Friedrich-Alexander-Universität
Naturwissenschaftliche Fakultät

Gas Network Modelling and Optimal Locations for Control under Uncertainty

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March 8, Nanzan University

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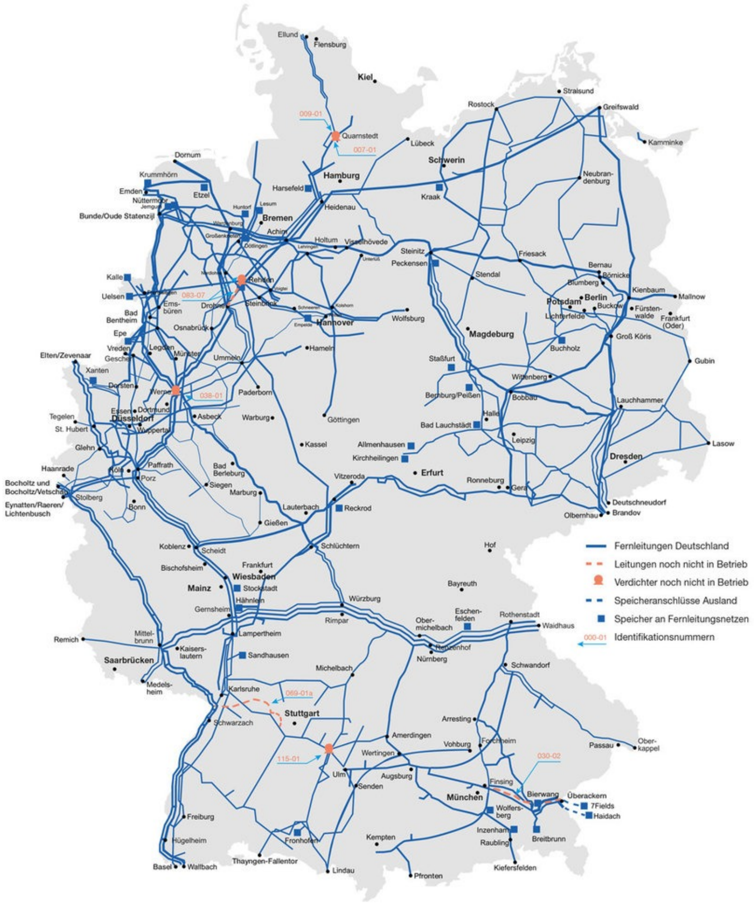
Motivation

Natural Gas Transport



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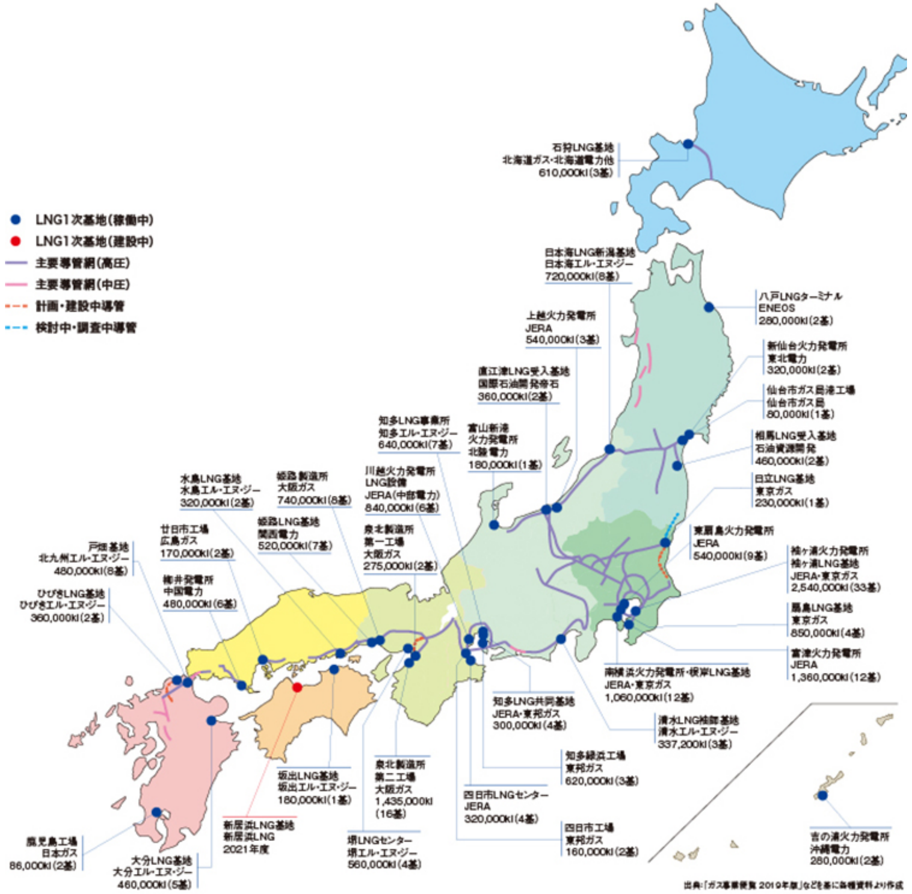

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Gas Network Modelling

The Isothermal Euler Equations

The isothermal Euler equations for ideal gases:

(ISO1)

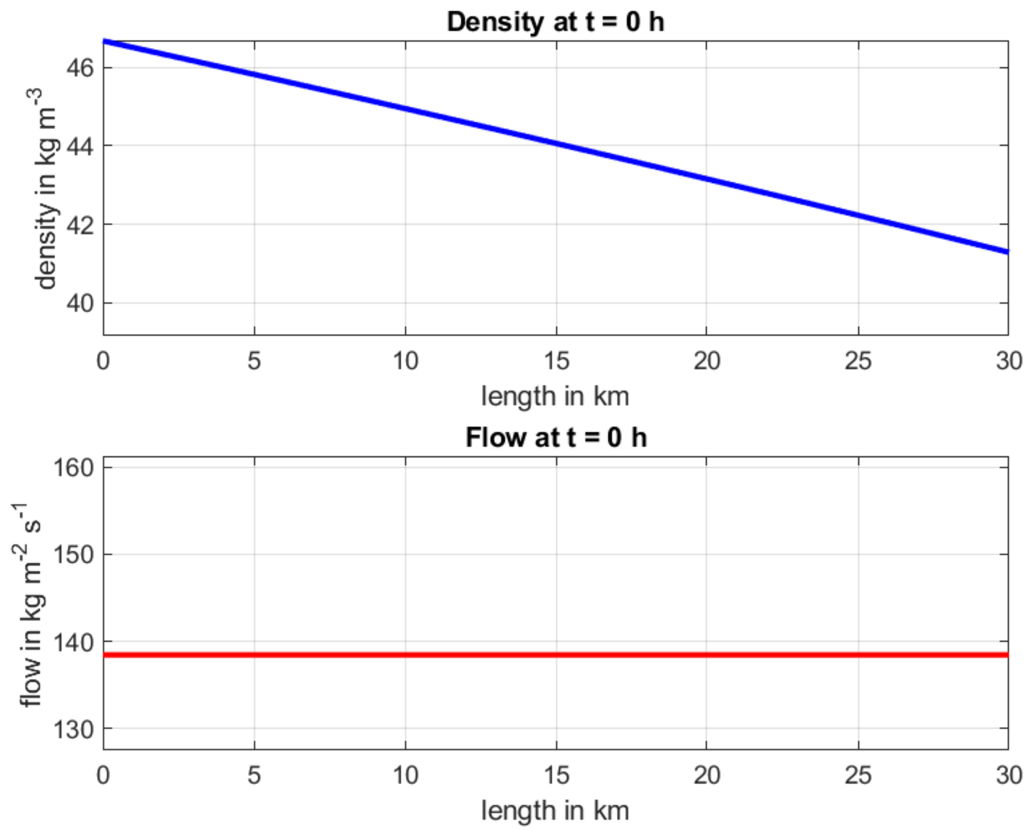
$$\rho_t + q_x = 0,$$
$$q_t + \left(p + \frac{q^2}{\rho}\right)_x = -\frac{\lambda}{2\,D}\,\frac{q|q|}{\rho},$$

Inlet density & Gas outflow

$$\rho(t,0) = \rho_0(t),$$
$$q(t,L) = b(t).$$

Initial condition

$$\rho(0,x) = \rho_{\text{ini}}(x),$$
$$q(0,x) = q_{ini}(x).$$



see Gugat and Ulbrich (2018): *Lipschitz solutions of initial boundary value problems for balance laws.* Math. Models Methods Appl. Sci., 28(5): 921–951

Gas Network Modelling

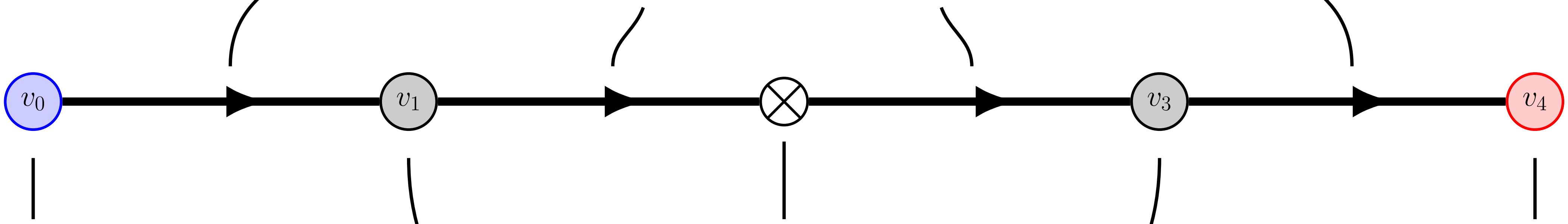
Gas Flow in Pipeline Networks

p	gas pressure	v	gas velocity	g	gravitational constant
ρ	gas density	λ/D	pipe friction	α	pipe slope

Isothermal Euler Equations

$$\rho_t + (\rho v)_x = 0$$

$$(\rho v)_t + \left(p + \rho v^2\right)_x = -\frac{\lambda}{2D}\rho v|v| - g\rho\sin(\alpha)$$



Source: Inlet Pressure

$$p(t) = p_0(t)$$

Compressor Station

$$\frac{p_{\text{out}}(t)}{p_{\text{in}}(t)} = u(t)$$

Sink: Gas Outflow

$$(\rho v)(t) = b(t)$$

Coupling Conditions

Conservation of Mass: $\sum(\rho v)_{\text{in}}(t) = \sum(\rho v)_{\text{out}}(t),$ Continuity in Pressure: $p_{\text{in}} = p_{\text{out}}$

Gas Network Modelling

Model Simplifications

(ISO1) - quasilinear model

$$\rho_t + q_x = 0$$

$$q_t + \left(p + \frac{q^2}{\rho}\right)_x = -\frac{\lambda^F}{2D}\frac{q|q|}{\rho}$$

subsonic flow: $|v| = \left|\frac{q}{\rho}\right| \ll c$

(ISO2) - semilinear model

$$\rho_t + q_x = 0$$

$$q_t + p_x = -\frac{\lambda^F}{2D}\frac{q|q|}{\rho}$$

stationary flow: $\frac{\partial}{\partial t} \equiv 0$

(ISO4) - stationary model

$$q_x = 0$$

$$p_x = -\frac{\lambda^F}{2D}\frac{q|q|}{\rho}$$

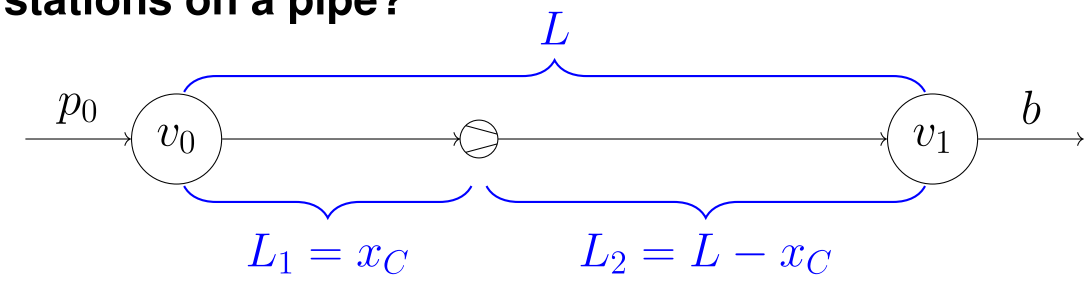
Optimal Compressor Location

Mathematical Modelling

- **The stationary gas flow for ideal gases on a single pipe is given by**

$$q(x) \equiv b \quad (const.), \qquad p(x) = p_0^2 - \phi\, b\, |b|\, x \quad \text{with} \quad \phi = \frac{\lambda}{D} R_S\, T, \quad x \in [0, L]$$

- **Where to place compressor stations on a pipe?**



- **The stationary gas flow with compressor station for ideal gas is given by**

$$p_1^2(x) = p_0^2 - \phi\, b\, |b|\, x \qquad x \in [0, L_1]$$

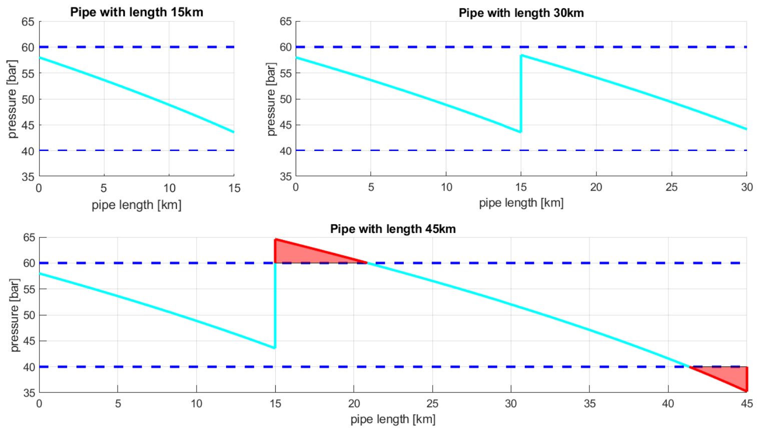
$$p_2^2(x) = u\, p_0^2 - \phi\, b\, |b|\, \big(u\, L_1 + x\big) \quad x \in [0, L_2]$$

- **Consider pressure bounds on the pipe**

$$\begin{array}{c} p_1(x) \in \big[p_{\min}, p_{\max}\big] \\ p_2(x) \in \big[p_{\min}, p_{\max}\big] \end{array} \iff \begin{array}{cc} p_1(0) \le p_{\max}, & p_1(L_1) \ge p_{\min}, \\ p_2(0) \le p_{\max}, & p_2(L_2) \ge p_{\min} \end{array}$$

Consider the following optimization problem:

(OPT 1)
$$\begin{cases} \min\limits_{u, x_C} & u^2, \\ \text{s.t.} & p_1(L_1) \geq p_{\text{min}}, \quad p_2(0) \leq p_{\text{max}}, \quad p_2(L_2) \geq p_{\text{min}}, \\ & u \geq 1, \\ & x_C \in [0, L]. \end{cases}$$

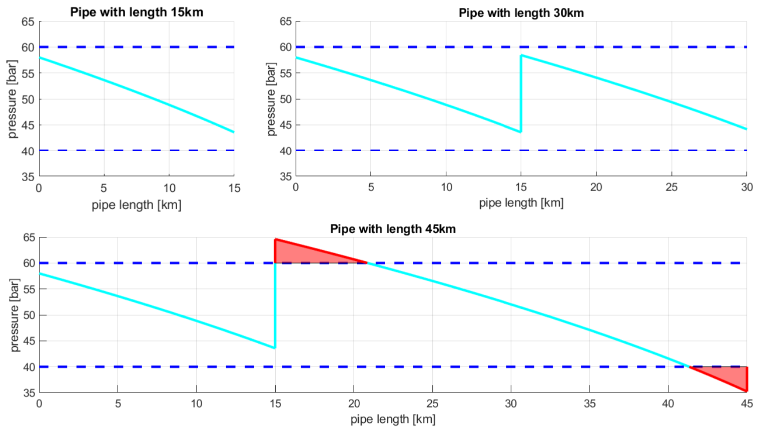


Optimal Compressor Location

Deterministic Optimization

Consider the following optimization problem:

$$
\text{(OPT 1)} \quad \begin{cases} \min\limits_{u, x_C} & u^2, \\ \text{s.t.} & p_1(L_1) \geq p_{\text{min}}, \quad p_2(0) \leq p_{\text{max}}, \quad p_2(L_2) \geq p_{\text{min}}, \\ & u \geq 1, \\ & x_C \in [0, L]. \end{cases}
$$



Lemma

Let $p_0 \in [p_{\text{min}}, p_{\text{max}}]$ and $b > 0$ be given.

(i) For $L \leq \frac{p_0^2 - p_{\text{min}}^2}{\phi\ b\ |b|}$ every point (u, x_C) with $u = 1$ and $x_C \in [0, L]$ is a solution of the optimization problem (OPT 1).

(ii) For $\frac{p_0^2 - p_{\text{min}}^2}{\phi\ b\ |b|} < L \leq \frac{p_0^2 + p_{\text{max}}^2 - 2\ p_{\text{min}}^2}{\phi\ b\ |b|}$ the optimization problem (OPT 1) has a unique solution (u^*, x_C^*) with $u^* > 1$ and $x_C \in [0, L]$.

(iii) For $L > \frac{p_0^2 + p_{\text{max}}^2 - 2\ p_{\text{min}}^2}{\phi\ b\ |b|}$ the optimization problem (OPT 1) does not have a solution.

Optimal Compressor Location

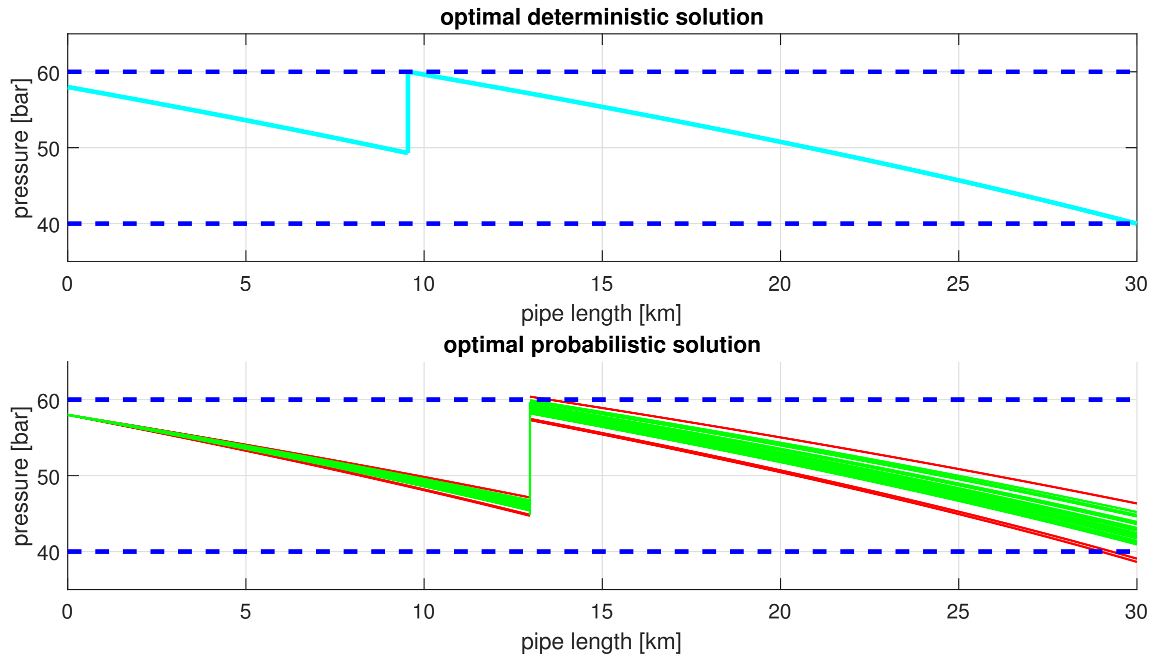
Probabilistic Optimization

Gas outflow b is random in the sense that

$$b = \xi(\omega), \quad \xi \sim \mathcal{N}(\mu, \sigma)$$

Consider the following optimization problem:

$$\text{(OPT 2)} \quad \begin{cases} \min_{u, x_C} \; u^2, \\ \text{s.t.} \; \mathbb{P}\begin{pmatrix} p_1(L_1) & \geq & p_{\text{min}} \\ p_2(0) & \leq & p_{\text{max}} \\ p_2(L_2) & \geq & p_{\text{min}} \\ p_2(L_2) & \leq & p_{\text{max}} \end{pmatrix} \geq \alpha, \\ u \geq 1, \\ x_C \in [0, L]. \end{cases}$$



Lemma

If (u^*, x_C^*) with $u^* > 1$ is a solution of (OPT 2), then the probabilistic constraint is active.

Optimal Compressor Location

Probabilistic Optimization

Theorem

Let $p_0 \in \left[\, p_{\text{min}},\ p_{\text{max}} \,\right]$ be given.

(i) If there exists a pair (u, x_C) with $u = 1$ and $x_C \in [0, L]$, that satisfies the constraints of (OPT 2), then every pair (u, x_C) with $u = 1$ and $x_C \in [0, L]$ is a solution of (OPT 2).

(ii) If there exist a pair (u, x_C), that satisfies the constraints of (OPT 2) and if (u, x_C) with $u = 1$ is infeasible for at least one $x_C \in [0, L]$, then there exists at least one solution (u^*, x_C^*) of (OPT 2) with $u^* > 1$ and $x_C^* \in [0, L]$.

Optimal Compressor Location

Probabilistic Optimization

Theorem

Let $p_0 \in \left[\, p_{\text{min}},\ p_{\text{max}} \,\right]$ be given.

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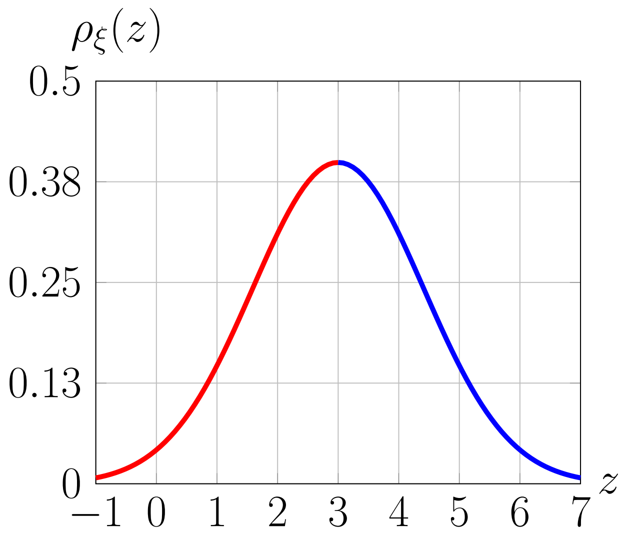
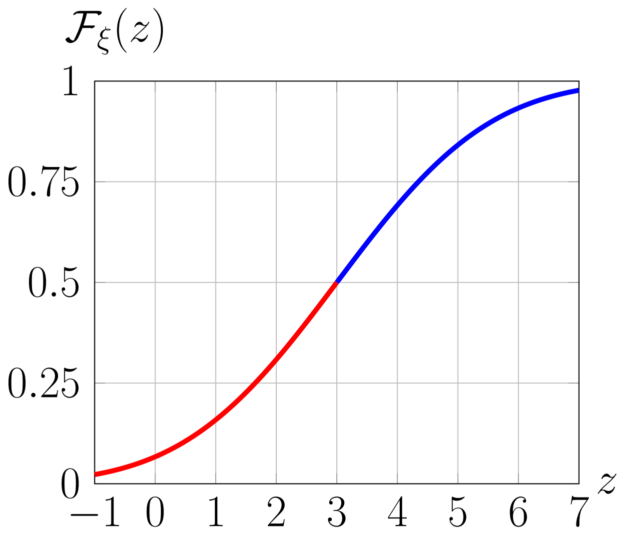
(ii) If there exist a pair (u, x_C), that satisfies the constraints of (OPT 2) and if (u, x_C) with $u = 1$ is infeasible for at least one $x_C \in [0, L]$, then there exists at least one solution (u^*, x_C^*) of (OPT 2) with $u^* > 1$ and $x_C^* \in [0, L]$.

Theorem

Let $\alpha > \frac{1}{2}$ be given. For a Gaussian distribution, *Statement (ii) in the last Theorem* guarantees the existence of a unique solution (u^*, x_C^*) of (OPT 2) with $u^* > 1$ and $x_C^* \in [0, L]$.

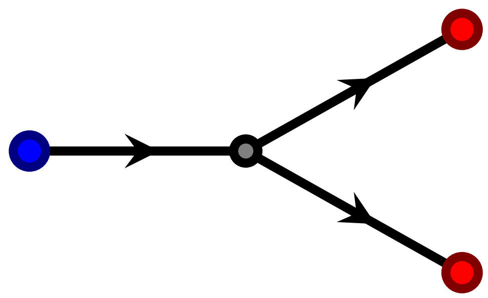




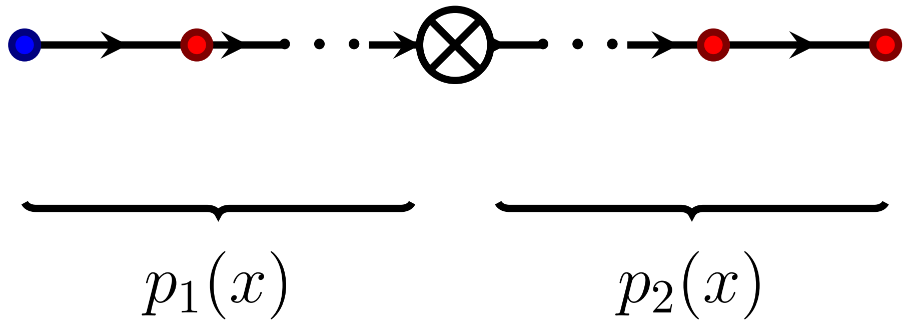
- **Consider a connected, directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$ and set of edges $\mathcal{E}$**
- **Binary variables δ_i states if a compressor location is located on edge e_i**
- **The stationary gas flow for ideal gas on pipe e_i is given by**

$$\begin{aligned} p_{i,1}^2(x) &= p_{i,1}^2(0) - \phi\, q_i\, |q_i|\, x && x \in [0, \delta_i\, x_{C,i}] \\ p_{i,2}^2(x) &= \big(1 - \delta_i + \delta_i\, u_i\big) p_{i,1}^2(L_{i,1}) - \phi\, q_i\, |q_i|\, x && x \in [0, L - \delta_i\, x_{C,i}] \end{aligned}$$

- **Uniqueness in general mainly depends on the graph topology**



(a) Scheme of a symmetric graph with one source and two sinks



(b) Scheme of a linear graph with one source and n sinks

Optimal Compressor Location

Deterministic Optimization on Networks

Consider the deterministic optimization problems

(OPT 3)
$$\begin{cases} \min\limits_{u,x_C,\delta} & \|u\|_2^2 = \sum_{i=1}^{m} u_i^2, \\ \text{s.t.} & \text{for all } i = 1,\cdots,m, \text{ we have} \\ & p_{i,1}(L_{i,1}) \geq p_{\min}, \\ & p_{i,2}(0) \leq p_{\max}, \\ & p_{i,2}(L_{i,2}) \geq p_{\min}, \\ & u_i \geq 1, \quad x_{C,i} \in [0, L_i], \quad \delta_i \in \{0,1\}, \\ & \sum_{j=1}^{m} \delta_j = n_C, \end{cases}$$

(OPT 4)
$$\begin{cases} \min\limits_{u,x_C,\delta,n_C} & \|u\|_2^2 = \sum_{i=1}^{m} u_i^2, \\ \text{s.t.} & \text{for all } i = 1,\cdots,m, \text{ we have} \\ & p_{i,1}(L_{i,1}) \geq p_{\min}, \\ & p_{i,2}(0) \leq p_{\max}, \\ & p_{i,2}(L_{i,2}) \geq p_{\min}, \\ & u_i \geq 1, \quad x_{C,i} \in [0, L_i], \quad \delta_i \in \{0,1\}, \\ & \sum_{j=1}^{m} \delta_j = n_C. \end{cases}$$

Optimal Compressor Location

Deterministic Optimization on Networks

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$$\begin{cases} \min_{u,x_C,\delta,n_C} & \|u\|_2^2 = \sum_{i=1}^m u_i^2, \\ \text{s.t.} & \text{for all } i = 1,\cdots,m, \text{ we have} \\ & p_{i,1}(L_{i,1}) \geq p_{\min}, \\ & p_{i,2}(0) \leq p_{\max}, \\ & p_{i,2}(L_{i,2}) \geq p_{\min}, \\ & u_i \geq 1, \quad x_{C,i} \in [0, L_i], \quad \delta_i \in \{0,1\}, \\ & \sum_{j=1}^m \delta_j = n_C. \end{cases}$$

Theorem

For all $v_i \in \mathcal{V}_{\text{in}}$ let $p_{i,0} \in [p_{\min}, p_{\max}]$ be given and for all $v_i \in \mathcal{V}_{\text{out}}$ let $b_i \geq 0$ be given. Further let a number $n_C \in \{0,\cdots,m\}$ be given.

(i) If a triple $(u, x_C, \delta) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$ with $u_j = 1$ for all $j = 1,\cdots,m$ satisfies the constraints of (OPT 3), every triple $(\mathbf{1}_m, x_C, \delta)$ with $x_{C,j} \in [0, L_j]$, $\delta_j \in \{0,1\}$ and $\sum_{j=1}^m \delta_j = n_C$ is a solution of (OPT 3).

(ii) If there exists a triple $(u, x_C, \delta) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$ that satisfies the constraints in (OPT 3), and if $(\mathbf{1}_m, x_C, \delta)$ is infeasible for at least one pair (x_C, δ) with $x_{C,j} \in [0, L_j]$ and $\delta_j \in \{0,1\}$, the optimization problem (OPT 3) has at least one solution.

Optimal Compressor Location

Probabilistic Optimization on Networks

Consider the probabilistic optimization problems

(OPT 6)
$$\begin{cases} \min\limits_{u,x_C,\delta} & \|u\|_2^2 = \sum_{i=1}^{m} u_i^2, \\ \text{s.t.} & \mathbb{P}\left(\begin{array}{lcl} p_{k,1}(L_{k,1}) & \geq & p_{\text{min}} \\ p_{k,2}(0) & \leq & p_{\text{max}} \\ p_{k,2}(L_{k,2}) & \geq & p_{\text{min}} \\ p_{k,2}(L_{k,2}) & \leq & p_{\text{max}} \end{array} \quad \forall k = 1,\cdots,m \right) \geq \alpha, \\ & \text{and for all } i = 1,\cdots,m, \text{ we have} \\ & u_i \geq 1, \quad x_{C,i} \in [0, L_i], \quad \delta_i \in \{0,1\}, \\ & \sum_{j=1}^{m} \delta_j = n_C, \end{cases}$$

(OPT 7)
$$\begin{cases} \min\limits_{u,x_C,\delta,n_C} & \|u\|_2^2 = \sum_{i=1}^{m} u_i^2, \\ \text{s.t.} & \mathbb{P}\left(\begin{array}{lcl} p_{k,1}(L_{k,1}) & \geq & p_{\text{min}} \\ p_{k,2}(0) & \leq & p_{\text{max}} \\ p_{k,2}(L_{k,2}) & \geq & p_{\text{min}} \\ p_{k,2}(L_{k,2}) & \leq & p_{\text{max}} \end{array} \quad \forall k = 1,\cdots,m \right) \geq \alpha, \\ & \text{and for all } i = 1,\cdots,m, \text{ we have} \\ & u_i \geq 1, \quad x_{C,i} \in [0, L_i], \quad \delta_i \in \{0,1\}, \\ & \sum_{j=1}^{m} \delta_j = n_C. \end{cases}$$

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Theorem

Let $p_{i,0} \in \left[\, p_{\text{min}},\ p_{\text{max}} \,\right]$ be given for every node $v_i \in \mathcal{V}_{\text{in}}$. Further let a number $n_C \in \{0, \cdots, m\}$ be given.

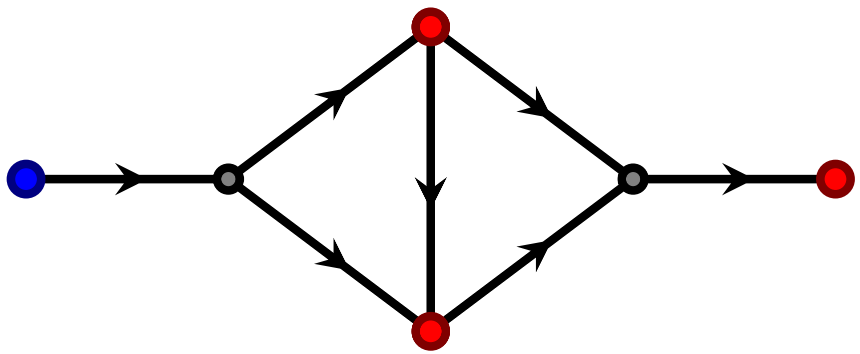
(i) If there exists a triple $(u, x_C, \delta) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$ with $u = \mathbf{1}_m$, that satisfies the constraints of (OPT 6), then every triple $(\mathbf{1}_m, x_C, \delta)$ with $x_{C,j} \in [0, L_j]$, $\delta_j \in \{0,1\}$ and $\sum_{j=1}^m \delta_j = n_C$ is a solution of (OPT 6).

(ii) If there exists a triple $(u, x_C, \delta) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$, that satisfies the constraints of (OPT 6), and if $(\mathbf{1}_m, x_C, \delta)$ is infeasible for at least one pair (x_C, δ) with $x_{C,j} \in [0, L_j]$ and $\delta \in \{0,1\}$, then (OPT 6) has at least one solution.

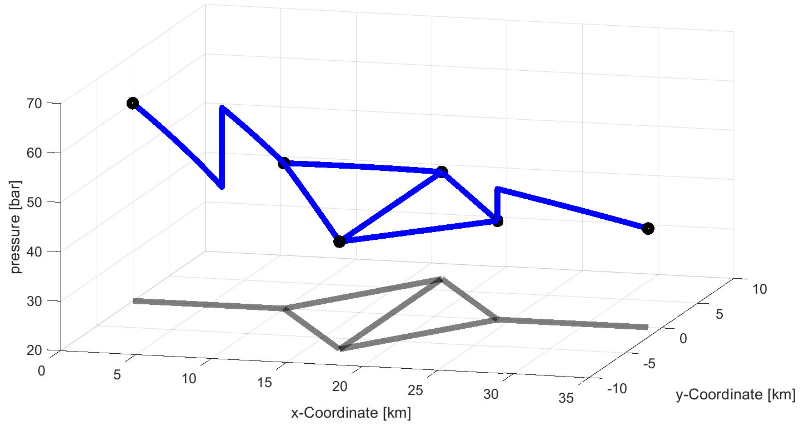
Optimal Compressor Location

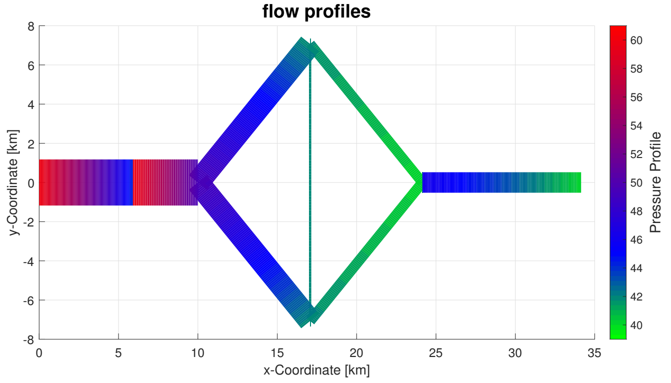
A Numerical Example on a Diamond Graph

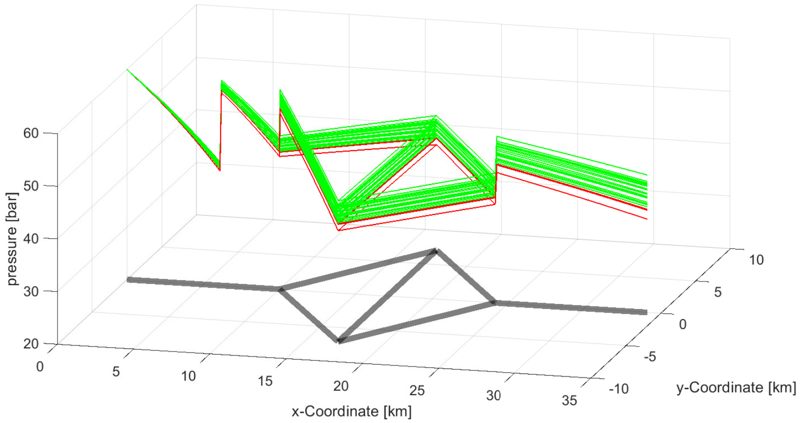
A scheme of a diamond graph with 1 source (blue) and 3 sinks (red):



Variable	Letter	Value	Unit
inlet pressure	p_0	60	bar
lower pressure bound	p_{min}	40	bar
upper pressure bound	p_{max}	60	bar
gas outflow (=mean value)	b (=μ)	$[90,\ 60,\ 120]$	kg/m^{2}s
covariance matrix	Σ	diag$(2.25,\ 2.25,\ 2.25)$	
speed of sound in the gas	c	343	m/s
pipe friction coefficient	λ^F	0.1	
pipe diameter	D	0.5	m
specific gas constant	R_S	515	J/kg K
gas temperature	T	293	K
probability level	α	0.8	







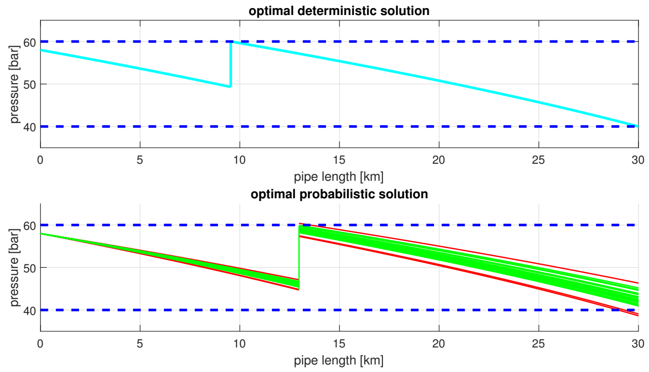
Optimal Compressor Location

A Numerical Example for Transient Gas Flow

- **Consider the probabilistic example for the stationary flow on a single pipe**
- **Solve the probabilistic constraint optimization problem for $\alpha = 90\%$**

(OPT 2)

$$\begin{cases} \min\limits_{u, x_C} \quad u^2, \\ \text{s.t.} \quad \mathbb{P}\begin{pmatrix} p_1(L_1) & \geq & p_{\text{min}} \\ p_2(0) & \leq & p_{\text{max}} \\ p_2(L_2) & \geq & p_{\text{min}} \\ p_2(L_2) & \leq & p_{\text{max}} \end{pmatrix} \geq \alpha \\ \quad u \geq 1, \\ \quad x_C \in [0, L]. \end{cases}$$



Optimal Compressor Location

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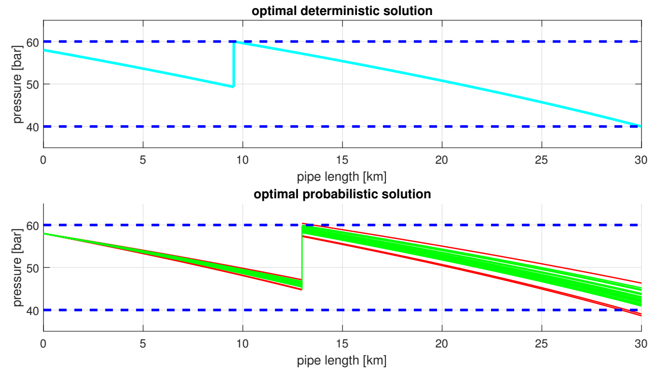
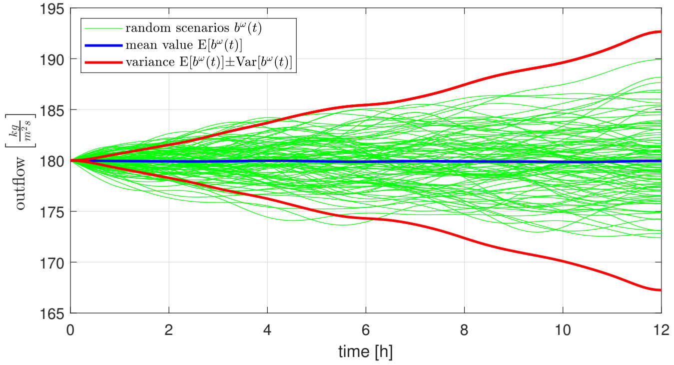
$$



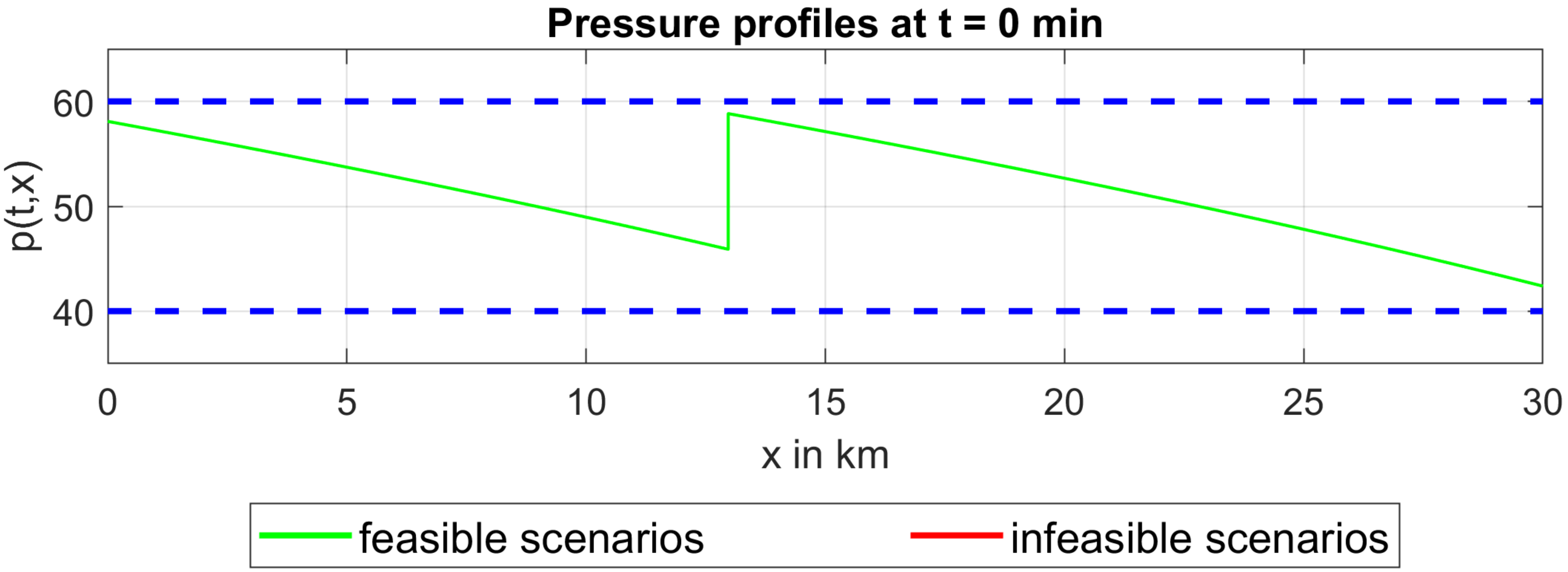
- **Randomize the boundary data in time by a *Wiener process***
- **The probabilistic robustness of the steady state control is**

$$
\mathbb{P}\begin{pmatrix} p_1(t, L_1) & \geq & p_{\text{min}} & \\ p_2(t, 0) & \leq & p_{\text{max}} & \forall t \in [0, T] \\ p_2(t, L_2) & \geq & p_{\text{min}} & \\ p_2(t, L_2) & \leq & p_{\text{max}} & \end{pmatrix} = 85.74\%
$$



Optimal Compressor Location

A Numerical Example for Transient Gas Flow



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