k_\ell)\partial_t\varphi_\ell + q_\ell(u_\ell,k_\ell)\partial_x\varphi_\ell\big)\,\mathrm{d}x\,\mathrm{d}t \geq 0,$$

where $\eta(u_\ell,k_\ell) \coloneqq |u_\ell - k_\ell|$ *and* $q_\ell(u_\ell,k) \coloneqq \operatorname{sign}(u_\ell-k_\ell)(f_\ell(u_\ell)-f_\ell(k_\ell))$. *Here,* $\mathcal{G}_{VV}$ *denotes the* vanishing viscosity germ, *defined as follows (see* [**19**, Definition 2.1]*):*

$$\mathcal{G}_{VV} \coloneqq \left\{ \begin{array}{l} u=(u_1,\dots,u_{m+n}) : \exists p\geq 0 \text{ such that} \\ \sum_{i=1}^{n} G_i(u_i,p) = \sum_{j=m+1}^{m+n} G_j(p,u_j) \text{ and} \\ G_i(u_i,p) = f_i(u_i), \qquad G_j(p,u_j) = f_j(u_j), \\ \forall i\in\{1,\dots,n\}, j\in\{n+1,\dots,n+m\} \end{array} \right\}.$$

Under assumptions (F1)–(F3), it can be proven that such entropy solutions exist and are the limit of a vanishing viscosity approximation process (see [**19**, Theorem 4.1]) and Godunov-type numerical schemes (see [**19**, Theorem 3.3] and also [**229**] for a more explicit implementation of the scheme). Moreover, with this entropy formulation, the following uniqueness result holds ([**19**, Proposition 3.1]).

THEOREM 9.1.1 (L^1-stability of entropy solutions). *Let us assume that (F1)–(F3) hold and let u and v be entropy solutions of* (1.2.5) *in the sense of Definition 9.1.1 with initial data $u_{0,\ell}, v_{0,\ell} \in L^\infty(I_\ell;\mathbb{R}_+)$ for $\ell \in \{1,\dots,n+m\}$, respectively, and same boundary data $u_{b,i} \in L^\infty((0,+\infty);\mathbb{R}_+)$ for $i \in \{1,\dots,n\}$. Then,*

$$\begin{aligned}\sum_{i=1}^{n} \|u_i(t,\cdot)-v_i(t,\cdot)\|_{L^1(I_i)} &+ \sum_{j=n+1}^{n+m} \|u_j(t,\cdot)-v_j(t,\cdot)\|_{L^1(I_j)}\\ \le \sum_{i=1}^{n} \|u_i(0,\cdot)-v_i(0,\cdot)\|_{L^1(I_i)} &+ \sum_{j=n+1}^{n+m} \|u_j(0,\cdot)-v_j(0,\cdot)\|_{L^1(I_j)}\end{aligned}$$

for every $t>0$. In particular, at most, one entropy solution exists for given initial and boundary data.

A comparison principle can also be established following [**19**], which implies that the inf in assumption (F3) is actually taken in a bounded interval. Moreover, due to the finite speed of propagation of the waves of hyperbolic conservation laws, these existence and uniqueness results can be extended inductively to more general networks (see [**138**]).

9.2. Proof of the controllability of the hyperbolic problem

Before going into the proof of Theorem 9.0.1, we shall outline the strategy with the following toy problem.[1]

REMARK 9.2.1 (A case study: the IBVP for the linear transport equation). *We consider*

$$\begin{cases}\partial_t u(t,x) + c\partial_x u(t,x) = 0, & t>0,\ x\in(0,L),\\ u(0,x)=u_0(x), & x\in(0,L),\\ u(t,0)=0, & t>0,\end{cases} \tag{9.2.1}$$

where $L>0$, $c>0$, and $u_0 \in L^2((0,L))$. Let us define the Lyapunov functional

$$\forall t\ge 0,\quad J_\nu(t) := \int_0^L u^2 e^{-\nu x}\,\mathrm{d}x, \tag{9.2.2}$$

with $\nu>0$, and compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}J_\nu(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u^2e^{-\nu x}\,\mathrm{d}x &= \int_0^L 2u\partial_t u e^{-\nu x}\,\mathrm{d}x = -\int_0^L 2cu\partial_x u e^{-\nu x}\,\mathrm{d}x\\ &= -\nu c\int_0^L u^2 e^{-\nu x}\,\mathrm{d}x \underbrace{-[u^2e^{-\nu x}]_0^L}_{\le 0}\\ &\le -\nu c J_\nu(t).\end{aligned}$$

Gronwall's lemma yields

$$J_\nu(t) \le e^{-c\nu t}J_\nu(0), \qquad t\ge 0.$$

We then observe that

$$e^{-\nu}\|u(t,\cdot)\|^2_{L^2((0,L))} \le J_\nu(t) \le \|u(t,\cdot)\|^2_{L^2((0,L))}.$$

Putting these together, we have

$$e^{-L\nu}\|u(t,\cdot)\|^2_{L^2((0,L))} \le e^{-c\nu t}\|u_0\|_{L^2((0,L))},$$

i.e.

$$\|u(t,\cdot)\|^2_{L^2((0,L))} \le e^{-c\nu t+L\nu}\|u_0\|_{L^2((0,L))} = e^{-c\nu\left(t-\frac{L}{c}\right)}\|u_0\|_{L^2((0,L))}.$$

Therefore, letting $\nu\to+\infty$, we conclude $\|u(t,\cdot)\|_{L^2((0,L))}=0$ for $t>L/c$.

[1]This example was presented by V. Perrollaz in the conference "VIII Partial Differential Equations, Optimal Design and Numerics", 2019.

In order to make the proof of Remark 9.2.1 rigorous for conservation laws, we need to rely on the entropy formulation (see [**122**]). Moreover, to adapt the argument to the case of networked systems, we need to take particular care of the transmission of information at the junction.

Proof of Theorem 9.0.1. Following the strategy in [**122**], we define, for each edge $i \in \{1, \dots, n\}$ and $j \in \{n+1, \dots, n+m\}$, the Lyapunov functionals

$$\forall t \geq 0, \quad J_{\nu,i}(t) \coloneqq \int_{-L_i}^{0} |u_i(t,x) - v_i(t,x)| e^{-\nu x} \, \mathrm{d}x, \quad J_{\nu,j}(t) \coloneqq \int_{0}^{L_j} |u_j(t,x) - v_j(t,x)| e^{-\nu x} \, \mathrm{d}x,$$

for a fixed $\nu > 0$.

Step 1. *Analysis of the incoming edges.* Given $\bar{t} \geq 0$, for any $i \in \{1, \dots, n\}$, the edge-wise entropy condition (see Point (1) of Definition 9.1.1) yields, by a "doubling of variables"-type argument (see [**122**]),

$$\begin{aligned} 0 \leq & \int_0^{\bar{t}} \int_{-L_i}^{0} |u_i(t,x) - v_i(t,x)| \partial_t \varphi_i(t,x) \, \mathrm{d}x \, \mathrm{d}t \\ & + \int_0^{\bar{t}} \int_{-L_i}^{0} \operatorname{sign}\big(u_i(t,x) - v_i(t,x)\big) \big(f_i(u_i(t,x)) - f_i(v_i(t,x))\big) \partial_x \varphi_i(t,x) \, \mathrm{d}x \, \mathrm{d}t, \\ & + \int_{-L_i}^{0} |u_i(0,x) - v_i(0,x)| \varphi_i(0,x) \, \mathrm{d}x, \end{aligned}$$

with $\varphi_i \in C_c^\infty(\mathbb{R}^2; \mathbb{R}_+)$. Here, we used the existence of a strong trace at the boundary to use point (2) of Definition 9.1.1.

We consider a sequence $\{\varphi_{i,k}\}_{k \in \mathbb{N}} \subset C_c^\infty([0, +\infty) \times (-L_i, 0); \mathbb{R}_+)$ such that

$$\varphi_{i,k}(t,x) \to \chi_{(-\infty, \bar{t}]}(t) e^{-\nu x} \quad \text{strongly in } L^1 \text{ as } k \to +\infty.$$

Then, letting $k \to \infty$, we obtain

$$J_{\nu,i}(\bar{t}) \leq J_{\nu,i}(0) - \nu \int_0^{\bar{t}} \int_{-L_i}^{0} e^{-\nu x} \operatorname{sign}\big(u_i(t,x) - v_i(t,x)\big) \big(f_i(u_i(t,x)) - f_i(v_i(t,x))\big) \, \mathrm{d}x \, \mathrm{d}t. \tag{9.2.3}$$

Here, we needed to use the existence of strong traces at the boundary guaranteed by (F2).

In order to estimate the last term of (9.2.3), we observe that, for all $(a,b) \in \mathbb{R}$,

$$\begin{aligned} \operatorname{sign}(a-b)(f_\ell(a) - f_\ell(b)) &= \operatorname{sign}(a-b) \left(\int_0^1 f_\ell'(b + s(a-b)) \, (a-b) \, \mathrm{d}s \right) \\ &= |a-b| \int_0^1 f_\ell'(b + s(a-b)) \, \mathrm{d}s \\ &\geq |a-b| \int_0^1 c_\ell \, \mathrm{d}s \geq c_\ell |a-b|, \end{aligned}$$

where we used assumption (F3) to bound the f_ℓ' from below. Therefore, we obtain

$$J_{\nu,i}(\bar{t}) \leq J_{\nu,i}(0) - \nu c_i \int_0^{\bar{t}} J_{\nu,i}(t) \, \mathrm{d}t.$$

A distributional (differential) Gronwall-type argument along these lines (using $\varphi_i(t,x) = \varphi(t) e^{-\nu x}$) then yields

$$J_{\nu,i}(\bar{t}) \leq e^{-c_i \nu \bar{t}} J_{\nu,i}(0). \tag{9.2.4}$$

As $\bar{t}$ was arbitrarily chosen, we can write, for all $t \geq 0$,

$$\|u_i(t,\cdot) - v_i(t,\cdot)\|_{L^1(I_i)} \leq J_{\nu,i}(t) \leq e^{\nu L_i} \|u_i(t,\cdot) - v_i(t,\cdot)\|_{L^1(I_i)}. \tag{9.2.5}$$

Thus, plugging (9.2.5) into (9.2.4), we compute

$$\|u_i(t,\cdot) - v_i(t,\cdot)\|_{L^1(I_i)} \leq J_{\nu,i}(t) \leq e^{\nu L_i - \nu c_i t} J_{\nu,i}(0) \leq e^{-\nu c_i \left(t - \frac{L_i}{c_i}\right)} \|u_{0,i} - v_{0,i}\|_{L^1(I_i)}$$

and, letting $\nu \to +\infty$, we conclude that $u_i(t,\cdot) - v_i(t,\cdot) = 0$ for $t > L_i/c_i$. Therefore, $u_i(t,\cdot) = v_i(t,\cdot)$ for all $i \in \{1, \dots, n\}$ if $t > \max_{i \in \mathcal{I}_{\text{in}}} \{L_i/c_i\}$.

Step 2. *Analysis of the outgoing edges.* By Definition 9.1.1 (and Remark 9.1.1), the traces of u and v at the junction satisfy

$$\begin{aligned}&f_j(u_j(t,0+)) - f_j(v_j(t,0+)) = f_{n+1}(u_{n+1}(t,0+)) - f_{n+1}(v_{n+1}(t,0+)),\\ &\forall j \in \{n+1,\dots,n+m\},\end{aligned} \tag{9.2.6}$$

$$\sum_{i=1}^{n} \left(f_i(u_i(t,0-)) - f_i(v_i(t,0-))\right) = \sum_{j=1}^{n+m} \left(f_j(u_j(t,0+)) - f_j(v_j(t,0+))\right). \tag{9.2.7}$$

From Step 1, for all $i \in \{1,\dots,n\}$, we have $u_i(t,0-) - v_i(t,0-) = 0$ for $t > \max_{i\in\mathcal{I}_{\rm in}} \{L_i/c_i\}$. Then, from (9.2.7), we have

$$\sum_{j=1}^{n+m} \left(f_j(u_j(t,0+)) - f_j(v_j(t,0+))\right) = 0.$$

By (9.2.6), this yields $u_j(t,0+) = v_j(t,0+)$ for $t > \max_{i\in\mathcal{I}_{\rm in}} \{L_i/c_i\}$ for all $j \in \{n+1,\dots,n+m\}$. Then, we can repeat the argument of Step 1: we consider the Lyapunov functional

$$J_{\nu,j}(t) = \int_0^{L_j} |u_j(t,x) - v_j(t,x)| e^{-\nu x}\,\mathrm{d}x$$

and prove that $u_j(t,\cdot) = v_j(t,\cdot)$ for all $j \in \{n+1,\dots,n+m\}$ if $t > \max_{j\in\mathcal{I}_{\rm out}} \{L_j/c_j\}$.

Step 3. *Conclusion of the argument.* Putting Step 1 and Step 2 together, we conclude that, for any

$$t > \widehat{T} \coloneqq \max_{i\in\mathcal{I}_{\rm in}}\{L_i/c_i\} + \max_{j\in\mathcal{I}_{\rm out}}\{L_j/c_j\},$$

it holds

$$u_\ell(t,x) = v_\ell(t,x) \qquad \text{for a.e. } x \in I_\ell,\ \forall \ell \in \{1,\dots,n+m\}.$$

□

9.3. Proof of the exponential stabilization of the viscous problem

In this Section, we prove the stabilization result for the viscous problem. As before, we first illustrate the strategy with a toy problem.

Remark 9.3.1 (The effect of viscosity in the toy problem). *Let us consider a viscous regularization of the toy problem* (9.2.1)*:*

$$\begin{cases} \partial_t u_\varepsilon(t,x) + c\partial_x u_\varepsilon(t,x) = \varepsilon\partial^2_{xx} u_\varepsilon(t,x), & t>0,\ x\in(0,L),\\ u_\varepsilon(0,x) = u_0(x), & x\in(0,L),\\ u_\varepsilon(t,0) = u_\varepsilon(t,L) = 0, & t>0, \end{cases}$$

where $\varepsilon > 0$, $L > 0$, $c > 0$, and $u_0 \in L^2((0,L))$. Then, we can estimate the Lyapunov functional (9.2.2) *as follows:*

$$J_\nu(t) \le e^{-(c\nu - \varepsilon\nu^2)t} J_\nu(0), \qquad t \ge 0.$$

This yields

$$\|u_\varepsilon(t,\cdot)\|_{L^2((0,L))} \le e^{-(c\nu-\varepsilon\nu^2)t + L\nu}\|u_0\|_{L^2((0,L))},$$

which only implies an exponential stabilization *result:*

$$\|u_\varepsilon(t,\cdot)\|_{L^2((0,L))} \le e^{-\nu(c-\varepsilon\nu)\left(t-\frac{L}{c-\varepsilon\nu}\right)} = C_1 e^{-C_2 t}\|u_0\|_{L^2((0,L))},$$

with $C_1 \coloneqq e^{L\nu}$ and $C_2 \coloneqq c\nu - \varepsilon\nu^2$ ($C_2 > 0$ for $\nu\varepsilon < c$).

As expected, the effect of viscosity prevents us from controlling exactly the state to zero by simply using null boundary data; instead, at the time $t \ge L/c$, still a small exponential tail remains. More precisely, we let $\alpha \in (0,1)$ and $\nu = -\frac{c\alpha}{2\varepsilon}$ and compute

$$\|u_\varepsilon(t,\cdot)\|_{L^2((0,L))} \le e^{-\frac{c\alpha}{2\varepsilon}\left(\left(1-\frac{\alpha}{2}\right)ct - 1\right)}\|u_0\|_{L^2((0,L))}.$$

For $t > \frac{1}{c(1-\alpha)}$*, we deduce*

$$\|u_\varepsilon(t,\cdot)\|_{L^2((0,L))} \leq e^{-\frac{c\alpha^2}{4\varepsilon(1-\alpha)}} \|u_0\|_{L^2((0,L))}.$$

This estimate is motivated by [**10**, Lemma 2.1]*: it is consistent with the decay of the free solution of advection-diffusion equations first used in* [**106**] *to prove a uniform controllability result.*

The same point can be made when considering the controllability/stabilization of numerical approximations of (9.1.1) *that introduce artificial viscosity.*

PROOF OF THEOREM 9.0.2. Let $u_\varepsilon = (u_{\varepsilon,1}, \dots, u_{\varepsilon,n+m})$ and $v_\varepsilon = (v_{\varepsilon,1}, \dots, v_{\varepsilon,n+m})$ be classical solutions of (1.2.6) and let us consider the following Lyapunov functional:

$$\forall t \geq 0, \quad J_\nu(t) \coloneqq \sum_{i=1}^{n} \int_{-L_i}^{0} |u_{\varepsilon,i}(t,x) - v_{\varepsilon,i}(t,x)| e^{-\nu x} \, \mathrm{d}x + \sum_{j=n+1}^{n+m} \int_{0}^{L_j} |u_{\varepsilon,j}(t,x) - v_{\varepsilon,j}(t,x)| e^{-\nu x} \, \mathrm{d}x.$$

Then, as in the proof of Theorem 9.0.1, but using the junction condition of (1.2.6) similarly to [**19**, Eq. (89)], we compute

$$\begin{aligned}
0 \leq &- \sum_{i=1}^{n} \int_{-L_i}^{0} |u_{\varepsilon,i}(\bar{t},x) - v_{\varepsilon,i}(\bar{t},x)| e^{-\nu x} \, \mathrm{d}x - \sum_{j=n+1}^{n+m} \int_{0}^{L_j} |u_{\varepsilon,j}(\bar{t},x) - v_{\varepsilon,j}(\bar{t},x)| e^{-\nu x} \, \mathrm{d}x \\
&+ \sum_{i=1}^{n} \int_{-L_i}^{0} |u_{\varepsilon,i}(0,x) - v_{\varepsilon,i}(0,x)| e^{-\nu x} \, \mathrm{d}x + \sum_{j=n+1}^{n+m} \int_{0}^{L_j} |u_{\varepsilon,j}(0,x) - v_{\varepsilon,j}(0,x)| e^{-\nu x} \, \mathrm{d}x \\
&- \nu \sum_{i=1}^{n} \int_{0}^{\bar{t}} \int_{-L_i}^{0} \operatorname{sign}\big(u_{\varepsilon,i}(t,x) - v_{\varepsilon,i}(t,x)\big)\big(f_i(u_{\varepsilon,i}(t,x)) - f_i(v_{\varepsilon,i}(t,x))\big) e^{-\nu x} \, \mathrm{d}x \, \mathrm{d}t \\
&- \nu \sum_{j=n+1}^{n+m} \int_{0}^{\bar{t}} \int_{0}^{L_j} \operatorname{sign}\big(u_{\varepsilon,j}(t,x) - v_{\varepsilon,j}(t,x)\big)\big(f_j(u_{\varepsilon,j}(t,x)) - f_j(v_{\varepsilon,j}(t,x))\big) e^{-\nu x} \, \mathrm{d}x \, \mathrm{d}t \\
&+ \varepsilon \nu^2 \sum_{i=1}^{n} \int_{0}^{\bar{t}} \int_{-L_i}^{0} |u_{\varepsilon,i}(t,x) - v_{\varepsilon,i}(t,x)| e^{-\nu x} \, \mathrm{d}x \, \mathrm{d}t \\
&+ \varepsilon \nu^2 \sum_{j=n+1}^{n+m} \int_{0}^{\bar{t}} \int_{0}^{L_j} |u_{\varepsilon,j}(t,x) - v_{\varepsilon,j}(t,x)| e^{-\nu x} \, \mathrm{d}x \, \mathrm{d}t,
\end{aligned}$$

where we got rid of an extra boundary term $\varepsilon\nu(n-m)\int_0^{\bar{t}} |u_{\varepsilon,1}(t,0) - v_{\varepsilon,1}(t,0)| \, \mathrm{d}t$ thanks to the assumption $n \leq m$.

From this, we deduce

$$\begin{aligned}
J_\nu(\bar{t}) \leq &\; J_\nu(0) + \varepsilon\nu^2 \int_0^{\bar{t}} J_\nu(t) \, \mathrm{d}t \\
&- \nu \int_0^{\bar{t}} \sum_{i=1}^{n} c_i \int_{-L_i}^{0} |u_{\varepsilon,i}(t,x) - v_{\varepsilon,i}(t,x)| e^{-\nu x} \, \mathrm{d}x \, \mathrm{d}t \\
&- \nu \int_0^{\bar{t}} \sum_{j=n+1}^{n+m} c_j \int_{0}^{L_j} |u_{\varepsilon,j}(t,x) - v_{\varepsilon,j}(t,x)| e^{-\nu x} \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}$$

Taking $c \coloneqq \min_{\ell \in \{1,\dots,n+m\}} c_\ell$, we get

$$J_\nu(\bar{t}) \leq J_\nu(0) - (c\nu - \varepsilon\nu^2) \int_0^{\bar{t}} J_\nu(t) \, \mathrm{d}t.$$

Along the same lines, a (differential) Gronwall-type argument then yields

$$J_\nu(\bar{t}) \leq e^{-(c\nu - \varepsilon\nu^2)\bar{t}} J_\nu(0).$$

This implies the claimed exponential stabilization result for a sufficiently small $\nu > 0$. Indeed, since $\bar{t} > 0$ was arbitrarily chosen, we have, for $t \geq 0$,

$$\sum_{\ell=1}^{n} \|u_{\varepsilon,\ell}(t,\cdot) - v_{\varepsilon,\ell}(t,\cdot)\|_{L^1(I_\ell)} \leq e^{-(c\nu-\varepsilon\nu^2)t+L\nu} \sum_{\ell=1}^{n} \|u_{\varepsilon,0,\ell} - v_{\varepsilon,0,\ell}\|_{L^1(I_\ell)},$$

where $L := \max_{i\in\{1,\dots,n\}} L_i + \max_{j\in\{n+1,\dots,n+m\}} L_j$. Therefore, by choosing $\nu = -\frac{c\alpha}{2\varepsilon}$ for any $\alpha \in (0,1]$, we compute

$$\sum_{\ell=1}^{n+m} \|u_{\varepsilon,\ell}(t,\cdot) - v_{\varepsilon,\ell}(t,\cdot)\|_{L^1(I_\ell)} \leq e^{-\frac{c\alpha}{2\varepsilon}\left(\left(1-\frac{\alpha}{2}\right)ct-L\right)} \sum_{\ell=1}^{n+m} \|u_{\varepsilon,0,\ell} - v_{\varepsilon,0,\ell}\|_{L^1(I_\ell)}. \tag{9.3.1}$$

□

9.4. Numerical experiments

In this Section, we present some numerical simulations to illustrate our main result. We consider a star-shaped graph with $n = 2$ incoming edges of length 1 and $m = 3$ outgoing edges of length 1 and let $f_\ell(\xi) := \xi/(1+\xi)$ for $\ell \in \{1,\dots,5\}$. We shall apply the Godunov numerical scheme proposed in [**202**] (and implemented by Musch in [**201**]). We simulate the evolution of the dynamics corresponding to the following sets of initial and boundary data.

Example I. *Oscillatory initial data vs. edge-wise constant entropy solution:*

$$u_0(x) := (|\sin(16x)|, |\sin(16x)|, |\cos(16x)|, |\cos(16x)|, |\cos(16x)|);$$
$$v_0(x) := (2, 1, 7/11, 7/11, 7/11);$$
$$u_{b,1} \equiv v_{b,1} \equiv 2, \quad u_{b,2} \equiv v_{b,2} \equiv 1.$$

Example II. *Initial data containing one shock in an incoming edge vs. edge-wise constant entropy solution:*

$$u_0(x) := (2\,\mathbb{1}_{(-1,-0.2)}(x) + 3\,\mathbb{1}_{(-0.2,0)}(x), 1, 1/2, 1/2, 1/2);$$
$$v_0(x) := (2, 1, 7/11, 7/11, 7/11);$$
$$u_{b,1} \equiv v_{b,1} \equiv 2, \quad u_{b,2} \equiv v_{b,2} \equiv 1.$$

The effect of "numerical viscosity" prevents finite-time exact controllability with these boundary controls; but, for sufficiently refined meshes, the exponential error tail is not distinguishable and, after a sufficiently long time, we get $u(T,\cdot) = v(T,\cdot) = v_0$ for both examples (v_0 being an edge-wise constant entropy-admissible solution, i.e. $v_0 \in \mathcal{G}_{VV}$).

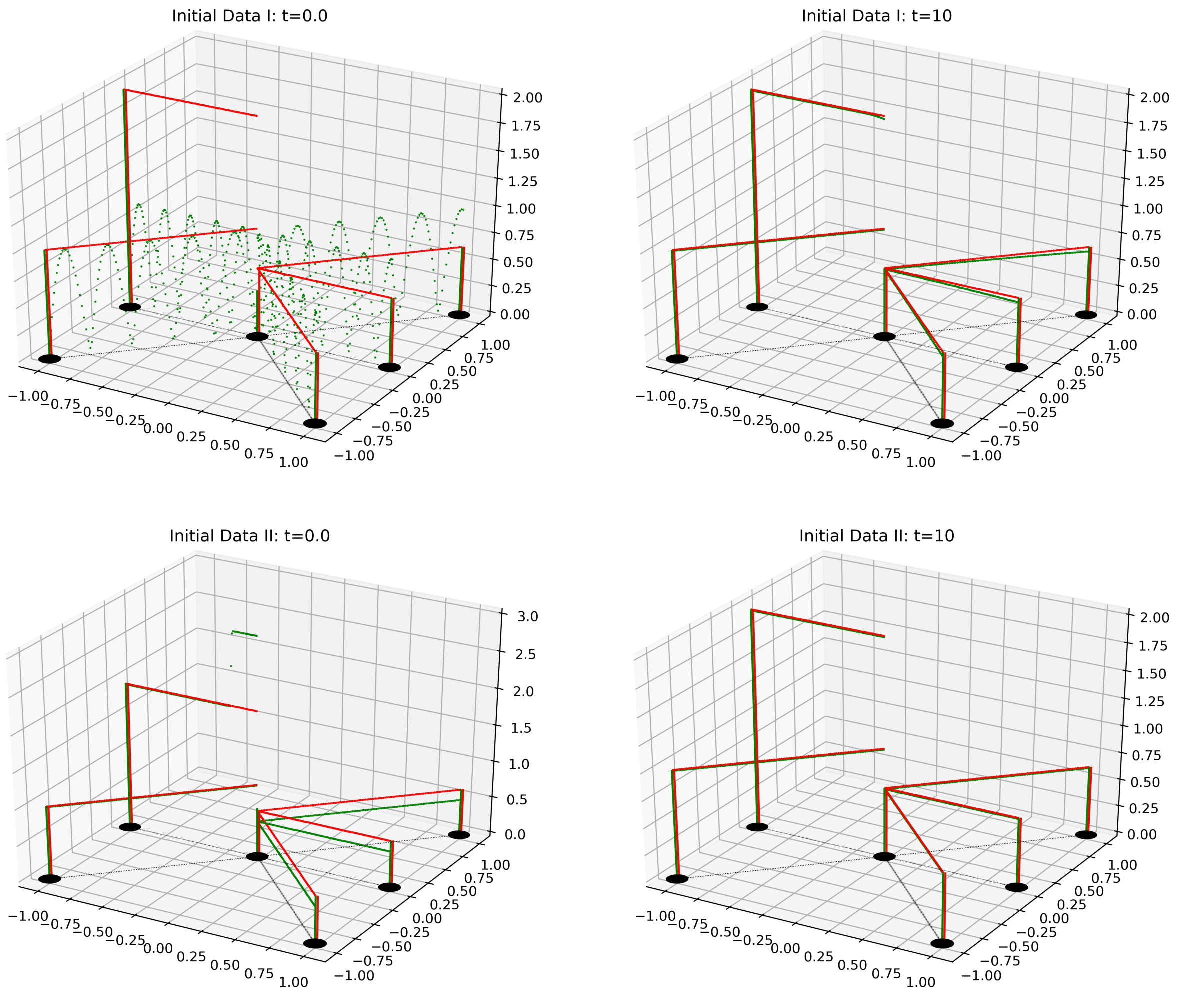

FIGURE 9.1. FIRST ROW: Simulation of Example I at times $t = 0$ and $t = 10$. SECOND ROW: Simulation of Example II at times $t = 0$ and $t = 10$. In both cases, the CFL (Courant–Friedrichs–Lewy) number is $C = 0.5$ and the space mesh size is $\Delta x = 2^{-6}$ (for each edge). We refer to [**201**] for the code that can be used to produce the figures and videos of the evolution.

CHAPTER 10

Conclusions and open problems

The results presented in this thesis open many avenues for future research at the confluence of nonlocal and viscous regularizations of conservation laws, singular limits, control theory, and networks. To guide future investigations, we have compiled a set of questions below.

(1) *Nonlocal–to–local convergence for more general weights.* In the case of a symmetric weight, the solution of the nonlocal conservation law does not satisfy a maximum principle (see [**168**, Example 7.3 and Figure 9]). From the examples in [**90**], it is also apparent that we cannot expect the nonlocal solution to converge in a strong or weak sense to the local entropy solution; however, numerical experiments (cf. [**168**, Example 7.3]) suggest that convergence in a measure-valued sense may hold. For asymmetric piecewise-constant weights (which are not covered by the results in [**89**]), the numerical simulations shown in Chapters 3 and 4 indicate that a positive result may also be true.

(2) *Oleĭnik-type inequalities for more general velocities.* The Oleĭnik-type inequalities of Chapter 4 have been obtained for rather specific classes of velocity functions. It remains an open problem to establish similar inequalities in more general cases. In particular, the case of a power-type velocity, as in $\partial_t\rho + \partial_x(W^{q-1}\rho) = 0$ (with $q > 2$), naturally arises in connection with the long-time convergence of the solution to the local N-wave profiles.

(3) *Nonlocality in the velocity.* A different type of nonlocal conservation law involves taking a weighted average of the velocity rather than the solution itself: namely, $\partial_t\rho + \partial_x((V(\rho) * \gamma)\rho) = 0$. A recent contribution on the nonlocal–to–local singular limit problem for this kind of model with BV data is contained in [**132**], but Oleĭnik-type inequalities have not yet been established.

(4) *Non-integrable initial data and long-time asymptotics.* Considering initial data merely in L^∞ instead of $L^1 \cap L^\infty$ would pose a significant issue for the study of long-time asymptotics carried out in Chapter 5: since solutions might then have infinite mass, the compactness arguments would need to be modified; moreover, the initial mass would no longer govern the limit profile. We refer to [**158**] for the study of this problem for the heat equation. In the periodic setting, further information is available for the long-time behavior of (local) conservation laws (see, e.g., [**182, 61**]).

(5) *Effect of viscosity on the long-time behavior.* In the framework of the study of the long-time behavior of Chapter 5, it would be of interest to analyze the competition between nonlocal advection and diffusion effects. Currently, the only known results concern the positive effect of viscosity on the convergence in the nonlocal–to–local limit in the case of initial data that are uniformly bounded in L^∞ with respect to the scaling parameter (as in Chapter 6).

(6) *Asymptotic preserving numerical schemes.* A rigorous justification of the qualitative reproduction of the nonlocal–to–local singular limit and the long-time asymptotics by numerical schemes (such as the ones presented in [**213**, Chapter 3] and [**173**] or those surveyed in [**134**]) appears to be unexplored. In future work, we aim to prove that the counterparts of the convergence results of Chapters 3, 4, and 5 also hold at the discrete level.

(7) *Nonlocal–to–local limit for the initial-boundary value problem.* The well-posedness of the IBVP associated with nonlocal conservation laws was studied in [**172**]. The natural question arises of whether the solution converges to the local entropy solution with boundary data achieved in the sense of Bardos–LeRoux–Nédélec (see [**27**]) as the nonlocal weight

approaches a Dirac delta distribution. This problem is also motivated by and strongly related to the study of the cost of the boundary controls for the nonlocal conservation law (see Chapter 7) in the nonlocal–to–local singular limit.

(8) *Nonlocal–to–local limit for multi-dimensional conservation laws.* In the theory of multi-dimensional nonlocal conservation laws (for which we refer to [**171**] and references therein), the study of the convergence of the solution to the entropy-admissible one of the corresponding local model has not yet been addressed. Some related results are available for the 2D incompressible α-Euler system, where the (nonlocal) vorticity equation plays a key role (see, e.g., [**188, 1**]).

(9) *Controllability for more general weights.* The controllability results of Chapter 7 focus on the case of an exponential weight. For suitable classes of more general weights, the solution to the corresponding nonlocal IBVP still exists, is unique, and satisfies a maximum principle (see [**172**, Corollary 5.9]). However, the proofs of some of our controllability and stabilization results appear to present many more technical difficulties. Similarly, it would be interesting to remove the assumption on the lower/upper-bound on the initial datum used in the stabilization results of Chapter 7 and consider constant boundary data such that $u_\ell \neq u_r$. In this case, we expect the dynamics to converge to the corresponding steady-state solutions (as suggested by the numerical simulations).

(10) *Nonlocal conservation laws on networks.* The study of nonlocal conservation laws on networks is still at its primordial stage. Namely, in [**155**] a class of nonlocal conservation laws modeling multi-commodity flow was studied and, more recently, some results on nonlocal traffic models were obtained in [**131**]. In the latter reference, the modeling framework includes suitable coupling conditions at intersections to either ensure maximum flux or distribution parameters and focuses, in particular, on the cases of 1–to–1, 2–to–1, and 1–to–2 junctions. The study of more general nonlocal models on tree-shaped networks and the related control and singular limit problems remains open for investigation.

(11) *Alternative junction conditions for advection-diffusion equations on networks.* In addition to the continuity condition that we have imposed at the junctions in Chapter 8, there are alternative transmission conditions that are physically relevant for the advection-diffusion problem. It would be interesting to see if analogous results on the cost of controllability can be obtained in the framework of the "membrane-type" junction conditions of [**148, 78**]. The main challenge to overcome is that, in our analysis, the continuity condition has been pivotal in estimating the boundary terms arising in the Carleman inequality. For the nonlinear problem of Chapter 9, similar questions arise when considering different realizations of viscous regularizations and different types of entropy conditions at the junction.

(12) *Networks containing loops.* In Chapter 8, it is essential to assume that our network contains no loops in several key points, such as the definition of the auxiliary functions needed for the Carleman estimate. It would be interesting to characterize the (uniform) controllability properties of the parabolic problem according to the topology or metric properties of the network (see, e.g., [**23**] and [**117**, Chapters 5 & 8] for some results in this direction).

(13) *Nonlinear conservation laws with vanishing viscosity and cost of controllability.* The full extension of the results of Chapter 8 to the case of nonlinear conservation laws (as done in [**141, 183**] in the case of a segment) remains to be addressed in the future—possibly also replacing the monotonicity assumption on the flux used in Chapter 9 with a convexity/concavity condition as in [**183**].

(14) *Dispersive effects and cost of controllability.* In Chapter 8, we have only considered a model that includes advection and diffusive phenomena, but we may also want to account for dispersive effects. In the case of zero dispersion limit, the uniform control properties of the linearized Korteweg–de Vries equation were studied in [**142**] (on the real line); subsequently, in [**143**], the authors addressed the case of zero diffusion-dispersion; further

recent works on the KdV equation with a vanishing parameter in the diffusive term include [**53, 54**]. In the setting of star-shaped graphs, well-posedness and controllability results for the KdV equation are also available in the literature (see [**56, 59, 58, 12**]), but the uniform controllability problem has not been addressed yet.

(15) *Optimal time for the decay of the cost of controllability.* Establishing sharp estimates on the time that separates blow-up and decay for the cost of controllability in the vanishing viscosity limit from Chapter 8 is an interesting problem; however, the question, raised in [**106**], remains unanswered even on segments (i.e., networks that consist of a single edge).

(16) *Controllability on networks under positivity constraints.* For physical applications, it is relevant to drive the dynamics of a system to rest while preserving the non-negativity of the initial datum along the evolution. Such controllability results under positivity constraints have been achieved only recently for the heat equation in Euclidean domains (see [**193, 216**]) and are missing on networks.

(17) *Numerical schemes and uniform controllability on networks.* A rigorous study of the controllability of semi-discrete advection-diffusion equations on networks and of the cost of boundary controls in the discrete–to–continuous limit has not yet been carried out. Some related results in this direction are available in the case of one-dimensional Euclidean domains (see, e.g., [**40, 41, 42, 199, 10, 11, 9**]).

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