Sums-of-squares Polyconvexity

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1 Introduction

Polyconvexity is a useful concept in the field of Calculus of Variations (CoV). Consider the integral (adapted from [6])

$$I(u) = \int_{\Omega} f(x, u(x), \nabla u(x))dx$$

where

- $\Omega \subset \mathbb{R}^n$ is an open set,
- $u : \Omega \to \mathbb{R}^N$ and hence $\nabla u \in \mathbb{R}^{N\times n}$, and
- $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}, f = f(x, u, \xi)$ is a measurable in x and continuous in (u, ξ). In other words, f is a Carathéodory function.

Associated to the functional I is the minimization problem

$$m := \min\{I(u) : u \in X\},$$

where X is the space of admissible functions. In other words, we wish to find $\bar{u} \in X$ such that

$$m = I(\bar{u}) \leq I(u) \text{ for every } u \in X.$$

This minimization problem arises in fields such as nonlinear elasticity and optimal design [6]. In most cases, the admissible set X is given by

$$X = u_0 + W_0^{1,p}(\Omega; \mathbb{R}^N),$$

where u_0 is a given function and the notation $u \in X$ is a shortcut meaning that $u = u_0$ on $\partial\Omega$ and $u \in W^{1,p}(\Omega; \mathbb{R}^N)$. The existence of minimizers in the space X relies on the fundamental property of (sequential) weak lower semicontinuity, which is closely related to the convexity of the mapping $\xi \to f(x, u, \xi)$ in the scalar case ($N = 1$ or $n = 1$). In the vectorial case ($N, n > 1$), the convexity of the same function is only a sufficient condition for the weak lower semicontinuity of I. A necessary condition in the vectorial case is the so-called *quasiconvexity* introduced by Morrey [4]. A Borel measurable and locally bounded function $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ is said to be *quasiconvex* if

$$f(\xi) \leq \frac{1}{\operatorname{meas} D} \int_D f(\xi + \nabla\varphi(x))dx$$

for every bounded open set $D \subset \mathbb{R}^n$, every $\xi \in \mathbb{R}^{N\times n}$, and every $\varphi \in W_0^{1,\infty}(D; \mathbb{R}^N)$.

As shown in [6],

$$f \text{ quasiconvex} \iff I \text{ weakly lower semi-continuous.}$$

Since the notion of quasiconvexity is not a pointwise condition, it is hard to verify if a given function is quasiconvex. Hence a stronger notion called *polyconvexity* was introduced by Ball [2]. A function $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ is said to be *polyconvex* if $f(\xi)$ is a convex function of minors of ξ. The complete definition is stated in Chapter 2.5. As stated in [6], the relation between the notions of convexity, polyconvexity and quasiconvexity can be realised as

$$f \text{ convex} \implies f \text{ polyconvex} \implies f \text{ quasiconvex.}$$

Thus, we can conclude that

$$f \text{ polyconvex} \implies I \text{ weakly lower semi-continuous.}$$

Checking if a given function f is polyconvex is often simpler than checking quasiconvexity, but it still remains difficult in general. Similarly, checking if a given function is convex is an NP-Hard problem ([11]), even in the case of polynomials. Polynomial optimization techniques such as *Sum-of-Squares (SOS) polynomials* (reviewed in Chapter 2.2), i.e. polynomials that can be expressed as sums of squares of other polynomials, allow us to study notions like *SOS-convexity*. SOS-convexity is a stronger form of convexity for polynomials (reviewed in Chapter 2.4) that can be formulated as a *semidefinite program (SDP)*, a tractable optimization problem that can be solved efficiently using computers.

The aim of this thesis is to explore if it is possible to do the same for polyconvexity by defining a Sums-Of-Squares strengthening to polyconvexity, which enables us with a tool to check if a polynomial is polyconvex in a tractable way using Semi-Definite Programs. The particular goal of the thesis is to show that there are two approaches to defining a Sums-of-Squares strengthening to polyconvexity, obtained by replacing non-negativity conditions with sums-of-squares strengthenings in two different characterizations of polyconvexity. We explore the relationship between these two approaches, showing that they are not equivalent through a counterexample. This leads us to select one of our two approaches as the ”right” definition of Sums-Of-Squares polyconvexity.

The thesis is organized as follows: In Chapter 2, we review the necessary background on sums-of-squares polynomials, convexity, sums-of-squares convexity, and classical polyconvexity. In Chapter 3, we introduce the two SOS strengthenings of polyconvexity and begin exploring the relation between them and their relation to Sums-Of-Squares Convexity. Chapter 4 is devoted to analyzing their relationship and proving that the two notions of sums-of-squares polyconvexity defined in Chapter 3 are not equivalent. Chapter 5 summarizes the main points of the thesis and outlines open problems that deserve further investigation.

2 Background

2.1 Polynomials and polynomial ideals

In this section, we recall the definitions of polynomials, homogeneous polynomials and generated polynomial ideals. These concepts will be used throughout the thesis. We begin by recalling the definition of polynomials and homogeneous polynomials.

Definition 1 (Polynomials and homogeneous polynomials)**.** A *polynomial* p in n variables $x = (x_1, \ldots, x_n)$ over the real numbers $\mathbb{R}$ is a finite linear combination of monomials:

$$p(x) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha x^\alpha = \sum_{\alpha \in \mathbb{N}^n} p_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

where each coefficient $p_\alpha \in \mathbb{R}$, the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a tuple of non-negative integers. The size of a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is defined as $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$.

A polynomial p of degree d is said to be *homogeneous* or a *form* if all monomials appearing in p have total degree exactly d. In other words,

$$p(x) = \sum_{\alpha \in \mathbb{N}^n, |\alpha| = d} p_\alpha x^\alpha.$$

The ring of all polynomials over $\mathbb{R}$ with indeterminate $x \in \mathbb{R}^n$ is denoted by $\mathbb{R}[x]$. We will henceforth refer to homogeneous polynomials as forms.

We now recall the definitions of ideals in $\mathbb{R}[x]$, real radical ideals and generated polynomial ideals.

Definition 2 (Ideal in $\mathbb{R}[x]$, adapted from [5])**.** An *ideal* $\mathscr{I}$ in the polynomial ring $\mathbb{R}[x]$ is a subset $\mathscr{I} \subseteq \mathbb{R}[x]$ such that:

1. $\mathscr{I}$ is an additive subgroup of $\mathbb{R}[x]$. In other words, for all $f, g \in \mathscr{I}$, the sum $f + g$ is in $\mathscr{I}$, and for every $f \in \mathscr{I}$, the additive inverse $-f$ is in $\mathscr{I}$.
2. For every $f \in \mathscr{I}$ and every $h \in \mathbb{R}[x]$, the product hf is in the subset $\mathscr{I}$. In other words, $\mathscr{I}$ is closed under multiplication by arbitrary polynomials in $\mathbb{R}[x]$.

Definition 3 (Generated polynomial ideal, adapted from [12])**.** Given $h_1, h_2, \ldots, h_m \in \mathbb{R}[x]$,

$$\mathscr{I}(h_1, h_2, \ldots, h_m) := \left\{ \sum_{i=1}^{m} u_j h_j | u_1, u_2, \ldots, u_m \in \mathbb{R}[x] \right\}$$

is the *ideal generated by* $h_1, h_2, \ldots, h_m$.

We now define a real radical ideal and an equivalent characteristic, which will be crucial in Chapter 4.

Definition 4 (Real radical ideal (adapted from [12])). The real radical of $\mathscr{I}$ is defined as

$$\sqrt[\mathbb{R}]{\mathscr{I}} = \{f \in \mathbb{R}[\mathbf{x}] \mid f^{2k} + \sum_{j=1}^{m} p_j^2 \in \mathscr{I} \text{ for some } k \geq 1, p_1, \ldots, p_m \in \mathbb{R}[\mathbf{x}]\}.$$

An ideal $\mathscr{I}$ is called real radical if $\mathscr{I} = \sqrt[\mathbb{R}]{\mathscr{I}}$.

We can also characterize real radical ideals using the The Real Nullstelenzatz [12] as follows:

$$\mathscr{I} \text{ is real radical}$$

$$\iff$$

$$\text{The only polynomials vanishing at all points of } V_{\mathbb{R}}(\mathscr{I}) \text{ are the polynomials in } \mathscr{I},$$

where $V_{\mathbb{R}}(\mathscr{I}) := \{x \in \mathbb{R}^n \mid f(x) = 0 \ \forall f \in \mathscr{I}\}$ is the real variety associated to $\mathscr{I}$.

We now state an important theorem relating to generated polynomial ideals. This defines a set of polynomials that agree on a set of defined polynomial equations. We use this concept to prove an important Lemma in Chapter 4.

Theorem 1. *If $\mathscr{I}(h_1, h_2, \ldots, h_m)$ is a real radical generated polynomial ideal, the following statements are equivalent:*

- $g(x) = f(x)$ *for all* $x \in \{x | h_i(x) = 0 \ \forall i = \{1, 2, \ldots, m\}\}$.
- $g(x) = f(x) + \sum_{i=1}^{m} u_i(x) h_i(x), \ u_i(x) \in \mathbb{R}[x]$.

The proof of Theorem 1 can be found in [12].

We now look at a Lemma from [12] that gives useful criterion to check if an ideal is real radical. This Lemma will play key role in connecting Sums-Of-Squares polynomials and real radical generated polynomial ideals.

Lemma 1. *Let $\mathscr{I}$ be an ideal in $\mathbb{R}[x]$. Then, $\mathscr{I}$ is real radical if and only if*

$$\forall p_1, ..., p_m \in \mathbb{R}[x], \ \sum_{j=1}^{m} p_j^2 \in \mathscr{I} \Rightarrow p_1, ..., p_m \in \mathscr{I}.$$

The proof of Lemma 1 can be found in [12].

2.2 Sums-Of-Squares polynomials

Problems with non-negativity constraints are NP-hard [11]. We will see in the following subsections that replacing a non-negativity constraint on a polynomial with a Sums-Of-Squares constraint makes a problem numerically tractable. So, we look into Sums-Of-Squares polynomials and some of their properties.

We first recall the definition of Sums-Of-Squares Polynomials.

Definition 5 (Sums of Squares (SOS))**.** A polynomial $\sigma(x) \in \mathbb{R}[x]$ is said to be a *Sum of Squares* if

$$\sigma(x) = f_1^2(x) + f_2^2(x) + \ldots + f_n^2(x)$$

for some polynomials $f_1(x), \ldots, f_n(x) \in \mathbb{R}[x]$.

The expression $f_1^2(x) + f_2^2(x) + \ldots + f_n^2(x)$ is called the *SOS decomposition* of σ. We denote the subset of degree d SOS polynomials in $\mathbb{R}_d[x]$ by $\Sigma_d[x]$.

It is clear that any SOS polynomial $\sigma = f_1^2 + f_2^2 + \ldots + f_n^2$ must be an even degree polynomial. It can also be shown that the polynomials $f_1, f_2, \ldots, f_n$ are of degree d or less [8]. In other words, if $\sigma \in \mathbb{R}_{2d}[x]$ then $f_1, f_2, \ldots, f_n \in \mathbb{R}_d[x]$.

We now recall the definition of a SOS polynomial matrix as defined in [1]. We will use this concept later to define a SOS strengthening to convexity.

Definition 6 (SOS polynomial matrix)**.** A symmetric polynomial matrix $U(x)_{m\times m}$ is said to be a *SOS-matrix* if there exists a polynomial matrix $V(x)_{s\times m}$ for some $s \in \mathbb{N}$ such that $U(x) = V(x)^T V(x)$.

We define the set of polynomial matrices of dimension $m \times m$ in the variable $x \in \mathbb{R}^n$ as $\Sigma\mathbb{R}[x]^{m\times m}$.

We now form a connection between SOS matrices and SOS polynomials in the following theorem.

Theorem 2. *A polynomial matrix $U(x) \in \Sigma\mathbb{R}^{m\times m}$ is SOS if and only if the polynomial $y^T U(x) y$ is SOS for all $x \in \mathbb{R}^n, y \in \mathbb{R}^m$.*

Proof. Assume $U(x)$ is an SOS matrix. Definition 6 implies that we can find a polynomial matrix $V(x) \in \mathbb{R}[x]^{s\times m}$ such that $U(x) = V(x)^T V(x)$. From this expansion, we get

$$y^T U(x) y = y^T V(x)^T V(x) y = \|V(x)y\|^2.$$

Hence we can conclude $y^T U(x) y$ is SOS.

Similarly, if $y^T U(x) y$ is SOS, we can obtain an SOS decomposition that is linear in y since $y^T U(x) y$ is quadratic in y. In other words,

$$y^T U(x) y = \|V(x)y\|^2 = y^T V(x)^T V(x) y.$$

Hence we can conclude that $U(x) = V(x)^T V(x)$. □

We now look into the relationship between non-negative and SOS polynomials. It is clear from Definition 5 that all SOS polynomials are non-negative. Hilbert's Theorem (1888) [9] talks about the degrees and dimensions for which a non-negative polynomial can be expressed as a sum of squares (SOS) of polynomials. The result of the theorem is as follows.

Theorem 3 (Hilbert's Theorem (1888))**.** *Every non-negative polynomial is SOS only in the following cases:*

1. $n = 1, d \in \mathbb{N}, d$ *is even*
2. $d = 2, n \in \mathbb{N}$
3. $n = 2, d = 4$

A common counterexample of a non-negative polynomial that is not SOS is the Motzkin polynomial [13] given by

$$p(x, y) = x^4 y^2 + y^4 x^2 + 1 - 3x^2 y^2.$$

2.3 Semidefinite Programs and SOS polynomials

We will now look at how an SOS constraint can be formulated as a Semidefinite Program (SDP). SDPs are tractable can be solved efficiently, hence this formulation of SOS constraints make a difficult problem tractable. This subsection is adapted from the lecture notes [8].

For convenience, we define

$$L := \dim \mathbb{R}_d[x] = \binom{n+d}{d},$$

$$U := \dim \mathbb{R}_{2d}[x] = \binom{n+2d}{2d}.$$

We fix a basis of $\mathbb{R}_d[x]$ and collect its elements in the polynomial vector

$$p(x) := \begin{pmatrix} p_1(x) \\ \vdots \\ p_L(x) \end{pmatrix}.$$

Similarly we collect the elements of a basis of $\mathbb{R}_{2d}[x]$ in the polynomial vector

$$q(x) := \begin{pmatrix} q_1(x) \\ \vdots \\ q_U(x) \end{pmatrix}.$$

In order to transform a SOS constraint into an SDP, we recall a proposition that defines a *Gram matrix*.

Proposition 1. A polynomial $\sigma \in \mathbb{R}_{2d}[x]$ is SOS if and only if there exists a symmetric positive semidefinite matrix S called the Gram matrix such that $\sigma = p(x)^T S p(x)$.

The proof can be realized from the proof of Theorem 2 with $U(x) = S, m = L$ and $y = p(x)$. We can conclude from Proposition 1 that checking if a polynomial σ is SOS reduces to checking if it has a positive semidefinite Gram matrix S.

We can now define a linear operator to have a relation between $p(x)$ and $q(x)$. This will be important to build the constraint of the SDP. In the following Lemma, $\mathbb{S}^L$ stands for the set of $L \times L$ symmetric matrices.

Lemma 2. *Let p, q, L, U be defined as above. There exists a unique operator $\Lambda : \mathbb{R}^U \to \mathbb{S}^L$ such that*

$$\Lambda(q(x)) = p(x)p(x)^T.$$

Proof. Since $p(x)$ is defined as a basis of $\mathbb{R}_d[x]$, we have that $p_i p_j \in \mathbb{R}_{2d}[x]$ for all pairs of indices $(i, j) \in \{1, 2, \ldots, L\}^2$. This implies that there exist coefficients $\lambda_1^{ij}, \lambda_2^{ij}, \ldots, \lambda_U^{ij}$ such that $p_i p_j = \lambda_1^{ij} q_1 + \ldots + \lambda_U^{ij} q_U$. Since $p_i p_j = p_j p_i$ we have that $\lambda_k^{ij} = \lambda_k^{ji}$, so we can construct symmetric $L \times L$ matrices

$$\Lambda_k := \begin{bmatrix} \lambda_k^{11} & \lambda_k^{12} & \ldots & \lambda_k^{1L} \\ \lambda_k^{21} & \lambda_k^{22} & \ldots & \lambda_k^{2L} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_k^{L1} & \lambda_k^{L2} & \ldots & \lambda_k^{LL} \end{bmatrix}, \quad k = 1, 2, \ldots, U.$$

We can now define a linear operator Λ for $y \in \mathbb{R}^U$ by

$$\Lambda(y) := \Lambda_1 y_1 + \ldots + \Lambda_U y_U,$$

which is the desired operator. □

Since Λ is a linear operator, we can define an adjoint operator $\Lambda^* : \mathbb{S}^L \to \mathbb{R}^{\mathbb{U}}$. It can be verified that Λ^* acts on $S \in \mathbb{S}^L$ to give

$$\Lambda^*(S) = \begin{pmatrix} \langle \Lambda_1, S \rangle \\ \vdots \\ \langle \Lambda_U, S \rangle \end{pmatrix},$$

where $\langle A, B \rangle$ denotes the Frobenius inner product between matrices. From the definition of Λ and Λ^*, we can infer that

$$p(x)^T S p(x) = \langle S, p(x)p(x)^T \rangle = \langle S, \Lambda(q(x)) \rangle = \Lambda^*(S) \cdot q(x).$$

In other words, $\Lambda^*(S)$ gives us the coefficients of $p(x)^T S p(x)$ with respect to the basis $q(x)$. From Proposition 1, we can see that the vector $\Lambda^*(S)$ must match with the coefficients of σ when represented in the basis q of $\mathbb{R}_{2d}[x]$. Defining the coefficients of σ as the vector

$$\text{coeff}_q(\sigma) = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_U \end{pmatrix},$$

we can define the SDP equivalent to checking if the polynomial σ is SOS:

$$\begin{aligned} &\text{Find} && S \in \mathbb{S}^L \\ &\text{such that} && \Lambda^*(S) = \text{coeff}_q(\sigma). \\ & && S \succeq 0 \end{aligned} \tag{1}$$

If there exists a solution to the SDP (1), then σ is SOS. The SDP (1) can be modified to have different objective functions such as $\langle 0, S \rangle$ or $\text{tr}(S)$, which will give different results for the SOS decomposition of σ if there is a solution.

The numerically tractability of SDPs is primarily due to their convexity and strong duality properties [8], hence allowing the SOS reformulation of NP-hard problems to be solved in an easier and faster way.

2.4 Convexity and SOS-Convexity

Next, we review a tractable notion of convexity based on SOS polynomials. We begin by recalling the definition of a convex function.

Definition 7 (Convex function). A function f is said to be *convex* if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \text{ for all } x, y \in \mathbb{R}^n \text{ and all } \lambda \in [0,1].$$

The next result shows that convexity can be established in a number of equivalent ways.

Proposition 2 (First order and second order characterization of convexity, adapted from [1]). Let $p := p(x)$ be a polynomial, $\nabla p := \nabla p(x)$ be its gradient, and $H(x) := \nabla^2 p(x)$ be its Hessian. Then the following are equivalent:

1. p is convex.

2. p lies above the supporting hyperplane at every point. In other words,

$$p(y) \geq p(x) + \nabla p(x)^T(y-x) \text{ for all } x, y \in \mathbb{R}^n. \tag{2}$$

3. $H(x)$ is a positive semidefinite polynomial matrix for all $x \in \mathbb{R}^n$. In other words,

$$y^T H(x) y \geq 0 \text{ for all } x, y \in \mathbb{R}^n. \tag{3}$$

We denote $\widetilde{C}_{n,d}$ as the set of convex polynomials of degree d in n variables and $C_{n,d}$ as the set of convex forms of degree d in n variables. For matrix valued polynomials, we define the set of convex polynomials of degree d on the matrices in $\mathbb{R}^{N\times n}$ as $\widetilde{C}_{N,n,d}$ and $C_{N,n,d}$ for non-homogeneous polynomials and forms respectively.

Convex functions are useful in optimization, but in general it is not computationally tractable to check if a function is convex. This is also true for functions as "nice" as polynomials. So, we now define a SOS strengthening of convexity as defined in [1].

Definition 8 (SOS-convexity)**.** A polynomial $p(x)$ is said to be *SOS-convex* if its Hessian $H(x) = \nabla^2 p(x)$ is an SOS-matrix.

If we replace the non-negativity constraints in the characterizations of convexity in (2) and (3) with SOS constraints, we obtain a non-trivial analogue of Proposition 2 for SOS-convexity.

Proposition 3 (Characterization of SOS-Convexity, adapted from [1])**.** Let $p := p(x)$ be a polynomial of degree d in n variables with its gradient and Hessian denoted respectively by $\nabla p := \nabla p(x)$ and $H := H(x)$. Let g_λ, g_∇, and g_{∇^2} be defined as

$$g_\lambda(x,y) = (1-\lambda)p(x) + \lambda p(y) - p((1-\lambda)x + \lambda y), \tag{4}$$

$$g_\nabla(x,y) = p(y) - p(x) - \nabla p(x)^T(y-x), \tag{5}$$

$$g_{\nabla^2}(x,y) = y^T H(x) y. \tag{6}$$

Then, $g_\lambda(x,y)$ is SOS $\iff$ $g_\nabla(x,y)$ is SOS $\iff$ $g_{\nabla^2}(x,y)$ is SOS (i.e., $H(x)$ is an SOS-matrix).

We denote $\widetilde{\Sigma C}_{n,d}$ as the set of SOS-convex polynomials of degree d in n variables and $\Sigma C_{n,d}$ as the set of SOS-convex forms of degree d in n variables.

Article [1] talks about the gap between Convexity and SOS-Convexity. It can be shown that $\widetilde{C}_{n,d} = \widetilde{\Sigma C}_{n,d}$ for the same values of n and d as in Hilbert's Theorem 3.

Proposition 4 (Adapted from [1])**.** The equality $\widetilde{\Sigma C}_{n,d} = \widetilde{C}_{n,d}$ holds if and only if:

- $n = 1, d \in \mathbb{N}$ even (univariate even degree polynomials).
- $n \in \mathbb{N}, d = 2$ (quadratic polynomials).

- $n = 2, d = 4$ (bivariate quartic polynomials).

Unlike Hilbert's theorem, the values of n are different for forms. This is summarized as follows.

Proposition 5 (Adapted from [1]). The equality $\Sigma C_{n,d} = C_{n,d}$ holds if and only if:

- $n = 2, d \in \mathbb{N}$ even (bivariate even degree forms).
- $n \in \mathbb{N}, d = 2$ (quadratic forms).
- $n = 3, d = 4$ (trivariate quartic forms).

For all other values of n and d, it can be shown that the inclusions $\widetilde{\Sigma C}_{n,d} \subset \widetilde{C}_{n,d}$ and $\Sigma C_{n,d} \subset C_{n,d}$ are strict. Examples of convex but not SOS-convex polynomials in $\widetilde{C}_{2,6} \backslash \widetilde{\Sigma C}_{2,6}, \widetilde{C}_{3,4} \backslash \widetilde{\Sigma C}_{3,4}, C_{3,6} \backslash \Sigma C_{3,6}$ and $C_{4,4} \backslash \Sigma C_{4,4}$ can be found in [1]. The same work also shows that having counter examples in these sets implies the strict inclusion for higher values of n and d.

2.5 Polyconvexity

We now recall the definition of a polyconvex function and some of its basic properties as stated in [6].

In the following definition, $\operatorname{adj}_s \xi$ stands for the matrix of all $s \times s$ minors of the matrix $\xi \in \mathbb{R}^{N \times n}, 2 \leq s \leq \min\{n, N\}$ and

$$\tau(N, n) = \sum_{s=1}^{\min\{N,n\}} \binom{N}{s} \binom{n}{s}. \tag{7}$$

Definition 9 (Polyconvex function). A function $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ is *polyconvex* if there exists a convex function $g : \mathbb{R}^{\tau(N,n)} \to \mathbb{R} \cup \{+\infty\}$ such that

$$f(\xi) = g(T(\xi))$$

where $T : \mathbb{R}^{N \times n} \to \mathbb{R}^{\tau(N,n)}$ is such that

$$T(\xi) = (\xi, \operatorname{adj}_2 \xi, \dots, \operatorname{adj}_{\min\{N,n\}} \xi). \tag{8}$$

We can see from this definition that if we have a convex function f, then taking $g = f$ gives that f is also polyconvex. Hence every convex function is polyconvex. We now look at some examples of polyconvex functions that are not convex.

Example 1. *Consider the functions*

$$f : \mathbb{R}^{2\times 2} \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}, g(\delta) = \delta^2,$$

$$f(\xi) = g(\det \xi) = (\det \xi)^2.$$

It is clear that the function $g(\delta) = \delta^2$ *is convex, hence* f *is polyconvex. Considering the hessian of* f *evaluated at* I_2 *(the* 2×2 *identity matrix), we obtain*

$$\nabla^2 f(\xi)|_{\xi=I_2} = \begin{bmatrix} 2 & 0 & 0 & -4 \\ 0 & 0 & -2 & 0 \\ 0 & -2 & 0 & 0 \\ -4 & 0 & 0 & 2 \end{bmatrix}. \tag{9}$$

We can see that the minor $\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$ *has a determinant of* -4*, which tells us that* f *is not convex. Hence* f *is polyconvex but not convex.*

Example 2. *Consider the functions*

$$f : \mathbb{R}^{2\times 2} \to \mathbb{R}, g : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R},$$

$$g(\mathbf{X}, \delta) = ||\mathbf{X}||^2 + \delta^4,$$
$$f(\xi) = g(\xi, \det \xi) = ||\xi||^2 + (\det \xi)^4.$$

It is clear that the polynomial $g(\mathbf{X}, \delta)$ *is a convex function as it is a sum of convex polynomials, which tells us that* $f(\xi)$ *is polyconvex. Again, computing the hessian of* $f(\xi)$ *evaluated at* I_2*, we obtain*

$$\nabla^2 f(\xi)|_{\xi=I_2} = \begin{bmatrix} 14 & 0 & 0 & 16 \\ 0 & 2 & -4 & 0 \\ 0 & -4 & 2 & 0 \\ 16 & 0 & 0 & 14 \end{bmatrix}.$$

We can see that the minor $\begin{bmatrix} 2 & -4 \\ -4 & 2 \end{bmatrix}$ *has a determinant of* -12*, which tells us that* f *is not convex. Hence* f *is polyconvex but not convex.*

It is also important to note that the convex function g is not always unique for a given polyconvex function f. For example, consider the function

$$f : \mathbb{R}^{2\times 2} \to \mathbb{R}, f(\xi) = ||\xi||^2.$$

The functions

$$g_1(\xi, a) = \xi_{11}^2 + \xi_{12}^2 + \xi_{21}^2 + \xi_{22}^2 \text{ and}$$
$$g_2(\xi, a) = (\xi_{11} - \xi_{22})^2 + (\xi_{12} + \xi_{21})^2 + 2a$$

are both convex and satisfy $g_1(T(\xi)) = f(\xi), g_2(T(\xi)) = f(\xi)$ but $g_1 \neq g_2$. Hence, the representation of the convex function is not unique.

A polyconvex function can also be defined in the following equivalent way for functions f that only take finite values.

Proposition 6 (Polyconvex function - first order definition, adapted from [6]). A function $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ is *polyconvex* if, for all matrices $\xi, \eta \in \mathbb{R}^{N\times n}$, there exists a vector-valued function $\beta : \mathbb{R}^{N\times n} \to \mathbb{R}^{\tau(N,n)}$ such that

$$f(\eta) \geq f(\xi) + \langle \beta(\xi); T(\eta) - T(\xi) \rangle. \tag{10}$$

We now look at an important example from [6].

Theorem 4 (The example of Alibert-Dacorogna-Marcellini). *Let $\gamma \in \mathbb{R}$ and let $f_\gamma : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be defined as*

$$f_\gamma(\xi) = \|\xi\|^2(\|\xi\|^2 - 2\gamma \det \xi). \tag{11}$$

Then

$$f_\gamma \text{ is convex } \iff |\gamma| \leq \gamma_c = \frac{2\sqrt{2}}{3}, \tag{12}$$

$$f_\gamma \text{ is polyconvex } \iff |\gamma| \leq \gamma_p = 1. \tag{13}$$

This example talks about a specific function f_γ that is dependent on the parameter γ. Having certain bounds on γ determines the convexity/polyconvexity of the function f_γ. This example will play a key role in our study of SOS-polyconvexity, which will be discussed in Chapters 3 and 4 below.

3 SOS-polyconvexity

We have seen in Chapter 2 how SOS-convexity was defined by replacing the non-negativity constraints of convexity in (2) with SOS constraints. The central contribution of this thesis is to define SOS-polyconvexity by repeating the same with the definitions of polyconvexity. As discussed in subsection 2.3, these SOS strengthenings will make it more tractable to check if a given polynomial is polyconvex.

Given the two equivalent definitions of a polyconvex function, the standard definition 9 and the first order definition in Proposition 6 , we can define an SOS strengthening to polyconvexity in two ways. Contrary to the equivalent definitions of SOS-convexity in Proposition 3, we will see that the two approaches to defining SOS-polyconvexity are not equivalent.

3.1 Approach 1

The first approach to defining SOS-polyconvex functions is to replace the convexity of g in Definition 9 with SOS-convexity.

Definition 10 (Type-1 SOS-polyconvexity)**.** A polynomial $f(\xi):\mathbb{R}^{N\times n}\to\mathbb{R}$ is said to be *Type-1 SOS-polyconvex (Type-1 SOS-PC)* if there exists an SOS-convex polynomial $g:\mathbb{R}^{\tau(N,n)}\to\mathbb{R}$ such that $f(\xi)=g(T(\xi))$.

We denote $\Sigma PC^1_{N,n,d}$ as the set of Type-1 SOS-PC polynomials f of degree d. When $N=n$, we simplify the notation to $\Sigma PC^1_{n,d}$.

From subsection 2.4, we know that any SOS convex function is also convex. Applying this to g in Definition 10, we can see that any Type-1 SOS-PC polynomial is also polyconvex.

It is easy to construct Type-1 SOS-PC polynomials that are not convex. The next example shows such a construction.

Example 3. *Consider the polynomial f defined as follows:*

$$f:\mathbb{R}^{2\times 2}\to\mathbb{R}, f(\xi)=||\xi||^2+(\det\xi)^2. \tag{14}$$

Defining $g:\mathbb{R}^{2\times 2}\times\mathbb{R}\to\mathbb{R}, g(\mathbf{X},\delta)=||\mathbf{X}||^2+\delta^2$, we get

$$\nabla^2 g(\mathbf{X},\delta)=2I_5=(\sqrt{2}I_5)^T(\sqrt{2}I_5). \tag{15}$$

From this decomposition of the hessian, we can conclude that g is SOS-convex by (6) *in Proposition 3. Hence $f(\xi)=g(\xi,\det\xi)\in\Sigma PC^1_{2,2}$.*

From (9) *in example 1, we can see that* $(\det\xi)^2$ *is not convex. Hence f is Type-1 SOS-PC but not convex.*

We now look into the relationship between SOS-convex and Type-1 SOS-PC polynomials.

Lemma 3. *For all $N, n, d \in \mathbb{N}$ with $N, n > 1, \Sigma C_{N,n,d} \subset \Sigma PC^1_{N,n,d}$ with strict inclusion.*

Proof. It is clear that any polynomial f in $\Sigma C_{N,n,d}$ can be identified as a polynomial in $\Sigma PC^1_{N,n,d}$ by taking $g = f$ in Definition 9.

To show that the inclusion is strict, consider the matrix $\xi \in \mathbb{R}^{N\times n}$ and consider the function $f(\xi) = (\xi_{11}\xi_{22} - \xi_{12}\xi_{21})^2$. The only non-zero principal submatrix of $\nabla^2 f$ is the 4×4 matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

which is not PSD. So, one can verify easily that we have that $f(\xi)$ is not convex and hence not SOS-convex. In contrast, it is clear that f is Type-1 SOS-PC. Thus we have a strict inclusion. □

3.2 Approach 2

We now look at defining an SOS-polyconvexity strengthening using the first order definition in Proposition 6.

Definition 11 (Type-2 SOS-polyconvexity)**.** A polynomial $f(\xi) : \mathbb{R}^{N\times n} \to \mathbb{R}$ is said to be *Type-2 SOS-polyconvex (Type-2 SOS-PC)* if for all matrices $\xi, \eta \in \mathbb{R}^{N\times n}$, there exists a vector-valued polynomial $\beta : \mathbb{R}^{N\times n} \to \mathbb{R}^{\tau(N,n)}$ such that

$$f(\eta) - f(\xi) - \langle \beta(\xi); T(\eta) - T(\xi) \rangle \tag{16}$$

is SOS in η and ξ.

We denote $\Sigma PC^2_{N,n,d}$ as the set of polynomials f of degree d that satisfy the definition of Type-2 SOS-PC. In the case of $N = n$, we simplify the notation to $\Sigma PC^2_{n,d}$.

From subsection 2.2, we know that any SOS polynomial is non-negative. Applying this to equation (16), we can see that any Type-2 SOS-PC polynomial satisfies the first order definition of polyconvexity. Hence, all Type-2 SOS-PC polynomials are also polyconvex.

We now look at an example of a Type-2 SOS-PC polynomial, using the same polynomial f from example 3.

Example 4. *Consider the Type-1 SOS-PC polynomial f from* (14) *in Example 3. We claim that f is Type-2 SOS-PC. We need to show that there exists a polynomial matrix $\beta_1(\xi) \in \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ and a polynomial scalar $\beta_2(\xi) : \mathbb{R}^{2\times 2} \to \mathbb{R}$ such that*

$$f(\eta) - f(\xi) - \langle \beta_1(\xi); \eta - \xi \rangle - \beta_2(\xi)(\det \eta - \det \xi)$$

is SOS in η and ξ.

With $\beta_1(\xi) = 2\xi$ and $\beta_2(\xi) = 2\det \xi$, we find that

$$f(\eta) - f(\xi) - \langle \beta_1(\xi); \eta - \xi \rangle - \beta_2(\xi)(\det \eta - \det \xi) = \|\eta - \xi\|^2 + (\det \eta - \det \xi)^2.$$

The last expression is evidently SOS, hence $f \in \Sigma PC^2_{2,2}$.

3.3 Relationship between SOS-polyconvex definitions

We have seen in Example 4 that a function can be both Type-1 SOS-PC and Type-2 SOS-PC. So, we now aim to explore the relation between the two definitions of SOS-polyconvexity.

Theorem 5. $\Sigma PC^1_{N,n,d} \subseteq \Sigma PC^2_{N,n,d}$.

Proof. Consider a polynomial $f \in \Sigma PC^1_{N,n,d}$. Then, there exists a polynomial g SOS-convex such that $f(\xi) = g(T(\xi))$. We aim to show that there exists a polynomial vector $\beta : \mathbb{R}^{N\times n} \to \mathbb{R}^{\tau(N,n)}$ such that

$$f(\eta) - f(\xi) - \langle \beta(\xi); T(\eta) - T(\xi) \rangle \text{ is SOS for all } \xi, \eta \in \mathbb{R}^{N\times n}.$$

Since g is SOS-convex, we get from (5) in Proposition 3 that

$$g_\nabla(X, Y) \text{ is SOS, where } X, Y \in \mathbb{R}^{\tau(N,n)}.$$

Since the maps $\xi \to T(\xi)$ and $\eta \to T(\eta)$ are polynomial maps, we conclude that $g_\nabla(T(\xi), T(\eta))$ is also an SOS polynomial of ξ and η. This tells us that

$$g_\nabla(T(\xi), T(\eta)) = g(T(\eta)) - g(T(\xi)) - \langle \nabla g|_{T(\xi)}; T(\eta) - T(\xi) \rangle$$

is SOS.

Since $f(\xi) = g(T(\xi))$ and $f(\eta) = g(T(\eta))$, we have

$$g_\nabla(T(\xi), T(\eta)) = f(\eta) - f(\xi) - \langle \nabla g|_{T(\xi)}, T(\eta) - T(\xi) \rangle.$$

Hence, the definition fo Type-2 SOS-PC holds with $\beta(\xi) := \nabla g|_{T(\xi)}$. □

We have seen that every Type-1 SOS-PC polynomial is also Type-2 SOS-PC. Our next aim is to determine that this inclusion is strict. We now state the main theorem of this thesis, which shows that the inclusion is strict at least when $N = n = 2$ and $d \geq 4$.

Theorem 6. *The inclusion* $\Sigma PC^1_{2,d} \subset \Sigma PC^2_{2,d}$ *is strict for all* $d \geq 4$.

The result of this theorem shows that the two approaches to defining an SOS strengthening of Polyconvexity are not equivalent, illustrating the key difference between SOS-Convexity and SOS-Polyconvexity. We will prove this theorem by exhibiting that the function f_γ defined in (11) is an element of $\Sigma PC^2_{2,4} \setminus \Sigma PC^1_{2,4}$ in the following chapter.

4 Proof of Theorem 6

Consider the function

$$f_\gamma(\xi) = \|\xi\|^2(\|\xi\|^2 - 2\gamma \det \xi),$$

as defined in Theorem 4. We will show that:

- $f_\gamma \in \Sigma PC^1_{2,4} \Rightarrow |\gamma| \leq \gamma_c = \frac{2\sqrt{2}}{3} \approx 0.94281 < 1.$
- $f_1 \in \Sigma PC^2_{2,4}.$

Then, $f_1 \in \Sigma PC^2_{2,4} \backslash \Sigma PC^1_{2,4}$, proving Theorem 6.

Remark 1. We can observe from [6] that

$$f_\gamma(Q\xi) = f_{-\gamma}(\xi) \text{ for every } \xi \in \mathbb{R}^{2\times 2} \text{ and } Q \in O(2) \text{ with } \det Q = -1.$$

Hence, f_γ is convex (respectively polyconvex) if and only if $f_{-\gamma}$ is convex (respectively polyconvex). Thus considering the value of γ, we can restrict the value to $\gamma \geq 0$ without loss of generality.

4.1 Defining the structure of $g(\xi, \det \xi)$

We first determine the value of γ such that

$$f_\gamma(\xi) \in \Sigma PC^1_{2,4}.$$

In other words, we seek γ such that

$$f_\gamma(\xi) = g(\xi, \det \xi) \text{ for an SOS-Convex polynomial } g. \tag{17}$$

To do this, we first try to derive a form for g.

Lemma 4. *Assume $f_\gamma(\xi) = g(\xi, \det \xi)$ for $g : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ polynomial. Then, there exists a polynomial $h(\xi, \delta)$ such that $g(\xi, \delta) = f_\gamma(\xi) + (\det \xi - \delta)h(\xi, \delta)$.*

Proof. Using Theorem 1, the proof follows from the fact $g(\xi, \delta) - f_\gamma(\xi) \in \mathscr{I}(\det \xi - \delta)$ if we can show that the polynomial ideal generated by $\det \xi - \delta$ is real radical. To do this we employ Lemma 1.

Define the ideal generated by $\det \xi - \delta$ as

$$\mathscr{I}(\det \xi - \delta) = \{(\det \xi - \delta)h(\xi, \delta) | h \in \mathbb{R}[\xi, \delta]\}.$$

Let $\sigma(\xi,\delta) \in \mathscr{I}(\det\xi - \delta)$ be such that σ is SOS. By the Definition of $\mathscr{I}(\det\xi - \delta)$,

$$\sigma(\xi,\delta) = \sum_{i=1}^{m} p_j^2 = (\det\xi - \delta)h(\xi,\delta). \tag{18}$$

Clearly, $\sigma(\xi,\delta) \geq 0$ for all ξ, δ. We can also see that $\sigma(\xi,\delta)$ vanishes when $\delta = \det\xi$. Substituting $\delta = \det\xi$ in (18), we can infer that

$$0 = \sum_{i=1}^{m} p_j^2(\xi, \det\xi) \geq p_j^2(\xi, \det\xi) \geq 0$$

for all polynomials $p_1, p_2, \ldots, p_m$. This implies that $p_j^2(\xi, \det\xi) = 0$, hence $p_j(\xi, \det\xi) = 0$ for all $j \in \{1, 2, \ldots, m\}$.

This tells us that $\delta = \det\xi$ is a root of the univariate polynomial $\delta \mapsto p_j(\xi,\delta)$ for fixed ξ, hence we can write p_j in the form $(\det\xi - \delta)\hat{p_j}$ for all $j \in \{1, 2, \ldots, m\}$.

So, we obtain $p_j \in \mathscr{I}(\det\xi - \delta)$ for all $j \in \{1, 2, \ldots, m\}$, which implies $\mathscr{I}(\det\xi - \delta)$ is a real radical ideal by Lemma 1.

□

4.1.1 Defining the structure of $h(\xi,\delta)$

Now that we have established that $g(\xi,\delta) = f_\gamma(\xi) + (\det\xi - \delta)h(\xi,\delta)$, we aim to explore the structure of $h(\xi,\delta)$. As we will be comparing the quantities $\det\xi$ and δ, we will henceforth take δ to have a weighted degree of two in comparison to the entries of ξ.

So, we define a weight vector $w = (1, 1, 1, 1, 2)$ and define $\widetilde{d} = \langle w, \alpha \rangle$ as weighted degree. With the help of this weighted degree, we split $h(\xi,\delta)$ into homogeneous parts of fixed weighted degree. Specifically, we write

$$h(\xi,\delta) = \sum_{d=0}^{p} h_d(\xi,\delta),$$

where

$$h_d(\xi,\delta) = \sum_{\langle w,\alpha \rangle = d} \xi_{11}^{\alpha_1}\xi_{12}^{\alpha_2}\xi_{21}^{\alpha_3}\xi_{22}^{\alpha_4}\delta^{\alpha_5} h_{d,\alpha}. \tag{19}$$

and $p \in \mathbb{N}$ such that

$$p \leq \max(\deg_\xi g - 2, \deg_\delta g - 1).$$

In particular, h_0 is defined as a constant, h_1 is defined as

$$h_1(\xi,\delta) = b_{11}\xi_{11} + b_{12}\xi_{12} + b_{21}\xi_{21} + b_{22}\xi_{22}, \tag{20}$$

and h_2 is defined as

$$\begin{aligned} h_2(\xi,\delta) = c_{11}\xi_{11}^2 + c_{10}\xi_{11}\xi_{12} + c_9\xi_{11}\xi_{21} + c_8\xi_{11}\xi_{22} + c_7\xi_{12}^2 + \\ c_6\xi_{12}\xi_{21} + c_5\xi_{12}\xi_{22} + c_4\xi_{21}^2 + c_3\xi_{21}\xi_{22} + c_2\xi_{22}^2 + c_1\delta. \end{aligned} \tag{21}$$

Note that linear terms in δ do not appear in h_1 because δ has a weighted degree of two. Similar considerations apply to h_2. Next, we give a more explicit structure for h_0, h_1 and h_2, which will show that f_γ is Type-1 SOS-PC only if $\gamma \leq \frac{2\sqrt{2}}{3}$. We proceed in three steps.

4.2 Step 1: $h_0 = h_1 = 0$.

According to assumption (17), we have that $g(\xi,\delta) = f_\gamma(\xi) + (\det\xi - \delta)h(\xi,\delta)$ is SOS-convex. Since any SOS-convex polynomial is also convex, we can conclude from (3) in proposition 2 that $\nabla^2 g(\xi,\delta) \succeq 0$ for all ξ,δ.

We can now use the structure of $h(\xi,\delta)$ to split $\nabla^2 g(\xi,\delta)$ into matrices $M_0, M_1, \ldots, M_p$. In other words,

$$\nabla^2 g(\xi,\delta) = M_0 + M_1(\xi,\delta) + \ldots + M_p(\xi,\delta),$$

where the polynomial matrix $M_k(\xi,\delta) \in \mathbb{R}[\xi,\delta]^{5\times 5}$ is the homogeneous part of $\nabla^2 g$ of weighted degree k for $k \in \{0,1,\ldots,p\}$. The explicit expressions are complicated and are not reported for brevity.

Now fix $\epsilon > 0$ arbitrary for a given ξ,δ. Plugging $\epsilon\xi, \epsilon^2\delta$ into $\nabla^2 g(\xi,\delta)$, we obtain

$$\nabla^2 g(\epsilon\xi,\epsilon^2\delta) = M_0 + \epsilon M_1(\xi,\delta) + \epsilon^2 M_2(\xi,\delta) + \sum_{d=3}^{p} \epsilon^d M_d(\xi,\delta) =: H(\xi,\delta,\epsilon). \tag{22}$$

This implies that $H(\xi,\delta,\epsilon)$ is positive semidefinite for all ξ, δ, and $\epsilon > 0$. By continuity of the matrix-valued polynomial $\epsilon \mapsto H(\xi,\delta,\epsilon)$, we must have $H(\xi,\delta,\epsilon) \succeq 0$ for $\epsilon = 0$. In other words, $H(\xi,\delta,0) = M_0 \succeq 0$. Written in full, this constraint is

$$M_0 = \begin{bmatrix} 0 & 0 & 0 & h_0 & -b_{11} \\ 0 & 0 & -h_0 & 0 & -b_{12} \\ 0 & -h_0 & 0 & 0 & -b_{21} \\ h_0 & 0 & 0 & 0 & -b_{22} \\ -b_{11} & -b_{12} & -b_{21} & -b_{22} & -2c_1 \end{bmatrix} \succeq 0.$$

This requires that

$$c_1 \leq 0. \tag{23}$$

Moreover, since the first four entries on the main diagonal are zero, we must also have that

$$h_0 = b_{11} = b_{12} = b_{21} = b_{22} = 0,$$

which, in particular, implies that $h_1 = 0$ by equation (20).

4.3 Step 2: $h_2 = -a(\det \xi + \delta)$

Now we look at the 4×4 top-left block of the hessian $\nabla^2 g$, where M_0 vanishes. Examining the 4×4 top-left block M_1, we obtain

$$\begin{bmatrix} 2\xi_{22}b_{11} & -\xi_{21}b_{11} + \xi_{22}b_{12} & -\xi_{12}b_{11} + \xi_{22}b_{21} & 2\xi_{11}b_{11} + \xi_{12}b_{12} + \xi_{21}b_{21} + 2\xi_{22}b_{22} \\ * & -2\xi_{21}b_{12} & -\xi_{11}b_{11} - 2\xi_{12}b_{12} - 2\xi_{21}b_{21} - \xi_{22}b_{22} & \xi_{11}b_{12} - \xi_{21}b_{22} \\ * & * & -2\xi_{12}b_{21} & \xi_{11}b_{21} - \xi_{12}b_{22} \\ * & * & * & 2\xi_{11}b_{22} \end{bmatrix},$$

where * denotes symmetric entries omitted for convenience.

Substituting $h_0 = 0$ and $h_1 = 0$, we can see that the 4×4 top-left block of M_1 vanishes. Then, looking at the same 4×4 top-left block of $H(\xi, \delta, \epsilon)$, we see that

$$[H(\xi, \delta, \epsilon)]_{4\times4} = \epsilon^2 \left([M_2(\xi, \delta)]_{4\times4} + \sum_{d=3}^{p} \epsilon^{d-2} [M_d(\xi, \delta)]_{4\times4} \right) \succeq 0 \ \forall \xi, \delta \text{ and } \epsilon > 0$$

Again, by continuity of the polynomial mapping $\epsilon \mapsto [H(\xi, \delta, \epsilon)]_{4\times4}$, we have that $[M_2(\xi, \delta)]_{4\times4} \succeq 0$ for all ξ, δ. Substituting $\xi = 0$, we obtain

$$[M_2(0, \delta)]_{4\times4} = \begin{bmatrix} -2c_{11}\delta & -c_{10}\delta & -c_9\delta & c_1\delta - c_8\delta \\ -c_{10}\delta & -2c_7\delta & -c_1\delta - c_6\delta & -c_5\delta \\ -c_9\delta & -c_1\delta - c_6\delta & -2c_4\delta & -c_3\delta \\ c_1\delta - c_8\delta & -c_5\delta & -c_3\delta & -2c_2\delta \end{bmatrix} \succeq 0$$

for all values of δ.
As $[M_2(0, \delta)]_{4\times4}$ is linear in δ, we obtain

$$c_2 = c_3 = c_4 = c_5 = c_7 = c_9 = c_{10} = c_{11} = 0 \text{ and}$$

$$c_1 = -c_6 = c_8.$$

By substituting $a = -c_1 = c_6 = -c_8$, we obtain $h_2 = -a(\det \xi + \delta)$, which is the desired result. We also note from (23) that

$$a = -c_1 \geq 0. \tag{24}$$

4.4 Step 3: f_γ is Type-1 SOS-PC only if $|\gamma| \leq \frac{2\sqrt{2}}{3}$

Finally, we use the structure of h_0, h_1 and h_2 obtained above to show that f_γ Type-1 SOS-PC only if $|\gamma| \leq \frac{2\sqrt{2}}{3}$. By substituting $\xi_{12} = \xi_{21} = 0$ and $\delta = 0$ into $[M_2(\xi, \delta)]_{4\times4}$ and looking at a specific 2×2 minor, we find that

$$H(\xi) = \begin{bmatrix} 12\xi_{11}^2 - 12\xi_{11}\xi_{22}\gamma - 2a\xi_{22}^2 + 4\xi_{22}^2 & -6\xi_{11}^2\gamma - 4a\xi_{11}\xi_{22} + 8\xi_{11}\xi_{22} - 6\xi_{22}^2\gamma \\ -6\xi_{11}^2\gamma - 4a\xi_{11}\xi_{22} + 8\xi_{11}\xi_{22} - 6\xi_{22}^2\gamma & -2a\xi_{11}^2 + 4\xi_{11}^2 - 12\xi_{11}\xi_{22}\gamma + 12\xi_{22}^2 \end{bmatrix} \succeq 0.$$

Substituting $\xi_{11} = r\cos\frac{\theta}{2}, \xi_{22} = r\sin\frac{\theta}{2}$, we obtain

$$\frac{1}{2r^2}\det(H(\xi)) = \Phi(\theta) = A\sin^2\theta + B\sin\theta + C \geq 0 \quad \text{for all values of } \theta \in [-\pi,\pi] \tag{25}$$

$$\text{where } \begin{cases} A = 9\gamma^2 + 6a - \frac{3}{4}a^2, \\ B = -12\gamma - 3a\gamma, \\ C = 12 - 6a - 9\gamma^2. \end{cases} \tag{26}$$

Substituting $\theta = 0$, we obtain that $C \geq 0$. In other words,

$$C = 12 - 6a - 9\gamma^2 \geq 0 \Rightarrow \gamma^2 \leq \frac{4}{3} - \frac{2}{3}a, \tag{27}$$

$$\gamma^2 \geq 0 \Rightarrow a \leq 2.$$

Thus, we must have

$$0 \leq |\gamma| \leq \sqrt{\frac{4}{3} - \frac{2}{3}a}, \tag{28}$$

$$0 \leq a \leq 2 \tag{29}$$

We can see from (27) that γ^2 attains its maximum at the smallest value of a. The bounds in (29) imply

$$|\gamma^{\max}| \leq \frac{2}{\sqrt{3}}. \tag{30}$$

From (25), we know that $\Phi(\theta) \geq 0$ for all values of θ. This is true *if and only if* $\Phi(\hat{\theta}) \geq 0$ for all minimizers $\hat{\theta}$ of Φ. In other words,

$$(25) \iff \Phi(\hat{\theta}) \geq 0 \ \forall \hat{\theta} \text{ such that } \Phi'(\hat{\theta}) = 0,$$

where

$$\Phi'(\theta) = \cos(\theta)(2A\sin(\theta) + B). \tag{31}$$

We now distinguish two cases.

Case 1: $-\frac{B}{2A} \geq 1$. In this case, the only θ satisfying (31) solves $\cos(\theta) = 0$ or $\sin(\theta) = 1$, so $\theta = \pm\frac{\pi}{2}$. Then, we need

$$\Phi(\frac{\pi}{2}) = A + B + C \geq 0,$$

$$\Phi(-\frac{\pi}{2}) = A - B + C \geq 0.$$

Adding these two inequalities, we obtain

$$A + C = 12 - \frac{3}{4}a^2 \geq 0 \Rightarrow a \leq 4,$$

which is already implied by (29).

From (28) and (29), we see that $B = -12\gamma - 3a\gamma \leq 0$. We can also see that the minimum of A is attained at $\gamma = 0$, which implies that the minimum of A ranges between 0 and 9 for a satisfying (29), hence $A \geq 0$.

By assumption, we also have that $|\frac{-B}{2A}| \geq 1$. From $B \leq 0$ and $A \geq 0$, we obtain that $\frac{-B}{2A} \geq 1$. Then,

$$2A + B \leq 0,$$

or equivalently,

$$p(\gamma, a) = 18\gamma^2 - \gamma(12 + 3a) + \left(12a - \frac{3}{2}a^2\right) \leq 0. \tag{32}$$

Since γ must be real, this requires that $p(\gamma, a)$ has real roots, which in turn means that we must have non-negative discriminant. Thus, we must have

$$13a^2 - 88a + 16 \geq 0$$

Thus, either

$$a \leq \frac{44 - \sqrt{1728}}{13} \approx 0.1869,$$

or

$$a \geq \frac{44 + \sqrt{1728}}{13} \approx 6.5822.$$

Since we know that $a \in [0, 2]$ from (29), only the first option is valid and, therefore

$$a \in [0, 0.1869]. \tag{33}$$

We now aim to find the largest root $\gamma_+(a)$ of $p(\gamma, a)$ and maximize it over a in the interval (33). The largest root of $p(\gamma, a)$ is given by

$$\gamma_+(a) = \frac{(4 + a) + \sqrt{13a^2 - 88a + 16}}{12}.$$

Computing the derivative with respect to a, we obtain

$$12\gamma_+(a) = 1 + \frac{13a-44}{\sqrt{13a^2-88a+16}}.$$

We claim that $\gamma_+(a)$ is decreasing in a, in other words, $\gamma_+(a) \leq 0$ for all $a \in [0, 0.1869]$. We can see that

$$\gamma_+(a) \leq 0 \iff 1 + \frac{13a-44}{\sqrt{13a^2-88a+16}} \leq 0.$$

This is true only if

$$\begin{cases} (i)\ a \leq \frac{44}{13} \approx 3.384, \\ (ii)\ 13a^2 - 88a + 16 \geq 0, \\ (iii)\ 13a^2 - 88a + 160 \geq 0. \end{cases}$$

From the bounds of a in (33), we can see that (i) and (ii) are true. Computing the discriminant of (iii), we see that it is negative. Hence for the values of a in the interval $[0, 0.1869]$, $\gamma_+(a)$ is decreasing in a. Thus we obtain

$$\gamma_1^{\max} = \gamma_+(0) = \frac{2}{3} < \frac{2}{\sqrt{3}}.$$

Case 2: $|-\frac{B}{2A}| \leq 1.$ In this case, there is an extra stationary point $\sin(\theta) = \frac{-B}{2A}$. Substituting this choice of θ in Φ, we must have

$$0 \leq A\left(\frac{-B}{2A}\right)^2 + B\left(\frac{-B}{2A}\right) + C = -\frac{B^2}{4A} + C$$

$$\Rightarrow B^2 - 4AC \leq 0.$$

Substituting the values of A, B and C from (26), we obtain

$$-81\gamma^4 + \frac{9}{2}(a^2 - 28a + 16)\gamma^2 + \frac{9}{2}a(a^2 - 10a + 16) \geq 0. \tag{34}$$

The left-hand side of this inequality is a quadratic in γ^2, and that it opens downwards. This tells us that the value of γ^2 lies between the two roots of (34). Computing the two roots and only considering the non-negative one, we obtain

$$\gamma^2 \leq \frac{1}{18}a^2 - \frac{5}{9}a + \frac{8}{9}.$$

As this is the only non-negative root, we can conclude $\gamma^2 \in [0, \frac{1}{18}a^2 - \frac{5}{9}a + \frac{8}{9}]$. Maximizing $\gamma = \sqrt{\frac{1}{18}a^2 - \frac{5}{9}a + \frac{8}{9}}$ with respect to $a \in [0, 2]$, we obtain that $\gamma_2^{\max} = \frac{2\sqrt{2}}{3}$ at $a = 0$.

Conclusion: Combining the two cases considered above, we can conclude that if f_γ is Type-1 SOS-PC then

$$|\gamma| \leq \max\{\gamma_1^{\max}, \gamma_2^{\max}\} = \gamma_2^{\max} = \frac{2\sqrt{2}}{3} = \gamma_c. \tag{35}$$

4.5 Proof of $f_1 \in \Sigma PC^2_{2,4}$

We now show that the function f_γ as defined in equation (11) is Type-2 SOS-polyconvex when $\gamma = 1$. Written in full, we aim to prove that

$$f_1(\xi) = \|\xi\|^2(\|\xi\|^2 - 2\det\xi) \in \Sigma PC^2_{2,4}.$$

By Definition 6, we try to find a polynomial vector $\beta(\xi) \in \mathbb{R}[\xi]^5$ such that

$$f_1(\eta) - f_1(\xi) - \langle \beta(\xi); T(\eta) - T(\xi)\rangle \tag{36}$$

is SOS in ξ and η. We can see that equation (36) has the structure of a first order approximation of $f_1(\eta)$ around $f_1(\xi)$. Hence we consider $\beta(\xi) = \nabla\phi|_{T(\xi)}$ where $\phi(\xi, \delta) = \|\xi\|^2(\|\xi\|^2 - 2\delta)$, and check if (36) is SOS. In other words, substituting

$$\beta(\xi) = \begin{pmatrix} 4\xi_{11}(\|\xi\|^2 - \det\xi) - 2\|\xi\|^2\xi_{22} \\ 4\xi_{12}(\|\xi\|^2 - \det\xi) + 2\|\xi\|^2\xi_{21} \\ 4\xi_{21}(\|\xi\|^2 - \det\xi) + 2\|\xi\|^2\xi_{12} \\ 4\xi_{22}(\|\xi\|^2 - \det\xi) - 2\|\xi\|^2\xi_{11} \\ -2\|\xi\|^2 \end{pmatrix} \tag{37}$$

in (36) with $\Delta = \eta - \xi$, we can see that

$$\begin{aligned} f_1(\eta) - f_1(\xi) - \langle \beta(\xi); T(\eta) - T(\xi)\rangle = {} & (\Delta_{11}^2 - \Delta_{12}\Delta_{21})^2 + (\Delta_{12}^2 - \Delta_{11}\Delta_{22})^2 \\ & + (\Delta_{21}^2 - \Delta_{11}\Delta_{22})^2 + (\Delta_{22}^2 - \Delta_{12}\Delta_{21})^2. \end{aligned} \tag{38}$$

It is clear from (38) that (36) is SOS in η and ξ. Hence we have that $f_1 \in \Sigma PC^2_{2,4}$.

5 Conclusion

In this thesis, we have seen that there are two ways to define an SOS-strengthening to polyconvexity and explored the relationship between them. We have established that all Type-1 SOS-PC polynomials are Type-2 SOS-PC but the reverse is not true, at least in the case of $N = n = 2$, using the example of Alibert-Dacorogna-Marcellini.

The strict inclusion between these SOS strenthenings remains an open problem for the general cases of all N, n. An interesting path to prove this strict inclusion for the general case is investigate a generalized version of the example of Alibert-Dacorogna-Marcellini. For example, one could consider the function

$$f_\gamma : \mathbb{R}^{N\times n} \to \mathbb{R} \text{ with } N, n > 1, f_\gamma = \|\xi\|^2(\|\xi\|^2 - 2\gamma\ (\xi_{11}\xi_{22} - \xi_{12}\xi_{21})).$$

One can further examine this function to formulate an argument analogous to the proof in Chapter 4.

Since Type-2 SOS-PC is more general than Type-1 SOS-PC, we suggest to adopt Type-2 SOS-PC as the standard definition of SOS-Polyconvexity and call it SOS-PC for brevity. As stated before, the SOS strenthening can be used to formulate an SDP that checks if a polynomial is SOS-PC and hence polyconvex in a numerically tractable way, which can be useful when solving minimization problems in the field of Calculus of Variations.

The possibility to check computationally if a polynomial is SOS-PC, of course, is only useful if SOS-PC polynomials are common. It would therefore be interesting to check how many polyconvex polynomials in literature are in fact SOS-PC. Such computations can also help investigate if there exist polyconvex polynomials that are not SOS-PC. We expect such polynomials to exist, just like there are convex polynomials that are not SOS-Convex [1], but we do not know of explicit examples yet.

The SOS-PC strengthening defined in this thesis can be used in other ways. In nonlinear elasticity, the stored energy density function $W(F)$ being polyconvex is sufficient for well-posedness ([2]). Direct modeling of W for composites or biological tissues is difficult. The definition of polyconvexity by Ball ([2]) has been used to formulate ways to perform regression to approximate the energy function W. The idea is to parametrize the the energy function G of the minors $T(F)$ and enforce convexity on G, while fitting G to data points $\{F_i, W_i\}$. This results in a polyconvex W with $W(F) = G(T(F))$. There have also been neural network based approaches, which can be found in [10]. However, we have also seen that solving a problem with convexity constraints are NP-Hard ([11]). One possible way could be to build a SOS-PC approximation to a given function, by the least-squares regression for example, at carefully chosen data points. This gives a way to construct SOS-PC polynomial energies from data, which could be useful in the modelling of composite materials. SOS-PC regression from simulation/experimental data gives a principled way to build valid energy models.

The method of using SOS and other polynomial optimization techniques in the fields of Calculus of Variation (CoV) can be explored further. The minimization of the energy functional of a body under deformation is the central problem in CoV, i.e.

$$\inf_{u\in H_0^1(\Omega)} I(u(x)) = \inf_{u\in H_0^1(\Omega)} \int_\Omega W(\nabla u(x))dx. \tag{39}$$

If W is bounded from below, then there exist the convex, polyconvex, quasiconvex envelopes of W, denoted by W^c, W^{pc} and W^{qc} respectively. Correspondingly, under certain technical assumptions, we can see from [7] that

$$\inf_u I(u) \equiv \min_u \int_\Omega W^{qc} \geq \min_u \int_\Omega W^{pc} \geq \min_u \int_\Omega W^c.$$

Hence, it is clear that one way to overcome the problem of nonexistence of minimizers of I due to lack of quasiconvexity is to replace E with its quasiconvex envelope W^{qc}. This is in particular useful for numerical discretizations of the variational problem (39). Unfortunately, computing the quasiconvex envelope W^{qc} is very hard. Computing the polyconvex envelope W^{pc} is simpler, hence it is commonly used in practice.

Bartels ([3]) provides an algorithm that uses an iterative method to approximate from above the polyconvex envelope W^{pc} of a given function $W : \mathbb{R}^{N\times n} \to \mathbb{R}$. It utilizes a linear programming approach to approximate the envelope by discretizing the space of matrices and evaluating the function at these discrete points. The method ensures that the resulting approximation is convex in all minors, adhering to the polyconvexity condition. This approximation from above grows large quickly due to the discretization of matrix spaces, making it harder to compute. However, it can be used to obtain upper bounds on the polyconvex envelope, even if the problem itself is intractable.

The PEPCE (Pointwise Evaluation of PolyConvex Envelopes) algorithm provided in [7] is designed to compute the polyconvex envelope W^{pc} of a given function $W : \mathbb{R}^{N\times n} \to \mathbb{R}$. It formulates this as a constrained optimization problem, where the goal is to find a combination of matrices $A_1, A_2, \ldots, A_{\tau(N,n)+1}$ such that the convex combination of the functions $W(A_i)$ and the associated polyconvex terms satisfy the constraints, thereby approximating $W^{pc}(A^0)$ for a given matrix A^0.

An interesting application of SOS and polynomial optimization techniques could be to use SOS methods to construct polyconvex envelopes. Dacorogna ([6]) provides a formula for computing the polyconvex envelope W^{pc} of a function $W : \mathbb{R}^{N\times n} \to \mathbb{R}$ at a given point $F \in \mathbb{R}^{N\times n}$, given by

$$
\begin{aligned}
W^{pc}(F) = \inf_{\mu \in \mathscr{M}_+(\mathbb{R}^{N\times n})} & \int_{\mathbb{R}^{N\times n}} W(\xi) d\mu \\
\text{such that} & \int T(\xi) d\mu = T(F), \\
& \int d\mu = 1.
\end{aligned} \tag{40}
$$

Denoting the Lagrange multiplier associated with the first constraint by λ and the Lagrange multiplier associated with the second constraint as γ, we obtain the dual of (40) to be

$$
\begin{aligned}
& W^{pc}(F) = \sup_{\lambda \in \mathbb{R}^{\tau(N,n)}, \gamma \in \mathbb{R}} \gamma + \langle \lambda, T(F) \rangle \\
& \text{such that } W(\xi) - \langle \lambda, T(\xi) \rangle - \gamma \geq 0 \ \forall \xi \in \mathbb{R}^{N\times n}
\end{aligned} \tag{41}
$$

Replacing the non-negativity constraint with an SOS constraint results in a tractable way to compute a lower bound of $W^{pc}(F)$. This is complementary to existing methods, which instead compute upper bounds by discretizing μ into sum of Diracs, $\mu = \delta_{A_0} + \delta_{A_1} + \ldots + \delta_{A_{\tau+1}}$ and looking for the matrices $A_0, A_1, \ldots, A_{\tau+1}$. This results in an interesting path to explore further.

Literatur

[1] AHMADI, Amir A. ; PARRILO, Pablo A.: A Complete Characterization of the Gap Between Convexity and SOS-Convexity. In: *SIAM Journal on Optimization* 23 (2013), Nr. 2, S. 811–833

[2] BALL, John M.: Convexity conditions and existence theorems in nonlinear elasticity. In: *Archive for Rational Mechanics and Analysis* 63 (1977), Nr. 4, S. 337–403. http://dx.doi.org/10.1007/BF00248899. – DOI 10.1007/BF00248899

[3] BARTELS, Sören: Reliable and efficient approximation of polyconvex envelopes. In: *SIAM Journal on Numerical Analysis* 43 (2005), Nr. 1

[4] CHARLES B. MORREY, Jr.: Quasi-convexity and the lower semicontinuity of multiple integrals. In: *Pacific Journal of Mathematics* 2 (1952), Nr. 1, 25–53. https://projecteuclid.org/journals/pacific-journal-of-mathematics/volume-2/issue-1/Quasi-convexity-and-the-lower-semicontinuity-of-multiple-integrals/pjm/1103051941.full

[5] CONRAD, Keith: *Notes on Ideals*. https://kconrad.math.uconn.edu/blurbs/ringtheory/ideals.pdf, . – Accessed: 2025-08-10

[6] DACOROGNA, Bernard: *Direct Methods in the Calculus of Variations*. 2nd. Springer, 2008. – ISBN 978–0–387–35779–9

[7] ENEYA, Levis ; BOSSE, Torsten ; GRIEWANK, Andreas: A method for pointwise evaluation of polyconvex envelopes. In: *Afrika Matematika* 24 (2013), Nr. 1

[8] FANTUZZI, Giovanni: *Polynomial Optimization: Theory and Applications*. 2024

[9] HILBERT, David: Ueber die Darstellung definiter Formen als Summe von Formenquadraten. In: *Mathematische Annalen* 32 (1888), 342-350. https://doi.org/10.1007/BF01443605

[10] KIRCHDOERFER, Tristan ; ORTIZ, Michael: Data-driven computational mechanics. In: *Computer Methods in Applied Mechanics and Engineering* 304 (2016). http://dx.doi.org/10.1016/j.cma.2016.02.001. – DOI 10.1016/j.cma.2016.02.001

[11] KLERK, Etienne de ; PASECHNIK, V. D. A Linear Programming Reformulation of the Standard Quadratic Optimization Problem. In: *Journal of Global Optimization* 37 (2007), Nr. 1, S. 75–84. http://dx.doi.org/10.1007/s10898-006-9037-9. – DOI 10.1007/s10898–006–9037–9

[12] LAURENT, Monique: Sums of Squares, Moment Matrices and Optimization over Polynomials / Centrum Wiskunde & Informatica (CWI). Version: 2010. https://homepages.cwi.nl/~monique/files/moment-ima-update-new.pdf. 2010. – Forschungsbericht

[13] MOTZKIN, TS: *The arithmetic-geometric inequality, Inequalities (Proc. Sympos. Wright-Patterson Air Force Base, Ohio, 1965)*. 1967

Erklärung

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