Qualitative analysis of nonregular differential-algebraic equations and applications in the gas dynamics

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Differential-Algebraic Equations (DAEs) are also called descriptor, algebraic-differential and degenerate differential equations.

Fields of application of the theory of semilinear DAEs are radio engineering, control problems, cybernetics, robotics technology, economics, mechanics, chemical kinetics, and gas industry.

Study of DAEs (solvability, structure, index, stability, numerical methods: K. Weierstrass (1867), L. Kronecker (1890), V.P. Skripnik (1964), Gear C.W. (1971), A.G. Rutkas (1975), R.E. Showalter (1975), S.L. Campbell (1976), Yu.E. Boyarintsev (1977), A. Favini (1977), V.F. Chistyakov (1980), L.R. Petzold, L.A. Vlasenko, E. Hairer, Ch. Lubich, P. Kunkel, V. Mehrmann, R. März, C. Tischendorf, A.A. Shcheglova A.M. Samoilenko, R. Riaza, Yu.E. Gliklikh, A. Yonchev and others.

Nonregular DAEs: A.G. Rutkas, P. Kunkel, V. Mehrmann, S.M. Chuiko, V.F. Chistyakov, E.V. Chistyakova, S.P. Zubova and others.

Consider the initial value (Cauchy) problem for an implicit differential equation

$$\frac{d}{dt}[Ax] + Bx = f(t,x), \tag{1}$$
$$x(t_0) = x_0, \tag{2}$$

where $t \in [t_+, \infty)$, $t_+ \geq 0$, $x \in \mathbb{R}^n$, $f \in C([t_+, \infty) \times \mathbb{R}^n, \mathbb{R}^m)$ and $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ are linear operators (or $m \times n$ matrices).

In the case when $m \neq n$ or $m = n$ and the operator A is noninvertible (degenerate), the equation (1) is called a **differential-algebraic equation** (**DAE**) or **degenerate differential equation**. In the DAE terminology, equations of the form (1) are called *semilinear*.

- **The pencil** $\lambda A + B$ is called **regular** if $n = m = \mathrm{rk}(\lambda A + B)$.

  Otherwise, if $n \neq m$ or $n = m$ and $\mathrm{rk}(\lambda A + B) < n$, the pencil is called **singular** or **nonregular** (irregular).
- The **semilinear DAE with the singular pencil** is called **singular** or **nonregular**.

The function $x(t)$ is called a *solution of the equation* (1) *on* $[t_0, t_1)$, $t_1 \leq \infty$, if $x(t) \in C([t_0, t_1), \mathbb{R}^n)$, $(Ax)(t) \in C^1([t_0, t_1), \mathbb{R}^m)$ and $x(t)$ satisfies (1) on $[t_0, t_1)$. If the function $x(t)$ additionally satisfies the initial condition (2), then it is called a *solution of the initial value problem* (*IVP*).

The IVP (1), (2): $\frac{d}{dt}[Ax]+Bx=f(t,x), \quad x(t_0)=x_0.$

A solution $x(t)$ of the IVP (1), (2) is called **global** (or defined in the future) if it exists on $[t_0,\infty)$.

A solution $x(t)$ of the IVP (1), (2) is called **Lagrange stable** if it is global and $\sup\limits_{t\in[t_0,\infty)} \|x(t)\| < \infty$.

A solution $x(t)$ of the IVP (1), (2) has a **finite escape time** (is **blow-up in finite time**) and is called **Lagrange unstable** if it exists on some finite interval $[t_0,T)$ and is unbounded, i.e., $\lim\limits_{t\to T-0} \|x(t)\| = \infty$.

The equation (1) is called **Lagrange stable** if every solution of the IVP (1), (2) is Lagrange stable (the DAE is Lagrange stable for every consistent initial point).

The equation (1) is called **Lagrange unstable** if every solution of the IVP (1), (2) is Lagrange unstable.

Solutions of the equation (1) are called **ultimately bounded**, if there exists a constant $K > 0$ (K is independent of the choice of t_0, x_0) and for each solution $x(t)$ with an initial point (t_0, x_0) there exists a number $\tau = \tau(t_0, x_0) \geq t_0$ such that $\|x(t)\| < K$ for all $t \in [t_0 + \tau, \infty)$.

The equation (1) is called **ultimately bounded** or **dissipative**, if for any consistent initial point (t_0, x_0) there exists a global solution of the IVP (1), (2) and all solutions are ultimately bounded.

If the number τ does not depend on the choice of t_0, then the solutions of (1) are called *uniformly ultimately bounded* and the equation (1) is called *uniformly ultimately bounded* or *uniformly dissipative*.

Main results:

- **Theorems on the existence and uniqueness of global solutions**

  Some advantages: the restrictions of the type of the global Lipschitz condition are not used and the requirements for the smoothness of the nonlinear part of the DAE are weakened.
- **Theorem on the Lagrange stability of the DAE** (the boundedness of solutions)
- **Theorem on the Lagrange instability of the DAE** (the blow-up of solutions in finite time)
- **Theorem on the ultimate boundedness (dissipativity) of the DAE** (the ultimate boundedness of solutions)
- The application of the obtained theorems to the study of isothermal models of gas networks are shown.

See [*M. Filipkovska, Qualitative analysis of nonregular differential-algebraic equations and the dynamics of gas networks*, `https://doi.org/10.48550/arXiv.2309.00186`].

Earlier, certain models of nonlinear electrical circuits, including, in the conditions of incomplete data, and inverse problems were considered [*M.S. Filipkovska, Lagrange stability and instability of irregular semilinear differential-algebraic equations and applications. Ukrainian Math. J.* ***70****(6), 947–979 (2018)*]

The block structure of the singular operator pencil $\lambda A + B$

There exist the direct decompositions of spaces

$$\mathbb{R}^n = X_s \dot{+} X_r = X_{s_1} \dot{+} X_{s_2} \dot{+} X_r, \quad \mathbb{R}^m = Y_s \dot{+} Y_r = Y_{s_1} \dot{+} Y_{s_2} \dot{+} Y_r \tag{3}$$

such that the **singular operator pencil** $\lambda A + B$ takes the block form

$$\begin{pmatrix} \lambda A_s + B_s & 0 \\ 0 & \lambda A_r + B_r \end{pmatrix}, \quad \begin{array}{l} \lambda A_s + B_s \text{ is a } \textit{purely singular pencil}, \\ \lambda A_r + B_r \text{ is a } \textit{regular pencil}. \end{array} \tag{4}$$

We introduce the projectors onto subspaces of the decompositions (3):
$S : \mathbb{R}^n \to X_s$, $F : \mathbb{R}^m \to Y_s$, $S_i : \mathbb{R}^n \to X_{s_i}$, $F_i : \mathbb{R}^m \to Y_{s_i}$, $P : \mathbb{R}^n \to X_r$, $Q : \mathbb{R}^m \to Y_r$.

$$A_s = \begin{pmatrix} A_{gen} & 0 \\ 0 & 0 \end{pmatrix}, \; B_s = \begin{pmatrix} B_{gen} & B_{und} \\ B_{ov} & 0 \end{pmatrix} : X_s = X_{s_1} \dot{+} X_{s_2} \to Y_s = Y_{s_1} \dot{+} Y_{s_2}, \tag{5}$$

$\exists A_{gen}^{-1}$, $A_{gen} = F_1 A S_1|_{X_{s_1}}$, $B_{gen} = F_1 B S_1|_{X_{s_1}}$, $B_{und} = F_1 B S_2|_{X_{s_2}}$, $B_{ov} = F_2 B S_1|_{X_{s_1}}$

- If $\mathrm{rk}(\lambda A + B) = m < n$, the corresponding system of equations is **underdetermined**:

$$A_s = (A_{gen}\; 0), \; B_s = (B_{gen}\; B_{und}) : X_{s_1} \dot{+} X_{s_2} \to Y_s, \quad Y_s = Y_{s_1}, \; Y_{s_2} = \{0\}. \tag{6}$$

- If $\mathrm{rk}(\lambda A + B) = n < m$, the corresponding system of equations is **overdetermined**:

$$A_s = \begin{pmatrix} A_{gen} \\ 0 \end{pmatrix}, \; B_s = \begin{pmatrix} B_{gen} \\ B_{ov} \end{pmatrix} : X_s \to Y_{s_1} \dot{+} Y_{s_2}, \quad X_s = X_{s_1}, \; X_{s_2} = \{0\}. \tag{7}$$

Suppose that $\lambda A_r + B_r$ **is a regular pencil of index not higher than 1** (index 0 or 1):

$$\exists C_1, C_2 > 0 : \left\| (\lambda A_r + B_r)^{-1} \right\| \le C_1, \quad |\lambda| \ge C_2. \tag{8}$$

Then there exist the real spectral projectors of Riss type $\tilde{P}_i : X_r \to X_i$, $\tilde{Q}_i : Y_r \to Y_i$, $i = 1, 2$, which decompose spaces X_r, Y_r into direct sums of subspaces

$$X_r = X_1 \dot{+} X_2, \quad Y_r = Y_1 \dot{+} Y_2. \tag{9}$$

$$A_r = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_r = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} : X_1 \dot{+} X_2 \to Y_1 \dot{+} Y_2. \tag{10}$$

By $P_i : \mathbb{R}^n \to X_i$, $Q_i : \mathbb{R}^m \to Y_i$ denote the extensions of the projectors $\tilde{P}_i$, $\tilde{Q}_i$.

Introduce the extensions of the operators from the block representations to $\mathbb{R}^n$:
$\mathcal{A}_s = FA$, $\mathcal{A}_r = QA$, $\mathcal{B}_s = FB$, $\mathcal{B}_r = QB$, $\mathcal{A}_{gen} = F_1 A$, $\mathcal{B}_{gen} = F_1 B S_1$, $\mathcal{B}_{und} = F_1 B S_2$, $\mathcal{B}_{ov} = F_2 B S_1$, $\mathcal{A}_j = Q_j$, $\mathcal{B}_j = Q_j B \in L(\mathbb{R}^n, \mathbb{R}^m)$, $j = 1, 2$.

Also, introduce the semi-inverse operators (the extensions of the inverse operators to $\mathbb{R}^m$): $\mathcal{A}_{gen}^{(-1)} \in L(\mathbb{R}^m, \mathbb{R}^n)$ (i.e., $\mathcal{A}_{gen}^{(-1)} \mathbb{R}^m = \mathcal{A}_{gen}^{(-1)} Y_{s_1} = X_{s_1}$ $(Y_{s_2} \dot{+} Y_r = \mathrm{Ker}(\mathcal{A}_{gen}^{(-1)}))$ and $A_{gen}^{-1} = \mathcal{A}_{gen}^{(-1)}\big|_{Y_{s_1}}$) and $\mathcal{A}_1^{(-1)}, \mathcal{B}_2^{(-1)} \in L(\mathbb{R}^m, \mathbb{R}^n)$.

[*M.S. Filipkovska, A block form of a singular pencil of operators and a method of obtaining it. Visnyk of V.N. Karazin Kharkiv National University. Ser. "Mathematics, Applied Mathematics and Mechanics"* ***89***, *33–58 (2019)*],

[*M.S. Filipkovska, Lagrange stability and instability of irregular semilinear differential-algebraic equations and applications. Ukrainian Math. J.* ***70****(6), 947–979 (2018)*].

Application of the block structure of the DAE operator coefficients

With respect to the decompositions (3), (9) **any vector** $x \in \mathbb{R}^n$ **can be uniquely represented as the sum**

$$x = x_{s_1} + x_{s_2} + x_1 + x_2, \quad x_{s_i} = S_i x \in X_{s_i},\ x_i = P_i x \in X_i,\ i = 1,2.$$

The DAE (1) $\frac{d}{dt}[Ax(t)] + Bx(t) = f(t,x)$ is equivalent to the system

$$\frac{d}{dt} x_{s_1} = \mathcal{A}_{gen}^{(-1)}\big(F_1 f(t,x) - \mathcal{B}_{gen} x_{s_1} - \mathcal{B}_{und} x_{s_2}\big), \tag{11}$$

$$\frac{d}{dt} x_{p_1} = \mathcal{A}_1^{(-1)}\big(Q_1 f(t,x) - \mathcal{B}_1 x_{p_1}\big), \tag{12}$$

$$0 = \mathcal{B}_2^{(-1)} Q_2 f(t,x) - x_{p_2}, \tag{13}$$

$$0 = F_2 f(t,x) - \mathcal{B}_{ov} x_{s_1}, \tag{14}$$

We introduce the manifold

$$L_{t_*} = \{(t,x) \in [t_*,\infty) \times \mathbb{R}^n \mid (F_2 + Q_2)[Bx - f(t,x)] = 0\} =$$
$$= \{(t,x) \in [t_*,\infty) \times \mathbb{R}^n \mid (t,x) \text{ satisfies the equations (13), (14) }\}.$$

The derivative $V'_{(11),(12)}(t, x_{s_1}, x_{p_1})$ of the function V along the trajectories of the system (or the derivative of V with respect to the system) (11), (12) has the form

$$V'_{(11),(12)}(t, x_{s_1}, x_{p_1}) = \frac{\partial V}{\partial t}(t, x_{s_1}, x_{p_1}) + \frac{\partial V}{\partial (x_{s_1}, x_{p_1})}(t, x_{s_1}, x_{p_1}) \cdot \Upsilon(t, x) =$$
$$= \frac{\partial V}{\partial t}(t, x_{s_1}, x_{p_1}) + \frac{\partial V}{\partial x_{s_1}}(t, x_{s_1}, x_{p_1}) \cdot \Big[\mathcal{A}_{gen}^{(-1)}\big(F_1 f(t, x) - \mathcal{B}_{gen} x_{s_1} - \mathcal{B}_{und} x_{s_2}\big)\Big] +$$
$$+ \frac{\partial V}{\partial x_{p_1}}(t, x_{s_1}, x_{p_1}) \cdot \Big[\mathcal{A}_1^{(-1)}\big(Q_1 f(t, x) - \mathcal{B}_1 x_{p_1}\big)\Big], \quad (15)$$

where

$$\Upsilon(t, x) = \begin{pmatrix} \mathcal{A}_{gen}^{(-1)}\big(F_1 f(t, x) - \mathcal{B}_{gen} x_{s_1} - \mathcal{B}_{und} x_{s_2}\big) \\ \mathcal{A}_1^{(-1)}\big(Q_1 f(t, x) - \mathcal{B}_1 x_{p_1}\big) \end{pmatrix} \quad (16)$$

is a vector consisting of the right-hand sides of the equations (11) and (12).

Theorem 1 (the global solvability). Let $f \in C([t_+,\infty)\times\mathbb{R}^n,\mathbb{R}^m)$, $\frac{\partial}{\partial x} f \in C([t_+,\infty)\times\mathbb{R}^n, L(\mathbb{R}^n,\mathbb{R}^m))$, $\lambda A + B$ is a singular pencil of operators such that its regular block $\lambda A_r + B_r$ from (4) has the index not higher than 1, and the following conditions are fulfilled:

1. For any fixed $t \in [t_+,\infty)$, $x_{s_1} \in X_{s_1}$, $x_{s_2} \in D_{s_2}$, where $D_{s_2} \subset X_{s_2}$ is a some set, and $x_{p_1} \in X_1$, there exists a unique $x_{p_2} \in X_2$ such that $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$;

2. For any fixed t_*, $x_* = x^*_{s_1} + x^*_{s_2} + x^*_{p_1} + x^*_{p_2}$ such that $(t_*, x_*) \in L_{t_+}$ and $x^*_{s_2} \in D_{s_2}$, the operator Φ_{t_*,x_*} defined by

$$\Phi_{t_*,x_*} = \left[\frac{\partial Q_2 f}{\partial x}(t_*,x_*) - B\right] P_2 : X_2 \to Y_2 \tag{17}$$

   is invertible.

3. There exists a number $R > 0$, a positive definite function $V \in C^1([t_+,\infty)\times D_{s_1}\times D_{p_1},\mathbb{R})$, where a set $D_{s_1}\times D_{p_1} \subset X_{s_1}\times X_1$ is such that $D_{s_1}\times D_{p_1} \supset \{\|(x_{s_1},x_{p_1})\| \ge R\}$, and a function $\chi \in C([t_+,\infty)\times(0,\infty),\mathbb{R})$ such that:

a. $V(t, x_{s_1}, x_{p_1}) \to \infty$ uniformly in t on each finite interval $[a,b) \subset [t_+, \infty)$ as $\|(x_{s_1}, x_{p_1})\| \to \infty$;

b. for all $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$, for which $x_{s_2} \in D_{s_2}$ and $\|(x_{s_1}, x_{p_1})\| \geq R$, the inequality

$$V'_{(11),(12)}(t, x_{s_1}, x_{p_1}) \leq \chi\big(t, V(t, x_{s_1}, x_{p_1})\big), \tag{18}$$

where $V'_{(11),(12)}(t, x_{s_1}, x_{p_1})$ has the form (15), is satisfied;

c. the differential inequality $dv/dt \leq \chi(t, v)$ $(t \in [t_+, \infty))$ does not have positive solutions with finite escape time.

Then for each initial point $(t_0, x_0) \in L_{t_+}$, where $S_2 x_0 \in D_{s_2}$, the initial value problem (1), (2) has a unique global (i.e., on $[t_0, \infty)$) solution $x(t)$ for which the choice of the function $\phi_{s_2} \in C([t_0, \infty), D_{s_2})$ with the initial value $\phi_{s_2}(t_0) = S_2 x_0$ uniquely defines the component $S_2 x(t) = \phi_{s_2}(t)$ when $\operatorname{rank}(\lambda A + B) < n$; when $\operatorname{rank}(\lambda A + B) = n$, the component $S_2 x$ is absent.

Let us choose the function χ in the form

$$\chi(t,v) = k(t)\,U(v), \tag{19}$$

where $k \in C([t_+,\infty),\mathbb{R})$ and $U \in C(0,\infty)$.

Then in Theorem 1 all conditions remain unchanged, except for condition 3 which takes the form:

3. There exists a number $R > 0$, a positive definite function $V \in C^1([t_+,\infty) \times D_{s_1} \times D_{p_1},\mathbb{R})$, where a set $D_{s_1} \times D_{p_1} \subset X_{s_1} \times X_1$ is such that $D_{s_1} \times D_{p_1} \supset \{\|(x_{s_1},x_{p_1})\| \geq R\}$, and functions $k \in C([t_+,\infty),\mathbb{R})$, $U \in C(0,\infty)$ such that:

    1. condition (a) of Theorem 1 holds, i.e., $V(t,x_{s_1},x_{p_1}) \to \infty$ uniformly in t on each finite interval $[a,b) \subset [t_+,\infty)$ as $\|(x_{s_1},x_{p_1})\| \to \infty$;
    2. for all $(t, x_{s_1}+x_{s_2}+x_{p_1}+x_{p_2}) \in L_{t_+}$, for which $x_{s_2} \in D_{s_2}$ and $\|(x_{s_1},x_{p_1})\| \geq R$, the following inequality holds:

    $$V'_{(11),(12)}(t,x_{s_1},x_{p_1}) \leq k(t)\,U\big(V(t,x_{s_1},x_{p_1})\big); \tag{20}$$

    3. $\int\limits_{v_0}^{\infty} \frac{dv}{U(v)} = \infty$ ($v_0 > 0$ is some constant).

Also, we can weaken some requirements of Theorem 1 and, as a consequence, some requirements of another theorems as well.

Let us choose the positive definite scalar function $V(t, x_{s_1}, x_{p_1})$, which will be called a *Lyapunov type function*, in the form

$$V(t, x_{s_1}, x_{p_1}) = \big((x_{s_1}, x_{p_1}), (x_{s_1}, x_{p_1})\big)_H = \big(H(t)(x_{s_1}, x_{p_1}), (x_{s_1}, x_{p_1})\big), \qquad (21)$$

where $H \in C([t_+, \infty), L(X_{s_1} \times X_1))$ is a positive definite self-adjoint operator function such that $H(t)\big|_{X_{s_1}} : X_{s_1} \to X_{s_1} \times \{0\}$ and $H(t)\big|_{X_1} : X_1 \to \{0\} \times X_1$ for any fixed t.

Due to the properties of the operator function H, the function (21) satisfies the conditions of Theorem 1 (however, of course, the conditions on the derivative $V'_{(11),(12)}(t, x_{s_1}, x_{p_1})$ in these theorems need to be checked).

The operator $H(t) \in L(X_{s_1} \times X_1)$ has the block structure

$$H(t) = \begin{pmatrix} H_{s_1}(t) & 0 \\ 0 & H_1(t) \end{pmatrix} : X_{s_1} \times X_1 \to X_{s_1} \times X_1, \qquad (22)$$

where $H_{s_1} \in C([t_+, \infty), L(X_{s_1}))$ and $H_1 \in C([t_+, \infty), L(X_1))$ are positive definite self-adjoint operator functions.

$$V'_{(11),(12)}(t, x_{s_1}, x_{p_1}) = \Big(\frac{d}{dt} H(t)(x_{s_1}, x_{p_1}), (x_{s_1}, x_{p_1})\Big) +$$
$$2\Big(H_{s_1}(t) x_{s_1}, \Big[\mathcal{A}_{gen}^{(-1)}\big(F_1 f(t,x) - \mathcal{B}_{gen} x_{s_1} - \mathcal{B}_{und} x_{s_2}\big)\Big]\Big) +$$
$$2\Big(H_1(t) x_{p_1}, \Big[\mathcal{A}_1^{(-1)}\big(Q_1 f(t,x) - \mathcal{B}_1 x_{p_1}\big)\Big]\Big).$$

Theorem 1(Lagrange stability). Let $f \in C([t_+,\infty)\times\mathbb{R}^n,\mathbb{R}^m)$, $\frac{\partial}{\partial x}f \in C([t_+,\infty)\times\mathbb{R}^n, L(\mathbb{R}^n,\mathbb{R}^m))$, $\lambda A + B$ is a singular pencil of operators such that its regular block $\lambda A_r + B_r$ from (4) has the index not higher than 1, and conditions 1, 2 of Theorem 1 as well as the following conditions hold:

3. There exists a number $R > 0$, a positive definite function $V \in C^1([t_+,\infty)\times D_{s_1}\times D_{p_1},\mathbb{R})$, where a set $D_{s_1}\times D_{p_1} \subset X_{s_1}\times X_1$ is such that $D_{s_1}\times D_{p_1} \supset \{\|(x_{s_1},x_{p_1})\| \geq R\}$, and a function $\chi \in C([t_+,\infty)\times(0,\infty),\mathbb{R})$ such that:
   1. $V(t,x_{s_1},x_{p_1}) \to \infty$ uniformly in t on $[t_+,\infty)$ as $\|(x_{s_1},x_{p_1})\| \to \infty$;
   2. for all $(t, x_{s_1}+x_{s_2}+x_{p_1}+x_{p_2}) \in L_{t_+}$, for which $x_{s_2} \in D_{s_2}$ and $\|(x_{s_1},x_{p_1})\| \geq R$, the inequality (18) is satisfied;
   3. the differential inequality $dv/dt \leq \chi(t,v)$ $(t \in [t_+,\infty))$, does not have unbounded positive solutions for $t \in [t_+,\infty)$.

Then for each initial point $(t_0,x_0) \in L_{t_+}$, where $S_2x_0 \in D_{s_2}$, the initial value problem (1), (2) has a unique global solution $x(t)$ for which the choice of the function $\phi_{s_2} \in C([t_0,\infty),D_{s_2})$ with the initial value $\phi_{s_2}(t_0) = S_2x_0$ uniquely defines the component $S_2x(t) = \phi_{s_2}(t)$ when $\operatorname{rank}(\lambda A + B) < n$.

Let, in addition to the above conditions, the following conditions also hold:

5. For all $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$, for which $x_{s_2} \in D_{s_2}$ and $\|x_{s_1} + x_{s_2} + x_{p_1}\| \le M < \infty$ (M is an arbitrary constant), the inequality

$$\|x_{p_2}\| \le K_M < \infty$$

or the inequality $\|Q_2 f(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2})\| \le K_M < \infty$, where $K_M = K(M)$ is some constant, is satisfied.

6. $\|F_2 f(t, x)\| < +\infty$ for all $(t, x) \in L_{t_+}$ such that $S_2 x \in D_{s_2}$ and $\|x\| \le C < \infty$ (C is an arbitrary constant).

Then, for the initial points $(t_0, x_0) \in L_{t_+}$ where $S_2 x_0 \in D_{s_2}$ and any function $\phi_{s_2} \in C([t_0, \infty), D_{s_2})$ satisfying the relations $\phi_{s_2}(t_0) = S_2 x_0$ and $\sup\limits_{t \in [t_0, \infty)} \|\phi_{s_2}(t)\| < +\infty$, the equation (1), where $S_2 x = \phi_{s_2}(t)$, is Lagrange stable; when $\mathrm{rank}(\lambda A + B) = n < m$, the component $S_2 x$ is absent.

A model of a gas flow for a single pipe (in the isothermal case)

Consider a mathematical model for a gas pipeline flow (the flow on a single pipe), assuming that the temperature is identically equal to $T_0 = \mathrm{const}$. The model consists of the **isothermal Euler equations**

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho v) = 0, \\ &\partial_t(\rho v) + \partial_x p = -\frac{\lambda_{fr}}{2D}\rho v|v| - g\rho s_{lope} \end{aligned} \tag{23}$$

and the **equation of state for a real gas** in the form

$$p = RT_0\rho z(p), \tag{24}$$

- $x \in [0, L]$, $t \in \mathscr{I} \subset [0, \infty)$, $\mathscr{I}$ is a time interval, $L < \infty$ is the pipe length
- $\rho = \rho(t, x)$, $v = v(t, x)$, $p = p(t, x)$ are respectively the density, velocity and pressure
- g is the gravitational constant, and R is the specific gas constant
- λ_{fr} is the pipe friction coefficient, and D is the pipe diameter
- $s_{lope} = s_{lope}(x)$ is the slope of the pipe
- $z = z(p)$ is the compressibility factor

[P. Domschke, B. Hiller, J. Lang, V. Mehrmann, R. Morandin, C. Tischendorf. *Gas Network Modeling: An Overview*, 2021 (Preprint)]

Further, we denote by $\mathrm{q} := \rho \mathrm{v}$ a mass flow by the cross-sectional area equal to 1, assume that the directions of gas flows in pipes are known and that $\mathrm{s_{lope}}(\mathrm{x}) \equiv \sin\theta$, where the parameter θ denotes the angle of the pipe slope, and discretize the equations (23). Then we obtain the spatially discretized equations

$$\frac{\mathrm{d}\rho_{\mathrm{r}}}{\mathrm{dt}} + \frac{\mathrm{q_r} - \mathrm{q_l}}{\mathrm{L}} = 0, \tag{25}$$

$$\frac{\mathrm{dq_l}}{\mathrm{dt}} + \frac{\mathrm{p_r} - \mathrm{p_l}}{\mathrm{L}} + \rho_{\mathrm{r}}\,\mathrm{g}\sin\theta = -\frac{\lambda_{\mathrm{fr}}}{2\mathrm{D}}\frac{\mathrm{q_l^2}}{\rho_{\mathrm{r}}}, \tag{26}$$

$$\mathrm{p_r} = \mathrm{R_s}\mathrm{T_0}\rho_{\mathrm{r}}\mathrm{z}(\mathrm{p_r}). \tag{27}$$

where $\mathrm{q_r}(\mathrm{t}) := \mathrm{q}(\mathrm{t},\mathrm{L})$, $\mathrm{p_r}(\mathrm{t}) := \mathrm{p}(\mathrm{t},\mathrm{L})$, $\rho_{\mathrm{r}}(\mathrm{t}) := \rho(\mathrm{t},\mathrm{L})$ and $\mathrm{q_l}(\mathrm{t}) := \mathrm{q}(\mathrm{t},0)$, $\mathrm{p_l}(\mathrm{t}) := \mathrm{p}(\mathrm{t},0)$. If we represent the pipe as a graph consisting of an edge and two vertices (nodes), define the vertices as the left and right nodes and fix the edge orientation from the left node to the right node, then $\mathrm{q_r}(\mathrm{t})$, $\mathrm{p_r}(\mathrm{t})$ and $\rho_{\mathrm{r}}(\mathrm{t})$ are defined at the right end of pipe and $\mathrm{q_l}(\mathrm{t})$, $\mathrm{p_l}(\mathrm{t})$ are defined at the left end of pipe. In general, previously, the pipe is divided into parts of a short length through the introduction of artificial nodes and the specified spatial discretization are performed on each part (subpipe).

Suppose that the functions $\mathrm{q_r}$ and $\mathrm{p_l}$ are given, that is, we consider the boundary conditions of the form

$$\mathrm{q}(\mathrm{t},\mathrm{L}) = \mathrm{q_r}(\mathrm{t}), \quad \mathrm{p}(\mathrm{t},0) = \mathrm{p_l}(\mathrm{t}), \qquad \mathrm{t} \in \mathscr{I}.$$

We introduce the variable vector $\mathrm{x} = (\rho_\mathrm{r}, \mathrm{q_l}, \mathrm{p_r})^\mathrm{T}$ (we denote it by x for convenience and comparison with further results, since the original variable x is already absent from the equations) and denote

$$\mathrm{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathrm{B} = \begin{pmatrix} 0 & -\mathrm{L}^{-1} & 0 \\ \mathrm{g}\sin\theta & 0 & \mathrm{L}^{-1} \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathrm{f}(\mathrm{t},\mathrm{x}) = \begin{pmatrix} -\mathrm{L}^{-1}\mathrm{q_r}(\mathrm{t}) \\ \mathrm{L}^{-1}\mathrm{p_l}(\mathrm{t}) - 0.5\lambda_{\mathrm{fr}}\mathrm{D}^{-1}\mathrm{q_l}^2\rho_\mathrm{r}^{-1} \\ \mathrm{R_s}\mathrm{T_0}\,\rho_\mathrm{r}\mathrm{z}(\mathrm{p_r}) \end{pmatrix}. \quad (28)$$

Then the system (25)–(27) can be written in the vector form

$$\frac{\mathrm{d}}{\mathrm{dt}}[\mathrm{Ax}] + \mathrm{Bx} = \mathrm{f}(\mathrm{t},\mathrm{x}), \qquad \mathrm{t} \in \mathscr{I}, \quad (29)$$

where $\mathrm{A}, \mathrm{B} \in \mathbb{R}^{3\times 3}$ and $\mathrm{f} \in \mathrm{C}(\mathscr{I} \times \mathbb{R}^3, \mathbb{R}^3)$. The initial condition for (29) can be given as

$$\mathrm{x}(\mathrm{t_0}) = \mathrm{x_0}, \qquad \mathrm{x_0} = (\rho_\mathrm{r}^0, \mathrm{q_l}^0, \mathrm{p_r}^0)^\mathrm{T}. \quad (30)$$

where ρ_r^0 and $\mathrm{p_r^0}$ have to satisfy the equation (27) for $\mathrm{t} = \mathrm{t_0}$, i.e., $\mathrm{p_r^0} = \mathrm{R_s}\mathrm{T_0}\rho_\mathrm{r}^0\mathrm{z}(\mathrm{p_r^0})$.

In general, the DAE (29) is regular (since the pencil $\lambda A + B$ is regular), but if any of the input parameters (i.e., $q_r(t)$ or $p_l(t)$) is not specified, then the system (25)–(27) is underdetermined and the corresponding DAE is singular (nonregular). Also, if it is required to realize the evolution of some variable (i.e., p_r, or ρ_r, or q_l) such that it becomes equal to the prescribed function, then this system is overdetermined and the corresponding DAE is singular.

A model of a gas network (in the isothermal case)

In [*M. Filipkovska, Qualitative analysis of nonregular differential-algebraic equations and the dynamics of gas networks*, `https://doi.org/10.48550/arXiv.2309.00186`], a mathematical model of a *gas network* in the form of the singular (nonregular) DAE (1) is presented. The gas network consists of pipes, valves, regulators and compressors, and is similar to that presented in [Kreimeier, T., Sauter, H., Streubel, S.T., Tischendorf, C., Walther, A. *Solving Least-Squares Collocated Differential Algebraic Equations by Successive Abs-Linear Minimization – A Case Study on Gas Network Simulation*, 2022 [Preprint]].

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Thank you for your attention!