Analysis of gas networks using mathematical models in the form of differential algebraic equations.

Part I

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Consider **implicit ordinary differential equations** (ODEs) of the form

$$\frac{d}{dt}[A(t)x(t)] + B(t)x(t) = f(t,x(t)), \qquad t \in [t_+,\infty), \tag{1}$$

$$A(t)\frac{d}{dt}x(t) + B(t)x(t) = f(t,x(t)), \tag{2}$$

where $t_+ \geq 0$, $A(t), B(t)$ $(t \in [t_+,\infty))$ are closed linear operators from X to Y with the domains $D_{A(t)}$, $D_{B(t)}$, $D = D_{A(t)} \cap D_{B(t)} \neq \{0\}$, X, Y are Banach spaces, $f\colon [t_+,\infty) \times X \to Y$.

The time-varying operators $A(t)$, $B(t)$ can be degenerate.

The differential equations (DEs) (1) and (2) with a degenerate (for some t) operator $A(t)$ are called **time-varying (nonautonomous) degenerate DEs** or **time-varying differential-algebraic equations (DAEs)**. In the terminology of DAEs, equations of the form (1), (2) are commonly referred to as **semilinear**.

We study the initial value problem (the Cauchy problem) for the DAEs (1), (2) with the initial condition

$$x(t_0) = x_0. \tag{3}$$

Fields of application of the theory of DAEs are control theory, radioelectronics, cybernetics, mechanics, robotics technology, economics, ecology and chemical kinetics.

In particular, semilinear DAEs are used in modelling

- transient processes in electrical circuits
- gas flow in networks
- dynamics of neural networks
- dynamics of complex mechanical and technical systems (e.g., robots)
- multi-sectoral economic models
- kinetics of chemical reactions

Notice that any type of a PDE can be represented as a DAE in infinite-dimensional spaces (an abstract DAE) and, possibly, a complementary boundary condition.

Assume that the characteristic operator pencil $\lambda A(t)+B(t)$ ($\lambda \in \mathbb{C}$ is a parameter), associated with the linear part of the DAE (1) or (2), is a **regular pencil of index not higher than 1**: for each $t \geq t_+$ the pencil $\lambda A(t)+B(t)$ be regular and there exist functions $C_1\colon [t_+,\infty) \to (0,\infty)$, $C_2\colon [t_+,\infty) \to (0,\infty)$ such that for every $t \in [t_+,\infty)$ the pencil resolvent $R(\lambda,t) = (\lambda A(t)+B(t))^{-1}$ satisfies the constraint

$$\|R(\lambda,t)\| \leq C_1(t), \quad |\lambda| \geq C_2(t). \tag{4}$$

Then for each $t \in [t_+,\infty)$, there exist the two pairs of mutually complementary projectors

$$P_j(t)\colon D \to D_j(t) \text{ and } Q_j(t)\colon Y \to Y_j(t), \;\; j=1,2,$$

which generate the direct decompositions

$$D = D_1(t)\dot{+}D_2(t), \quad Y = Y_1(t)\dot{+}Y_2(t) \quad \text{such that} \tag{5}$$

the pair of subspaces $X_1(t)$, $Y_1(t)$ and $X_2(t)$, $Y_2(t)$ are invariant under the operators $A(t)$, $B(t)$ (i.e., $A(t), B(t)\colon X_j(t) \to Y_j(t)$) and $A_j(t) = A(t)|_{D_j(t)}$, $B_j(t) = B(t)|_{D_j(t)}\colon D_j(t) \to Y_j(t)$, $j=1,2$, are such that $A_2(t)=0$ and there exist $A_1^{-1}(t)$, $B_2^{-1}(t)$ if $D_1(t) \neq \{0\}$, $D_2(t) \neq \{0\}$ respectively ($D_2(t) = \operatorname{Ker} A(t) \cap D$, $Y_1(t) = A(t)D$)

$$A(t)=A_1(t)\dot{+}A_2(t),\; B(t)=B_1(t)\dot{+}B_2(t)\colon D_1(t)\dot{+}D_2(t) \to Y_1(t)\dot{+}Y_2(t) \tag{6}$$

[Rutkas A.G., Vlasenko L.A. Existence of solutions of degenerate nonlinear differential operator equations, *Nonlinear Oscillations*, 2001]

For each $t \in [t_+,\infty)$ the projectors can be determined by the formulas [Rutkas A.G., Vlasenko L.A. *Nonlinear Oscillations*, 2001]

$$\begin{aligned} P_1(t) &= \frac{1}{2\pi i} \oint\limits_{|\lambda|=C_2(t)} R(\lambda,t)A(t)\,d\lambda, \quad P_2(t) = I_X - P_1(t), \\ Q_1(t) &= \frac{1}{2\pi i} \oint\limits_{|\lambda|=C_2(t)} A(t)R(\lambda,t)\,d\lambda, \quad Q_2(t) = I_Y - Q_1(t). \end{aligned} \tag{7}$$

and the auxiliary operator $G(t) = A(t) + B(t)P_2(t) \colon D \to Y$ has the bounded inverse $G^{-1}(t) \in (Y,X)$.

Let $X = Y = D = \mathbb{R}^n$.

For each t any $x \in \mathbb{R}^n$ can be uniquely represented in the form

$$x = x_{p_1}(t) + x_{p_2}(t), \quad x_{p_i}(t) = P_i(t)x \in X_i(t).$$

The DAE (1) $[A(t)x(t)]' + B(t)x(t) = f(t,x(t))$ is reduced to the equivalent system

$$[P_1(t)x(t)]' = \big[P_1'(t) - G^{-1}(t)Q_1(t)[A'(t) + B(t)]\big]P_1(t)x(t) + G^{-1}(t)Q_1(t)f(t,x(t)),$$
$$G^{-1}(t)Q_2(t)[f(t,x(t)) - A'(t)P_1(t)x(t)] - P_2(t)x(t) = 0 \qquad \text{or}$$

$$x_{p_1}'(t) = \big[P_1'(t) - G^{-1}(t)Q_1(t)[A'(t) + B(t)]\big]x_{p_1}(t) + G^{-1}(t)Q_1(t)f(t,x), \tag{8}$$
$$G^{-1}(t)Q_2(t)[f(t,x_{p_1}(t) + x_{p_2}(t)) - A'(t)x_{p_1}(t)] - x_{p_2}(t) = 0. \tag{9}$$

Introduce the manifold

$$L_{t_+} = \{(t,x) \in [t_+,\infty) \times \mathbb{R}^n \mid Q_2(t)[B(t)x + A'(t)P_1(t)x - f(t,x)] = 0\}. \tag{10}$$

The consistency condition $(t_0,x_0) \in L_{t_+}$ for the initial point (t_0,x_0) is one of the necessary conditions for the existence of a solution of the initial value problem (1), (3).

$V'_{(8)}(t,x_{p_1}(t)) = \frac{\partial V}{\partial t}(t,x_{p_1}(t)) + \Big(\frac{\partial V}{\partial z}(t,x_{p_1}(t)), \big[P_1'(t) - G^{-1}(t)Q_1(t)[A'(t) + B(t)]\big]x_{p_1}(t) + G^{-1}(t)Q_1(t)f(t,x_{p_1}(t) + x_{p_2}(t))\Big)$ is the derivative of the function $V(t,z)$ along the trajectories of the equation (8), where $V(t,z)$ is a continuously differentiable and positive definite scalar function.

The IVP (1), (3): $\dfrac{d}{dt}[A(t)x(t)] + B(t)x(t) = f(t,x(t)), \quad x(t_0) = x_0.$

Definitions

A solution $x(t)$ of the initial value problem (IVP) (1), (3) is called **global** or **defined in the future** if it exists on $[t_0,\infty)$.

A solution $x(t)$ of the IVP (1), (3) is called **Lagrange stable** if it is global and bounded, i.e., $\sup\limits_{t\in[t_0,\infty)} \|x(t)\| < \infty$.

A solution $x(t)$ of the IVP (1), (3) has a **finite escape time** (is **blow-up in finite time**) and is called **Lagrange unstable** if it exists on some finite interval $[t_0,T)$ and is unbounded, i.e., $\lim\limits_{t\to T-0} \|x(t)\| = \infty$.

The equation (1) is called **Lagrange stable** if every solution of the IVP (1), (3) is Lagrange stable (the DAE is Lagrange stable for every consistent initial point).

The equation (1) is called **Lagrange unstable** if every solution of the IVP (1), (3) is Lagrange unstable.

J. La Salle obtained the theorems on the global solvability, the Lagrange stability and instability of the explicit ODE $x' = f(t,x)$ [J. La Salle, S. Lefschetz, *Stability by Liapunov's Direct Method with Applications*, 1961].

Solutions of the equation (1) are called **ultimately bounded**, if there exists a constant $K > 0$ (K is independent of the choice of t_0, x_0) and for each solution $x(t)$ with an initial point (t_0, x_0) there exists a number $\tau = \tau(t_0, x_0) \geq t_0$ such that $\|x(t)\| < K$ for all $t \in [t_0 + \tau, \infty)$.

The equation (1) is called **ultimately bounded** or **dissipative**, if for any consistent initial point (t_0, x_0) there exists a global solution of the initial value problem (1), (3) and all solutions are ultimately bounded.

If the number τ does not depend on the choice of t_0, then the solutions of (1) are called *uniformly ultimately bounded* and the equation (1) is called *uniformly ultimately bounded* or *uniformly dissipative*.

Ultimately bounded systems of explicit ODEs $x' = f(t, x)$, which are also called dissipative systems and D-systems, were studied in [Yoshizawa T., *Stability theory by Liapunov's second method*, 1966] and [La Salle J., Lefschetz S., 1961].

Main results:

- **Theorems on the existence and uniqueness of global solutions**

  Some advantages: the restrictions of the type of the global Lipschitz condition (including contractive mapping) are not used.
- **Theorem on the Lagrange stability of the DAE** (the boundedness of solutions)
- **Theorem on the Lagrange instability of the DAE** (solutions have finite escape time)
- **Theorem on the ultimate boundedness (dissipativity) of the DAE** (the ultimate boundedness of solutions)
- **Theorems on the Lyapunov stability and instability of the equilibrium state of the DAE**
- **Theorems on asymptotic stability and asymptotic stability in the large of the equilibrium state** (complete stability of the DAE)
- **Numerical methods**

The application of the obtained theorems to the study of certain mathematical models of electrical circuits with nonlinear and time-varying elements are shown.

Theorem 1 (the global solvability). Let $f \in C([t_+,\infty)\times\mathbb{R}^n,\mathbb{R}^n)$, $\frac{\partial}{\partial x}f \in C([t_+,\infty)\times\mathbb{R}^n,L(\mathbb{R}^n))$, $A,B \in C^1([t_+,\infty),L(\mathbb{R}^n))$, the pencil $\lambda A(t)+B(t)$ satisfy (4), where $C_2 \in C^1([t_+,\infty),(0,\infty))$, and the following conditions be satisfied:

1) for each $t\in[t_+,\infty)$ and each $x_{p_1}(t)\in X_1(t)$ there exists a unique $x_{p_2}(t)\in X_2(t)$ such that $(t,x_{p_1}(t)+x_{p_2}(t))\in L_{t_+}$;

2) for each $t_*\in[t_+,\infty)$, $x^*_{p_i}(t_*)\in X_i(t_*)$, $i=1,2$, such that $(t_*,x^*_{p_1}(t_*)+x^*_{p_2}(t_*))\in L_{t_+}$ the operator $\Phi_{t_*,x^*_{p_1}(t_*),x^*_{p_2}(t_*)}\colon X_2(t_*)\to Y_2(t_*)$, $\Phi_{t_*,x^*_{p_1}(t_*),x^*_{p_2}(t_*)} = \left[\frac{\partial}{\partial x}\left[Q_2(t_*)f(t_*,x^*_{p_1}(t_*)+x^*_{p_2}(t_*))\right]-B(t_*)\right]P_2(t_*)$, is invertible;

3) there exist a number $R>0$, a positive definite function $V\in C^1([t_+,\infty)\times U^c_R(0),\mathbb{R})$, where $U^c_R(0)=\{z\in\mathbb{R}^n \mid \|z\|\geq R\}$, and a function $\chi\in C([t_+,\infty)\times(0,\infty),\mathbb{R})$ such that:

3.1) $V(t,z)\to\infty$ uniformly in t on every finite interval $[a,b)\subset[t_+,\infty)$ as $\|z\|\to\infty$,

3.2) for all t, $x_{p_1}(t)$, $x_{p_2}(t)$ such that $(t,x_{p_1}(t)+x_{p_2}(t))\in L_{t_+}$, $\|x_{p_1}(t)\|\geq R$, the inequality $V'_{(8)}(t,x_{p_1}(t))\leq\chi\big(t,V(t,x_{p_1}(t))\big)$ holds,

3.3) the inequality $v'\leq\chi(t,v)$, $t\geq t_+$, has no positive solutions $v(t)$ with finite escape time.

Then **for each initial point** $(t_0,x_0)\in L_{t_+}$ **there exists a unique global solution of the IVP** (1), (3).

Statement 1.

Theorem 1 remains valid if the conditions 1), 2) are replaced by the following: there exists a constant $0 \le \alpha < 1$ such that

$$\left\|G^{-1}(t)\,Q_2(t)\,f\big(t,x_{p_1}(t)+x^1_{p_2}(t)\big)-G^{-1}(t)\,Q_2(t)\,f\big(t,x_{p_1}(t)+x^2_{p_2}(t)\big)\right\| \le \\ \le \alpha\left\|x^1_{p_2}(t)-x^2_{p_2}(t)\right\| \quad (11)$$

for any $t\in[t_+,\infty)$, $x_{p_1}(t)\in X_1(t)$ and $x^i_{p_2}(t)\in X_2(t)$, $i=1,2$.

Theorem 2 (the global solvability).

Theorem 1 remains valid if the conditions 1), 2) are replaced by the following:

1) for each $t\in[t_+,\infty)$, $x_{p_1}(t)\in X_1(t)$ there exists $x_{p_2}(t)\in X_2(t)$ such that $(t,x_{p_1}(t)+x_{p_2}(t))\in L_{t_+}$;

2) for each $t_*\in[t_+,\infty)$, $x^*_{p_1}(t_*)\in X_1(t_*)$, $x^i_{p_2}(t_*)\in X_2(t_*)$ such that $(t_*,x^*_{p_1}(t_*)+x^i_{p_2}(t_*))\in L_{t_+}$, $i=1,2$, the operator function $\Phi_{t_*,x^*_{p_1}(t_*)}(x_{p_2}(t_*))$ defined by

$$\Phi_{t_*,x^*_{p_1}(t_*)}\colon X_2(t_*)\to L(X_2(t_*),Y_2(t_*)),$$

$$\Phi_{t_*,x^*_{p_1}(t_*)}(x_{p_2}(t_*))=\left[\frac{\partial}{\partial x}\big[Q_2(t_*)f(t_*,x^*_{p_1}(t_*)+x_{p_2}(t_*))\big]-B(t_*)\right]P_2(t_*), \quad (12)$$

is basis invertible on $[x^1_{p_2}(t_*),x^2_{p_2}(t_*)]$.

Theorem 3 (Lagrange stability). Let $f \in C([t_+,\infty)\times\mathbb{R}^n,\mathbb{R}^n)$, $\frac{\partial}{\partial x}f \in C([t_+,\infty)\times\mathbb{R}^n,L(\mathbb{R}^n))$, $A,B \in C^1([t_+,\infty),L(\mathbb{R}^n))$, the pencil $\lambda A(t)+B(t)$ satisfy (4), where $C_2 \in C^1([t_+,\infty),(0,\infty))$, the requirements 1), 2) of Theorem 1 or 2 be fulfilled, and

3) there exists a number $R>0$, a positive definite function $V \in C^1([t_+,\infty)\times U_R^c(0),\mathbb{R})$ and a function $\chi \in C([t_+,\infty)\times(0,\infty),\mathbb{R})$ such that:

3.1) $V(t,z)\to\infty$ uniformly in t on $[t_+,\infty)$ as $\|z\|\to\infty$;

3.2) for all t, $x_{p_1}(t)$, $x_{p_2}(t)$ such that $(t,x_{p_1}(t)+x_{p_2}(t))\in L_{t_+}$, $\|x_{p_1}(t)\|\geq R$, the inequality $V'_{(8)}(t,x_{p_1}(t))\leq\chi\big(t,V(t,x_{p_1}(t))\big)$ holds;

3.3) the differential inequality $v'\leq\chi(t,v)$, $t\geq t_+$, has no unbounded positive solutions $v(t)$ for $t\in[t_+,\infty)$.

Let one of the following conditions be also satisfied:

4.a) for all $(t,x_{p_1}(t)+x_{p_2}(t))\in L_{t_+}$, $\|x_{p_1}(t)\|\leq M<\infty$ (M is an arbitrary constant), the inequality $\|G^{-1}(t)Q_2(t)[f(t,x_{p_1}(t)+x_{p_2}(t))-A'(t)x_{p_1}(t)]\|\leq K_M<\infty$, where $K_M=K(M)$ is some constant, holds;

4.b) for all $(t,x_{p_1}(t)+x_{p_2}(t))\in L_{t_+}$, $\|x_{p_1}(t)\|\leq M<\infty$, the inequality $\|x_{p_2}(t)\|\leq K_M<\infty$, where $K_M=K(M)$ is some constant, holds.

Then **the equation** (1) **is Lagrange stable**.

Theorem 4 (Lagrange instability). Let $f \in C([t_+,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$, $\frac{\partial}{\partial x} f \in C([t_+,\infty) \times \mathbb{R}^n, L(\mathbb{R}^n))$, $A, B \in C^1([t_+,\infty), L(\mathbb{R}^n))$, the pencil $\lambda A(t) + B(t)$ satisfy (4), where $C_2 \in C^1([t_+,\infty),(0,\infty))$, the requirements 1), 2) of Theorem 1 or 2 be fulfilled, and

3) there exists a region $\Omega \subset \mathbb{R}^n$, $0 \notin \Omega$, such that the component $P_1(t)x(t)$ of each existing solution $x(t)$ with the initial point $(t_0,x_0) \in L_{t_+}$, where $P_1(t_0)x_0 \in \Omega$, remains all the time in Ω;

4) there exist a positive definite function $V \in C^1([t_+,\infty) \times \Omega, \mathbb{R})$ and a function $\chi \in C([t_+,\infty) \times (0,\infty), \mathbb{R})$ such that:

4.1) for all t, $x_{p_1}(t)$, $x_{p_2}(t)$ such that $(t, x_{p_1}(t) + x_{p_2}(t)) \in L_{t_+}$, $x_{p_1}(t) \in \Omega$, the inequality $V'_{(8)}(t, x_{p_1}(t)) \geq \chi\big(t, V(t, x_{p_1}(t))\big)$ holds,

4.2) the inequality $v' \geq \chi(t,v)$, $t \geq t_+$, has no positive solutions defined in the future (i.e., defined for all $t \geq t_+$).

Then **for each initial point** $(t_0,x_0) \in L_{t_+}$ **such that** $P_1(t_0)x_0 \in \Omega$**, there exists a unique solution of the IVP** (1), (3) **and this solution is Lagrange unstable**.

Remarks on the form of the functions χ

It is usually convenient to choose $\chi \in C([t_+,\infty)\times(0,\infty),\mathbb{R})$ in the form

$$\chi(t,v) = k(t)U(v), \tag{13}$$

where $U \in C(0,\infty)$, $k \in C([t_+,\infty),\mathbb{R})$. Then the theorem conditions can be changed as follows:

- in Theorems 1, 2 on the global solvability, it suffices to require that $\int\limits_c^\infty \frac{dv}{U(v)} = \infty$ ($c > 0$ is some constant) instead of the condition 3.3);
- in Theorem 3 on the Lagrange stability, it suffices to require that $\int\limits_c^\infty \frac{dv}{U(v)} = \infty$ and $\int\limits_{t_0}^\infty k(t)dt < \infty$ ($t_0 \ge t_+$ is some number) instead of the condition 3.3);
- in Theorem 4 on the Lagrange instability, it suffices to require that $\int\limits_c^\infty \frac{dv}{U(v)} < \infty$ and $\int\limits_{t_0}^\infty k(t)dt = \infty$ instead of the condition 4.2).

[Filipkovskaya M. S. Global solvability of time-varying semilinear differential-algebraic equations, boundedness and stability of their solutions. I, *Differential Equations*, 2021]
[Filipkovskaya M. S. Global solvability of time-varying semilinear differential-algebraic equations, boundedness and stability of their solutions. II, *Differential Equations*, 2021]

Theorem 5 (uniform dissipativity (ultimate boundedness)). Let $f \in C([t_+,\infty)\times\mathbb{R}^n,\mathbb{R}^n)$, $\frac{\partial}{\partial x}f \in C([t_+,\infty)\times\mathbb{R}^n, L(\mathbb{R}^n))$, $A,B \in C^1([t_+,\infty),L(\mathbb{R}^n))$, the pencil $\lambda A(t)+B(t)$ satisfy (4), where $C_2 \in C^1([t_+,\infty),(0,\infty))$, the requirements 1), 2) of Theorem 1 or 2 be fulfilled, and

3) there exist a number $R>0$, a positive definite function $V \in C^1([t_+,\infty)\times U_R^c(0),\mathbb{R})$ and functions $U_j \in C([0,\infty))$, $j=0,1,2$, such that $U_0(r)$ is non-decreasing and $U_0(r)\to+\infty$ as $r\to+\infty$, $U_1(r)$ is increasing, $U_2(r)>0$ for $r>0$, and for all $t\in[t_+,\infty)$, $x_{p_1}(t)\in X_1(t)$, $x_{p_2}(t)\in X_2(t)$ such that $(t,x_{p_1}(t)+x_{p_2}(t))\in L_{t_+}$, $\|x_{p_1}(t)\|\geq R$ the condition $U_0(\|x_{p_1}(t)\|)\leq V(t,x_{p_1}(t))\leq U_1(\|x_{p_1}(t)\|)$ and one of the following inequalities hold:

3.a) $V'_{(8)}(t,x_{p_1}(t))\leq -U_2\big(\|x_{p_1}(t)\|\big)$;

3.b) $V'_{(8)}(t,x_{p_1}(t))\leq -U_2\big((H(t)x_{p_1}(t),x_{p_1}(t))\big)$, where $H\in C([t_+,\infty),L(\mathbb{R}^n))$ is some self-adjoint positive definite operator function such that $\sup\limits_{t\in[t_+,\infty)}\|H(t)\|<\infty$;

3.c) $V'_{(8)}(t,x_{p_1}(t))\leq -C\,V(t,x_{p_1}(t))$, where $C>0$ is some constant;

4) there exist a constant $c>0$ and a number $T>t_+$ such that $\|G^{-1}(t)Q_2(t)[f(t,x_{p_1}(t)+x_{p_2}(t))-A'(t)x_{p_1}(t)]\|\leq c\,\|x_{p_1}(t)\|$ for all $(t,x_{p_1}(t)+x_{p_2}(t))\in L_T$.

Then **the DAE** (1) **is uniformly ultimately bounded (uniformly dissipative)**.

Remarks on the form of the functions V

It is often convenient to choose the positive definite scalar function $V(t,z)$ in the form

$$V(t,z) = (H(t)z,z), \tag{14}$$

where $H \in C^1([t_+,\infty),L(\mathbb{R}^n))$ is a self-adjoint positive definite operator function. The function $V(t,z)$ (14) satisfies the conditions (except for the conditions on the derivative of the function along the trajectories of (8)) of Theorems 1–4 on the global solvability, the Lagrange stability and the Lagrange instability, and if additionally $\sup\limits_{t\in[t_+,\infty)} \|H(t)\| < \infty$, then the function (14) also satisfies the conditions of Theorem 5 on the dissipativity.

[Filipkovska (Filipkovskaya) M. S. Global boundedness and stability of solutions of nonautonomous degenerate differential equations, *Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan*, 2020]

The IVP (1), (3): $\dfrac{d}{dt}[A(t)x(t)]+B(t)x(t)=f(t,x(t)), \quad x(t_0)=x_0.$

Introduce the uniform mesh $\omega_h=\{t_i=t_0+ih,\, i=0,\ldots,N,\, t_N=T\}$ on $[t_0,T]$ with the step size $h=(T-t_0)/N$. The values of an approximate solution at the points t_i are denoted by x_i, $i=0,\ldots,N$.

Theorem (on the convergence of the method). Let the conditions of Theorem 1 or 2 be satisfied and, additionally, the operator $\Phi_{t,P_1(t)z_*,P_2(t)u_*}=\Phi_{t,P_1(t)z_*}(P_2(t)u_*)\colon X_2(t)\to Y_2(t)$, which is defined by the formula (12) for each (fixed) t, $x^*_{p_1}(t)=P_1(t)z_*$, $x^*_{p_2}(t)=P_2(t)u_*$, be invertible for $(t,P_1(t)z_*+P_2(t)u_*)\in[t_0,T]\times\mathbb{R}^n$. If $A,B\in C^2([t_0,T],L(\mathbb{R}^n))$, $C_2\in C^2([t_0,T],(0,\infty))$, $f\in C^1([t_0,T]\times\mathbb{R}^n,\mathbb{R}^n)$ and the initial value x_0 are chosen so that the consistency condition $Q_2(t_0)\big[A'(t_0)P_1(t_0)x_0+B(t_0)x_0-f(t_0,x_0)\big]=0$ (i.e., $(t_0,x_0)\in L_{t_+}$) holds, then numerical method

$$z_0 = P_1(t_0)x_0, \quad u_0 = P_2(t_0)x_0, \tag{15}$$

$$\begin{aligned} z_{i+1} &= \Big(I + h\big[P_1'(t_i) - G^{-1}(t_i)Q_1(t_i)[A'(t_i) + B(t_i)]\big]P_1(t_i)\Big)z_i + \\ &+ h\,G^{-1}(t_i)Q_1(t_i)f\big(t_i,x_i\big), \end{aligned} \tag{16}$$

$$\begin{aligned} u_{i+1} &= u_i - \\ &- \left[I - G^{-1}(t_{i+1})Q_2(t_{i+1})\frac{\partial f}{\partial x}\big(t_{i+1},P_1(t_{i+1})z_{i+1} + P_2(t_{i+1})u_i\big)\,P_2(t_{i+1})\right]^{-1} \times \\ &\times \Big[u_i - G^{-1}(t_{i+1})Q_2(t_{i+1})\Big[f\big(t_{i+1},P_1(t_{i+1})z_{i+1} + P_2(t_{i+1})u_i\big) - \\ &- A'(t_{i+1})P_1(t_{i+1})z_{i+1}\Big]\Big], \end{aligned} \tag{17}$$

$$x_{i+1} = P_1(t_{i+1})z_{i+1} + P_2(t_{i+1})u_{i+1}, \quad i = 0,\ldots,N-1, \tag{18}$$

approximating the IVP (1), (3) on $[t_0,T]$, converges and has the first order of accuracy: $\max\limits_{0\le i\le N}\|x(t_i) - x_i\| = O(h)$, $h\to 0$ $\big(\max\limits_{0\le i\le N}\|z(t_i) - z_i\| = O(h)$, $\max\limits_{0\le i\le N}\|u(t_i) - u_i\| = O(h)$, $h\to 0\big)$.

Remark. If in Theorem we do not require the additional smoothness for f, A, B and C_2 (i.e., $f \in C([t_+,\infty)\times\mathbb{R}^n,\mathbb{R}^n)$, $\dfrac{\partial f}{\partial x} \in C([t_+,\infty)\times\mathbb{R}^n, L(\mathbb{R}^n))$, $A, B \in C^1([t_+,\infty), L(\mathbb{R}^n))$ and $C_2 \in C^1([t_+,\infty),(0,\infty))$), then the method (15)–(18) converges, but may not have the first order of accuracy: $\max\limits_{0\le i\le N} \|x(t_i) - x_i\| = o(1)$, $h \to 0$.

The model of a radio engineering device

A voltage source $\mathrm{e}(\mathrm{t})$,
nonlinear resistances φ, φ_0, ψ,
a nonlinear conductance h,
a linear resistance r,
a linear conductance g,
an inductance L and
a capacitance C are given.

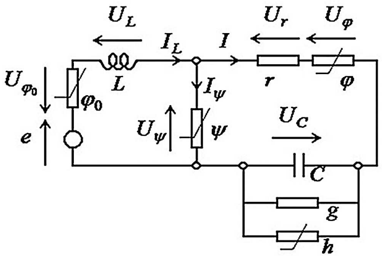

Fig. 1. The diagram of the electric circuit

Let $\mathrm{e}(\mathrm{t}) \in \mathrm{C}([0,\infty),\mathbb{R})$,
$\varphi(\mathrm{y}), \varphi_0(\mathrm{y}), \psi(\mathrm{y}), \mathrm{h}(\mathrm{y}) \in \mathrm{C}^1(\mathbb{R},\mathbb{R})$,
r, g, L, $\mathrm{C} > 0$.

The model of the circuit Fig. 1 is described by the system with the variables
$x_1 = I_L$, $x_2 = U_C$, $x_3 = I$:

$$L\frac{d}{dt}x_1 + x_2 + r\,x_3 = e(t) - \varphi_0(x_1) - \varphi(x_3), \quad (19)$$

$$C\frac{d}{dt}x_2 + g x_2 - x_3 = -h(x_2), \quad (20)$$

$$x_2 + r x_3 = \psi(x_1 - x_3) - \varphi(x_3). \quad (21)$$

The vector form of the system is the DAE

$$\frac{d}{dt}[Ax] + Bx = f(t,x), \qquad (22)$$

where $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$

$$A = \begin{pmatrix} L & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 & r \\ 0 & g & -1 \\ 0 & 1 & r \end{pmatrix}$$

$$f(t,x) = \begin{pmatrix} e(t) - \varphi_0(x_1) - \varphi(x_3) \\ -h(x_2) \\ \psi(x_1 - x_3) - \varphi(x_3) \end{pmatrix}$$

Lagrange stability of the model of a radio engineering device. The particular cases.

$$\varphi_0(y) = \alpha_1 y^{2k-1},\ \varphi(y) = \alpha_2 y^{2l-1},\ \psi(y) = \alpha_3 y^{2j-1},\ h(y) = \alpha_4 y^{2s-1}, \tag{23}$$
$$\varphi_0(y) = \alpha_1 y^{2k-1},\ \varphi(y) = \alpha_2 \sin y,\ \psi(y) = \alpha_3 \sin y,\ h(y) = \alpha_4 \sin y, \tag{24}$$

$k, l, j, s \in \mathbb{N}$, $\alpha_i > 0$, $i = \overline{1,4}$, $y \in \mathbb{R}$.

For each initial point (t_0, x^0) satisfying $x_2^0 + r x_3^0 = \psi(x_1^0 - x_3^0) - \varphi(x_3^0)$, there exists a unique global solution of the IVP (22), $x(t_0) = x^0$ $(x(t_0) = (I_L(t_0), U_C(t_0), I(t_0))^T)$ for the functions of the form (23), if $j \le k$, $j \le s$ and α_3 is sufficiently small, and for the functions of the form (24), if $\alpha_2 + \alpha_3 < r$.

If, additionally, $\sup\limits_{t\in[0,\infty)} |e(t)| < +\infty$ or $\int\limits_{t_0}^{+\infty} |e(t)|\,dt < +\infty$, then for the initial points (t_0, x^0) the DAE (22) is Lagrange stable (in both cases), i.e., every solution of the DAE is bounded. In particular, these requirements are fulfilled for voltages of the form

$$e(t) = \beta (t+\alpha)^{-n},\ e(t) = \beta e^{-\alpha t},\ e(t) = \beta e^{-\frac{(t-\alpha)^2}{\sigma^2}},\ e(t) = \beta \sin(\omega t + \theta), \tag{25}$$

where $\alpha > 0$, $\beta, \sigma, \omega \in \mathbb{R}$, $n \in \mathbb{N}$, $\theta \in [0, 2\pi]$.

[M.S. Filipkovska, Lagrange stability of semilinear differential-algebraic equations and application to nonlinear electrical circuits, *Journal of Mathematical Physics, Analysis, Geometry*, 2018]

Lagrange stability. The numerical solution

$L=500\cdot 10^{-6}$, $C=0.5\cdot 10^{-6}$, $r=2$, $g=0.2$, $t_0=0$, $x_0=(10,-10{,}5)^T$
$\varphi_0(y)=y^3$, $\varphi(y)=\sin y$, $\psi(y)=\sin y$, $h(y)=\sin y$, $e(t)=(2t+10)^{-2}$

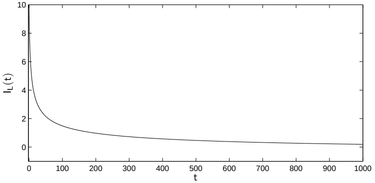

Fig. 2. The current $I_L(t)$

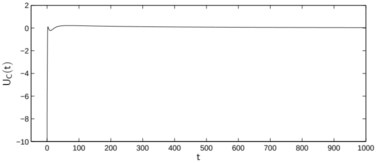

Fig. 3. The voltage $U_C(t)$

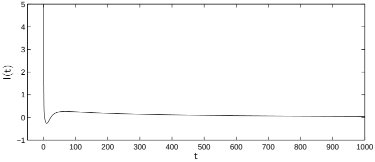

Fig. 4. The current $I(t)$

Lagrange stability. The numerical solution

$L = 500 \cdot 10^{-6}$, $C = 0.5 \cdot 10^{-6}$, $r = 2$, $g = 0.2$, $t_0 = 0$, $x_0 = (0,0,0)^T$,
$\varphi_0(y) = y^3$, $\varphi(y) = y^3$, $h(y) = y^3$, $\psi(y) = y^3$, $e(t) = 100\,e^{-t}\sin(5t)$

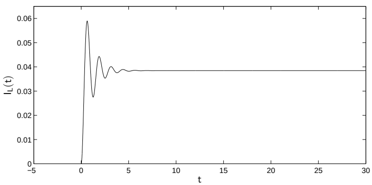

Fig. 5. The current $I_L(t)$

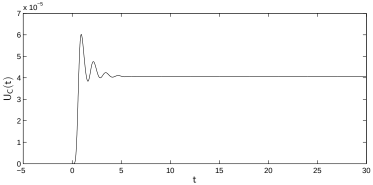

Fig. 6. The voltage $U_C(t)$

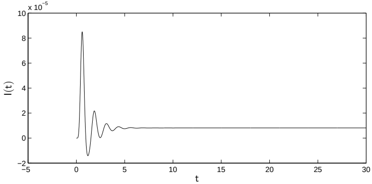

Fig. 7. The current $I(t)$

Lagrange stability. The numerical solution

$L = 300 \cdot 10^{-6}$, $C = 0.5 \cdot 10^{-6}$, $r = 2.6$, $g = 0.2$, $t_0 = 0$, $x_0 = (\pi/6, 0.5, 0)^T$, $\varphi_0(y) = y^3$, $\varphi(y) = \sin y$, $\psi(y) = \sin y$, $h(y) = \sin y$, $e(t) = 200\sin(0.5t) - 0.2$

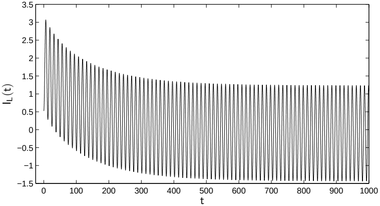

Fig. 8. The current $I_L(t)$

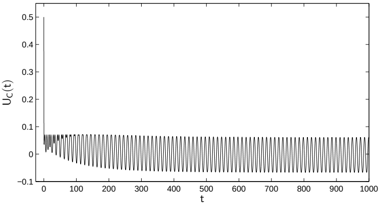

Fig. 9. The voltage $U_C(t)$

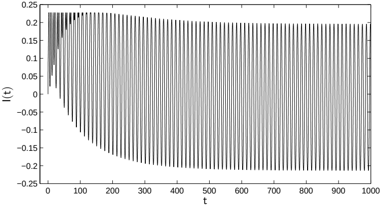

Fig. 10. The current $I(t)$

The global solution. The numerical solution

$L = 1000 \cdot 10^{-6}$, $C = 0.5 \cdot 10^{-6}$, $r = 2$, $g = 0.3$, $t_0 = 0$, $x^0 = (0,0,0)^T$
$\varphi_0(y) = y^3$, $\varphi(y) = y^3$, $\psi(y) = y^3$, $h(y) = y^3$, $e(t) = -t^2$

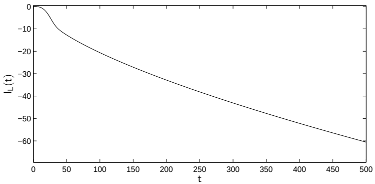

Fig. 11. The current $I_L(t)$

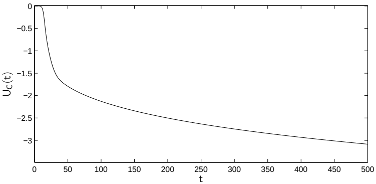

Fig. 12. The voltage $U_C(t)$

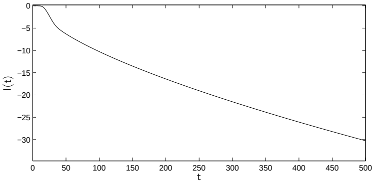

Fig. 13. The current $I(t)$

Lagrange instability of the radio engineering device model

Consider the system (19)–(21) with the nonlinear resistances and conductance

$$\varphi_0(y) = -y^2,\ \varphi(y) = y^3,\ \psi(y) = y^3,\ h(y) = y^2. \tag{26}$$

It is assumed that there exists $M_e = \sup\limits_{t\in[t_0,\infty)} |e(t)| < +\infty$. Choose

$$\Omega = \Big\{(x_1,x_2)^T \in \mathbb{R}^2 \mid x_1 > m_1, m_1 = \max\Big\{1+\sqrt{M_e},\ \sqrt[3]{g+r^{-1}}, 3CL^{-1},$$
$$\sqrt{\max\big\{3^{-1}(L(rC)^{-1}-r),0\big\}}\Big\}, x_2 < -rx_1 - x_1^3 - m_2, \tag{27}$$
$$m_2 = \max\big\{g - 2CL^{-1}r, 0\big\}\Big\}.$$

Then by Theorem 3 for any initial moment t_0 and any initial currents and voltage $I_L(t_0)$, $U_C(t_0)$, $I(t_0)$ satisfying $U_C(t_0) + rI(t_0) = \psi(I_L(t_0) - I(t_0)) - \varphi(I(t_0))$ and such that $(I_L(t_0), U_C(t_0))^T \in \Omega$ there exists a unique distribution of the currents and voltages in the circuit Fig. 1 only for $t_0 \le t < T$ ($[t_0, T)$ is some finite interval) and the currents and voltages are unbounded.

It means that there exists a unique solution of the Cauchy problem for the DAE (22) with the functions (26), $e(t)$ such that $\sup\limits_{t\in[t_0,\infty)} |e(t)| < +\infty$, and the initial condition $x(t_0) = (I_L(t_0), U_C(t_0), I(t_0))^T$, and this *solution has a finite escape time*.

Lagrange instability. The numerical solution

$L = 10 \cdot 10^{-6}$, $C = 0.5 \cdot 10^{-6}$, $r = 2$, $g = 0.2$,
$\varphi_0(x_1) = -x_1^2$, $\varphi(x_3) = x_3^3$, $h(x_2) = x_2^2$, $\psi(x_1 - x_3) = (x_1 - x_3)^3$, $e(t) = 2\sin t$,
$t_0 = 0$, $x_0 = (2.45, -20.625125, 2.5)^T$

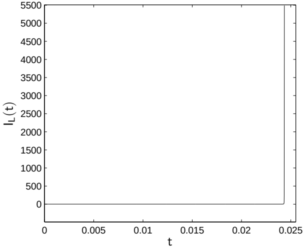

Fig. 14. The current $I_L(t)$

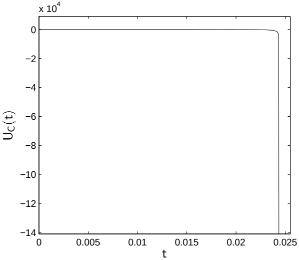

Fig. 15. The voltage $U_C(t)$

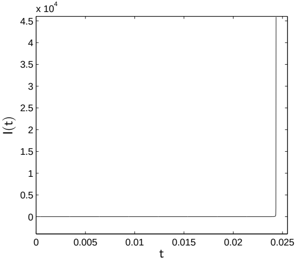

Fig. 16. The current $I(t)$

It is known that the dynamics of electrical circuits is modeled using DAEs which, in general, cannot be reduced to an explicit ODE.

The mathematical model of a time-varying nonlinear electrical circuit

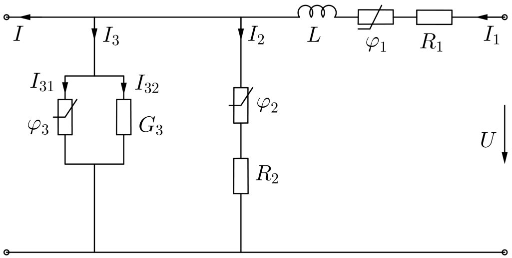

Fig. 17. The diagram of the electric circuit

A current $\mathrm{I}(\mathrm{t})$, a voltage $\mathrm{U}(\mathrm{t})$, resistances $\mathrm{R}_1(\mathrm{t})$, $\mathrm{R}_2(\mathrm{t})$, $\varphi_1(\mathrm{I}_1)$, $\varphi_2(\mathrm{I}_2)$, $\varphi_3(\mathrm{I}_{31})$, a conductance $\mathrm{G}_3(\mathrm{t})$, an inductance $\mathrm{L}(\mathrm{t})$ and a capacitance C are given.

A transient process in the electrical circuit (Fig. 17) is described by the system

$$\frac{\mathrm{d}}{\mathrm{dt}}[\mathrm{L}(\mathrm{t})\mathrm{I}_1(\mathrm{t})] + \mathrm{R}_1(\mathrm{t})\mathrm{I}_1(\mathrm{t}) = \mathrm{U}(\mathrm{t}) - \varphi_1(\mathrm{I}_1(\mathrm{t})) - \varphi_3(\mathrm{I}_{31}(\mathrm{t})), \tag{28}$$

$$\mathrm{I}_1(\mathrm{t}) - \mathrm{I}_{31}(\mathrm{t}) - \mathrm{I}_2(\mathrm{t}) = \mathrm{I}(\mathrm{t}) + \mathrm{G}_3(\mathrm{t})\varphi_3(\mathrm{I}_{31}(\mathrm{t})), \tag{29}$$

$$\mathrm{R}_2(\mathrm{t})\mathrm{I}_2(\mathrm{t}) = \varphi_3(\mathrm{I}_{31}(\mathrm{t})) - \varphi_2(\mathrm{I}_2(\mathrm{t})), \tag{30}$$

Denote $x_1(t)=I_1(t)$, $x_2(t)=I_{31}(t)$ and $x_3(t)=I_2(t)$.

The vector form of the system (28)–(30) is the time-varying semilinear DAE (1):

$$\frac{d}{dt}[A(t)x]+B(t)x=f(t,x),$$

where

$$x=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix},A(t)=\begin{pmatrix}L(t)&0&0\\0&0&0\\0&0&0\end{pmatrix},B(t)=\begin{pmatrix}R_1(t)&0&0\\1&-1&-1\\0&0&R_2(t)\end{pmatrix},$$

$$f(t,x)=\begin{pmatrix}U(t)-\varphi_1(x_1)-\varphi_3(x_2)\\I(t)+G_3(t)\varphi_3(x_2)\\\varphi_3(x_2)-\varphi_2(x_3)\end{pmatrix}.$$

The initial condition (3): $x(t_0)=x_0$, $x_0=(I_1(t_0),I_{31}(t_0),I_2(t_0))^T$.

It is assumed that the functions $L(t)$, $R_1(t)$, $R_2(t)$ and $G_3(t)$ are positive for all $t\in[t_+,\infty)$.

The projections $x_{p_j}(t)=P_j(t)x\in X_j(t)$ of a vector x have the form $x_{p_1}(t)=x_{p_1}=(x_1,x_1,0)^T$, $x_{p_2}(t)=x_{p_2}=(0,x_2-x_1,x_3)^T$.

Denote $z=x_1$, $u=x_2-x_1$, $w=x_3$, then $x_{p_1}=(z,z,0)^T$, $x_{p_2}=(0,u,w)^T$.

Using the introduced notation, the equations (29)–(30) can be rewritten as

$$w = -I(t) - u - G_3(t)\,\varphi_3(u+z), \tag{31}$$

$$u = \psi(t,z,u), \quad \text{where} \quad \psi(t,z,u) = -I(t) - \big(G_3(t) + R_2^{-1}(t)\big)\varphi_3(u+z) + \\ + R_2^{-1}(t)\,\varphi_2\big(-I(t) - u - G_3(t)\varphi_3(u+z)\big). \tag{32}$$

By **Theorem 1** for each initial point $(t_0,x_0) \in [t_+,\infty) \times \mathbb{R}^3$ satisfying the algebraic equations (i.e., $(t_0,x_0) \in L_{t_+}$)

$$x_1 - x_2 - x_3 = I(t) + G_3(t)\,\varphi_3(x_2), \tag{33}$$

$$R_2(t)\,x_3 = \varphi_3(x_2) - \varphi_2(x_3), \tag{34}$$

there exists a unique global solution $x(t)$ of the IVP (1), (3) if
$L, R_1, R_2 \in C^1([t_+,\infty),\mathbb{R})$, $I, U, G_3 \in C([t_+,\infty),\mathbb{R})$, $\varphi_j \in C^1(\mathbb{R})$, $j = 1,2,3$;
$L(t) > 0$, $R_1(t) > 0$, $R_2(t) > 0$, $G_3(t) > 0$ for all $t \in [t_+,\infty)$;

1) for each $t \in [t_+,\infty)$ and each $z \in \mathbb{R}$ there exists a unique $u \in \mathbb{R}$ satisfying the equality (32);

2) for each $t_* \in [t_+,\infty)$, $z_* \in \mathbb{R}$ and each $u_*, w_* \in \mathbb{R}$ satisfying the equalities (31), (32), one has the relation

$$\varphi_3'(u_* + z_*) + \big[\varphi_2'(w_*) + R_2(t_*)\big]\big[1 + G_3(t_*)\,\varphi_3'(u_* + z_*)\big] \neq 0; \tag{35}$$

3) there exists $R > 0$ such that $-\big(\varphi_1(z) + \varphi_3(u+z)\big)z \le R_1(t)z^2$ for all $t \in [t_+,\infty)$, $u, w \in \mathbb{R}$, $z \in \mathbb{R}$, $|z| \ge R$, satisfying the equalities (31), (32).

A similar assertion takes place according to **Theorem 2**, if the above conditions are satisfied with the following changes: the condition 1) does not contain the requirement that u be unique; the condition 2) is replaced by the following: 2*) for each $t_* \in [t_+, \infty)$, $z_* \in \mathbb{R}$ and each $u_*^j, w_*^j \in \mathbb{R}$, $j = 1,2$, satisfying the equalities (31), (32), the relation

$$\varphi_3'(u_2 + z_*) + \left[\varphi_2'(w_2) + R_2(t_*)\right]\left[1 + G_3(t_*)\,\varphi_3'(u_1 + z_*)\right] \neq 0$$

holds for any $u_k \in [u_*^1, u_*^2]$, $w_k \in [w_*^1, w_*^2]$, $k = 1,2$.

If, additionally, $\int\limits_{t_0}^{\infty} k(t)dt < \infty$, where $k(t) = 2L^{-1}(t)\,(|L'(t| + |U(t)|)$, the functions $I(t)$, $R_2^{-1}(t)$, $G_3(t)$ are bounded for all $t \in [t_+, \infty)$, and $\varphi_3(x_2)$, $\varphi_2(x_3)$ are bounded for $x_2 \in \mathbb{R}$ and $x_3 \in \mathbb{R}$ respectively, then *the DAE* (1) *is Lagrange stable* by **Theorem 3**.

The particular cases.

The the conditions 1), 2) are satisfied for the functions φ_2, φ_3 which are increasing (nondecreasing) on $\mathbb{R}$, for example,

$$\varphi_2(y) = a\,y^{2k-1},\, \varphi_3(y) = b\,y^{2m-1},\, \varphi_1(y) = c\,y^{2l-1}, \quad a,b,c > 0,\, k,m,l \in \mathbb{N}, \tag{36}$$

and if b is sufficiently small, $m \le l$, $\sup\limits_{t\in[t_+,\infty)} |I(t)| < \infty$ and $R_2(t) \ge K_0 = \mathrm{const} > 0$, $t \in [t_+,\infty)$, then the condition 3) is also fulfilled.

Note that in this case the mapping $\psi(t,z,u)$ is not globally contractive with respect to u. Obviously, the condition 1) is satisfied, if $\psi(t,z,u)$ is globally contractive with respect to u for any t, z, i.e., there exists a constant $\alpha < 1$ such that $\left|\psi(t,z,u_1) - \psi(t,z,u_2)\right| \le \alpha|u_1 - u_2|$ for any $t \in [t_+,\infty)$, $z \in \mathbb{R}$, $u_1, u_2 \in R$.

The numerical solution

$L(t) = 500$, $R_1(t) = e^{-t}$, $R_2(t) = 2 + e^{-t}$, $G_3(t) = (t+1)^{-1}$,
$I(t) = \sin t$, $U(t) = (t+1)^{-1}$,
$\varphi_i(y) = y^3$, $i = 1,2,3$, $\quad t_0 = 0$, $x_0 = (0,0,0)^T$

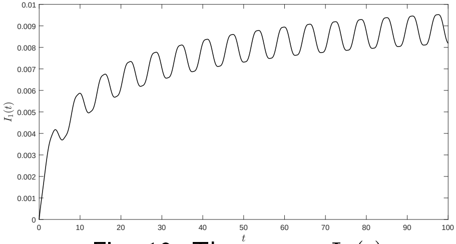

Fig. 18. The current $I_1(t)$

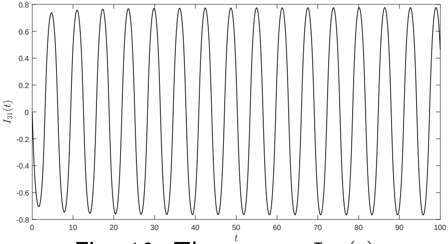

Fig. 19. The current $I_{31}(t)$

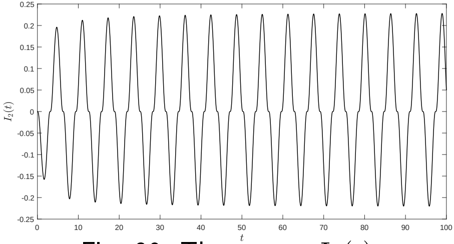

Fig. 20. The current $I_2(t)$

The numerical solution

$\mathrm{L}(\mathrm{t}) = 500\,(\mathrm{t}+1)^{-1}$, $\mathrm{R}_1(\mathrm{t}) = \mathrm{e}^{-\mathrm{t}}$, $\mathrm{R}_2(\mathrm{t}) = 2+\mathrm{e}^{-\mathrm{t}}$, $\mathrm{G}_3(\mathrm{t}) = (\mathrm{t}+1)^{-1}$,
$\mathrm{I}(\mathrm{t}) = (\mathrm{t}+1)^{-1} - 1$, $\mathrm{U}(\mathrm{t}) = (\mathrm{t}+1)^{-1}$,
$\varphi_{\mathrm{i}}(\mathrm{y}) = \mathrm{y}^3$, $\mathrm{i} = 1,2,3$, $\quad \mathrm{t}_0 = 0$, $\mathrm{x}_0 = (0,0,0)^{\mathrm{T}}$

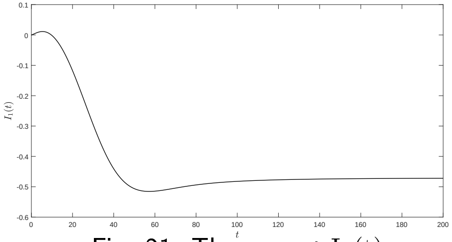

Fig. 21. The current $\mathrm{I}_1(\mathrm{t})$

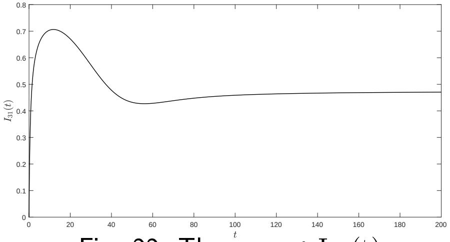

Fig. 22. The current $\mathrm{I}_{31}(\mathrm{t})$

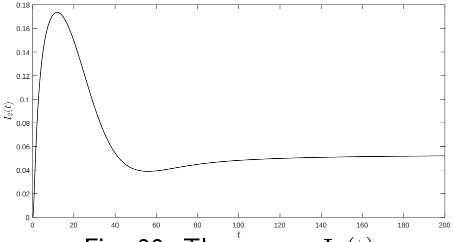

Fig. 23. The current $\mathrm{I}_2(\mathrm{t})$

A model for a gas pipeline flow

Consider a mathematical model for a gas pipeline flow (the flow on a single pipe), assuming that the temperature is identically equal to $\mathrm{T_0} = \mathrm{const.}$ The model consists of the **isothermal Euler equations**

$$\begin{aligned} &\rho_{\mathrm{t}} + (\rho \mathrm{v})_{\mathrm{x}} = 0, \\ &(\rho \mathrm{v})_{\mathrm{t}} + (\mathrm{p} + \rho \mathrm{v}^2)_{\mathrm{x}} = -\frac{\lambda}{2\mathrm{D}} \rho \mathrm{v}|\mathrm{v}| - \mathrm{g} \rho \mathrm{h}_{\mathrm{x}}, \end{aligned} \tag{ISO1}$$

and the **equation of state for a real gas** in the form

$$\mathrm{p} = \mathrm{R T_0} \rho \mathrm{z}(\mathrm{p}), \tag{ISGE}$$

- $\mathrm{x} \in [0,\mathrm{L}]$, $\mathrm{t} \in [0,\mathrm{t_1}) \subseteq [0,\infty]$, $\mathrm{L} < \infty$ is the length of the pipe
- $\rho = \rho(\mathrm{t,x})$, $\mathrm{v} = \mathrm{v}(\mathrm{t,x})$, $\mathrm{p} = \mathrm{p}(\mathrm{t,x})$ are respectively the density, velocity and pressure
- g is the gravitational constant, and R is the specific gas constant
- λ is the pipe friction coefficient, and D is the pipe diameter
- $\mathrm{h} = \mathrm{h}(\mathrm{x})$ is the height profile of the pipe over ground
- $\mathrm{z} = \mathrm{z}(\mathrm{p})$ is the compressibility factor

[P. Domschke, B. Hiller, J. Lang, V. Mehrmann, R. Morandin, C. Tischendorf. *Gas Network Modeling: An Overview*, 2021 (Preprint)]

Assuming $(\rho v^2)_x$ to be negligibly small and *introducing the variable* $\varphi = \rho v$ (the mass flow rate by cross section area), we obtain the **semilinear model** *for the isothermal Euler equations* with *the same gas state equation*:

$$\rho_t = -\varphi_x, \tag{37}$$
$$\varphi_t = -p_x - g\rho h_x - 0.5\lambda D^{-1}\varphi|\varphi|\rho^{-1}, \tag{38}$$
$$0 = -p + RT_0\rho z(p). \tag{39}$$

Denote $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -\frac{d}{dx} & 0 \\ -g\,h_x & 0 & -\frac{d}{dx} \\ 0 & 0 & -1 \end{pmatrix}$, $f(u) = \begin{pmatrix} 0 \\ -\frac{\lambda}{2D}\frac{\varphi|\varphi|}{\rho} \\ RT_0\rho z(p) \end{pmatrix}$ and $u = (\rho, \varphi, p)^T$. Then we can write the system (37)–(39) as:

$$A\frac{d}{dt}u(t) + Bu(t) = f(u(t)), \tag{40}$$

where $u = u(t)(x) = (\rho(t,x), \varphi(t,x), p(t,x))^T$, $x \in [0,L]$, $t \in [0,T] \subset [0,t_1)$. The initial condition has the form:

$$u(0) = u_0, \qquad u_0 = u_0(x) = (\rho(0,x), \varphi(0,x), p(0,x))^T, \quad x \in [0,L], \tag{41}$$

where $p(0,x)$ is chosen so as to satisfy the equation (39) for $t = 0$, $x \in [0,L]$. We will assume that $u(t,x)$ satisfies suitable boundary conditions, for example,

$$\varphi(t,0) = \varphi_l(t), \quad p(t,0) = p_l(t), \quad t \in [0,T], \tag{42}$$

i.e., $u(t)(0) = u_l(t) = (\rho(t,0), \varphi_l(t), p_l(t))^T$, where $\varphi_l(t)$ and $p_l(t)$ are given.

Consider the IVP for **time-invariant semilinear DAE**

$$\frac{d}{dt}[Au(t)] + Bu(t) = f(t,u(t)), \qquad t \in [t_0,T], \tag{43}$$

$$u(t_0) = u_0, \tag{44}$$

where $A, B \colon X \to Y$ are closed linear operators with the domains D_A, D_B, $D = D_A \cap D_B \neq \{0\}$, and X, Y are real Banach spaces.

Theorem [L.A. Vlasenko, Evolution models with implicit and degenerate differential equations, 2006]. Let the pencil $\lambda A + B$ satisfy (4), where $C_1 > 0$ and $C_2 > 0$ are constants. Assume that $f \in C([t_0,T] \times X, Y)$ satisfies the Lipschitz condition

$$\|f(t,u) - f(t,v)\| \leq M\|u - v\|, \quad u, v \in X, \tag{45}$$

where the constant M is such that

$$M\|Q_2\| \, \|G^{-1}| < 1. \tag{46}$$

Then for each initial point $u_0 \in D$ satisfying

$$Q_2Bu_0 = Q_2f(t_0,u_0),$$

there exists a unique global solution of the IVP (43), (44) on $[t_0,T]$.

Outlooks

1. The extension of the results to the case when X, Y are Banach spaces, $A(t)$, $B(t)$ $(t \in [t_+,\infty))$ are closed linear operators from X to Y with the domains $D_{A(t)}$, $D_{B(t)}$, $D = D_{A(t)} \cap D_{B(t)} \neq 0$.

2. $\lambda A(t) + B(t)$ is a regular pencil of **index** ν $(\nu \in \mathbb{N})$, i.e., there exist functions $C_1\colon [t_+,\infty) \to (0,\infty)$, $C_2\colon [t_+,\infty) \to (0,\infty)$ such that for every $t \in [t_+,\infty)$
$$\|R(\lambda,t)\| \le C_1(t)|\lambda|^{\nu-1}, \quad |\lambda| \ge C_2(t). \tag{47}$$

   [Rutkas A.G., Vlasenko L.A. Existence of solutions of degenerate nonlinear differential operator equations, *Nonlinear Oscillations*, 2001]

   Then for each $t \in [t_+,\infty)$ there exist the two pairs of mutually complementary projectors $P_j(t)$, $Q_j(t)$, $j = 1,2$, (7) which generate the direct decompositions of D and Y (5) such that the operators $A(t)$, $B(t)$ have the block representations (6), where $A_1^{-1}(t)$ and $B_2^{-1}(t)$ exist.

   In general, the order of pole of the resolvent $(A(t) + \mu B(t))^{-1}$ at the point $\mu = 0$ is called the *index* of the regular pencil $\lambda A(t) + B(t)$.

Example. Let $\nu = 2$ and $X = Y = D = \mathbb{R}^n$ or $\mathbb{C}^n$. Then there exist the direct decompositions

$$D_2(t) = D_{20}(t) \dot{+} D_{21}(t), \quad Y_2 = Y_{20}(t) \dot{+} Y_{21}(t) \quad (D_{20} = D \cap \operatorname{Ker} A) \tag{48}$$

that generate the two pairs of mutually complementary projectors $P_{2k}(t)\colon D \to X_j(t)$, $Q_{2k}(t)\colon Y \to Y_{2k}$, $k = 0,1$, such that the DAE (2) $A(t)x'(t) + B(t)x(t) = f(t,x(t))$ is reduced to the equivalent system

$$\begin{aligned}
&P_1(t)\frac{d}{dt}x(t) = -G^{-1}(t)B(t)P_1(t)x(t) + P_1(t)G^{-1}(t)f(t,x(t))],\\
&G^{-1}(t)A(t)P_{21}(t)\frac{d}{dt}x(t) = -P_{20}(t)x(t) + P_{20}(t)G^{-1}(t)f(t,x(t)),\\
&P_{21}(t)G^{-1}(t)f(t,x(t)) - P_{21}(t)x(t) = 0.
\end{aligned} \tag{49}$$

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Thank you for your attention!

Analysis of gas networks using mathematical models in the form of differential algebraic equations.

Part II

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Consider **implicit ordinary differential equations** (ODEs) of the form

$$\frac{d}{dt}[A(t)x(t)] + B(t)x(t) = f(t,x(t)), \qquad t \in [t_+,\infty), \tag{1}$$

$$A(t)\frac{d}{dt}x(t) + B(t)x(t) = f(t,x(t)), \tag{2}$$

where $t_+ \geq 0$, $A(t), B(t)$ $(t \in [t_+,\infty))$ are closed linear operators from X to Y with the domains $D_{A(t)}$, $D_{B(t)}$, $D = D_{A(t)} \cap D_{B(t)} \neq \{0\}$, X, Y are Banach spaces, $f\colon [t_+,\infty) \times X \to Y$.

The time-varying operators $A(t)$, $B(t)$ can be degenerate.

The differential equations (DEs) (1) and (2) with a degenerate (for some t) operator $A(t)$ are called **time-varying (nonautonomous) degenerate DEs** or **time-varying differential-algebraic equations (DAEs)**. In the terminology of DAEs, equations of the form (1), (2) are commonly referred to as **semilinear**.

We study the initial value problem (the Cauchy problem) for the DAEs (1), (2) with the initial condition

$$x(t_0) = x_0. \tag{3}$$

Fields of application of the theory of DAEs are control theory, radioelectronics, cybernetics, mechanics, robotics technology, economics, ecology and chemical kinetics.

In particular, semilinear DAEs are used in modelling

- transient processes in electrical circuits
- gas flow in networks
- dynamics of neural networks
- dynamics of complex mechanical and technical systems (e.g., robots)
- multi-sectoral economic models
- kinetics of chemical reactions

Notice that any type of a PDE can be represented as a DAE in infinite-dimensional spaces (an abstract DAE) and, possibly, a complementary boundary condition.

Model of a gas flow for a single pipe

We consider the mathematical model of a gas pipeline which consists of the **isothermal Euler equations** of the form

$$\rho_t = -\varphi_x, \tag{4}$$
$$\varphi_t = -p_x - g\rho h_x - 0.5\lambda D^{-1}\varphi|\varphi|\rho^{-1} \tag{5}$$

and the **equation of state for a real gas** in the form

$$p = RT_0\rho z(p), \tag{6}$$

- $x \in [0,L]$, $t \in [0,t_1) \subseteq [0,\infty]$, where $[t_0,t_1)$ is the time interval, $L < \infty$ is the pipe length and T_0 is the temperature
- $\rho = \rho(t,x)$, $\varphi = \varphi(t,x)$ ($\varphi := \rho v$, v is the velocity) and $p = p(t,x)$ are respectively the density, flow rate and pressure
- g is the gravitational constant, and R is the specific gas constant
- λ is the pipe friction coefficient, and D is the pipe diameter
- $h = h(x)$ is the height profile of the pipe over ground
- $z = z(p)$ is the compressibility factor

[P. Benner, S. Grundel, C. Himpe, C. Huck, T. Streubel, C. Tischendorf. *Gas Network Benchmark Models*, 2018]

Denote $A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$, $B=\begin{pmatrix}0&-\frac{d}{dx}&0\\-g\,h_x&0&-\frac{d}{dx}\\0&0&-1\end{pmatrix}$, $f(u)=\begin{pmatrix}0\\-\frac{\lambda}{2D}\frac{\varphi|\varphi|}{\rho}\\RT_0\rho z(p)\end{pmatrix}$ and $u=(\rho,\varphi,p)^T$. Then we can write the system (4)–(6) as:

$$A\frac{d}{dt}u(t)+Bu(t)=f(u(t)), \tag{7}$$

where $u=u(t)(x)=(\rho(t,x),\varphi(t,x),p(t,x))^T$, $x\in[0,L]$, $t\in[0,T]\subset[0,t_1)$. The initial condition has the form:

$$u(0)=u_0,\qquad u_0=u_0(x)=(\rho(0,x),\varphi(0,x),p(0,x))^T,\quad x\in[0,L], \tag{8}$$

where $p(0,x)$ is chosen so as to satisfy the equation (6) for $t=0$, $x\in[0,L]$. We will assume that $u(t,x)$ satisfies suitable boundary conditions, for example,

$$\varphi(t,0)=\varphi_l(t),\quad p(t,0)=p_l(t),\quad t\in[0,T], \tag{9}$$

i.e., $u(t)(0)=u_l(t)=(\rho(t,0),\varphi_l(t),p_l(t))^T$, where $\varphi_l(t)$ and $p_l(t)$ are given.

Then we consider the IVP (1), (3), where $X=Y=L_2$ and $A(t),B(t)\colon H_0^1[0,L]=\{u(x)\in H^1[0,L]\mid u(t)(0)=u_l(t)\}\to L_2$ for each $t\in[0,t_1)$.

[P. Benner, S. Grundel, C. Himpe, C. Huck, T. Streubel, C. Tischendorf. *Gas Network Benchmark Models*, 2018]

[Azevedo-Perdicoulis, T.-P., Jank, G. Modelling aspects of describing a gas network through a DAE system. 2007]

The gas network is considered to be described by a connected finite graph $\mathrm{G} = (\mathrm{V},\mathrm{E},\psi)$, where V denotes a set of vertices with $|\mathrm{V}| = \mathrm{n}$, and E denotes a set of edges with $|\mathrm{E}| = \mathrm{m}$. The mapping $\psi\colon \mathrm{E} \to \mathrm{V} \times \mathrm{V}$ is called the incidence map, where $\psi_1(\mathrm{e}) = \mathrm{v}_1$ is the initial vertex and $\psi_2(\mathrm{e}) = \mathrm{v}_2$ is the final vertex.

We define a flow $\varphi\colon \mathrm{E} \to \mathbb{R}$ and a pressure-drop $\mathrm{p}\colon \mathrm{E} \to \mathbb{R}$ on every edge, and a nodal-pressure $\Phi\colon \mathrm{V} \to \mathbb{R}$ and a nodal-flow $\mathrm{F}\colon \mathrm{V} \to \mathbb{R}$ on every vertex.

The Kirchhoff First Law (KFL) says that the flow rate vanishes at any vertex of the graph: *The Kirchhoff Second Law* (KSL) says that the pressure drop vanishes at every fundamental circuit of the graph.

Notice that the network may comprise valves, reservoirs, compressor stations, supplying sources, and regulators.

The isothermal Euler equations are linearised around the operation levels (p_*,φ_*), whence we set $\mathrm{p} = \mathrm{p}_* + \Delta\mathrm{p}$ and $\varphi = \varphi_* + \Delta\varphi$, with $\Delta\mathrm{p}$ and $\Delta\varphi$ as the deviations from the pressure-drop and flow, respectively, from the reference values p_* and φ_*.

Discretizing the gas net with respect to the space variable x, we obtain the gas network model in the form of a DAE of a type

$$\frac{\partial}{\partial \mathrm{t}}(\mathrm{Au}) = \mathrm{Bu} + \mathrm{f}(\mathrm{t},\mathrm{u}), \tag{10}$$

where $\mathrm{u} = (\Delta\varphi, \Phi, \mathrm{u_q}, \mathrm{u_p})^{\mathrm{T}}$, $\mathrm{u_q}$ denotes a control device controlling the flow in some edges by a "flow", $\mathrm{u_p}$ denotes a control device controlling the flow in some edges by a "pressure" and A is a certain degenerate matrix. The gas network model also may include the parameters denoting control devices controlling the pressure in the edges or nodes either through flow or pressure.

Assume that the characteristic operator pencil $\lambda A(t)+B(t)$ ($\lambda \in \mathbb{C}$ is a parameter), associated with the linear part of the DAE (1) or (2), is a **regular pencil of index not higher than 1**: for each $t \geq t_+$ the pencil $\lambda A(t)+B(t)$ be regular and there exist functions $C_1\colon [t_+,\infty) \to (0,\infty)$, $C_2\colon [t_+,\infty) \to (0,\infty)$ such that for every $t \in [t_+,\infty)$ the pencil resolvent $R(\lambda,t)=(\lambda A(t)+B(t))^{-1}$ satisfies the constraint

$$\|R(\lambda,t)\| \leq C_1(t), \quad |\lambda| \geq C_2(t). \tag{11}$$

Then for each $t \in [t_+,\infty)$, there exist the two pairs of mutually complementary projectors

$$P_j(t)\colon D \to D_j(t) \text{ and } Q_j(t)\colon Y \to Y_j(t),\ \ j=1,2,$$

which generate the direct decompositions

$$D = D_1(t)\dot{+}D_2(t), \quad Y = Y_1(t)\dot{+}Y_2(t) \quad \text{such that} \tag{12}$$

the pair of subspaces $X_1(t)$, $Y_1(t)$ and $X_2(t)$, $Y_2(t)$ are invariant under the operators $A(t)$, $B(t)$, and $A_j(t)=A(t)|_{D_j(t)}$, $B_j(t)=B(t)|_{D_j(t)}\colon D_j(t) \to Y_j(t)$, $j=1,2$, are such that $A_2(t)=0$, and there exist $A_1^{-1}(t)$ and $B_2^{-1}(t)$ if $D_1(t) \neq \{0\}$, $D_2(t) \neq \{0\}$ respectively ($D_2(t)=\operatorname{Ker} A(t) \cap D$, $Y_1(t)=A(t)D$)

$$A(t)=A_1(t)\dot{+}A_2(t),\ B(t)=B_1(t)\dot{+}B_2(t)\colon D_1(t)\dot{+}D_2(t) \to Y_1(t)\dot{+}Y_2(t) \tag{13}$$

[Rutkas A.G., Vlasenko L.A. Existence of solutions of degenerate nonlinear differential operator equations, *Nonlinear Oscillations*, 2001]

For each $t \in [t_+,\infty)$ the projectors can be determined by the formulas [Rutkas A.G., Vlasenko L.A. *Nonlinear Oscillations*, 2001]

$$\begin{aligned} P_1(t) &= \frac{1}{2\pi i} \oint\limits_{|\lambda|=C_2(t)} R(\lambda,t)\,A(t)\,d\lambda, \quad P_2(t) = I_X - P_1(t), \\ Q_1(t) &= \frac{1}{2\pi i} \oint\limits_{|\lambda|=C_2(t)} A(t)\,R(\lambda,t)\,d\lambda, \quad Q_2(t) = I_Y - Q_1(t). \end{aligned} \tag{14}$$

and the auxiliary operator $G(t) = A(t) + B(t)P_2(t)\colon D \to Y$ has the bounded inverse $G^{-1}(t) \in (Y,X)$.

Let $X = Y = D = \mathbb{R}^n$.

For each t any $x \in \mathbb{R}^n$ can be uniquely represented in the form

$$x = x_{p_1}(t) + x_{p_2}(t), \quad x_{p_i}(t) = P_i(t)x \in X_i(t).$$

The DAE (1) $[A(t)x(t)]' + B(t)x(t) = f(t,x(t))$ is reduced to the equivalent system

$$[P_1(t)x(t)]' = \big[P_1'(t) - G^{-1}(t)Q_1(t)[A'(t) + B(t)]\big]P_1(t)x(t) + G^{-1}(t)Q_1(t)f(t,x(t)),$$
$$G^{-1}(t)Q_2(t)[f(t,x(t)) - A'(t)P_1(t)x(t)] - P_2(t)x(t) = 0 \qquad \text{or}$$

$$x_{p_1}'(t) = \big[P_1'(t) - G^{-1}(t)Q_1(t)[A'(t) + B(t)]\big]x_{p_1}(t) + G^{-1}(t)Q_1(t)f(t,x), \tag{15}$$
$$G^{-1}(t)Q_2(t)[f(t,x_{p_1}(t) + x_{p_2}(t)) - A'(t)x_{p_1}(t)] - x_{p_2}(t) = 0. \tag{16}$$

Introduce the manifold

$$L_{t_+} = \{(t,x) \in [t_+,\infty) \times \mathbb{R}^n \mid Q_2(t)[B(t)x + A'(t)P_1(t)x - f(t,x)] = 0\}. \tag{17}$$

The consistency condition $(t_0,x_0) \in L_{t_+}$ for the initial point (t_0,x_0) is one of the necessary conditions for the existence of a solution of the initial value problem (1), (3).

$V'_{(15)}(t,x_{p_1}(t)) = \frac{\partial V}{\partial t}(t,x_{p_1}(t)) + \Big(\frac{\partial V}{\partial z}(t,x_{p_1}(t)), \big[P_1'(t) - G^{-1}(t)Q_1(t)[A'(t) + B(t)]\big]x_{p_1}(t) + G^{-1}(t)Q_1(t)f(t,x_{p_1}(t) + x_{p_2}(t))\Big)$ is the derivative of the function $V(t,z)$ along the trajectories of the equation (15), where $V(t,z)$ is a continuously differentiable and positive definite scalar function.

Main results:

- **Theorems on the existence and uniqueness of global solutions**

  Some advantages: the restrictions of the type of the global Lipschitz condition (including contractive mapping) are not used.
- **Theorem on the Lagrange stability of the DAE** (the boundedness of solutions)
- **Theorem on the Lagrange instability of the DAE** (solutions have finite escape time)
- **Theorem on the ultimate boundedness (dissipativity) of the DAE** (the ultimate boundedness of solutions)
- **Theorems on the Lyapunov stability and instability of the equilibrium state of the DAE**
- **Theorems on asymptotic stability and asymptotic stability in the large of the equilibrium state** (complete stability of the DAE)
- **Numerical methods**

The obtained theorems were used for the study of certain mathematical models of electrical circuits with nonlinear and time-varying elements.

The IVP (1), (3): $\frac{d}{dt}[A(t)x(t)] + B(t)x(t) = f(t,x(t)), \quad x(t_0) = x_0.$

Definitions

Let $f(t,0) \equiv 0$ and $f\colon [t_+,\infty) \times U_R^x(0) \to \mathbb{R}^n$, where

$$U_R^x(0) = \{x \in \mathbb{R}^n \mid \|x\| < R\}.$$

An *equilibrium position* (a *stationary solution*) $x_*(t) \equiv 0$ of the DAE (1) ($f(t,0) \equiv 0$) is called **Lyapunov stable**, or simply **stable**, if for each $\varepsilon > 0$ ($\varepsilon < R$) and each $t_0 \in [t_+,\infty)$ there exists a number $\delta = \delta(\varepsilon, t_0) > 0$ ($\delta \le \varepsilon$) such that for any consistent initial point (t_0,x_0) satisfying the condition $\|x_0\| < \delta$ there exists a global solution $x(t)$ of the IVP (1), (3) and this solution satisfies the inequality $\|x(t)\| < \varepsilon$ for all $t \in [t_0,\infty)$.
If, in addition, there exists $\widetilde{\delta} = \widetilde{\delta}(t_0) > 0$ ($\widetilde{\delta} \le \delta$) such that for each solution $x(t)$ with an initial point (t_0,x_0) satisfying the condition $\|x_0\| < \widetilde{\delta}$ the requirement $\lim\limits_{t\to\infty} x(t) = 0$ is fulfilled, then $x_*(t) \equiv 0$ is called **asymptotically Lyapunov stable**, or simply **asymptotically stable**.

If in the previous definition the number δ is independent of t_0, then the solution is called **uniformly Lyapunov stable** or **uniformly stable** (on $[t_+,\infty)$).

An *equilibrium position* $x_*(t) \equiv 0$ of the DAE (1) $(f(t,0) \equiv 0)$ is called **Lyapunov unstable**, or simply **unstable**, if for some $\varepsilon > 0$ $(\varepsilon < R)$, $t_0 \in [t_+,\infty)$ and any $\delta > 0$ there exist a solution $x_\delta(t)$ of the IVP (1), (3) and a time moment $t_1 > t_0$ such that $\|x_0\| < \delta$ and $\|x_\delta(t_1)\| \geq \varepsilon$.

Let $f(t,0) \equiv 0$ and $f\colon [t_+,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$.
If the equilibrium position $x_*(t) \equiv 0$ of the DAE (1) $(f(t,0) \equiv 0)$ is asymptotically stable and, moreover, for each point $(t_0,x_0) \in L_{t_+}$ there exists a global solution $x(t)$ of the IVP (1), (3) and $\lim\limits_{t\to\infty} x(t) = 0$, then $x_*(t) \equiv 0$ is called **asymptotically stable in the large**, and the DAE is called **completely stable** or **asymptotically stable**.

Remark. Since the Lagrange instability of a solution implies its Lyapunov instability, the theorems on the Lagrange instability of DAEs can also be treated as Lyapunov instability theorems.

Asymptotic stability in the large for explicit ODEs was considered in [Krasovsky N.N. *Some problems of the theory of stability of motion*, 1959].

$$B_{r_1}(0)=\{z\in\mathbb{R}^n \mid \|z\|\le r_1\},$$
$$B_{r_1,r_2}^{x_{p_1},x_{p_2}}(0)=\{x\in\mathbb{R}^n \mid \|x_{p_i}(t)\|\le r_i,\, x_{p_i}(t)=P_i(t)x, i=1,2\}. \tag{18}$$

Theorem 6 (Lyapunov stability and asymptotic stability of equilibrium position of the DAE). Let $f\in C([t_+,\infty)\times U_R^x(0),\mathbb{R}^n)$, $f(t,0)\equiv 0$, $\partial f/\partial x\in C([t_+,\infty)\times U_R^x(0),L(\mathbb{R}^n))$, $A,B\in C^1([t_+,\infty),L(\mathbb{R}^n))$, and the pencil $\lambda A(t)+B(t)$ satisfy (11), where $C_2\in C^1([t_+,\infty),(0,\infty))$. Assume that for each $t_*\in[t_+,\infty)$ and $x_{p_1}^*(t_*)=0$, $x_{p_2}^*(t_*)=0$ the operator
$\Phi_{t_*,x_{p_1}^*(t_*),x_{p_2}^*(t_*)}\colon X_2(t_*)\to Y_2(t_*)$,
$\Phi_{t_*,x_{p_1}^*(t_*),x_{p_2}^*(t_*)}=\left[\frac{\partial}{\partial x}\left[Q_2(t_*)f(t_*,x_{p_1}^*(t_*)+x_{p_2}^*(t_*))\right]-B(t_*)\right]P_2(t_*)$, is invertible. Then the following statements are true:

1. Let there exist numbers $r_1,r_2>0$, $r_1+r_2<R$, and a positive definite function $V\in C^1([t_+,\infty)\times B_{r_1}(0),\mathbb{R})$ such that for all $t\in[t_+,\infty)$, $x\in B_{r_1,r_2}^{x_{p_1},x_{p_2}}(0)$ the following inequality holds:

$$V'_{(15)}(t,x_{p_1}(t))\le 0. \tag{19}$$

Then the equilibrium position $x_*(t)\equiv 0$ of the DAE (1) is Lyapunov stable.

2. Let there exist numbers $r_1, r_2 > 0$, $r_1 + r_2 < R$, and positive definite functions $V \in C^1([t_+, \infty) \times B_{r_1}(0), \mathbb{R})$, $W \in C(B_{r_1}(0), \mathbb{R})$, and $U \in C(B_{r_1}(0), \mathbb{R})$ such that $V(t,z) \le W(z)$ for all $t \in [t_+, \infty)$, $z \in B_{r_1}(0)$, and

$$V'_{(15)}(t, x_{p_1}(t)) \le -U\big(x_{p_1}(t)\big) \tag{20}$$

for all $t \in [t_+, \infty)$, $x \in B_{r_1, r_2}^{x_{p_1}, x_{p_2}}(0)$, $x_{p_1}(t) \ne 0$;
also, let

$$G^{-1}(t)Q_2(t)[f(t, P_1(t)x + P_2(t)x) - A'(t)P_1(t)x] \to 0$$
$$\text{as } x \to 0 \text{ uniformly in } t \text{ on } [T, \infty) \text{ for some } T > t_+. \tag{21}$$

Then the equilibrium position $x_*(t) \equiv 0$ of the DAE (1) is asymptotically Lyapunov stable.

Theorem 7 (Lyapunov instability of equilibrium position of the DAE).
Let $f \in C([t_+,\infty)\times U_R^x(0),\mathbb{R}^n)$, $f(t,0)\equiv 0$, $\partial f/\partial x \in C([t_+,\infty)\times U_R^x(0),L(\mathbb{R}^n))$, $A,B \in C^1([t_+,\infty),L(\mathbb{R}^n))$, and the pencil $\lambda A(t)+B(t)$ satisfy (11), where $C_2 \in C^1([t_+,\infty),(0,\infty))$. Assume that for each $t_* \in [t_+,\infty)$ and $x_{p_1}^*(t_*)=0$, $x_{p_2}^*(t_*)=0$ the operator $\Phi_{t_*,x_{p_1}^*(t_*),x_{p_2}^*(t_*)}\colon X_2(t_*)\to Y_2(t_*)$, $\Phi_{t_*,x_{p_1}^*(t_*),x_{p_2}^*(t_*)} = \Big[\frac{\partial}{\partial x}\big[Q_2(t_*)f(t_*,x_{p_1}^*(t_*)+x_{p_2}^*(t_*))\big]-B(t_*)\Big]P_2(t_*)$, is invertible. Let there exist numbers $T\ge t_+$ and $r_1,r_2>0$, $r_1+r_2<R$, and a function $V\in C^1([T,\infty)\times B_{r_1}(0),\mathbb{R})$ such that

1. $V(t,z)\to 0$ uniformly in t on $[T,\infty)$ as $\|z\|\to 0$;
2. there exists a positive function $U\in C(B_{r_1}(0),[0,\infty))$ such that

$$V'_{(15)}(t,x_{p_1}(t)) \ge U\big(x_{p_1}(t)\big) > 0 \quad \text{or} \quad V'_{(15)}(t,x_{p_1}(t)) \le -U\big(x_{p_1}(t)\big) < 0 \qquad (22)$$

for all $t\in[T,\infty)$, $x\in B_{r_1,r_2}^{x_{p_1},x_{p_2}}(0)$, $x_{p_1}(t)\neq 0$;

3. for each $\Delta_1>0$ and each $\Delta_2>0$, $\Delta_i\le r_i$, there exist $x_{p_1}(T)\neq 0$ and $x_{p_2}(T)$ such that $\|x_{p_i}(T)\|<\Delta_i$, $i=1,2$, and $V(T,x_{p_1}(T))\,V'_{(15)}(T,x_{p_1}(T))>0$ (i.e., the sign of the function V coincides with the sign of the derivative $V'_{(15)}$ at $(T,x_{p_1}(T))$).

Then the equilibrium position $x_*(t)\equiv 0$ of the DAE (1) is Lyapunov unstable.

Theorem 8 (asymptotic stability in the large or complete stability of the DAE). Let $f \in C([t_+,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$, $f(t,0) \equiv 0$, $\partial f/\partial x \in C([t_+,\infty) \times \mathbb{R}^n, L(\mathbb{R}^n))$, $A, B \in C^1([t_+,\infty), L(\mathbb{R}^n))$, the pencil $\lambda A(t) + B(t)$ satisfy (11), where $C_2 \in C^1([t_+,\infty),(0,\infty))$. Let the conditions 1), 2) of Theorem 1 or 1), 2) of Theorem 2 (on the global solvability), as well as condition (21), be satisfied. Assume that there exist positive definite functions $V \in C^1([t_+,\infty) \times \mathbb{R}^n, \mathbb{R})$, $W \in C(\mathbb{R}^n, \mathbb{R})$ and $U \in C(\mathbb{R}^n, \mathbb{R})$ such that

1. $V(t,z) \le W(z)$ for all $t \in [t_+,\infty)$, $z \in \mathbb{R}^n$;
2. $V(t,z) \to \infty$ uniformly in t on $[t_+,\infty)$ as $\|z\| \to \infty$;
3. for all $(t, x_{p_1}(t) + x_{p_2}(t)) \in L_{t_+}$, $x_{p_1}(t) \ne 0$ $\big(x_{p_i}(t) = P_i(t)x$, $i = 1,2\big)$, the inequality (20) holds.

Then the equilibrium position $x_*(t) \equiv 0$ of the DAE (1) is asymptotically stable in the large (the DAE (1) is completely stable).

The conditions 1), 2) of Theorem 1 (on the global solvability):

1) for each $t \in [t_+,\infty)$ and each $x_{p_1}(t) \in X_1(t)$ there exists a unique $x_{p_2}(t) \in X_2(t)$ such that $(t,x_{p_1}(t)+x_{p_2}(t)) \in L_{t_+}$;

2) for each $t_* \in [t_+,\infty)$, $x^*_{p_i}(t_*) \in X_i(t_*)$, $i=1,2$, such that $(t_*,x^*_{p_1}(t_*)+x^*_{p_2}(t_*)) \in L_{t_+}$ the operator $\Phi_{t_*,x^*_{p_1}(t_*),x^*_{p_2}(t_*)}\colon X_2(t_*) \to Y_2(t_*)$, $\Phi_{t_*,x^*_{p_1}(t_*),x^*_{p_2}(t_*)} = \Big[\frac{\partial}{\partial x}\big[Q_2(t_*)f(t_*,x^*_{p_1}(t_*)+x^*_{p_2}(t_*))\big] - B(t_*)\Big] P_2(t_*)$, is invertible;

The conditions 1), 2) of Theorem 2 (on the global solvability):

1) for each $t \in [t_+,\infty)$, $x_{p_1}(t) \in X_1(t)$ there exists $x_{p_2}(t) \in X_2(t)$ such that $(t,x_{p_1}(t)+x_{p_2}(t)) \in L_{t_+}$;

2) for each $t_* \in [t_+,\infty)$, $x^*_{p_1}(t_*) \in X_1(t_*)$, $x^i_{p_2}(t_*) \in X_2(t_*)$ such that $(t_*,x^*_{p_1}(t_*)+x^i_{p_2}(t_*)) \in L_{t_+}$, $i=1,2$, the operator function $\Phi_{t_*,x^*_{p_1}(t_*)}(x_{p_2}(t_*))$ defined by $\Phi_{t_*,x^*_{p_1}(t_*)}\colon X_2(t_*) \to L(X_2(t_*),Y_2(t_*)), \Phi_{t_*,x^*_{p_1}(t_*)}(x_{p_2}(t_*)) = \Big[\frac{\partial}{\partial x}\big[Q_2(t_*)f(t_*,x^*_{p_1}(t_*)+x_{p_2}(t_*))\big] - B(t_*)\Big] P_2(t_*)$, is basis invertible on $[x^1_{p_2}(t_*),x^2_{p_2}(t_*)]$.

Outlooks

1. It is planned to extend the obtained results to the case when X, Y are Banach spaces, $f\colon [t_+,\infty)\times X \to Y$ and $A(t), B(t)\colon X \to Y$ ($t \in [t_+,\infty)$) are closed linear operators with the domains $D_{A(t)}$, $D_{B(t)}$, $D = D_{A(t)} \cap D_{B(t)} \neq \{0\}$.

   In this case we will require that $A(t)$, $B(t)$ be strongly continuously differentiable on $[t_+,\infty)$ (i.e., for each $d \in D$ the functions $A(t)d$, $B(t)d$ be continuously differentiable on $[t_+,\infty)$).

2. It is planned to consider semilinear time-varying DAEs of index higher than 1.

   A regular pencil $\lambda A(t) + B(t)$ is a regular pencil of **index** ν ($\nu \in \mathbb{N}$) if there exist functions $C_1, C_2\colon [t_+,\infty) \to (0,\infty)$ such that for every $t \in [t_+,\infty)$
   $$\|R(\lambda,t)\| \le C_1(t)|\lambda|^{\nu-1}, \quad |\lambda| \ge C_2(t). \qquad (23)$$

   [Rutkas A.G., Vlasenko L.A. Existence of solutions of degenerate nonlinear differential operator equations, *Nonlinear Oscillations*, 2001]

   Then for each $t \in [t_+,\infty)$ there exist the two pairs of mutually complementary projectors $P_j(t)$, $Q_j(t)$, $j = 1,2$, (14) which generate the direct decompositions of D and Y (12) such that the operators $A(t)$, $B(t)$ have the block representations (13), where $A_1^{-1}(t)$ and $B_2^{-1}(t)$ exist.

   In general, the order of pole of the resolvent $(A(t) + \mu B(t))^{-1}$ at the point $\mu = 0$ is called the *index* of the regular pencil $\lambda A(t) + B(t)$.

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Thank you for your attention!